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Long waves on a beach

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Equations of motion are derived for long waves in water of varying depth. The equations are for small amplitude waves, but do include non-linear terms. They correspond to the Boussinesq equations for water of constant depth. Solutions have been calculated numerically for a solitary wave on a beach of uniform slope. These solutions include a reflected wave, which is also derived analytically by using the linearized long-wave equations.

1. Introduction

When water waves approach a beach they usually increase in amplitude and break. An increase in amplitude can be derived theoretically from linearized equations of motion (e.g. Lamb 1932, §185); however, breaking has not been satisfactorily explained by any approximation. The finite-amplitude shallow-water equations (Airy equations) have solutions which become so steep that the water surface becomes vertical (e.g. Stoker 1957). However, an essential part of that approximation is that the water surface should have a very gentle slope, thus the approximation becomes invalid before waves start to break. The work presented here does not apply to breaking waves, but it should provide a better approximation for steep waves than the Airy equations.

If relatively gentle waves approach a shore it is observed that the crests behave in some ways like separate waves so that they sometimes look like solitary waves. Munk (1949*b*) attempted to describe the motion of waves up a beach by making use of the properties of the solitary wave; however as Ippen & Kulin (1954) pointed out, it is not possible for a solitary wave to maintain the same total energy and volume in water of varying depth.

The Boussinesq equations describe a solitary wave in water of constant depth: compared with the Airy equations they include an extra term due to the effect of the vertical acceleration of the water on the pressure but they are limited to small amplitude waves. Equations corresponding to Boussinesq's are derived here for water of variable depth using an expansion similar to that used by Keller (1948). They are derived for three-dimensional motion, but are only used for a two-dimensional example. The particular example of a solitary wave approaching a beach of uniform slope has been computed by using a numerical approximation and the results are presented in §3. These solutions show a wave reflected from the beach, and the linearized long-wave equations are used in §4 to give a qualitative description of reflexion.

2. Equations of motion

Non-dimensional variables are used as follows:

$$(x,y,z) = h_0^{-1}(x^*,y^*,z^*), \quad t = t^*\, g/h_0, \quad p = p^*/\rho g h_0,$$
$$(u,v,w) = (gh_0)^{-\frac{1}{2}}(u^*,v^*,w^*),$$

where * indicates a dimensional variable and h_0 is a length representative of the depth of water. The z-axis is taken vertically upwards, the free surface is $z = \zeta\ (x,y,t)$ and the bottom is $z = -H(x,y)$.

The x- and y-directions are treated in the same way, so, to simplify expressions occurring in the equations, a two-dimensional vector notation is used. A symbol in Clarendon type indicates a vector with x- and y-components only, e.g. $\mathbf{u} = (u,v)$ and $\mathbf{x} = (x,y)$. The two-dimensional vector operator $(\partial/\partial x, \partial/\partial y)$ is denoted by $\boldsymbol{\nabla}$.

Euler's equations of motion for an inviscid fluid are

$$\partial\mathbf{u}/\partial t + (\mathbf{u}\,.\,\boldsymbol{\nabla})\,\mathbf{u} + w(\partial\mathbf{u}/\partial z) + \boldsymbol{\nabla}p = 0, \tag{1}$$

and
$$\partial w/\partial t + (\mathbf{u}\,.\,\boldsymbol{\nabla})\,w + w(\partial w/\partial z) + (\partial p/\partial z) + 1 = 0. \tag{2}$$

The continuity equation is

$$\boldsymbol{\nabla}\,.\,\mathbf{u} + (\partial w/\partial z) = 0, \tag{3}$$

or, in integrated form,
$$(\partial\zeta/\partial t) + \boldsymbol{\nabla}\,.\,\mathbf{Q} = 0, \tag{4}$$

where
$$\mathbf{Q} = \int_{-H}^{\zeta} \mathbf{u}\,dz.$$

The boundary conditions are

$$p = 0 \quad \text{at} \quad z = \zeta(\mathbf{x},t),$$

and
$$(\mathbf{u}\,.\,\boldsymbol{\nabla})\,H + w = 0 \quad \text{at} \quad z = -H(\mathbf{x}).$$

The kinematic boundary condition at $z = \zeta$ is used in (4) and is not needed explicitly here. Irrotational motion is assumed so that

$$\partial u/\partial y = \partial v/\partial x \quad \text{and} \quad \partial\mathbf{u}/\partial z = \boldsymbol{\nabla}w. \tag{5}$$

Note that in the presence of a free surface the vorticity of an inviscid fluid does not necessarily remain zero if it is zero initially. The free surface can intersect itself, as happens when a wave breaks and vortex sheets are formed. However, the following theory is not expected to hold when a wave starts to break.

