POINCARE WAVE MODEL NOTES

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1. Equations of Motion

Consider the following equations of motion,

$$u_t - fv - g\eta_x = 0 \tag{1}$$

$$v_t + fu - g\eta_y = 0 \tag{2}$$

$$\eta_t - Hu_x - Hv_y = 0. \tag{3}$$

1.1. **Trial solution method.**

Take a trial solution of

$$\begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} e^{i(\omega t + kx + ly)} \tag{4}$$

And the equations must satisfy,

$$\begin{bmatrix} i\omega & -f & igk \\ f & i\omega & igl \\ iHk & iHl & i\omega \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} = 0 \tag{5}$$

A solution is possible when $\omega = 0$, $\omega = \sigma$ or $\omega = -\sigma$ where,

$$\sigma = \sqrt{gH(k^2 + l^2) + f_0^2}. \tag{6}$$

We thus have the general solution,

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix} = A \begin{bmatrix} -i\frac{g}{f_0} l \\ i\frac{g}{f_0} k \\ 1 \end{bmatrix} e^{i(kx+ly)} + B \begin{bmatrix} ilf_0 - \sigma k \\ -ikf_0 - \sigma l \\ K^2 H \end{bmatrix} e^{i(\sigma t + kx + ly)} + C \begin{bmatrix} -ilf_0 - \sigma k \\ ikf_0 - \sigma l \\ K^2 H \end{bmatrix} e^{i(-\sigma t + kx + ly)} \tag{7}$$

1.2. **Fourier transform method.**

Or alternatively, we take the the Fourier transformation, using the definitions,

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k,l) e^{i(kx+ly)} \, dk \, dl \tag{8}$$

$$\hat{f}(k,l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x,y) e^{-i(kx+ly)} \, dx \, dy \tag{9}$$

so the equations become,

$$\hat{u}_t - f\hat{v} - igk\hat{\eta} = 0 \tag{10}$$

$$\hat{v}_t + f\hat{u} - igl\hat{\eta} = 0 \tag{11}$$

$$\hat{\eta}_t - Hik\hat{u} - Hil\hat{v} = 0. \tag{12}$$

In matrix form this is,

$$\frac{d}{dt}\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} 0 & f & igk \\ -f & 0 & igl \\ ikH & ilH & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{bmatrix} \tag{13}$$

1.3. **Special Solution.** When $k \neq 0$, $l = 0$

$$\eta(t) = UD\frac{k}{\omega}\cos(kx + \omega t + \phi) \tag{14}$$

$$u(t) = U\cos(kx + \omega t + \phi) \tag{15}$$

$$v(t) = -U\frac{f}{\omega}\sin(kx + \omega t + \phi) \tag{16}$$

Rotated to a more general wave vector,

$$\eta(t) = U_{kl}D\frac{\sqrt{k^2 + l^2}}{\omega}\cos(kx + ly + \omega t + \phi) \tag{17}$$

$$u(t) = U_{kl}\cos(kx + ly + \omega t + \phi)\cos(\alpha) + U_{kl}\frac{f}{\omega}\sin(kx + ly + \omega t + \phi)\sin(\alpha) \tag{18}$$

$$v(t) = U_{kl}\cos(kx + ly + \omega t + \phi)\sin(\alpha) - U_{kl}\frac{f}{\omega}\sin(kx + ly + \omega t + \phi)\cos(\alpha) \tag{19}$$

where $\alpha = \tan^{-1}\frac{l}{k}$.

We want to use an FFT algorithm to quickly compute the spatial field resulting from the superposition of all these waves. In terms of wave vector components $m_{k,l}$, we need to fill in the following matrix,

$$\begin{array}{c|ccccc|ccc}
 & 0 & 1 & 2 & 3 & 4 & -3 & -2 & -1 \\
\hline
0 & m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & m^*_{0,3} & m^*_{0,2} & m^*_{0,1} \\
1 & m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m^*_{-1,3} & m^*_{-1,2} & m^*_{-1,1} \\
2 & m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m^*_{-2,3} & m^*_{-2,2} & m^*_{-2,1} \\
3 & m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m^*_{-3,3} & m^*_{-3,2} & m^*_{-3,1} \\
-4 & m_{-4,0} & m_{-4,1} & m_{-4,2} & m_{-4,3} & m_{-4,4} & m^*_{-4,3} & m^*_{-4,2} & m^*_{-4,1} \\
-3 & m^*_{3,0} & m_{-3,1} & m_{-3,2} & m_{-3,3} & m^*_{3,4} & m^*_{3,3} & m^*_{3,2} & m^*_{3,1} \\
-2 & m^*_{2,0} & m_{-2,1} & m_{-2,2} & m_{-2,3} & m^*_{2,4} & m^*_{2,3} & m^*_{2,2} & m^*_{2,1} \\
-1 & m^*_{1,0} & m_{-1,1} & m_{-1,2} & m_{-1,3} & m^*_{1,4} & m^*_{1,3} & m^*_{1,2} & m^*_{1,1}
\end{array}. \tag{20}$$

