Theory Comput. Systems **33**, 295–312 (2000)
DOI: 10.1007/s002240010003

Theory of Computing Systems

On Multidimensional Curves with Hilbert Property*

J. Alber and R. Niedermeier

Wilhelm-Schickard-Institut für Informatik, Universität Tübingen,
Sand 13, D-72076 Tübingen, Germany
{alber,niedermr}@informatik.uni-tuebingen.de

Abstract. Indexing schemes for grids based on space-filling curves (e.g., Hilbert curves) find applications in numerous fields, ranging from parallel processing over data structures to image processing. Because of an increasing interest in discrete multidimensional spaces, indexing schemes for them have won considerable interest. Hilbert curves are the most simple and popular space-filling indexing schemes. We extend the concept of curves with Hilbert property to arbitrary dimensions and present first results concerning their structural analysis that also simplify their applicability.

We define and analyze in a precise mathematical way r-dimensional Hilbert curves for arbitrary $r \geq 2$. Moreover, we generalize and simplify previous work and clarify the concept of Hilbert curves for multidimensional grids. As we show, curves with Hilbert property can be completely described and analyzed by "generating elements of order 1," thus, in comparison with previous work, reducing their structural complexity decisively. Whereas there is basically *one* Hilbert curve in the two-dimensional world, our analysis shows that there are 1536 structurally different simple three-dimensional Hilbert curves. Further results include generalizations of locality results for multidimensional indexings and an easy recursive computation scheme for multidimensional curves with Hilbert property. In addition, our formalism lays the groundwork for potential mechanized analysis of locality properties of multidimensional Hilbert curves.

* An extended abstract of this work appeared in the *Proceedings of the Fourth Annual International Computing and Combinatorics Conference* (*COCOON* '98), pages 329–338, held in Taipei, Taiwan, August 12–14, 1998. LNCS 1449, Springer-Verlag, Berlin. The work of the first author was partially supported by the DFG-research project PEAL (Parameterized complexity and Exact ALgorithms) NI 369/1-1. The second author's work was partially supported by a Feodor Lynen fellowship of the Alexander von Humboldt-Stiftung, Bonn, and the Center for Discrete Mathematics, Theoretical Computer Science, and Applications (DIMATIA), Prague.

1. Introduction

Discrete multidimensional spaces are of increasing importance in computer science. They appear in various settings such as combinatorial optimization, parallel processing, image processing, geographic information systems, data base systems, and data structures. For many applications, it is necessary to number the points of a discrete multidimensional space (which, equivalently, can be seen as a grid) by an indexing scheme mapping each point bijectively to a natural number in the range between 1 and the total number of points in the space. Often it is desirable that this indexing scheme preserves some kind of locality, that is, close-by points in the space are mapped to close-by numbers or vice versa. For this purpose, indexing schemes based on space-filling curves have shown to be of high value [2], [4]–[9], [11]–[17], [20].

In this paper we study Hilbert curves [10], perhaps the most popular space-filling indexing schemes. Properties of two- and three-dimensional (2D and 3D) Hilbert curves have been extensively studied recently [5]–[7], [9], [12], [14], [15], [18]. However, most of the work so far has focused on empirical studies. Up to now, little attention has been paid to the theoretical study of structural properties of multidimensional Hilbert curves, the focus of this paper. Whereas with "modulo symmetry" there is only one 2D Hilbert curve, there are many possibilities to define Hilbert curves in the 3D setting [5], [15], [18]. The advantage of Hilbert curves is their (compared with other curves) simple structure that may easily outweigh the asymptotically slightly better (concerning constant factors) locality properties of other space-filling curves. Also note that in defining indexing schemes for multidimensional grids, descriptional simplicity as provided by curves with Hilbert property is a desirable property.

Our results can shortly be sketched as follows. We generalize the notion of Hilbert curves to arbitrary dimensions. We clarify the concept of Hilbert curves in multidimensional spaces by providing a natural and simple mathematical formalism that allows combinatorial studies of multidimensional Hilbert indexings. For reasons of (geometrical) clearness, we base our formalism on permutations instead of, e.g., matrices or other formalisms [3]–[5], [18]. So we obtain the following insight: Space-filling curves with Hilbert property can be completely described by simple generating elements and permutations operating on them. Structural questions for Hilbert curves in arbitrary dimensions can be decided by reducing them to basic generating elements. Putting it in catchy terms, one might say that, for curves with Hilbert property, what holds "in the large" (i.e., for large side-length) can already be detected "in the small" (i.e., for side-length 2). In particular, this provides a basis for mechanized proofs of locality of curves with Hilbert property (see [15]). In addition, this observation allows the identification of seemingly different 3D Hilbert curves [5], the generalization of a locality result of Gotsman and Lindenbaum [9] to a larger class of multidimensional indexing schemes, and the determination that there are exactly $6 \cdot 2^8 = 1536$ structurally different simple 3D Hilbert curves. The latter clearly generalizes and answers Sagan's quest for describing 3D Hilbert curves [18]. Finally, we provide an easy recursive formula for computing Hilbert indexings in arbitrary dimensions and sketch a recipe for how to construct an r-dimensional Hilbert curve for arbitrary r in an easy way from two $(r-1)$-dimensional ones.

