CSCI 3022

intro to data science with probability & statistics

Lecture 5

January 29, 2018

2/2/18 101

1. HW 1 due this Fri. ~~9/15/17~~
2. Mathematical probability

Office Hours

Dan: Weds 11:00-1:00; Fri 8:00-9:50. Fleming 417.
Sofie (CA): Mon 11:00 -1:30, Tues 2:00-3:30, Thurs 2:00-5:00. CSEL.
Kyle (CA): Sun 2:00-4:45, MW 3:00-5:15, Tues 4:00-6:45 in CSEL (front or back depending on space)

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Last Time

\Omega

- **Sample Space** Ω: set of all possible outcomes of an experiment.
- **Event**: a *set* of one or more outcomes.
- **Probability Function P**: assigns value in [0,1] to each outcome or event.
  - Two requirements:
    1. Probability of the sample space is 1. $P(\Omega) = 1$
    2. $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint.
  - If A and B are not disjoint then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - If results of two trials don't affect each other, we say they are *independent*.

Warming up

- Suppose you draw one card from a standard 52-card deck

Question: What is the probability that the card is **A**♦?

$$\frac{1}{52} \quad \begin{array}{l} \leftarrow \text{\# of A}\diamondsuit \\ \leftarrow \text{total number of disjoint outcomes (cards)} \end{array}$$

Question: What is the probability that the card is an **A** *or* a ♦?

$$P(A \cup \diamondsuit) = P(A) + P(\diamondsuit) - P(A \cap \diamondsuit)$$

$$\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \boxed{\frac{16}{52}}$$

A rigorous way to compute probabilities

- Suppose we know $P(\omega)$ for each outcome ω in Ω.
- We can compute the probability of an event A (1 or more outcomes) as the sum of the probabilities of the outcomes in A

$A = \{\omega_1, \omega_2, \omega_3\}$ then $P(A) = P(\omega_1) + P(\omega_2) + P(\omega_3)$

Question: suppose we flip a *biased* coin P({H,T}) = {p,1-p} exactly 3 times. What is the probability that we get two or more T?

$$\left.\begin{Bmatrix} TTT, \\ TTH, \\ THT, \\ HTT, \end{Bmatrix}\right\} = A \qquad \begin{matrix} \to (1-p)^3 \\ \to (1-p)^2 p \\ \to (1-p)^2 p \\ \to (1-p^2) p \end{matrix}$$

$P(A) = (1-p)^3 + 3p(1-p)^2$

To Infinity and Beyond!

- Suppose you flip a biased coin until a H comes up. Prove that the probability that you flip a H eventually is 1. $p > 0$

Question 1: what is the sample space for this experiment?

$\Omega = \{ \{H\}, \{T, H\}, \{T, T, H\}, \{T, T, T, H\}, \ldots \}$

$|\Omega| = \infty$

Question 2: what is the probability that you flip a heads eventually?

$P(E) = P(out_1) + P(out_2) + P(out_3) + \ldots$

$P(E) = p + p(1-p) + p(1-p)^2 + p(1-p)^3 + \ldots$

$P(E) = p\sum_{i=0}^{\infty} (1-p)^i = p \frac{1}{1-(1-p)} = p \frac{1}{p} = 1$

$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$

$0 < r < 1$

Conditional probability

Question: Stop a random person on the street and ask them what month they were born. What is the probability that they were born in a 31-day month?

$L = \{ J, Mar, May, Jul, Aug, Oct, Dec \}$ $\quad P(L) = \frac{7}{12}$

Question: What is the probability that they were born in a month with an *r* in the name?

$R = \{ Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec \}$ $\quad P(R) = \frac{8}{12}$

bc. lazy

* assuming each month is equally probable.

Conditional probability

Question: Suppose that the person tells you that they were born in a 31-day month. *Now* what is the probability that they were born in a month with an *r* in it?

