Modified shallow water equations for large bathymetry variations

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Acknowledgements

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Classical nonlinear shallow water equations

A. de Saint-Venant, CRAS (1871) [dSV71]

- Non-conservative form (in 2D for simplicity):

$$\begin{aligned} \eta_t + \big((\eta + d)u\big)_x &= -d_t, \\ u_t + uu_x + g\eta_x &= 0. \end{aligned}$$

- Conservative form

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \big(hu^2 + \tfrac{1}{2}gh^2\big)_x &= ghd_x. \end{aligned}$$

$\eta(x,t)$: free surface elevation

$d(x,t)$: bottom bathymetry

$h(x,t) := \eta + d$: total water depth

Dispersive (non-hydrostatic) effects

- Boussinesq regime: $\varepsilon = o(1)$, $\mu^2 = o(1)$, $\mathrm{S} := \frac{\varepsilon}{\mu^2} \sim 1$
- Literature is countless: Peregrine [Per67], Bona-Smith [BS76], Nwogu [Nwo93], Bona & Chen [BC98]
- We do not deal with these effects here!

Hydrostatic models: Saint-Venant / Savage-Hutter

- Valid for small slopes only!
- How to relax the restrictions on topography?

Classical asymptotic expansion method

Courtesy of R. Dressler (1978) [Dre78]

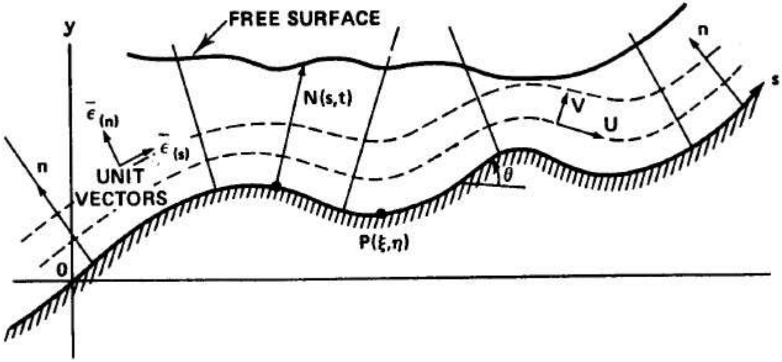

Figure: Local curvilinear coordinates defined by the bottom topography.

Classical asymptotic expansion method

Courtesy of R. Dressler (1978) [Dre78]

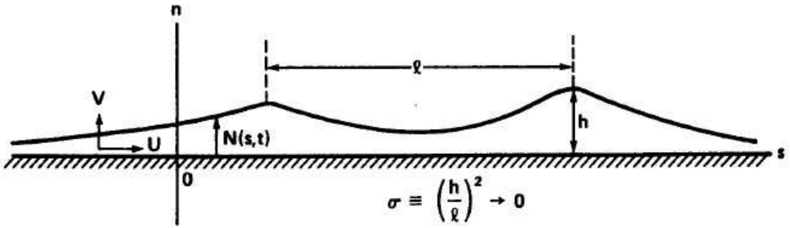

Figure: Fluid domain in new coordinates.

- The bottom is flattened in new coordinates
- We make the shallow water approximation
- Average the flow in the local vertical direction

Shallow water equations on arbitrary slopes

Model derived by Bouchut et al. (2003); Keller (2003)

Existing literature:

R.F. Dressler: JHR (1978), [Dre78]

F. Bouchut *et al*: CRAS (2003), [BMCPV03]

J.B. Keller: JFM (2003), [Kel03]

Model by Bouchut & Keller (2003):

$$\left(h - \tfrac{1}{2}\theta_x h^2\right)_t + \left(\frac{\log(1-\theta_x h)}{-\theta_x} u\right)_x = 0$$

$$u_t + \left(\frac{1}{(1-\theta_x h)^2}\frac{u^2}{2} + gh\cos\theta + gd\right)_x = 0$$

- $h(x,t) := d(x) + \eta(x,t)$, $\tan\theta(x) := d_x(x)$
- Dressler: $\kappa(x)$; Bouchut & Keller: $\theta_x(x)$

Shallow water equations on arbitrary slopes

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Relaxed variational principle

D. Clamond & D. Dutykh (2012), [CD12]

Relaxed variational principle:

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla}\eta - \tilde{\nu})\tilde{\phi} + (d_t + \check{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla} d + \check{\nu})\check{\phi} - \frac{1}{2} g\eta^2 + \int_{-d}^{\eta} \left[\boldsymbol{\mu} \cdot \boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^2 + \nu v - \frac{1}{2} v^2 + (\boldsymbol{\nabla} \cdot \boldsymbol{\mu} + \nu_y)\phi \right] \mathrm{d}y$$

Classical formulation (for comparison):

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi} d_t - \frac{1}{2} g\eta^2 - \int_{-d}^{\eta} \left[\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}\phi_y^2 \right] \mathrm{d}y$$

Degrees of freedom: $\eta, \phi; \boldsymbol{u}, v; \boldsymbol{\mu}, \nu$

Choice of the ansatz:

$$\phi \approx \bar{\phi}(\boldsymbol{x}, t), \quad \boldsymbol{u} = \boldsymbol{\mu} \approx \bar{\boldsymbol{u}}(\boldsymbol{x}, t), \quad v = \nu \approx \check{v}(\boldsymbol{x}, t) = -d_t - \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} d$$

Lagrangian:

$$\mathscr{L} = \left(h_t + \bar{\boldsymbol{u}} \cdot (h\bar{\boldsymbol{u}})\right) \bar{\phi} - \tfrac{1}{2}\, g\, \eta^2 + \tfrac{1}{2}\, h \left(\bar{\boldsymbol{u}}^{\,2} + \check{v}^{\,2}\right)$$

Euler-Lagrange equations:

$$h_t + \boldsymbol{\nabla} \cdot [\, h\, \bar{\boldsymbol{u}}\,] = 0,$$

$$[\, \bar{\boldsymbol{u}} - \check{v}\, \boldsymbol{\nabla} d\,]_t + \boldsymbol{\nabla} \left[\, g\, \eta + \tfrac{1}{2}\, \bar{\boldsymbol{u}}^{\,2} + \tfrac{1}{2}\, \check{v}^{\,2} + \check{v}\, d_t\, \right] = 0.$$

Variational structure

Lagrangian and Hamiltonian structures [DC11]

Lagrangian density (by derivation):

$$\mathscr{L} = \bigl(h_t + \bar{\boldsymbol{u}} \cdot (h\bar{\boldsymbol{u}})\bigr)\, \bar{\phi} \;-\; \tfrac{1}{2}\, g\, \eta^2 \;+\; \tfrac{1}{2}\, h\, (\bar{\boldsymbol{u}}^{\,2} + \check{v}^{\,2})$$

Hamiltonian form:

$$\frac{\partial h}{\partial t} = \frac{\delta \mathscr{H}}{\delta \bar{\phi}}, \qquad \frac{\partial \bar{\phi}}{\partial t} = -\frac{\delta \mathscr{H}}{\delta h}$$

$$\mathscr{H} = \tfrac{1}{2} \int \left\{ g(h-d)^2 + h|\boldsymbol{\nabla}\bar{\phi}|^2 - \frac{h[d_t + \boldsymbol{\nabla}\bar{\phi} \cdot \boldsymbol{\nabla} d]^2}{1 + |\boldsymbol{\nabla} d|^2} \right\} \mathrm{d}^2\boldsymbol{x}$$

Or equivalently:

$$\mathscr{H} = \tfrac{1}{2} \int \left\{ g\eta^2 + h\bar{\boldsymbol{u}}^{\,2} + h(\check{v} + d_t)^2 - h{d_t}^{\,2} \right\} \mathrm{d}^2\boldsymbol{x},$$

Conservation laws

- Mass conservation:

$$h_t + \boldsymbol{\nabla} \cdot [h \bar{\boldsymbol{u}}] = 0$$

- Momentum conservation:

$$[h \bar{\boldsymbol{u}}]_t + \boldsymbol{\nabla} [h \bar{\boldsymbol{u}}^2 + \tfrac{1}{2} g h^2] = (g + \gamma) h \boldsymbol{\nabla} d + \underbrace{h\bar{\boldsymbol{u}} \wedge \boldsymbol{\nabla} \check{v} \wedge \boldsymbol{\nabla} d}_{\equiv 0 \text{ in } 2D}$$

$$\gamma(x, t) := \frac{\mathrm{d}\check{v}}{\mathrm{d}t} = \check{v}_t + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\check{v}$$