The neglect of viscosity, which is reasonable for water more than about a foot in depth, should be a very good approximation for waves invading still water, because the effects of viscosity are at first confined to thin boundary layers at the surface and on the bottom, and since the waves travel much faster than the water itself, the vorticity in the boundary layers is not carried forward with the waves.

There are two important parameters associated with long waves. One is the ratio of amplitude to depth, and the other is the ratio of depth to wavelength; they will be represented by ϵ and σ respectively. For all long-wave theories $\sigma \ll 1$. For finite-amplitude theory $\epsilon = O(1)$ but for the solitary wave and the Boussinesq equations ϵ and σ^2 are of the same order, and we shall make this assumption here. In fact, by suitable choice of the horizontal length scale, $\epsilon = \sigma^2$ will be assumed. Ursell (1953) discusses the importance of the ratio $\epsilon:\sigma^2$.

The variables ζ, p, $\mathbf{u}$, $\mathbf{Q}$ are expanded in the form

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \tag{6}$$

and w as

$$w = \sigma(w_0 + \epsilon w_1 + \dots). \tag{7}$$

The independent variables are scaled so that

$$\partial/\partial x = \sigma(\partial/\partial x_1), \quad \partial/\partial y = \sigma(\partial/\partial y_1), \quad \text{or} \quad \boldsymbol{\nabla} = \sigma\boldsymbol{\nabla}_1, \quad \text{and} \quad \partial/\partial t = \sigma(\partial/\partial t_1). \tag{8}$$

The variables ζ_i, p_i, $\mathbf{u}_i$, $\mathbf{Q}_i$, $(i = 0, 1, \dots)$ and their derivatives with respect to x_1, y_1, z and t_1 are all assumed to be $O(1)$ so that the order of magnitude of terms in the equations appears explicitly when the relations (6), (7) and (8) are substituted. $H(\mathbf{x}_1)$ and its derivatives must also be assumed $O(1)$, otherwise the variations in depth of water would be shorter than the incident waves and tend to generate shorter waves, thus upsetting the scheme of the approximations.

The zero-order solution is taken to be still water, so that

$$p_0 = -z,$$

and all other zero-order variables are zero. If, instead, the zero-order equations are worked out, they are the Airy equations. Note, Keller (1948) calls them the first approximation.

The first-order relation from (5) is

$$\partial\mathbf{u}_1/\partial z = 0,$$

and hence,

$$\mathbf{u}_1 = \mathbf{u}_1(\mathbf{x}_1, t_1),$$

and therefore,

$$\mathbf{Q}_1 = H\mathbf{u}_1.$$

From (2) $\partial p_1/\partial z = 0$, which combined with the boundary condition $p_0 + \epsilon p_1 = 0$ at $z = \epsilon\zeta_1$, gives $p_1 = \zeta_1$. The first-order equations from (1) and (4) then give

$$(\partial\mathbf{u}_1/\partial t_1) + \boldsymbol{\nabla}_1\zeta_1 = 0 \tag{9}$$

and

$$(\partial\zeta_1/\partial t_1) + \boldsymbol{\nabla}_1.(H\mathbf{u}_1) = 0, \tag{10}$$

which are the linearized long-wave equations.

The vertical velocity w_1 is found by integrating the first-order equation from (3) with respect to z and applying the boundary condition at $z = -H$,

$$w_1 = -\boldsymbol{\nabla}_1.(H\mathbf{u}_1) - z\boldsymbol{\nabla}_1.\mathbf{u}_1.$$

For the second-order terms the same pattern is followed. Equation (5) gives

$$\partial\mathbf{u}_2/\partial z = \boldsymbol{\nabla}_1 w_1,$$

so that

$$\mathbf{u}_2 = \mathbf{U}_2(\mathbf{x}_1, t_1) - z\boldsymbol{\nabla}_1[\boldsymbol{\nabla}_1.(H\mathbf{u}_1)] - \tfrac{1}{2}z^2\boldsymbol{\nabla}_1(\boldsymbol{\nabla}_1.\mathbf{u}_1),$$

where $\mathbf{U}_2(\mathbf{x}_1, t_1)$ is an arbitrary function arising from the integration. Equation (2) now includes the vertical acceleration

$$(\partial w_1/\partial t_1) + (\partial p_2/\partial z) = 0,$$

and integrates to

$$p_2 = \zeta_2(\mathbf{x}_1, t_1) + z\frac{\partial}{\partial t_1}\boldsymbol{\nabla}_1.(H\mathbf{u}_1) + \tfrac{1}{2}z^2\frac{\partial}{\partial t_1}\boldsymbol{\nabla}_1.\mathbf{u}_1,$$

where the boundary condition at $z = \epsilon\zeta_1 + \epsilon^2\zeta_2$ has been used. On substituting for $\mathbf{u}_2$ and p_2 the second-order momentum equation, from (1), becomes

$$(\partial \mathbf{U}_2/\partial t) + (\mathbf{u}_1 . \nabla_1)\mathbf{u}_1 + \nabla_1 \zeta_2 = 0. \tag{11}$$

The form of this equation depends on the origin of z; the higher derivatives cancel out only when $z = 0$ is taken as the undisturbed free surface. Similarly, there are equivalent terms in the continuity equation which depend on the origin of z.