As a whole, our work lays foundations for future work dealing with combinatorial properties of multidimensional Hilbert curves and, in particular, a mechanized analysis of locality properties of multidimensional Hilbert curves. The main focus of this paper, however, is to provide a theoretical study of nice combinatorial properties of Hilbert curves in arbitrary dimensions, and it is not to study, e.g., locality properties in great depth, which may be the subject of future study. In this sense, our locality results shall be considered to underline the usefulness of Hilbert curves for practical issues.

The paper is organized as follows. Section 2 presents some basic facts on space-filling curves and grid indexings and, in particular, gives the construction scheme of 2D Hilbert curves. Section 3 contains our method of describing multidimensional curves with Hilbert property by "generators" and permutations operating on a given corner-labeling of a cube. One of our main results shows that the structural analysis of multidimensional Hilbert curves can be completely reduced to the analysis of their (small) generating elements. In Section 4 we apply the methodology of Section 3 to derive several results concerning the structural analysis and computation of curves with Hilbert property. Moreover, we give some locality results for 2D Hilbert curves. Finally, Section 5 draws some conclusions, outlines further generalizations, and gives some directions for future work.

2. Preliminaries

We focus our attention on cubic grids where, in the r-dimensional case, we have n^r points arranged in an r-dimensional grid with side-length n. An *r-dimensional* (*discrete*) *indexing C* is simply a bijective mapping $C\colon \{1, \ldots, n^r\} \to \{1, \ldots, n\}^r$, thus providing a total ordering of the grid points. An indexing which is continuous with respect to the discrete topology given by the Manhattan metric is called a *curve*.

We say that an r-dimensional cubic grid has *order k* if it has side-length 2^k. Analogously, a curve C has order k if its range is a cubic grid of order k.

Figure 1 shows a 2D curve indexing a grid of size 4. This curve can be found in Hilbert's original work [10], [19] as a constructing unit for a whole family of curves. Figure 2 shows the general construction principle for these so-called Hilbert curves: For any $k \geq 1$, four Hilbert curves of size 4^k are combined into a curve of size 4^{k+1} by rotating and reflecting them in such a way that concatenating the indexings yields a Hamiltonian path through the grid. As indicated in Figure 2, we only need to keep track of the orientation of the edge which contains the start and end of the curve. As we will see later, the above rule uniquely defines the 2D Hilbert curve up to global rotation and reflection.

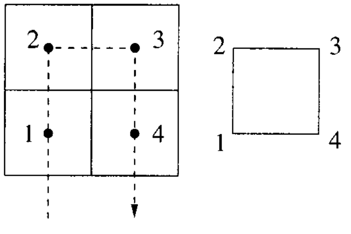


Fig. 1. The generator Hil_1^2 and its canonical corner-labeling $\widetilde{\mathrm{Hil}_1^2}$.

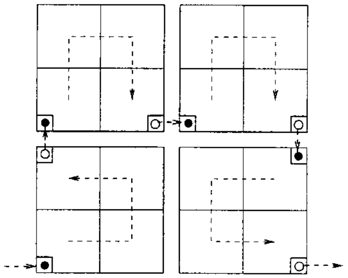


Fig. 2. Construction scheme for the 2D Hilbert curve.

One of the main features of the Hilbert curve is its "self-similarity." Here "self-similar" simply means that the curve can be generated by putting together identical (basic construction) units and only applying rotation and reflection to these units. In a sense, the Hilbert curve is the "simplest" self-similar, recursive, locality-preserving indexing scheme for square meshes of size $2^k \times 2^k$.