- Energy conservation:

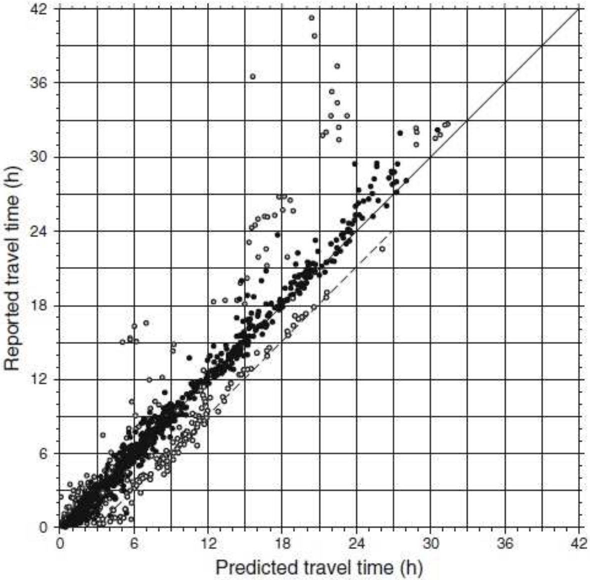
$$\left[h\frac{|\bar{\boldsymbol{u}}|^2 + \check{v}^2}{2} + g\frac{\eta^2 - d^2}{2}\right]_t + \boldsymbol{\nabla} \cdot \left[\left(\frac{|\bar{\boldsymbol{u}}|^2 + \check{v}^2}{2} + g\eta\right) h\bar{\boldsymbol{u}}\right] = -(g+\gamma)hd_t$$

- Gravity wave propagation speed in SV and mSV:

$$c_{SV} := \sqrt{gh}, \qquad c_{mSV} := \frac{\sqrt{gh}}{\sqrt{1 + |\boldsymbol{\nabla} d|^2}}$$

Real world tsunamis travel times

Source: P. Wessel, Pure Appl. Geophys. (2009) [Wes09]: 1476 records



Numerical discretization

Finite volume method – natural choice for hyperbolic systems

- Conservative form:

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \left(hu^2 + \tfrac{1}{2}gh^2\right)_x &= (g+\gamma)hd_x. \end{aligned}$$

- Semi-conservative form:

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (u - v_b d_x)_t + \left(g\eta + \tfrac{1}{2}u^2 + \tfrac{1}{2}v_b^2 + v_b d_t\right)_x &= 0. \end{aligned}$$

They are equivalent for smooth solutions!