From the definition of $\mathbf{Q}$,

$$\epsilon^2 \mathbf{Q}_2 = \int_{-H}^{0} \epsilon^2 \mathbf{u}_2 \, dz + \int_{0}^{\epsilon\zeta_1} \epsilon \mathbf{u}_1 \, dz,$$

which gives

$$\mathbf{Q}_2 = \zeta_1 \mathbf{u}_1 + H\mathbf{U}_2 + \tfrac{1}{2}H^2 \nabla_1[\nabla_1 . (H\mathbf{u}_1)] - \tfrac{1}{6}H^3 \nabla_1(\nabla_1 . \mathbf{u}_1).$$

The second-order terms of the continuity equation (4) are

$$(\partial \zeta_2/\partial t_1) + \nabla_1 . \mathbf{Q}_2 = 0. \tag{12}$$

At this point a change in approach is required. It is possible to find a solution for $\mathbf{u}_1$ and ζ_1 and then to proceed by putting these values in (11) and (12) to find $\mathbf{U}_2$ and ζ_2. However, this is only practicable for small values of t_1, and solutions of the equations for a flat bottom show that the actual solution can vary substantially from that of the linearized equations, as, for example, with the undular bore (Peregrine 1966). The second-order terms have first-order effects over moderate times. To include these effects first-order variables incorporating the second-order terms are used. For example, for the wave amplitude the obvious variable is

$$\eta = \epsilon\zeta_1 + \epsilon^2\zeta_2.$$

For the velocity variable there are more possibilities, in particular, the mean velocity,

$$\bar{\mathbf{u}} = \mathbf{Q}/(H+\eta) = \epsilon u_1 + \epsilon^2\{\mathbf{U}_2 + \tfrac{1}{2}H\nabla_1[\nabla_1 . (H\mathbf{u}_1)] - \tfrac{1}{6}H^2 \nabla_1(\nabla_1 . \mathbf{u}_1)\};$$

and the velocity at $z = 0$, $\qquad \mathbf{u}' = \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{U}_2.$

In terms of $\bar{\mathbf{u}}$ the momentum and continuity equations, formed by adding ϵ times (11) and (12) to (9) and (10) respectively and by changing back to the variables x, y and t, are

$$\frac{\partial \bar{u}}{\partial t} + (\bar{\mathbf{u}} . \nabla)\bar{\mathbf{u}} + \nabla\eta = \tfrac{1}{2}H \frac{\partial}{\partial t} \nabla[\nabla . (H\bar{\mathbf{u}})] - \tfrac{1}{6}H^2 \frac{\partial}{\partial t} \nabla(\nabla . \bar{\mathbf{u}}), \tag{13}$$

and

$$(\partial\eta/\partial t) + \nabla . [(H+\eta)\bar{\mathbf{u}}] = 0. \tag{14}$$

The variable used in the next section is $\bar{\mathbf{u}}$; however, it may not be the most convenient variable for three-dimensional problems since the irrotationality condition becomes

$$\frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x} = \frac{1}{2}\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\nabla . (H\bar{\mathbf{u}}) - \frac{1}{2}\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\nabla . (H\bar{\mathbf{u}}) - \tfrac{1}{3}H\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\nabla . \bar{\mathbf{u}} + \tfrac{1}{3}H\frac{\partial H}{\partial x}\frac{\partial}{\partial y}\nabla . \bar{\mathbf{u}}. \tag{15}$$

The equations for $\bar{\mathbf{u}}'$ may be better for three-dimensional motion: they are

$$(\partial \mathbf{u}'/\partial t) + (\mathbf{u}'.\nabla)\,\mathbf{u}' + \nabla\eta = 0, \tag{16}$$

$$(\partial\eta/\partial t) + \nabla.[(H+\eta)\,\mathbf{u}'] + \tfrac{1}{2}\nabla.\{H^2\nabla[\nabla.(H\mathbf{u}')] - \tfrac{1}{3}H^3\nabla(\nabla.\mathbf{u}')\} = 0 \tag{17}$$

and

$$(\partial u'/\partial y) - (\partial v'/\partial x) = 0. \tag{18}$$

A different derivation of equivalent equations for two-dimensional motion has been given recently by Mei & Le Méhauté (1966), the velocity at $z = -H(x)$ is the velocity variable they use.

