POINCARE WAVE MODEL NOTES

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1. Equations of Motion

Consider the following equations of motion,

$$u_t - fv - g\eta_x = 0 \tag{1}$$

$$v_t + fu - g\eta_y = 0 \tag{2}$$

$$\eta_t - Hu_x - Hv_y = 0. \tag{3}$$

1.1. **Trial solution method.**

Take a trial solution of

$$\begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} e^{i(\omega t + kx + ly)} \tag{4}$$

And the equations must satisfy,

$$\begin{bmatrix} i\omega & -f & igk \\ f & i\omega & igl \\ iHk & iHl & i\omega \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} = 0 \tag{5}$$

A solution is possible when $\omega = 0$, $\omega = \sigma$ or $\omega = -\sigma$ where,

$$\sigma = \sqrt{gH(k^2 + l^2) + f_0^2}. \tag{6}$$

We thus have the general solution,

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix} = A \begin{bmatrix} -i\frac{g}{f_0} l \\ i\frac{g}{f_0} k \\ 1 \end{bmatrix} e^{i(kx+ly)} + B \begin{bmatrix} ilf_0 - \sigma k \\ -ikf_0 - \sigma l \\ K^2 H \end{bmatrix} e^{i(\sigma t + kx + ly)} + C \begin{bmatrix} -ilf_0 - \sigma k \\ ikf_0 - \sigma l \\ K^2 H \end{bmatrix} e^{i(-\sigma t + kx + ly)} \tag{7}$$

1.2. **Fourier transform method.**

Or alternatively, we take the the Fourier transformation, using the definitions,

$$f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k,l) e^{i(kx+ly)}\, dk\, dl \tag{8}$$

$$\hat{f}(k,l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x,y) e^{-i(kx+ly)}\, dx\, dy \tag{9}$$

so the equations become,

$$\hat{u}_t - f\hat{v} - igk\hat{\eta} = 0 \tag{10}$$
$$\hat{v}_t + f\hat{u} - igl\hat{\eta} = 0 \tag{11}$$
$$\hat{\eta}_t - Hik\hat{u} - Hil\hat{v} = 0. \tag{12}$$

In matrix form this is,

$$\frac{d}{dt}\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} 0 & f & igk \\ -f & 0 & igl \\ ikH & ilH & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{bmatrix} \tag{13}$$

1.3. **Special Solution.** When $k \neq 0$, $l = 0$

$$\eta(t) = UD\frac{k}{\omega}\cos(kx + \omega t + \phi) \tag{14}$$
$$u(t) = U\cos(kx + \omega t + \phi) \tag{15}$$
$$v(t) = -U\frac{f}{\omega}\sin(kx + \omega t + \phi) \tag{16}$$

Rotated to a more general wave vector,

$$\eta(t) = U_{kl}D\frac{\sqrt{k^2 + l^2}}{\omega}\cos(kx + ly + \omega t + \phi) \tag{17}$$
$$u(t) = U_{kl}\cos(kx + ly + \omega t + \phi)\cos(\alpha) + U_{kl}\frac{f}{\omega}\sin(kx + ly + \omega t + \phi)\sin(\alpha) \tag{18}$$
$$v(t) = U_{kl}\cos(kx + ly + \omega t + \phi)\sin(\alpha) - U_{kl}\frac{f}{\omega}\sin(kx + ly + \omega t + \phi)\cos(\alpha) \tag{19}$$

where $\alpha = \tan^{-1}\frac{l}{k}$.

We want to use an FFT algorithm to quickly compute the spatial field resulting from the superposition of all these waves. In terms of wave vector components $m_{k,l}$, we need to fill in the following matrix,

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & -4 & -3 & -2 & -1 \\
\hline
0 & m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,-4} & m^*_{0,3} & m^*_{0,2} & m^*_{0,1} \\
1 & m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,-4} & m_{1,-3} & m_{1,-2} & m_{1,-1} \\
2 & m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,-4} & m_{2,-4} & m_{2,-2} & m_{2,-1} \\
3 & m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,-4} & m_{3,-4} & m_{3,-2} & m_{3,-1} \\
-4 & m_{-4,0} & m_{-4,1} & m_{-4,2} & m_{-4,3} & m_{-4,-4} & m^*_{-4,3} & m^*_{-4,2} & m^*_{-4,1} \\
-3 & m^*_{3,0} & m^*_{3,-1} & m^*_{3,-2} & m^*_{3,-3} & m^*_{3,-4} & m^*_{3,3} & m^*_{3,2} & m^*_{3,1} \\
-2 & m^*_{2,0} & m^*_{2,-1} & m^*_{2,-2} & m_{2,-3}* & m^*_{2,-4} & m^*_{2,3} & m^*_{2,2} & m^*_{2,1} \\
-1 & m^*_{1,0} & m^*_{1,-1} & m^*_{1,-2} & m^*_{1,-3} & m^*_{1,-4} & m^*_{1,3} & m^*_{1,2} & m^*_{1,1}
\end{array}. \tag{20}$$