Notice, however, that many of the terms are redundant, given the require hermitian symmetry. Four of the components are their own conjugate, and therefore must be real. In order to construct a reasonably efficient algorithm, we should separate out the positive wave vectors and negative wave vectors,

$$u(t) = U_{kl}\cos(\theta)\cos(\alpha) + U_{kl}\frac{f}{\omega}\sin(\theta)\sin(\alpha) \tag{21}$$

$$= \frac{U_{kl}}{2}\left(e^{i\theta} + e^{-i\theta}\right)\cos\alpha - i\frac{U_{kl}}{2}\frac{f}{\omega}\left(e^{i\theta} - e^{-i\theta}\right)\sin\alpha \tag{22}$$

$$= \frac{U_{kl}}{2}e^{i(kx+ly)}e^{i(\omega t+\phi)}\left(\cos\alpha - i\frac{f}{\omega}\sin\alpha\right) \tag{23}$$

$$+ \frac{U_{kl}}{2}e^{-i(kx+ly)}e^{-i(\omega t+\phi)}\left(\cos\alpha + i\frac{f}{\omega}\sin\alpha\right)$$

where $\theta = kx + ly + \omega t + \phi$.

Let's redo the above table, but treat each component as the magnitude of the resulting wave.

$$\begin{array}{c|ccccc|ccc}
 & 0 & 1 & 2 & 3 & 4 & -3 & -2 & -1 \\
\hline
0 & m_{0,0} & \frac{m_{0,1}}{2} & \frac{m_{0,2}}{2} & \frac{m_{0,3}}{2} & m_{0,4} & m^*_{0,3} & m^*_{0,2} & m^*_{0,1} \\
1 & m_{1,0} & \frac{m_{1,1}}{2} & \frac{m_{1,2}}{2} & \frac{m_{1,3}}{2} & m_{1,4} & m^*_{-1,3} & m^*_{-1,2} & m^*_{-1,1} \\
2 & m_{2,0} & \frac{m_{2,1}}{2} & \frac{m_{2,2}}{2} & \frac{m_{2,3}}{2} & m_{2,4} & m^*_{-2,3} & m^*_{-2,2} & m^*_{-2,1} \\
3 & m_{3,0} & \frac{m_{3,1}}{2} & \frac{m_{3,2}}{2} & \frac{m_{3,3}}{2} & m_{3,4} & m^*_{-3,3} & m^*_{-3,2} & m^*_{-3,1} \\
-4 & m_{-4,0} & \frac{m_{-4,1}}{2} & \frac{m_{-4,2}}{2} & \frac{m_{-4,3}}{2} & m_{-4,4} & m^*_{-4,3} & m^*_{-4,2} & m^*_{-4,1} \\
-3 & m^*_{3,0} & \frac{m_{-3,1}}{2} & \frac{m_{-3,2}}{2} & \frac{m_{-3,3}}{2} & m^*_{3,4} & m^*_{3,3} & m^*_{3,2} & m^*_{3,1} \\
-2 & m^*_{2,0} & \frac{m_{-2,1}}{2} & \frac{m_{-2,2}}{2} & \frac{m_{-2,3}}{2} & m^*_{2,4} & m^*_{2,3} & m^*_{2,2} & m^*_{2,1} \\
-1 & m^*_{1,0} & \frac{m_{-1,1}}{2} & \frac{m_{-1,2}}{2} & \frac{m_{-1,3}}{2} & m^*_{1,4} & m^*_{1,3} & m^*_{1,2} & m^*_{1,1}
\end{array}. \tag{24}$$

This means that if you want a wave of $\cos(2x+2y)$, insert 0.5 into $(k,l) = (2,2)$. However, if you want a wave of $\cos(2x+0y)$, you can insert either 1.0 into $(k,l) = (2,0)$, or 0.5 into both $(k,l) = (2,0)$ and $(k,l) = (-2,0)$, for example. For the middle column, the other half of the energy is implicitly assumed to be in the hermitian part that gets neglected. For the components that explicitly have their hermitian conjugates, you have to manually separate the energy.

We only need to set U_{kl} for half of the wave numbers, since the other half is found by hermitian symmetry. Thus, we will only define the positive k wave numbers. The coefficient $\hat{u}(k^+, l^\pm)$ has magnitude,

$$\hat{u}(k^+, l) = \frac{U_{kl}}{2}\left(\cos\alpha - i\frac{f}{\omega}\sin\alpha\right)e^{i(\omega t+\phi)}. \tag{25}$$