In all these equations the form of the second-order terms may be varied considerably by using the first-order relations

$$(\partial \mathbf{u}/\partial t) + \nabla\eta = 0, \quad \text{and} \quad (\partial\eta/\partial t) + \nabla.(H\mathbf{u}) = 0. \tag{19}$$

3. A solitary wave on a beach

For a beach of uniform slope α and with water in the region $x > 0$, $H(x) = \alpha x$, and (13) and (14) become

$$\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + \frac{\partial\eta}{\partial x} = \tfrac{1}{3}\alpha^2x^2\frac{\partial^3\bar{u}}{\partial x^2\,\partial t} + \alpha^2x\frac{\partial^2\bar{u}}{\partial x\,\partial t},$$

and

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(\alpha x+\eta)\bar{u}] = 0,$$

for two-dimensional waves with crests parallel to the shore line.

No analytical solution of these equations has been found, so approximate solutions have been calculated by making a finite-difference approximation to the equations and solving the resulting equations numerically with an electronic computer. The approximation was very similar to that which had already been used for the equations for a uniform depth of water (Peregrine 1966). The finite-difference equations are given in an appendix.

The initial condition was chosen to correspond to a realizable physical situation—a solitary wave meeting a beach after travelling in a channel of uniform depth. Experimental measurements of such an arrangement were made by Ippen & Kulin (1954). Another easily realizable situation is a train of waves approaching a beach, but this raises a number of problems. If the waves are assumed to come from deep water their shape in shallow water has not been determined, though for very gentle waves a reasonable approximation might be found from the linearized Stokes's wave approximation. More important is the fact that each wave is influenced by its predecessors and there is, at present, no way to determine this once waves have broken.

The initial water profile was

$$\eta = a_0\,\mathrm{sech}^2\,\tfrac{1}{2}(3a_0)^{\frac{1}{2}}(x-\alpha^{-1}).$$

That is, the same as a solitary wave in water of unit depth with its crest at $x = \alpha^{-1}$, which is where the undisturbed depth of water is unity. The wave was taken to have a constant initial velocity of $-(1+\frac{1}{2}a_0)$ so that $\bar{u}$ could be found from the continuity equation to be

$$\bar{u} = -(1+\tfrac{1}{2}a_0)\,\eta/(\alpha x+\eta),$$

thus making some initial allowance for the sloping bottom.

Results were calculated for a number of different beach slopes and initial wave amplitudes. An example is shown in figure 1. It shows the features one would expect, the wave steepens and grows in amplitude. Although the forward slope of the wave is similar to a solitary wave, the wave as a whole ceases to look like one. (Note that there is considerable vertical exaggeration in the figure.) The range of integration was $0 < x \leqslant \alpha^{-1}+10$, that is, the wave was initially well within the range of integration. The boundary condition at $x = \alpha^{-1}+10$ was designed to allow waves to travel out of the range of integration away from the beach, but not to generate waves approaching the beach. It appeared to be successful.

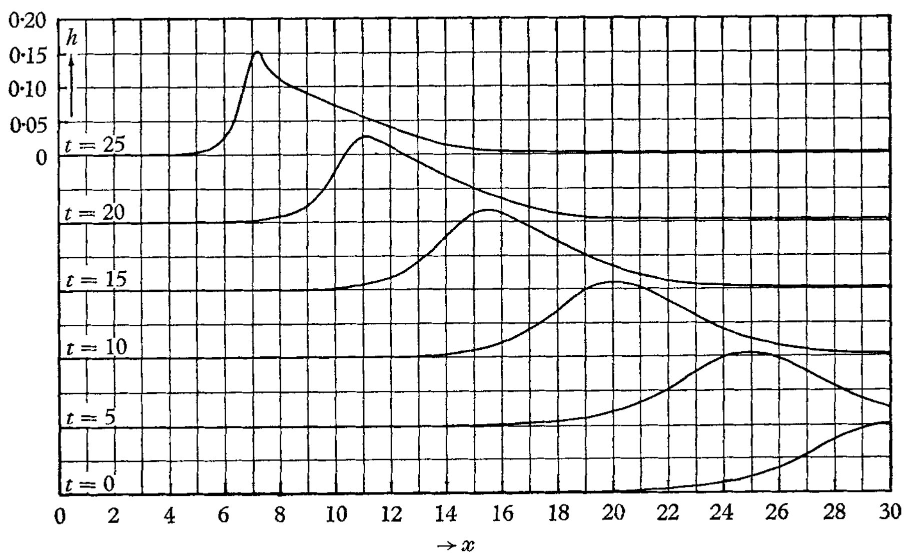

FIGURE 1. A solitary wave approaching a beach. $\alpha = 1/30$, $a_0 = 0{\cdot}1$.

As the wave approaches the shore it gets higher and steeper so that the two parameters, ϵ and σ, which were assumed to be small in the derivation of the equations, continually increase. It is clear that the equations are not valid for all time since the wave will ultimately break. It would be surprising if the approximation were good right up to breaking since ϵ and σ become $O(1)$. As a rough criterion for ending the calculations, the ratio of the amplitude to the initial depth of water at the position of the crest was allowed to increase to 0·6. In the experiments performed by Ippen & Kulin (1954) this ratio sometimes increased to 2 before the waves broke, while for periodic waves a value around 1 is more usual at breaking.

