Lecture 3

Set-Builder Notation is a notation used to describe the elements of a set. In its most abstract form, it can be represented as $\{x \mid \lambda(x)\}$. x is the element that would make up the set. The $\mid$ symbol represents "such that" and $\lambda(x)$ is a predicate (a function that returns true or false). λ defines the properties that x needs to be included in the set. Additionally, you may specify the domain of the elements of the set. For example, $\{x \in \mathbb{Z} \mid \lambda(x)\}$ meaning the set contains all elements of the set of integers that satisfy λ.[1]

Examples:

1. $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$. The complex numbers are the set whose elements are in the form $a + bi$ such that a, b are real numbers and i is the square root of -1.

2. $K = \{x \mid \sqrt{x} \in \mathbb{Z}\}$. This is the set of integer squares. This reads K is the set that has elements x such that the square root of x is an integer. In other words, this is the set of integer squares.

3. $\mathbb{E} = \{x \in \mathbb{Z} : 2 \mid x\}$. Here I use : for "such that" to differentiate from the "divides" operator. This is the set of evens.

4. $\mathcal{P}(S) = \{X \mid X \subseteq S\}$. The power set of S is all the sets X such that X is a subset of S.

5. The residue class of 3 modulo 4, denoted as [3], is defined as follows: $[3] = \{x \in \mathbb{Z} \mid x \bmod 4 = 3\}$. Therefore, the residue class of 3 modulo 4 is integers x such that the modulo of x by 4 is 3.[2]

6. $\mathrm{GL}_n(\mathbb{R})$ is the set[3] of invertible $n \times n$ matrices with real entries. It can be defined as $\mathrm{GL}_n(\mathbb{R}) = \{M \in \mathcal{M} \mid \det(M) \neq 0\}$ where $\mathcal{M}$ is the set of all $n \times n$ matrices with $\mathbb{R}$ entries.

7. $S \times R = \{(s, r) \mid s \in S, r \in R\}$. This set is called the cartesian product of set S and R. For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is set of pairs of real numbers.

Side note on Boolean Algebra: The fundamental operations defined in boolean algebra are:

1. *Conjunction.* This is "logical and" and it is denoted with $\wedge$.

2. *Disjunction.* This is "logical or" and it is denoted with $\vee$.

3. *Negation.* This is "logical not" and it is denoted with $\neg$.

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[1] Some set-builder notation uses : instead of | for "such that."

[2] This is not the typical definition of a residue class. The typical definition involves modular congruences and the equivalence classes that it forms.

[3] More specifically, it is a group under matrix multiplication.

These can be used to create more complicated sets using set-builder notation. There are also other operations in boolean algebra that are built off of these core ones such as exclusive or, material conditional, and material biconditional.

Operation: The binary operation **union** $\cup$ takes two sets and outputs a set containing all the elements from both sets. It can be defined in set-builder notation as $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Operation: The binary operation **intersection** $\cap$ takes two sets and outputs a set containing all the elements which are a part of both sets. In set-builder notation, it can be $A \cap B = \{a \in A \mid a \in B\}$ or $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

Operation: The binary operation **set difference** is denoted as either $A - B$ or $A \backslash B$. It outputs the set containing elements of S that is not in set B. It can be denoted in set builder notation as $A - B = \{a \in A \mid a \notin B\}$ or $\{x \mid x \in A \wedge x \notin B\}$. A simple consequence of this is $A - B \subseteq A$.

A note on notation: There is two representations of intersection and set difference in set builder notation. Although they define the same set, they are not semantically the same. In $A \cap B = \{a \in A \mid a \in B\}$, it reads as the set $A \cap B$ is the set containing elements a that are in A such that a is also in B. However, $A \cap B = \{x \mid x \in A \wedge x \in B\}$ reads as the set $A \cap B$ is the set containing elements x such that x are in A and x are in B. In the second case, x belongs to a set, but it is implicit. For example, consider the case that this implicit set does not contain subsets A or B. Thus, it would mean $A \cap B = \emptyset$ because all elements of this implicit set does not belong to either A or B. However, it is often assumed that this implicit set contains A and B. The same is said for the set difference representations.

Examples:

1. $\{a, b\} \cup \{b, c\} = \{a, b, c\}$
2. $\{a, b\} \cap \{b, c\} = \{b\}$
3. $\{a, b\} - \{b, c\} = \{a\}$
4. $S \cup \emptyset = S$
5. $S \cap \emptyset = \emptyset$
6. $S - \emptyset = S$
7. $\emptyset - S = \emptyset$
8. $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers.

Operation: The unary operation **complement** is a specific case of set difference. $\bar{A}$ is the complement set of an implicit set S. The complement is the subset of S such that elements of the set are not in A. With notation, $\bar{A} = S - A = \{x \in S \mid x \notin A\}$. There is an additional requirement that $A \subseteq S$. Note that this is not necessarily the case for set difference (See example 3 above where $\{b, c\} \nsubseteq \{a, b\}$).

Examples:

1. Given implicit set $\{1, 2, 3, 4, 5\}$, $\overline{\{2, 3, 4\}} = \{1, 5\}$.
2. Define the even integers with $\mathbb{E}$ and odd with $\mathbb{O}$. Then $\overline{\mathbb{E}} = \mathbb{O}$.

Definition: Two sets A, B are said to be disjoint if and only if their intersection is the empty set, i.e. $A \cap B = \emptyset$. In other words, they do not share any elements with each other. A simple example is that $\mathbb{E}$ is disjoint with $\mathbb{O}$.

Definition: The cardinality of set S is the number of unique elements in the set. It is denoted as $|S|$. Sometimes, the cardinality of a set is also called the order of the set.

Examples:

1. $|\{1, 2, 3\}| = 3$
2. $|\{\{1, 2, 3\}\}| = 1$
3. $|\emptyset| = 0$
4. $|\mathbb{Z}| = \infty$
5. $|\mathrm{Sym}_n| = n!$. Sym_n is the set of permutations on set $\{x \in \mathbb{N} \mid x \leq n\} = \{1, 2, \ldots, n\}$,[4] i.e. the different ways which the set can be rearranged.

Definition: The power set of set S denoted with $\mathcal{P}(S)$ is the set of all subsets of S. In set builder notation, it is $\mathcal{P}(S) = \{X \mid X \subseteq S\}$.

Examples:

1. $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
2. $\mathcal{P}(\emptyset) = \{\emptyset\}$

[4] I am not including 0 in $\mathbb{N}$.