The first value is simply set outright, but then it is evolved with $e^{i\omega t}$. This is similar to the negative wave numbers,

$$\hat{u}(k^-, l) = \frac{U_{kl}}{2}\left(\cos\alpha + i\frac{f}{\omega}\sin\alpha\right)e^{-i(\omega t+\phi)}. \tag{26}$$

but these evolve with the negative component. A couple of special cases arise,

$$\hat{u}(0,0) = U\cos(\omega t + \phi) \tag{27}$$
$$\hat{u}(0,l^{max}) = 0 \tag{28}$$
$$\hat{u}(k^{max},0) = 0 \tag{29}$$
$$\hat{u}(k^{max},l^{max}) = 0 \tag{30}$$

In practice then, we take the matrix ω_{kl} and compute $a_{kl}\cos(\omega_{kl}t+\phi)+b_{kl}\sin(\omega_{kl}t+\phi)$ where a_{kl} is ones for everything except the zeroth frequency and the Nyquist frequency components. The matrix b_{kl} is i and $-i$ depending on which column its in.

$$v(t) = U_{kl}\cos(\theta)\sin(\alpha) - U_{kl}\frac{f}{\omega}\sin(\theta)\cos(\alpha) \tag{31}$$
$$= \frac{U_{kl}}{2}\left(e^{i\theta}+e^{-i\theta}\right)\sin\alpha + i\frac{U_{kl}}{2}\frac{f}{\omega}\left(e^{i\theta}-e^{-i\theta}\right)\cos\alpha \tag{32}$$
$$= \frac{U_{kl}}{2}e^{i(kx+ly)}e^{i(\omega t+\phi)}\left(\sin\alpha + i\frac{f}{\omega}\cos\alpha\right) + \frac{U_{kl}}{2}e^{-i(kx+ly)}e^{-i(\omega t+\phi)}\left(\sin\alpha - i\frac{f}{\omega}\cos\alpha\right) \tag{33}$$

$$\hat{v}(0,0) = -U\sin(\omega t + \phi) \tag{34}$$
$$\hat{u}(0,l^{max}) = 0 \tag{35}$$
$$\hat{u}(k^{max},0) = 0 \tag{36}$$
$$\hat{u}(k^{max},l^{max}) = 0 \tag{37}$$

2. Solutions

Take a trial solution of

$$\begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} e^{-i(\omega t - kx - ly)} \tag{38}$$

And the equations must satisfy,

$$\begin{bmatrix} -i\omega + \frac{\delta+\sigma_n}{2} & \frac{\sigma_s-\zeta}{2} - f & igk \\ \frac{\sigma_s+\zeta}{2} + f & -i\omega + \frac{\delta-\sigma_n}{2} & igl \\ iH_1 k & iH_1 l & -i\omega \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} = 0 \tag{39}$$

2.1. **Special Solution.** If we look for purely inertial oscillations with no height perturbation, then we find that,

$$\omega = -i\frac{\delta}{2} \pm \sqrt{\left(f + \frac{\zeta}{2}\right)^2 - \frac{\sigma^2}{4}} \tag{40}$$

where $\sigma^2 = \sigma_s^2 + \sigma_n^2$.

3. Spatially Variable Background

$$u_t - f(x,y)v - g\eta_x = 0 \tag{41}$$
$$v_t + f(x,y)u - g\eta_y = 0 \tag{42}$$
$$\eta_t - H_1 u_x - H_1 v_y = 0 \tag{43}$$

This system is quasilinear. It definitely should be solvable, no?

The time rate change of the divergence is,

$$\frac{\partial}{\partial t}\nabla\cdot\mathbf{u} - f\zeta - g\nabla^2\eta - f_x v + f_y u = 0. \tag{44}$$

The time rate change of the curl is,

$$\frac{\partial}{\partial t}\zeta + f\nabla\cdot\mathbf{u} + u f_x + v f_y = 0. \tag{45}$$

Take a time derivative of the divergence equation to find that,

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u} - f\zeta_t - g\nabla^2\eta_t - f_x v_t + f_y u_t = 0. \tag{46}$$

Now plop in the variation of ζ equation,

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u} + f^2\nabla\cdot\mathbf{u} + f u f_x + f v f_y - g\nabla^2\eta_t - f_x v_t + f_y u_t = 0. \tag{47}$$

Reorganize,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(fu - v_t) + f_y(fv + u_t) - g\nabla^2\eta_t = 0. \tag{48}$$

Substitute,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(2fu - g\eta_y) + f_y(2fv + g\eta_x) - g\nabla^2\eta_t = 0 \tag{49}$$

or,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(g\eta_y - 2v_t) + f_y(2u_t - g\eta_x) - g\nabla^2\eta_t = 0. \tag{50}$$

Hmmm, not sure which I like best. Either way,

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)\eta_t+Hf_x(g\eta_y-2v_t)+Hf_y(2u_t-g\eta_x)-gH\nabla^2\eta_t=0. \tag{51}$$

Let $f=f_0+f_1x$, then

$$\left(\frac{\partial^2}{\partial t^2}+f_0^2+2f_0f_1x+f_1^2x^2\right)\eta_t+Hf_1(g\eta_y-2v_t)-gH\nabla^2\eta_t=0. \tag{52}$$