The maximum height of a wave is an easy thing to measure and is also a parameter of some practical importance. Results for the variation in height of the waves are plotted in figure 2 for a variety of initial wave amplitudes with the same beach slope. There is no systematic variation with wave amplitude. Similarly, there is no systematic variation with slope. The full line in figure 2 represents Green's Law, which is derived from the linearized equations of motion and gives a variation of amplitude of $H^{-\frac{1}{4}}$ (Lamb 1932, §185). It appears to give a reasonable approximation.

The experimental results of Ippen & Kulin show considerable scatter. However, they fitted curves of the type H^{-n} to their data and found $n = 0{\cdot}49$, $0{\cdot}26$, $0{\cdot}19$ for $\alpha = 0{\cdot}023$, $0{\cdot}050$ and $0{\cdot}065$ respectively ($n = 0{\cdot}25$ for Green's Law).

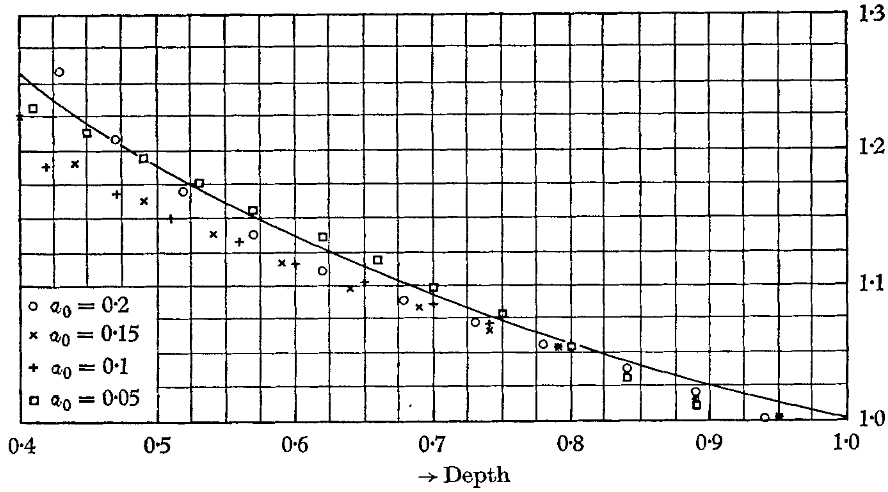

FIGURE 2. Change of amplitude with depth. Solitary waves of different initial amplitudes. $\alpha = 1/20$.

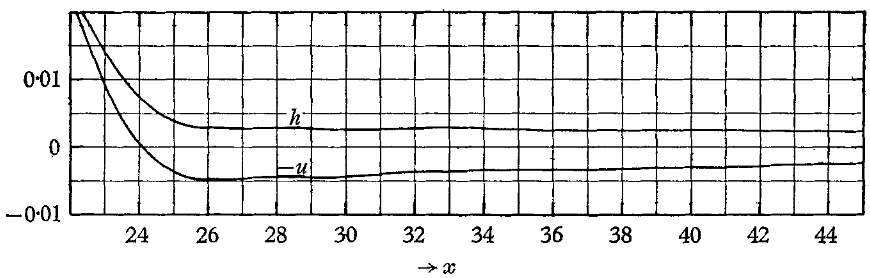

FIGURE 3. Reflexion from a solitary wave. $\alpha = 1/40$, $a_0 = 0{\cdot}1$, $t = 25$.

There is one unexpected property common to all the calculations, the wave amplitude does not start increasing at once. This could be because the initial conditions may not correspond to the appropriate solution of the equations, or, it may represent a real effect. The latter explanation is possible since Ippen & Kulin observed a slight decrease in the amplitude of solitary waves as they passed from water of a uniform depth on to a sloping beach. They surmised that it might be due to a reflexion from the foot of the beach. However, there is no evidence in the calculated results for any reflexion other than that due to the beach as a whole.

The reflected wave took the form of a long low elevation behind the incident wave. It was an outward travelling wave since the water velocities associated with it were directed away from the shore. A typical example is shown in figure 3 with substantially greater vertical exaggeration than in figure 1. In the next section such a reflexion is derived from the linearized long-wave equations. A similar derivation can be made from the Airy equations.

4. Reflexion

When waves run up a gently sloping beach, they break and lose much of their energy in turbulence, unless they are so small that surface tension and viscosity are important; hence, there cannot be total reflexion. Some reflexion may occur before the wave breaks, in the region where shallow-water equations are valid; this reflexion is considered here. The linearized equations are used and a wave of limited extent is taken as an example.