3. Formalizing Hilbert Curves in *r* Dimensions

In this section we generalize the construction principle of 2D Hilbert curves to arbitrary dimensions in a rigorous, mathematically precise way. Our attention is restricted to indexing schemes of cubes with side-lengths 2^k for any natural number k, although generalizations to sidelengths of the form b^k are straightforward (see Section 5). We generate an r-dimensional curve filling a cubic grid with side-length 2^k with a sequence of 2^r subcurves filling grids with side-length 2^{k-1} each. For the generating subcurves, we claim a certain similarity as given by the 2D Hilbert version of the Hilbert curve. By "similar" we mean that the subcurves can be transformed by a symmetry mapping (i.e., reflections or rotations) into each other. We need a certain formalism to express these symmetry mappings. This, for example, can be done by means of permutations. Fixing a certain indexing of the corners in a multidimensional grid, such a symmetry transformation can be expressed by the action of a permutation on the given indexing. This is one of the most intuitive approaches to describe such automorphisms on the grid. Furthermore, there turns out to be a very simple relation between curves of the lowest order possible and such corner-labelings. This, at first sight, seemingly strange formalism used by us was the basis for deriving structural results on, e.g., 3D Hilbert curves as presented in Section 4. So we can hardly imagine a comparatively simple presentation of all structurally different 3D Hilbert curves as given in Table 1 (see Section 4.1) using other formalisms.

3.1. *Classes of Self-Similar Indexings and Their Generators*

Let $V_r := \{x_1x_2 \cdots x_{r-1}x_r \mid x_i \in \{0, 1\}\}$ be the set of all 2^r corners of an r-dimensional cube coded in binary. Moreover, let $\mathcal{I}\colon V_r \longrightarrow \{1, \ldots, 2^r\}$ denote an arbitrary labeling of these corners. To describe the orientation of subindexings inside an indexing of higher order, we want to use symmetry mappings, which can be expressed via suitable permutations operating on the labels $\{1, \ldots, 2^r\}$ that represent the corners of the grid via $\mathcal{I}$. Observe that any r-dimensional indexing C_1 of order 1 naturally induces such a corner-labeling (see Figures 1 and 3). We call it the *canonical* corner-labeling and denote it by $\widetilde{C_1}\colon\ V_r \longrightarrow \{1, \ldots, 2^r\}$. Furthermore, for fixed corner-labeling $\mathcal{I}$, let $W_{\mathcal{I}}$ $(\subset Sym(2^r))$ denote all permutations on $\{1, \ldots, 2^r\}$ such that $\mathcal{I}^{-1} \circ W_{\mathcal{I}} \circ \mathcal{I}$ is the group of rotations and reflections of the r-dimensional cube. In other words, $W_{\mathcal{I}}$ is the set of all permutations that preserve the neighborhood-relations $n(i, j)$ of the corner-labeling $\mathcal{I}$:

$$W_{\mathcal{I}} := \left\{\pi \in Sym(2^r)\colon\ n(i, j) = n(\pi(i), \pi(j)), \forall i, j \in \{1, \ldots, 2^r\}\right\},$$

where

$$n(i, j) := \begin{cases} 1 & \text{if} \quad d_{\mathrm{h}}(\mathcal{I}^{-1}(i), \mathcal{I}^{-1}(j)) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $d_{\mathrm{h}}(\cdot, \cdot)$ denotes the Hamming-distance of two binary numbers. For a given permutation $\tau \in W_{\mathcal{I}}$, we sometimes write $(\tau : \mathcal{I})$ in order to emphasize that τ is operating on the corner-labels given by $\mathcal{I}$. The point here is that once we have fixed a certain corner-labeling $\mathcal{I}$, the set $W_{\mathcal{I}}$ will provide all necessary transformations to describe a construction principle of how to generate indexings of higher order by piecing together a suitable indexing of lower order. Obviously, each permutation $(\tau : \mathcal{I})$ canonically induces a bijective mapping on a cubic grid of order k. In the following, we do not distinguish between a permutation and the corresponding mapping on a grid.

