## Step 1:

$$\text{Mean vector} = \begin{pmatrix} \dfrac{6+9+4+10}{4} \\ \dfrac{6+10+11+5}{4} \end{pmatrix} = \begin{pmatrix} 7.25 \\ 8 \end{pmatrix}$$

For data $(6, 6)$, difference from mean vector $= \begin{pmatrix} 6-7.25 \\ 6-8 \end{pmatrix} = \begin{pmatrix} -1.25 \\ -2 \end{pmatrix}$

For data $(9, 10)$, difference from mean vector $= \begin{pmatrix} 9-7.25 \\ 10-8 \end{pmatrix} = \begin{pmatrix} 1.75 \\ 2 \end{pmatrix}$

For data $(4, 11)$, difference from mean vector $= \begin{pmatrix} 4-7.25 \\ 11-8 \end{pmatrix} = \begin{pmatrix} -3.25 \\ 3 \end{pmatrix}$

For data $(10, 5)$, difference from mean vector $= \begin{pmatrix} 10-7.25 \\ 5-8 \end{pmatrix} = \begin{pmatrix} 2.75 \\ -3 \end{pmatrix}$

## Step 2:

$$Y = \begin{pmatrix} -1.25 & 1.75 & -3.25 & 2.75 \\ -2 & 2 & 3 & -3 \end{pmatrix}$$

$$\Sigma = 0.25YY^T = 0.25 \begin{pmatrix} -1.25 & 1.75 & -3.25 & 2.75 \\ -2 & 2 & 3 & -3 \end{pmatrix} \begin{pmatrix} -1.25 & -2 \\ 1.75 & 2 \\ -3.25 & 3 \\ 2.75 & -3 \end{pmatrix} = \begin{pmatrix} 5.6875 & -3 \\ -3 & 6.5 \end{pmatrix}$$

## Step 3:

$$\begin{vmatrix} 5.6875 - \lambda & -3 \\ -3 & 6.5 - \lambda \end{vmatrix} = 0$$

$$(5.6875 - \lambda)(6.5 - \lambda) - (-3)(-3) = 0$$

Solving, we obtain the eigenvalues $\lambda = 3.0664$ and $\lambda = 9.1211$.

For $\lambda = 3.0664$,

$$\begin{pmatrix} 5.6875 - 3.0664 & -3 \\ -3 & 6.5 - 3.0664 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2.6211 & -3 \\ -3 & 3.4336 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1.1445 \\ -1.1445 & 1.31 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = 1.1445x_2$$

Choosing the eigenvector with unit length, we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.7531 \\ -0.658 \end{pmatrix}$.

For $\lambda = 9.1211$,

$$\begin{pmatrix} 5.6875 - 9.1211 & -3 \\ -3 & 6.5 - 9.1211 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3.4336 & -3 \\ -3 & -2.6211 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.8737 \\ 0.8737 & 0.7634 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = -0.8737x_2$$

Choosing the eigenvector with unit length, we have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.658 \\ -0.7531 \end{pmatrix}$.

## Step 4:

$$\Phi = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix}$$

## Step 5:

$$Y = \Phi^T X = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix} X$$

For data $(6, 6)$, $Y = \Phi^T X = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -0.5706 \\ -8.4661 \end{pmatrix}$

For data $(9, 10)$, $Y = \Phi^T X = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix} \begin{pmatrix} 9 \\ 10 \end{pmatrix} = \begin{pmatrix} -1.609 \\ -13.3571 \end{pmatrix}$

For data $(4, 11)$, $Y = \Phi^T X = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix} \begin{pmatrix} 4 \\ 11 \end{pmatrix} = \begin{pmatrix} -5.6518 \\ -10.2497 \end{pmatrix}$

For data $(10, 5)$, $Y = \Phi^T X = \begin{pmatrix} 0.658 & -0.7531 \\ -0.7531 & -0.658 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 2.8143 \\ -10.8203 \end{pmatrix}$

$$\text{Mean vector} = \begin{pmatrix} \dfrac{(-0.5706) + (-1.609) + (-5.6518) + 2.8143}{4} \\ \dfrac{(-8.4661) + (-13.3571) + (-10.2497) + (-10.8203)}{4} \end{pmatrix} = \begin{pmatrix} -1.2543 \\ -10.7233 \end{pmatrix}$$

For data $(6, 6)$, final transformed vector $= \begin{pmatrix} -0.5706 - (-1.2543) \\ -8.4661 - (-10.7233) \end{pmatrix} = \begin{pmatrix} 0.6837 \\ 2.2572 \end{pmatrix}$

For data $(9, 10)$, final transformed vector $= \begin{pmatrix} -1.609 - (-1.2543) \\ -13.3571 - (-10.7233) \end{pmatrix} = \begin{pmatrix} -0.3547 \\ -2.6338 \end{pmatrix}$

For data $(4, 11)$, final transformed vector $= \begin{pmatrix} -5.6518 - (-1.2543) \\ -10.2497 - (-10.7233) \end{pmatrix} = \begin{pmatrix} -4.3975 \\ 0.4736 \end{pmatrix}$

For data $(10, 5)$, final transformed vector $= \begin{pmatrix} 2.8143 - (-1.2543) \\ -10.8203 - (-10.7233) \end{pmatrix} = \begin{pmatrix} 4.0685 \\ -0.097 \end{pmatrix}$

## Step 6:

Thus,
$(6, 6)$ is reduced to $(0.6837)$;
$(9, 10)$ is reduced to $(-0.3547)$;
$(4, 11)$ is reduced to $(-4.3975)$;
$(10, 5)$ is reduced to $(4.0685)$;