The linearized long-wave equations (19) for two-dimensional motion are

$$(\partial u/\partial t)+(\partial \eta/\partial x) = 0,$$

and

$$(\partial \eta/\partial t)+H(\partial u/\partial x)+\alpha u = 0,$$

where $\alpha = dH/dx$. A characteristic form of these equations is

$$\left\{\frac{\partial}{\partial t}+H^{\frac{1}{2}}\frac{\partial}{\partial x}\right\}(\eta+H^{\frac{1}{2}}u) = -\tfrac{1}{2}\alpha u,$$

$$\left\{\frac{\partial}{\partial t}-H^{\frac{1}{2}}\frac{\partial}{\partial x}\right\}(\eta-H^{\frac{1}{2}}u) = -\tfrac{1}{2}\alpha u.$$

Characteristic co-ordinates (X, Y) are introduced such that

$$dX = dt-H^{-\frac{1}{2}}dx$$

and

$$dY = dt+H^{-\frac{1}{2}}dx,$$

so that

$$2(\partial/\partial Y)(\eta+H^{\frac{1}{2}}u) = -\tfrac{1}{2}\alpha u$$

and

$$2(\partial/\partial X)(\eta-H^{\frac{1}{2}}u) = -\tfrac{1}{2}\alpha u.$$

If the characteristics X = constant and Y = constant through a point P are denoted by C_1 and C_2 respectively, the solution of these equations may be written in the form

$$\eta+H^{\frac{1}{2}}u = [\eta+H^{\frac{1}{2}}u]_{\text{on } C_1 \text{ at } t=0}-\int_{C_1}\tfrac{1}{4}\alpha u\,dY, \tag{20}$$

$$\eta-H^{\frac{1}{2}}u = [\eta-H^{\frac{1}{2}}u]_{\text{on } C_2 \text{ at } t=0}-\int_{C_2}\tfrac{1}{4}\alpha u\,dX, \tag{21}$$

where the integrals are from $t = 0$ to P.

Now, suppose α to be sufficiently small for the integrals in (20) and (21) to be small relative to both η and u. This implies that $\alpha < \sigma$ and that the integration is over a distance small compared with α^{-1}. The second condition means that the total change of depth must be small in the region where u differs from zero. With these assumptions a first approximation is

$$\eta+H^{\frac{1}{2}}u = \text{constant along } C_1, \tag{22}$$

$$\eta-H^{\frac{1}{2}}u = \text{constant along } C_2. \tag{23}$$

If a wave travels in the $(-x)$-direction into still water the constant in (22) is zero, and thus

$$\eta = -H^{\frac{1}{2}}u = \text{constant},$$

along C_2 characteristics. Figure 4 shows the track of a wave of limited extent propagating in the $(-x)$-direction.

Reflexions will occur in the region to the right of the wave and are found by looking at terms of $O(\alpha)$ while assuming the first-order terms to be unchanged. Consider a point in this area, such as P. The C_2 characteristic, PQ, through P starts in water behind the wave, and if the reflexion is as small as $O(\alpha)$, (21) can be written as

$$\eta - H^{\frac{1}{2}}u = O(\alpha^2) \quad \text{at} \quad P. \tag{24}$$

On the other hand, the C_1 characteristic through P, PR, starts in still water and passes through the wave where the integral in (20) is significant: that is, at P,

$$\eta + H^{\frac{1}{2}}u = -\int_{AB} \tfrac{1}{4}\alpha u\, dY + O(\alpha^2).$$

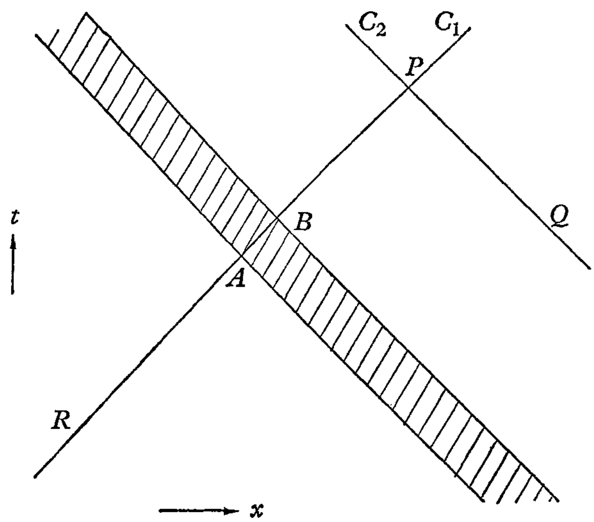

FIGURE 4

When this is combined with (24), it is seen that there is reflexion from the wave and it has height

$$\eta = -\int_{AB} \tfrac{1}{8}\alpha u\, dY + O(\alpha^2). \tag{25}$$

An approximate value for this integral may be found by assuming the depth along AB to vary so little that it may be taken as constant. This is consistent with the approximations already made. The integral is changed from one along AB to one at a constant time $t = t_0$.