Notice, however, that many of the terms are redundant, given the require hermitian symmetry. Four of the components are their own conjugate, and therefore must be real. In order to construct a reasonably efficient algorithm, we should separate out the positive wave vectors and negative wave vectors,

$$u(t) = U_{kl}\cos(\theta)\cos(\alpha) + U_{kl}\frac{f}{\omega}\sin(\theta)\sin(\alpha) \tag{21}$$

$$= \frac{U_{kl}}{2}\left(e^{i\theta} + e^{-i\theta}\right)\cos\alpha - i\frac{U_{kl}}{2}\frac{f}{\omega}\left(e^{i\theta} - e^{-i\theta}\right)\sin\alpha \tag{22}$$

$$= \frac{U_{kl}}{2}e^{i(kx+ly)}e^{i(\omega t+\phi)}\left(\cos\alpha - i\frac{f}{\omega}\sin\alpha\right) \tag{23}$$

$$+ \frac{U_{kl}}{2}e^{-i(kx+ly)}e^{-i(\omega t+\phi)}\left(\cos\alpha + i\frac{f}{\omega}\sin\alpha\right)$$

where $\theta = kx + ly + \omega t + \phi$.

Let's redo the above table for the proper ordering,

$$\begin{array}{c|ccccc|ccc}
 & 0 & 1 & 2 & 3 & 4 & -3 & -2 & -1 \\
\hline
0 & m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & m^*_{0,3} & m^*_{0,2} & m^*_{0,1} \\
1 & m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m^*_{1,3} & m^*_{1,2} & m^*_{1,1} \\
2 & m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m^*_{2,3} & m^*_{2,2} & m^*_{2,1} \\
3 & m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m^*_{3,4} & m^*_{3,2} & m^*_{3,1} \\
-4 & m_{-4,0} & m_{-4,1} & m_{-4,2} & m_{-4,3} & m_{-4,4} & m^*_{-4,3} & m^*_{-4,2} & m^*_{-4,1} \\
-3 & m^*_{3,0} & m_{-3,1} & m_{-3,2} & m_{-3,3} & m^*_{-3,4} & m^*_{-3,3} & m^*_{-3,2} & m^*_{-3,1} \\
-2 & m^*_{2,0} & m_{-2,1} & m_{-2,2} & m_{-2,3} & m^*_{-2,4} & m^*_{-2,3} & m^*_{-2,2} & m^*_{-2,1} \\
-1 & m^*_{1,0} & m_{-1,1} & m_{-1,2} & m_{-1,3} & m^*_{-1,4} & m^*_{-1,3} & m^*_{-1,2} & m^*_{-1,1}
\end{array}. \tag{24}$$

We only need to set U_{kl} for half of the wave numbers, since the other half is found by hermitian symmetry. Thus, we will only define the positive k wave numbers. The coefficient $\hat{u}(k^+, l^\pm)$ has magnitude,

$$\hat{u}(k^+, l) = \frac{U_{kl}}{2}\left(\cos\alpha - i\frac{f}{\omega}\sin\alpha\right)e^{i(\omega t+\phi)}. \tag{25}$$

The first value is simply set outright, but then it is evolved with $e^{i\omega t}$. This is similar to the negative wave numbers,

$$\hat{u}(k^-, l) = \frac{U_{kl}}{2}\left(\cos\alpha + i\frac{f}{\omega}\sin\alpha\right)e^{-i(\omega t+\phi)}. \tag{26}$$

but these evolve with the negative component. A couple of special cases arise,

$$\hat{u}(0,0) = U\cos(\omega t + \phi) \tag{27}$$

$$\hat{u}(0, l^{max}) = 0 \tag{28}$$

$$\hat{u}(k^{max}, 0) = 0 \tag{29}$$

$$\hat{u}(k^{max}, l^{max}) = 0 \tag{30}$$

In practice then, we take the matrix ω_{kl} and compute $a_{kl}\cos(\omega_{kl}t+\phi)+b_{kl}\sin(\omega_{kl}t+\phi)$ where a_{kl} is ones for everything except the zeroth frequency and the Nyquist frequency components. The matrix b_{kl} is i and $-i$ depending on which column its in.