$P(R) = \frac{8}{12}$

$P(L) = \frac{7}{12}$ ← we know this to be true.

outcome that we are conditioning on.

$P(R \mid L)$

given, conditioned on

Conditional probability

Definition: The conditional probability of A given C is

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

provided that $P(C) > 0$

Example: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it contains *at least* two consecutive 1s given that the first bit is a 1?

$$P(\text{at least 2 consec 1s} \mid \text{first bit is 1}) = \frac{P(\text{at least 2 consec 1s AND 1st bit is one})}{P(\text{1st bit is 1})}$$

$$= \frac{5/16}{1/2} = 5/8$$

Conditional probability

Example2: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it **does not** contain two consecutive 1s given that the first bit is a 1?

$\frac{3}{8}$ $\quad P(E|C) = 1 - P(E^c|C)$

NB: the conditional probability $P(\,\cdot\,|C)$ is a valid probability function.

The product rule of probability

- The definition of conditional probability can be manipulated!

$= P(C \cap A) = P(C|A)P(A)$

- In this form, we call this the **product rule**:

$$P(A \cap C) = P(A|C)P(C)$$

- The product rule is useful when the conditional probability is easy to compute, but the probability of intersections of events is difficult.

Example: you draw 2 cards from a deck. What is the probability that they are both black?

$P(1^{st}\text{ Black} \cap 2^{nd}\text{ Black}) = P(2^{nd}\text{ Black} | 1^{st}\text{ Black})\, P(1^{st}\text{ Black})$

$= \frac{25}{51} \cdot \frac{1}{2} = \boxed{\frac{25}{102}}$

Independent events - intuition

Example: you draw 2 cards from a deck. What is the probability that they are both black?

Are the events of drawing the first black card and the second black card **independent?**

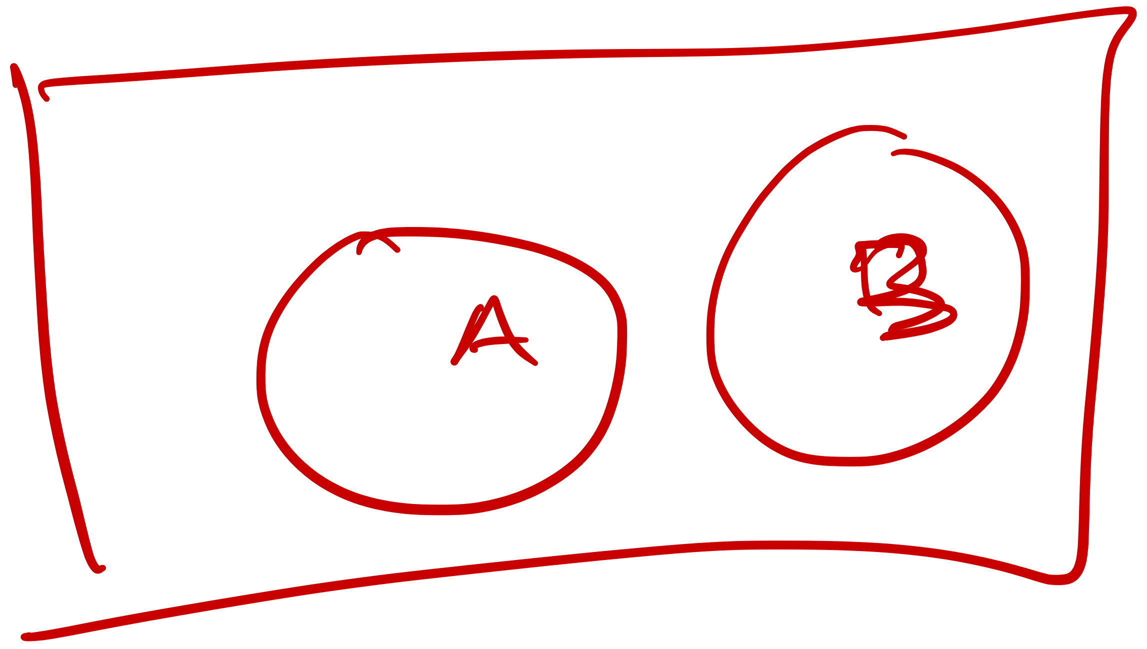
NO! :(

Independent events - math

Definition: An event A is said to be independent of event B if $P(A|B) = P(A)$

This definition, combined with the product rule (or the definition of conditional probability) gives us many equivalent definitions (or tests!) for independence:

1. if $P(A|B) = P(A)$ $\iff$ independent A & B
2. if $P(B|A) = P(B)$ $\iff$
3. if $P(A \cap B) = P(A)P(B)$ $\iff$



Subtleties of Independence

- **Def**: Events $A_1, A_2, \ldots, A_m$ are independent if

$$P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1)P(A_2)\ldots P(A_m)$$

Question: is independence of A, B, and C the same as: A and B are independent; B and C are independent; A and C are independent?

$P(C \mid A, B) = 1$

Example: Flip a fair coin twice. Let A be "Heads on flip 1", let B be "Heads on flip 2", let C be "the two flips match".

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{4}$

$P(A) \cdot P(B) \cdot P(C) = \frac{1}{8}$

$P(C \mid B) = \frac{1}{2} = P(C)$

C and B indep.
C and A are indep.
A and B are indep

Law of Total Probability

- Suppose I have a bag of marbles. The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles.
- Now suppose I put the two bags in a box. If I close my eyes, grab a bag from the box, and then grab a marble from that bag, what is the probability that it is black?

$$P(\text{Black}) = P(\text{Black} \mid 1^{st}\text{ bag})\,\underline{P(1^{st}\text{ bag})} + P(\text{Black} \mid 2^{nd}\text{ Bag})\,\underline{P(2^{nd}\text{ Bag})}$$

$$= \frac{4}{10}\cdot\frac{1}{2} + \frac{7}{10}\cdot\frac{1}{2}$$

$$= \frac{11}{20}$$

Law of Total Probability

- Same as before: The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles. But suppose the first bag is much larger than the second bag—I'm twice as likely to grab the first bag as the second bag. What's the probability that I get a black marble?