Along AB

$$dX = dt - H^{-\frac{1}{2}}dx = 0$$

and

$$dY = dt + H^{-\frac{1}{2}}dx.$$

So that, if ds is an element of distance along AB with

$$ds^2 = dx^2 + dt^2,$$

dY may be changed to $2(1+H)^{-\frac{1}{2}}ds$. Corresponding to an element of distance ds on AB there is an element of length dx at time t_0 which has the same C_2 characteristics and, hence, the same values of η and $H^{\frac{1}{2}}u$ to the first order (see figure 5). If $t = t_0$ is close enough to both A and B for all lines in figure 5 to be assumed straight, then

$$(ds)_{AB} = \tfrac{1}{2}(1+H)^{\frac{1}{2}}H^{-\frac{1}{2}}(dx)_{t=t_0},$$

and thus

$$\int_{AB} u\,dY = H^{-\frac{1}{2}}\int_{t=t_0} u\,dx.$$

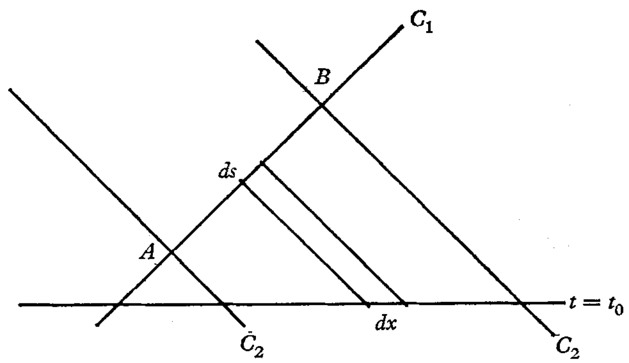

FIGURE 5

This gives an expression for the height of the reflexion, which may be written as

$$-\tfrac{1}{8}\alpha H^{-\frac{1}{2}}\int_{t=t_0} u\,dx \quad \text{or} \quad \tfrac{1}{8}\alpha H^{-1}\int_{t=t_0} \eta\,dx.$$

For example, with a solitary wave of amplitude a in unit depth of water

$$\eta = \tfrac{1}{2}\alpha(\tfrac{1}{3}a)^{\frac{1}{2}}.$$

For the example illustrated in figure 3, $\eta = 0{\cdot}002$, which is similar to the calculated amplitude.

A qualitative description of the form that reflexion takes can now be given for a few examples. First, consider a single wave of elevation, such as a solitary wave, passing over a gentle slope from water of one depth to water of a lesser depth (figure 6). By drawing the appropriate characteristics a good idea of the reflexion is obtained. While the wave is crossing the slope it is continually giving rise to a reflexion. If the slope has length L times its average depth, the reflexion has a length of approximately $2L$. If the wave were travelling up a beach, the reflexion occurring before the wave breaks would take the form of one very long wave: its amplitude would increase towards its end, corresponding to the incident wave travelling in shallower water. If the incident wave were a depression instead of an elevation, the reflexion would also be a depression. On the other hand if a wave of elevation were travelling into deeper water its reflexion would be a depression. For example, a solitary wave passing over a hump in a uniform channel would give rise to a reflected wave consisting of a long elevation followed by a depression.

Now consider a sinusoidal train of waves passing over the same bottom topography as in figure 6. In this case the amount of reflexion depends on how many waves a characteristic meets when it crosses the sloping strip. Successive troughs and crests will tend to cancel one another, so that, depending on the number of waves along a characteristic, reflexion will have minima and maxima. This is similar in its effects to the reflexion of light off thin films of transparent material when interference occurs. Such a situation has been described analytically by Kajiura (1961). He considered a transition between levels with the depth varying as x^2 (the solution was originally used by Rayleigh 1894, §148*b*, for acoustic waves). The analytic solution shows that interference is important in determining the amount of reflexion. The reflexion diminishes as the number of waves on the transition strip is increased, as well as becoming zero at regular intervals. Dean (1964) has considered a linear transition and found similar results.