- We apply a 2nd order finite volume scheme to the semi-conservative system

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

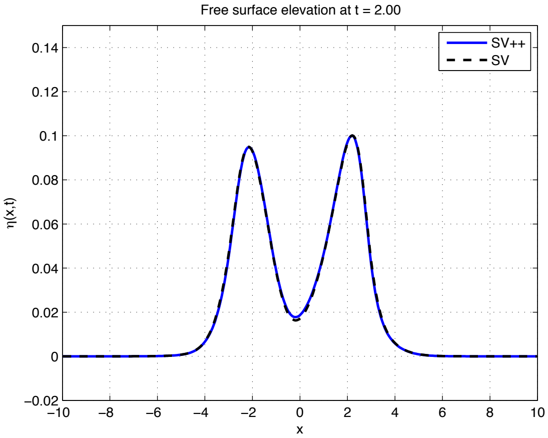


Figure: $t = 2$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

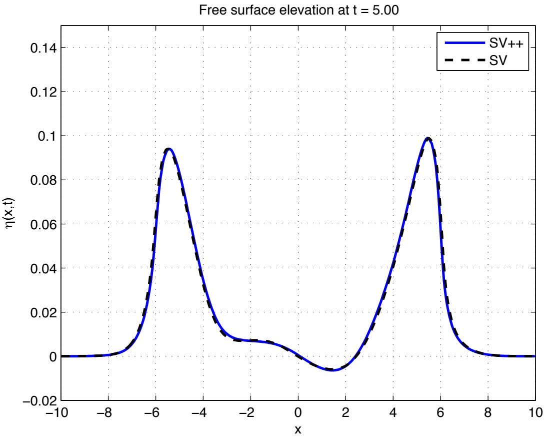


Figure: $t = 5$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

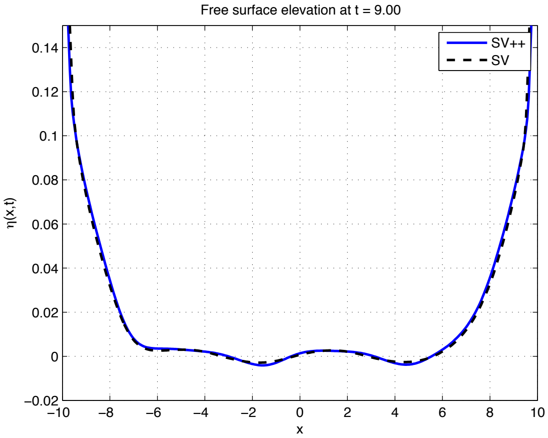


Figure: $t = 9$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

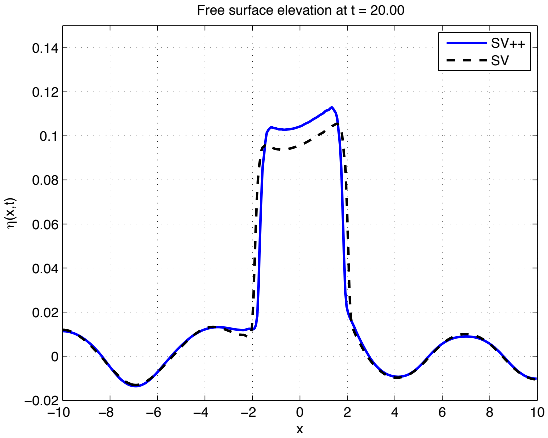


Figure: $t = 20$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

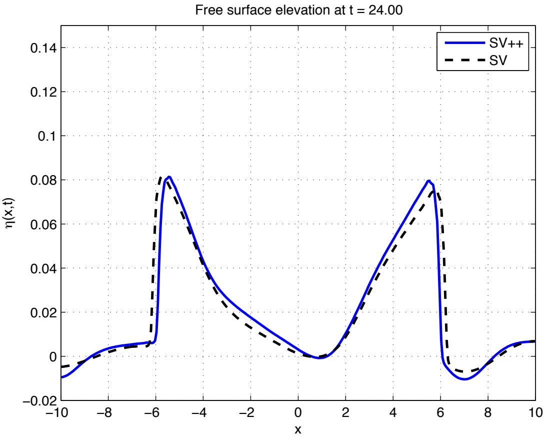


Figure: $t = 24$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

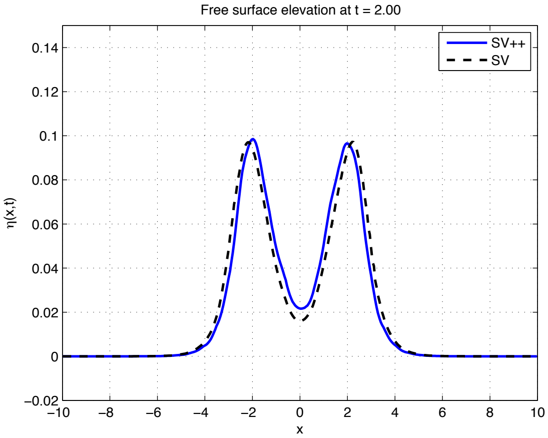


Figure: $t = 2$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

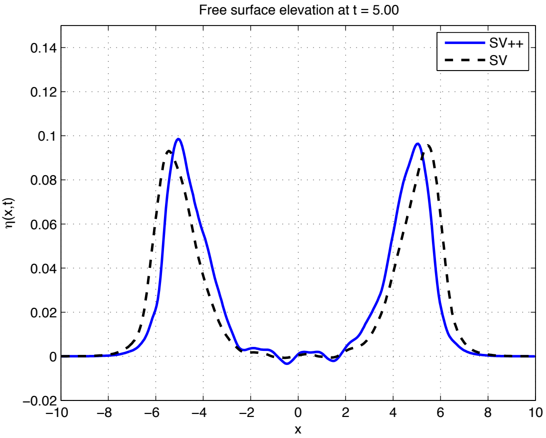
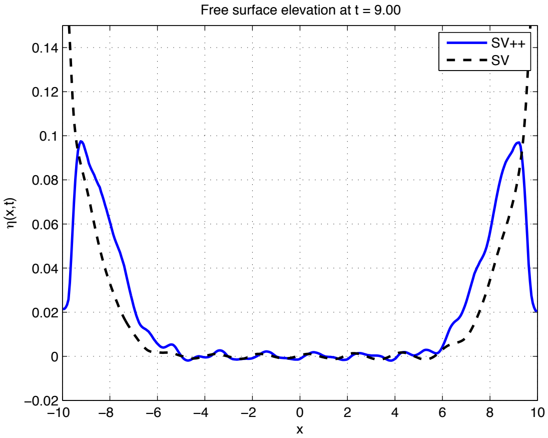


Figure: $t = 5$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations



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Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