We partition an r-dimensional cubic grid of order k into 2^r subcubes of order $k-1$. For each $x_1 \cdots x_r \in V_r$ we, therefore, set

$$p^{(k)}_{(x_1 \cdots x_r)} := \left(x_1 \cdot 2^{k-1}, \ldots, x_r \cdot 2^{k-1}\right) \in \{0, 2^{k-1}\} \times \cdots \times \{0, 2^{k-1}\}$$

to be the "lower-left corner" of such a subcube. Let C_{k-1} be an r-dimensional indexing of order $k-1$ $(k \geq 2)$. Our goal is to define a "self-similar" indexing C_k of order k by putting together 2^r pieces of type C_{k-1}. Let $\mathcal{I}\colon\ V_r \longrightarrow \{1, \ldots, 2^r\}$ be a corner-labeling. We intend to arrange the 2^r subindexings of type C_{k-1} "along" $\mathcal{I}$. The position of the i'th (where $i' \in \{1, \ldots, 2^r\}$) subindexing inside C_k can formally be described with the help of the grid-points $p^{(k)}_{(x_1 \cdots x_r)}$. Bearing in mind the classical construction principle for the 2D Hilbert curve, the orientation of the constructing indexing C_{k-1} inside C_k can be expressed by using symmetric transformations (i.e., reflections and rotations). For any sequence of permutations $\tau_1, \ldots, \tau_{2^r} \in W_{\mathcal{I}}$ we, therefore, define

$$C_k(i) := (\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')}, \tag{1}$$

where $i \in \{1, \ldots, (2^k)^r\}$ and $i' = (i-1) \operatorname{div} (2^{k-1})^r + 1$. The geometric intuition behind this is that the indexing C_k can be partitioned into 2^r components of the form C_{k-1} (reflected or rotated in a suitable way). These subindexings are arranged inside C_k "along" the given corner-labeling $\mathcal{I}$. The orientation of the i'th subindexing inside C_k is described by the effect of $(\tau_{i'} : \mathcal{I})$.

Definition 1. Whenever two r-dimensional indexings C_{k-1} of order $k-1$ and C_k of order k satisfy (1) for a given sequence of permutations $\tau_1, \ldots, \tau_{2^r} \in W_{\mathcal{I}}$, where $\mathcal{I}\colon V_r \longrightarrow \{1, \ldots, 2^r\}$ is a corner-labeling, we write

$$C_{k-1} \underset{(\tau_1,\ldots,\tau_{2^r})}{\overset{\mathcal{I}}{\ll}} C_k,$$

and call C_{k-1} the *constructor* of C_k.

Our final goal is to iterate this process starting with an indexing C_1 of order 1. It is only natural and, in our opinion, "preserves the spirit of Hilbert" to fix the corner-labeling according to the structure of the defining indexing C_1. Hence, in this situation, we can specify our $\mathcal{I}$ to be the canonical corner-labeling $\widetilde{C_1}$. By successively repeating the construction principle in (1) $k-1$ times, we obtain a curve of order k.

Definition 2. Let $\mathcal{C} = \{C_k \mid k \geq 1\}$ be a family of r-dimensional indexings of order k. We call $\mathcal{C}$ a *Class of Self-Similar Indexings* (*CSSI*) if there exists a sequence of permutations $\tau_1, \ldots, \tau_{2^r} \in W_{\widetilde{C_1}}$ for the canonical corner-labeling $\widetilde{C_1}$, such that, for each indexing C_k, it holds that

$$C_1 \underset{(\tau_1,\ldots,\tau_{2^r})}{\overset{\widetilde{C_1}}{\ll}} C_2 \underset{(\tau_1,\ldots,\tau_{2^r})}{\overset{\widetilde{C_1}}{\ll}} \cdots \underset{(\tau_1,\ldots,\tau_{2^r})}{\overset{\widetilde{C_1}}{\ll}} C_{k-1} \underset{(\tau_1,\ldots,\tau_{2^r})}{\overset{\widetilde{C_1}}{\ll}} C_k.$$

In this case, C_1 is called the *generator of the CSSI* $\mathcal{C}$ and we call

$$\mathcal{H}(C_1, (\tau_1, \ldots, \tau_{2^r})) := \{C_k \mid k \geq 1\}$$

the CSSI generated by C_1 and $\tau_1, \ldots, \tau_{2^r}$. A CSSI $\mathcal{C} = \{C_k \mid k \geq 1\}$ is called a *Class with Hilbert Property* (*CHP*) if all indexings C_k are curves, i.e., they are continuous.

Note that the CSSI $\mathcal{H}(C_1, (\tau_1, \ldots, \tau_{2^r}))$ is well-defined, since any CSSI is uniquely determined by its generator C_1 and the choice of the permutations $\tau_1, \ldots, \tau_{2^r} \in W_{\widetilde{C_1}}$. The nomenclature "Curve with Hilbert Property" is due to the fact that the constructing principle for a CHP grew out of the classical one for 2D Hilbert curves. Our concept for multidimensional CHPs only makes use of the very essential tools which can be found in Hilbert's context (see [10]) as rotation and reflection. We deliberately avoid more complicated structures (e.g., the use of different sequences of permutations in each inductive step, or the use of several generators for the constructing principle) in order to maintain conceptual simplicity and ease of construction and analysis. However, the theory which we develop in this paper is not necessarily restricted to the continuous case. That is the reason why all our definitions and theorems in Section 3.2 are formulated in the more general setting of non-continuous indexings. In Section 3.2 we provide a necessary and sufficient condition on the generating elements of a CSSI (generator and sequence of permutations) such that the entire family consists of continuous curves only, i.e., is a CHP. We end this subsection with an example.