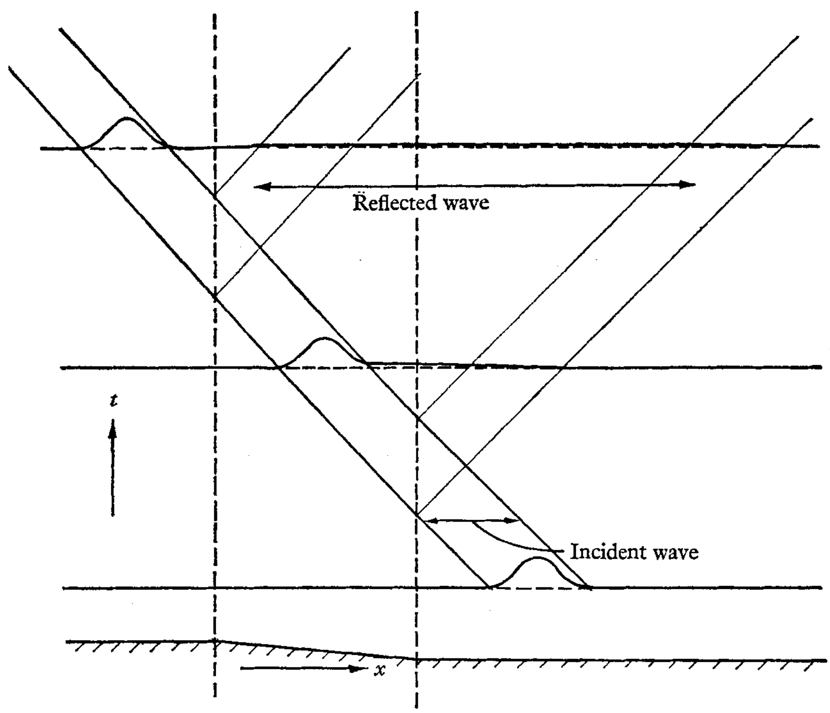

FIGURE 6. Reflexion from a single wave passing over a finite slope.

In practice the most usual situation is a train of waves approaching a shore from deep water. The results of this section indicate that there will be some reflexion from the shore before the waves break. The approximate evaluation of the integral in (25) assumes small changes in depth, but it is reasonable to suppose there will be little qualitative difference for large changes of depth. On a gently sloping beach a large number of waves would be in shallow-water at once and reflexion would be very small because of interference. However, on a steep beach, with only one or two waves in shallow water the reflexion can be much larger.

'Surf beats' (Munk 1949*a*; Tucker 1950) are a form of long-wave reflexion which appears to take place in the zone of breaking waves. Nevertheless, parts of this reflexion may occur in a similar manner to that outlined here. The mean water level in a group of high waves is lowered by 'radiation stress' (Longuet-

Higgins & Stewart 1962) and the effect on the integral in (25) would be the same as that from a long wave of depression. From wave records and details of bottom topography one might evaluate this integral and compare it with the observed reflexion; the remainder might then be attributed to the surf zone.

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Appendix

The finite-difference equations used in the calculations described in §3 are as follows, using the same notation as Peregrine (1966).

First, an approximation to the continuity equation which gives $\eta^*_{r,s+1}$, a provisional value for $\eta_{r,s+1}$,

$$\frac{\eta^*_{r,s+1}-\eta_{r,s}}{\Delta t}+(\alpha x+\eta_{r,s})\frac{u_{r+1,s}-u_{r-1,s}}{2\Delta x}+u_{r,s}\left[\frac{\eta_{r+1,s}-\eta_{r-1,s}}{2\Delta x}+\alpha\right]=0.$$

Secondly, an approximation to the momentum equation to find $u_{r,s+1}$,

$$\begin{aligned}\frac{u_{r,s+1}-u_{r,s}}{\Delta t}+u_{r,s}\frac{u_{r+1,s+1}-u_{r-1,s+1}+u_{r+1,s}-u_{r-1,s}}{4\Delta x}&\\ +\frac{\eta^*_{r+1,s+1}-\eta^*_{r-1,s+1}+\eta_{r+1,s}-\eta_{r-1,s}}{4\Delta x}&\\ =\tfrac{1}{3}(\alpha x)^2\frac{u_{r+1,s+1}-2u_{r,s+1}+u_{r-1,s+1}-u_{r+1,s}+2u_{r,s}-u_{r-1,s}}{\Delta x^2\Delta t}&\\ +\alpha^2 x\frac{u_{r+1,s+1}-u_{r-1,s+1}-u_{r+1,s}+u_{r-1,s}}{\Delta x\Delta t}&.\end{aligned}$$

Finally, the continuity equation was used again to give an improved value for $\eta_{r,s+1}$,

$$\begin{aligned}\frac{\eta_{r,s+1}-\eta_{r,s}}{\Delta t}+(\alpha x+\eta_{r,s})\frac{u_{r+1,s+1}-u_{r-1,s+1}+u_{r+1,s}-u_{r-1,s}}{\Delta x}&\\ +\frac{u_{r,s+1}+u_{r,s}}{2}\left[\frac{\eta_{r+1,s}-\eta_{r-1,s}}{2\Delta x}+\alpha\right]&=0.\end{aligned}$$

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CORRIGENDUM

'Subcritical convective instability. Part 1. Fluid layers', by D. D. JOSEPH and C. C. SHIR, *J. Fluid Mech.* vol. 26, 1966, p. 753.

The words 'decreasing' and 'decrease' in the fourth and fifth lines of the summary should be replaced by 'increasing' and 'increase' respectively.