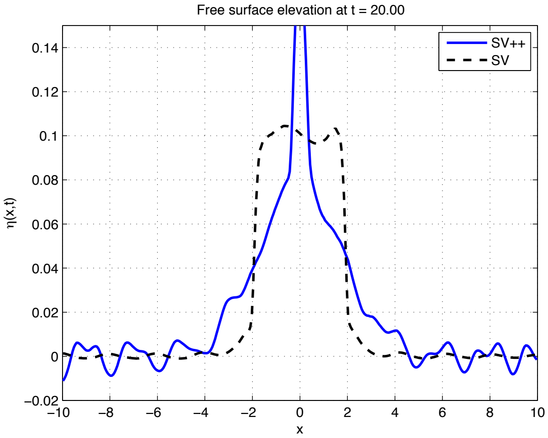

Figure: $t = 20$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

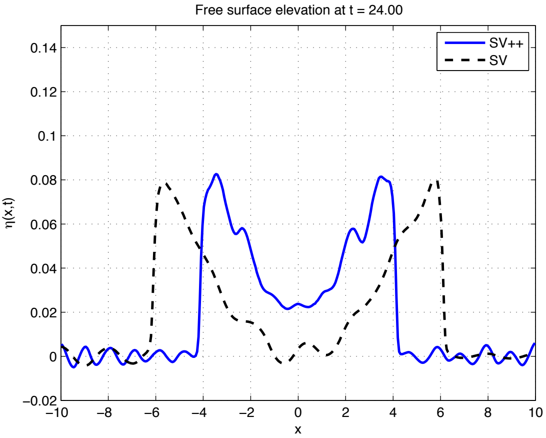

Figure: $t = 24$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

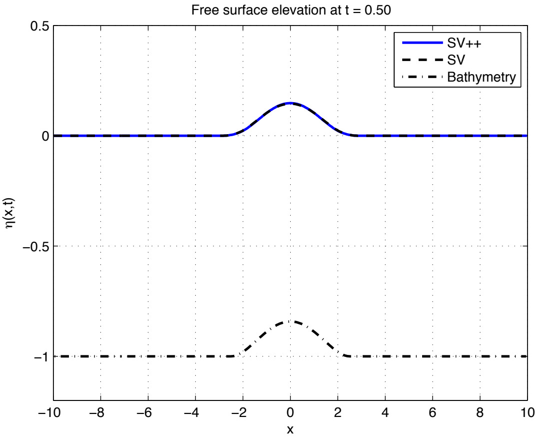

Figure: $t = 0.5$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

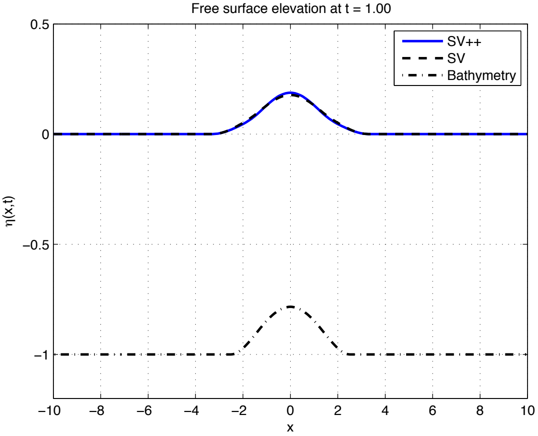

Figure: $t = 1.0$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

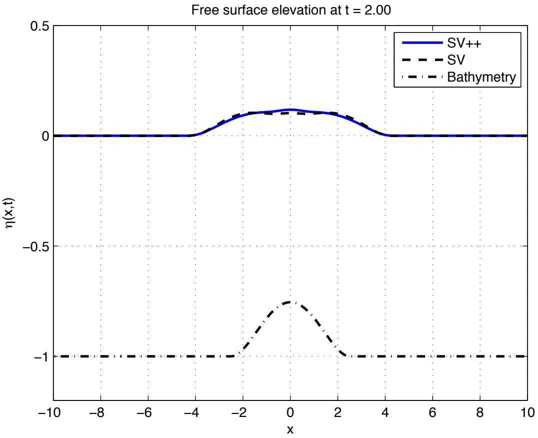

Figure: $t = 2.0$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