Example. One easily checks that the classical 2D Hilbert curve can be described via

$$\mathcal{H}\left(\mathrm{Hil}_1^2, ((2\,4), \mathrm{id}, \mathrm{id}, (1\,3))\right) = \left\{\mathrm{Hil}_k^2 \mid k \geq 1\right\},$$

where the generator Hil_1^2 is given in Figure 1.

As Theorem 4 will show, this is the only CHP of dimension 2 "modulo symmetry," which, once again, justifies the name "Curve with Hilbert Property."

3.2. *Disturbing the Generator of a CSSI*

In this subsection we analyze the effects of disturbing the generator of a CSSI by a symmetric mapping. We will see that any disturbance of the generator will be hereditary to the whole CSSI in a very canonical way. Conversely, if two different CSSIs show a certain similarity in one of their members, this similarity can already be found in the structure of the corresponding generators. We illustrate this by the following diagram. Let $\mathcal{H}(C_1, (\tau_1, \ldots, \tau_{2^r})) = \{C_k \mid k \geq 1\}$ and $\mathcal{H}(D_1, (\tau_1, \ldots, \tau_{2^r})) = \{D_k \mid k \geq 1\}$ be two CSSIs.[1] Suppose there is a similarity at a certain stage of the construction, i.e., for some k_0, the curves C_{k_0} and D_{k_0} can be obtained from each other by a similarity transformation Φ. Can we conclude a vertical link between the curves of other orders?

$$\begin{tikzcd}[column sep=huge]
C_1 \arrow[d, "\Phi"] \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{C_1}}{\ll}"] \arrow[dr, phantom, "\circlearrowright"] & C_2 \arrow[d, "\Phi"] \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{C_1}}{\ll}"] \arrow[dr, phantom, "\circlearrowright"] & \cdots \arrow[d, "\Phi"] \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{C_1}}{\ll}"] \arrow[dr, phantom, "\circlearrowright"] & C_k \arrow[d, "\Phi"] \\
D_1 \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{D_1}}{\ll}"] & D_2 \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{D_1}}{\ll}"] & \cdots \arrow[r, phantom, "{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{D_1}}{\ll}"] & D_k
\end{tikzcd}$$

The investigations in this section will show that the inner structure of CSSIs are strong enough to yield the same behavior at the stage of any order. As a consequence, it will be sufficient to analyze the generating elements of a CSSI. Since all the information is encoded in the generator and the defining permutations, questions like continuity of a CSSI and structural similarity with other CSSIs can be answered by considering the generating elements only. These observations greatly simplify our complete classification of CHPs, which are given in Section 4.

We split the proof of the main theorem of this section into three steps, since each of these already contains some nice structural behavior of CSSIs. As a first step, we make a simple observation concerning the behavior of the construction principle of Definition 1 under the "symmetric disturbance" of a constructor:

Lemma 1. *Let C_{k-1} and C_k be r-dimensional indexings of order $k-1$ and k, respectively. Suppose C_{k-1} is the constructor of C_k, i.e., $C_{k-1}\,{}_{(\tau_1,\ldots,\tau_{2^r})}\overset{\widetilde{\mathcal{I}}}{\ll} C_k$, for a sequence*

[1] Note that the τ's used in the definition of both CSSIs yield completely different automorphisms on the grid. Whereas in the first case they refer to the corner-labeling $\widetilde{C_1}$, in the second case they act on the corner-labeling $\widetilde{D_1}$, given by the generator D_1.

of permutations $\tau_1, \ldots, \tau_{2^r} \in W_{\mathcal{I}}$, where $\mathcal{I}$ is a corner-labeling. Then, for arbitrary $\varphi \in W_{\mathcal{I}}$, we have

$$(\varphi : \mathcal{I}) \circ {C_{k-1}}_{(\tau_1 \circ \varphi^{-1}, \ldots, \tau_{2^r} \circ \varphi^{-1})}^{\mathcal{I}} \ll C_k.$$