$$P(1^{st}\text{ bag}) + P(2^{nd}\text{ bag}) = 1$$

$$2x + x = 1$$

$$3x = 1$$

$$x = \frac{1}{3}$$

$$P(2^{nd}\text{ bag}) = \frac{1}{3}, \quad P(1^{st}\text{ bag}) = \frac{2}{3}$$

$$P(\text{black} \mid 1^{st})P(1^{st}) + P(\text{black} \mid 2^{nd})P(2^{nd})$$

$$\frac{4}{10} \cdot \frac{2}{3} + \frac{7}{10} \cdot \frac{1}{3}$$

$$= \frac{8}{30} + \frac{7}{30}$$

$$= 15/30 = \frac{1}{2}$$

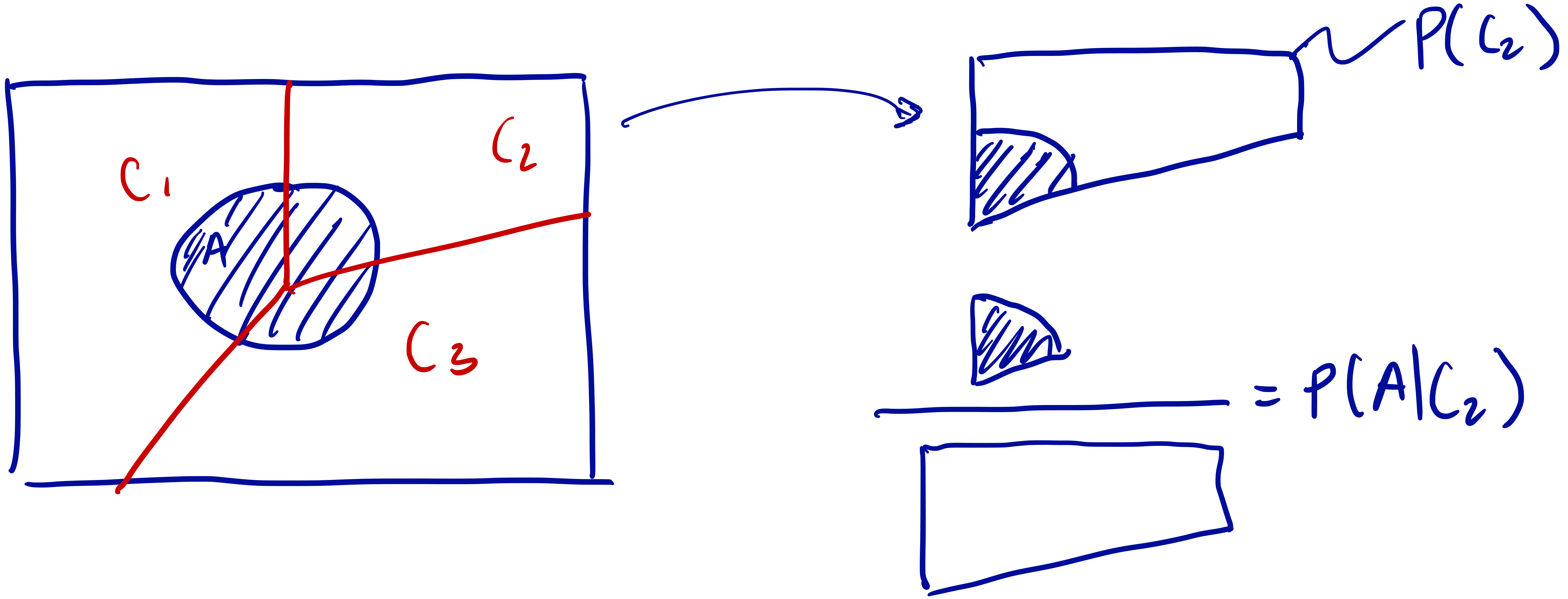
Law of Total Probability

- **Definition**: Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that

$$C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$$

Then the probability of an arbitrary event A can be expressed as:

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m)P(C_m)$$



The Birthday Paradox

- Say there are two random people in a room. What is the probability that they have different birthdays?

Let B_2 = event that 2 people in the same room have diff bdays.

The first person in the room must have a unique birthday... until the second person enters.

Then, the probability that the 2nd bday is diff. from first is:

$$1 - \frac{1}{365}$$

$\frac{1}{365}$ is prob. that they have the same bday.

The Birthday Paradox

- Say there are n random people in a room. What is the probability that they all have different birthdays?

Let B_n be the event that n people in a room all have diff. bdays.

$$P(B_n) = P(\underbrace{n^{th} \text{ birthday is diff from } n-1 \text{ other unique bdays}}_{\text{call this } A_n} \mid B_{n-1})P(B_{n-1})$$

$$= P(A_n \mid B_{n-1})P(B_{n-1})$$

$$= \left(1-\frac{n-1}{365}\right)P(B_{n-1})$$

$$P(B_{n-1}) = P(A_{n-1} \mid B_{n-2})P(B_{n-2})$$

$$= \left(1-\frac{n-2}{365}\right)P(B_{n-2})$$

$$= \left(1-\frac{n-1}{365}\right)\left(1-\frac{n-2}{365}\right)P(B_{n-2}) = \left(1-\frac{n-1}{365}\right)\left(1-\frac{n-2}{365}\right)\left(1-\frac{n-3}{365}\right)\cdots \cancel{P(B_1)}^{1}$$

The Birthday Paradox

- 23 people in the room: P(shared bday) = 0.5973
- 58 people in the room: P(shared bday) = 0.9917
- 120 people in the room: P(shared bday) = 0.9999999998