$$
\begin{aligned}
v(t) =& U_{kl}\cos(\theta)\sin(\alpha) - U_{kl}\frac{f}{\omega}\sin(\theta)\cos(\alpha) && (31)\\
=& \frac{U_{kl}}{2}\left(e^{i\theta}+e^{-i\theta}\right)\sin\alpha + i\frac{U_{kl}}{2}\frac{f}{\omega}\left(e^{i\theta}-e^{-i\theta}\right)\cos\alpha && (32)\\
=& \frac{U_{kl}}{2}e^{i(kx+ly)}e^{i(\omega t+\phi)}\left(\sin\alpha + i\frac{f}{\omega}\cos\alpha\right) && (33)\\
&+ \frac{U_{kl}}{2}e^{-i(kx+ly)}e^{-i(\omega t+\phi)}\left(\sin\alpha - i\frac{f}{\omega}\cos\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{v}(0,0) =& -U\sin(\omega t+\phi) && (34)\\
\hat{u}(0,l^{max}) =& 0 && (35)\\
\hat{u}(k^{max},0) =& 0 && (36)\\
\hat{u}(k^{max},l^{max}) =& 0 && (37)
\end{aligned}
$$

2. Solutions

Take a trial solution of

$$
\begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} e^{-i(\omega t - kx - ly)} \tag{38}
$$

And the equations must satisfy,

$$
\begin{bmatrix} -i\omega + \frac{\delta+\sigma_n}{2} & \frac{\sigma_s-\zeta}{2} - f & igk \\ \frac{\sigma_s+\zeta}{2} + f & -i\omega + \frac{\delta-\sigma_n}{2} & igl \\ iH_1 k & iH_1 l & -i\omega \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \eta_0 \end{bmatrix} = 0 \tag{39}
$$

2.1. Special Solution.

Special Solution. If we look for purely inertial oscillations with no height perturbation, then we find that,

$$
\omega = -i\frac{\delta}{2} \pm \sqrt{\left(f+\frac{\zeta}{2}\right)^2 - \frac{\sigma^2}{4}} \tag{40}
$$

where $\sigma^2 = \sigma_s^2 + \sigma_n^2$.

3. Spatially Variable Background

$$u_t - f(x,y)v - g\eta_x = 0 \tag{41}$$

$$v_t + f(x,y)u - g\eta_y = 0 \tag{42}$$

$$\eta_t - H_1 u_x - H_1 v_y = 0 \tag{43}$$

This system is quasilinear. It definitely should be solvable, no?

The time rate change of the divergence is,

$$\frac{\partial}{\partial t}\nabla\cdot\mathbf{u} - f\zeta - g\nabla^2\eta - f_x v + f_y u = 0. \tag{44}$$

The time rate change of the curl is,

$$\frac{\partial}{\partial t}\zeta + f\nabla\cdot\mathbf{u} + u f_x + v f_y = 0. \tag{45}$$

Take a time derivative of the divergence equation to find that,

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u} - f\zeta_t - g\nabla^2\eta_t - f_x v_t + f_y u_t = 0. \tag{46}$$

Now plop in the variation of ζ equation,

$$\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{u} + f^2\nabla\cdot\mathbf{u} + f u f_x + f v f_y - g\nabla^2\eta_t - f_x v_t + f_y u_t = 0. \tag{47}$$

Reorganize,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(fu - v_t) + f_y(fv + u_t) - g\nabla^2\eta_t = 0. \tag{48}$$

Substitute,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(2fu - g\eta_y) + f_y(2fv + g\eta_x) - g\nabla^2\eta_t = 0 \tag{49}$$

or,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\mathbf{u} + f_x(g\eta_y - 2v_t) + f_y(2u_t - g\eta_x) - g\nabla^2\eta_t = 0. \tag{50}$$

Hmmm, not sure which I like best. Either way,

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta_t + Hf_x(g\eta_y - 2v_t) + Hf_y(2u_t - g\eta_x) - gH\nabla^2\eta_t = 0. \tag{51}$$

Let $f = f_0 + f_1 x$, then

$$\left(\frac{\partial^2}{\partial t^2} + f_0^2 + 2f_0 f_1 x + f_1^2 x^2\right)\eta_t + Hf_1(g\eta_y - 2v_t) - gH\nabla^2\eta_t = 0. \tag{52}$$