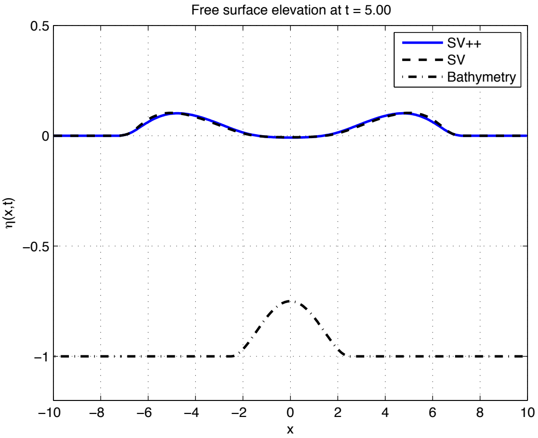

Figure: $t = 5.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

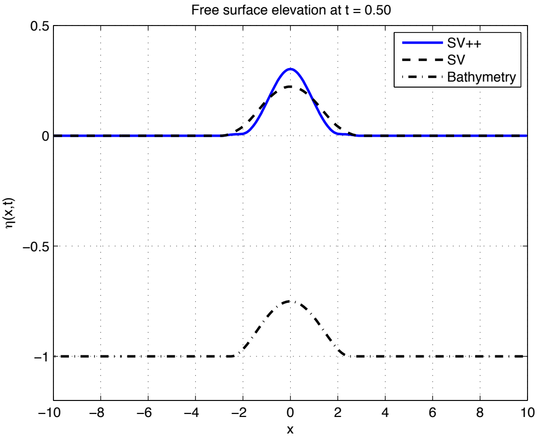

Figure: $t = 0.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

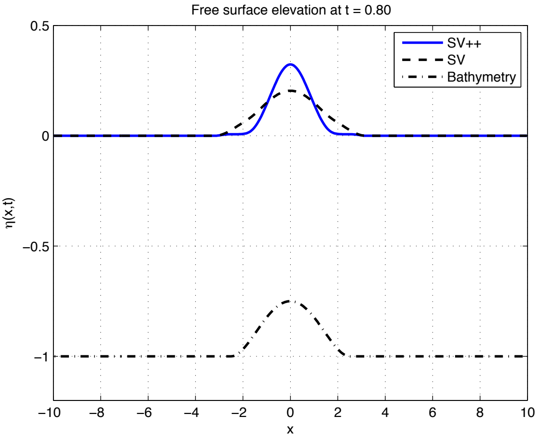

Figure: $t = 0.8$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

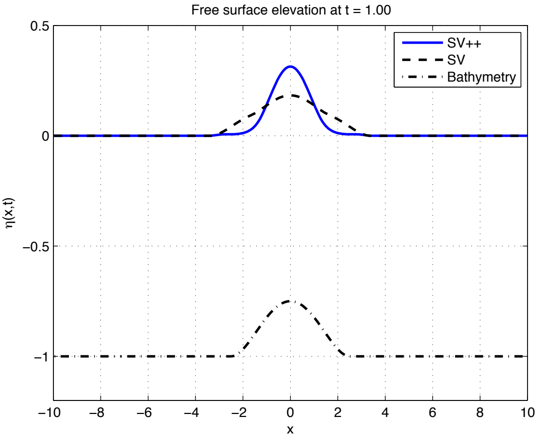

Figure: $t = 1.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

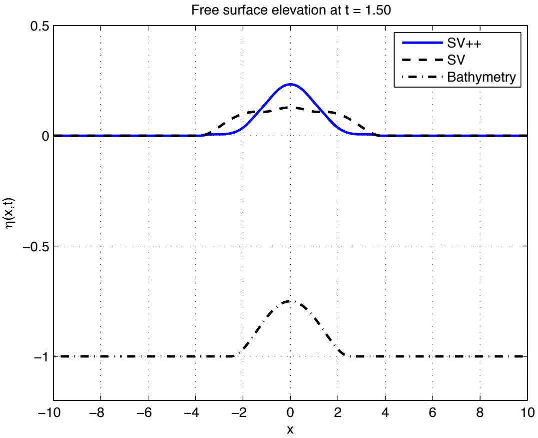

