Shoaling Waves Model

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1 Equations

We start with the Boussinesq equations from Nwogu, 1993. In his paper these are equations (25) and can be written as,

$$\eta_t + ((h+\eta)u)_x + \partial_x \left[\left(\frac{z^2}{2} - \frac{h^2}{6} \right) h u_{xx} + \left(z + \frac{h}{2} \right) h (hu)_{xx} \right] = 0 \tag{1}$$

$$u_t + g\eta_x + u u_x + \left[\frac{z^2}{2} u_{xxt} + z (h u_t)_{xx} \right] = 0 \tag{2}$$

where z is the depth of integration and can be set to some function of h, the bathymetry. Let's set $z = -\beta h$, so that, after some rearrangement,

$$\eta_t + ((h+\eta)u)_x + \partial_x \left[\left(\frac{\beta^2}{2} - \beta + \frac{1}{3} \right) h^3 u_{xx} + \left(-\beta + \frac{1}{2} \right) h^2 h_{xx} u \right] = 0 \tag{3}$$

$$u_t + g\eta_x + u u_x + \left[\left(\frac{\beta^2}{2} - \beta \right) h^2 u_{xxt} - \beta h h_{xx} u_t \right] = 0. \tag{4}$$

In his paper Nwogu uses the parameter α instead of β, the relationship between the two parameters is $\alpha = \frac{\beta^2}{2} - \beta$ and $\beta = 1 - \sqrt{1+2\alpha}$. Note that when u is found at the bottom of the ocean, this requires that $\beta = 1$, and $\alpha = -\frac{1}{2}$. If u is found at the surface, then $\beta = 0$ and $\alpha = 0$.

One of the key points of Nwogu's paper was that an 'optimal' value α can be chosen in which the linear frequency dispersion relationship most closely matches the true frequency dispersion relationship (for both the phase and group speed). The linear dispersion relation is,

$$\frac{\omega^2}{k^2} = gh \left[\frac{1 - \left(\alpha + \frac{1}{3}\right)(kh)^2}{1 - \alpha (kh)^2} \right]. \tag{5}$$

Nwogu find the optimal choice of α to be $\alpha = -0.393$, corresponding to $\beta = 0.537$ in our notation.

2 Model

For our model, we need to write the equations in the form $\frac{dy}{dt} = F$. So, rearranging we find that,

$$\eta_t = -\left((h+\eta)u\right)_x - \partial_x \left[\left(\alpha + \frac{1}{3}\right) h^3 u_{xx} + \left(-\beta + \frac{1}{2}\right) h^2 h_{xx} u\right] \tag{6}$$

$$\left[1 + \alpha h^2 \partial_{xx} - \beta h h_{xx}\right] u_t = -g\eta_x - u u_x. \tag{7}$$

where α and β are being intermingled freely to make the notation more compact. We can continue to write this in a more computationally efficient form, by reducing the number of transformations that have to be made.

$$\eta_t = -\partial_x \left[\left(h + \left(-\beta + \frac{1}{2}\right) h^2 h_{xx} + \eta\right) u + \left(\alpha + \frac{1}{3}\right) h^3 u_{xx}\right] \tag{8}$$

$$\left[1 + \alpha h^2 \partial_{xx} - \beta h h_{xx}\right] u_t = -g\eta_x - u u_x. \tag{9}$$

The tricky part is the operator,

$$L = 1 + \alpha h^2 \partial_{xx} - \beta h h_{xx} \tag{10}$$

which must be inverted at each time step. Using C to denote the forward cosine transform (or transform to whatever basis desired), and D_{xx} to denote the differentiation matrix in that basis, this becomes,

$$L = I + \operatorname{diag}\left(\alpha h^2\right) C^{-1} D_{xx} C - \operatorname{diag}\left(\beta h h_{xx}\right). \tag{11}$$

where the operator $\operatorname{diag} a(x_i)$ indicates a matrix with the the coefficients of a along the diagonal.