Figure: $t = 1.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

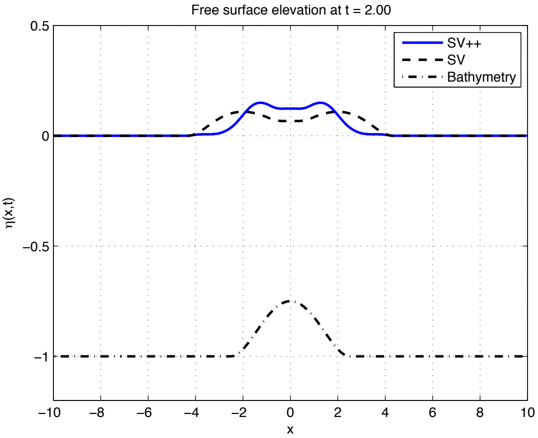

Figure: $t = 2.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

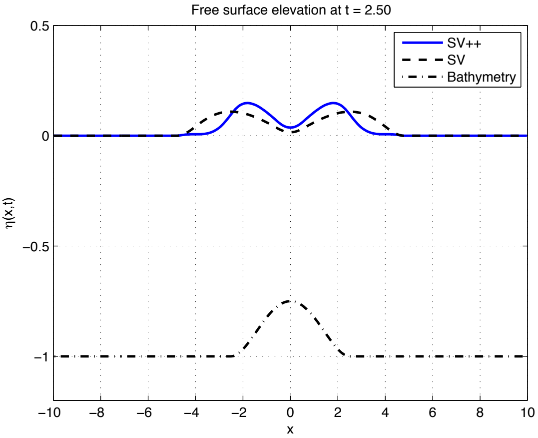

Figure: $t = 2.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

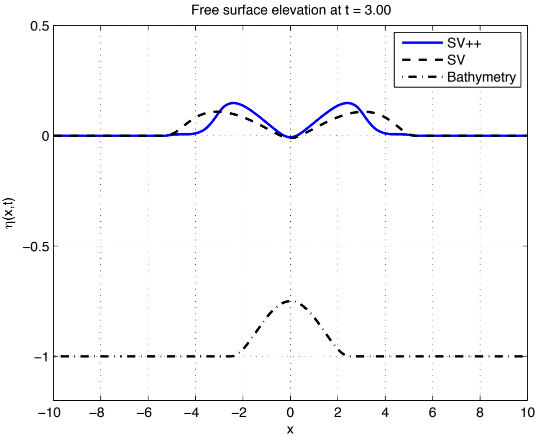

Figure: $t = 3.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

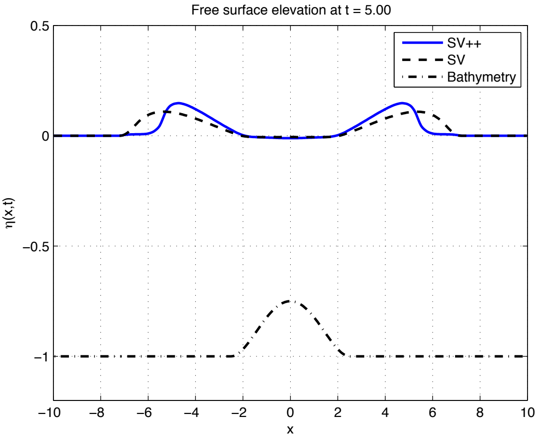

Figure: $t = 5.0$ s

Thank you for your attention!

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