Proof. Since C_{k-1} is the constructor of C_k, by Definition 1, resp. (1), we have

$$\begin{aligned} C_k(i) &= (\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')} \\ &= (\tau_{i'} : \mathcal{I}) \circ (\varphi^{-1} : \mathcal{I}) \circ (\varphi : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')} \\ &= (\tau_{i'} \circ \varphi^{-1} : \mathcal{I}) \circ ((\varphi : \mathcal{I}) \circ C_{k-1})\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')}, \end{aligned}$$

where $i \in \{1, \ldots, (2^k)^r\}$ and $i' = (i-1) \operatorname{div} (2^{k-1})^r + 1$, proving the claim by Definition 1. □

Whereas, by Lemma 1, we investigated the influence of disturbing the constructor, we now, in a second step, analyze how transforming the underlying corner-labeling influences the construction principle. We will need such a result, since two different CSSIs (by definition) come up with two different corner-labelings, each of which is given by the underlying generator.

Lemma 2. *Given the assumptions of Lemma 1, i.e., ${C_{k-1}}_{(\tau_1, \ldots, \tau_{2^r})}^{\mathcal{I}} \ll C_k$ for two r-dimensional indexings C_{k-1} and C_k of successive order, and let $\varphi \in W_{\mathcal{I}}$ for a corner-labeling $\mathcal{I}$. Then, for the modified corner-labeling $\mathcal{K} := \varphi^{-1} \circ \mathcal{I}$, we get $(\varphi : \mathcal{I}) = (\varphi : \mathcal{K}) =: \Phi$.[2] Moreover, with this transformation it holds that*

$${C_{k-1}}_{(\tau_1 \circ \varphi, \ldots, \tau_{2^r} \circ \varphi)}^{\mathcal{K}} \ll \Phi \circ C_k.$$

Proof. First we deduce a simple transformation-rule for permutations out of our given relation $\mathcal{K} = \varphi^{-1} \circ \mathcal{I}$. The effect of a given permutation $\pi \in W_{\mathcal{I}}$ acting on the labels given by the corner-labeling $\mathcal{I}$ is equivalent to the effect of the transformed permutation $\varphi^{-1} \circ \pi \circ \varphi$ operating on the labels of the transformed corner-labeling $\mathcal{K}$, i.e., $(\pi : \mathcal{I}) = (\varphi^{-1} \circ \pi \circ \varphi : \mathcal{K})$. More specifically, setting $\pi = \varphi$, this shows $(\varphi : \mathcal{I}) = (\varphi : \mathcal{K}) = \Phi$.

By assumption, C_{k-1} is the constructor of C_k which for all $i \in \{1, \ldots, (2^k)^r\}$ and $i' = (i-1) \operatorname{div} (2^{k-1})^r + 1$ yields

$$\begin{aligned} C_k(i) &= (\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')} \\ \Rightarrow \quad \Phi \circ C_k(i) &= \Phi \circ \left((\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')}\right) \\ &= \Phi \circ (\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(\varphi(i'))}, \end{aligned}$$

where the last equation is true because the effect of the symmetry mapping Φ on a CSSI-indexing C_k of order k can be split into its effect on the 2^r subindexings of order

[2] The fact that the corner-labeling is disturbed by φ^{-1} instead of φ is due only to technical reasons.

$k-1$ and the effect on the arrangement of these subindexings inside C_k. Whereas the i'th subindexing of C_k lies next to the corner $\mathcal{I}^{-1}(i')$, the position of the i'th subindexing of $\Phi \circ C_k$ is transformed according to φ. Therefore, its new position is given by the corner $\mathcal{I}^{-1}(\varphi(i'))$. Thus we get

$$\begin{aligned}\Phi \circ C_k(i) &= (\varphi \circ \tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{(\mathcal{I}^{-1}\circ\varphi)(i')} \\ &= (\tau_{i'} \circ \varphi : \mathcal{K}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{K}^{-1}(i')}\end{aligned}$$

by applying the transformation-rule treated at the beginning with $\pi = \varphi \circ \tau_{i'}$. By Definition 1, the last equation proves our claim. □

We now use Lemmas 1 and 2 to prove the main result of this section. For its illustration, we refer to the diagram at the beginning of this section. Also, recall the point made in the footnote there.

Theorem 3. *Let C_1 be the generator of the CSSI $\mathcal{H}(C_1, (\tau_1, \ldots, \tau_{2^r})) = \{C_k \mid k \geq 1\}$ and let D_1 be the generator of the CSSI $\mathcal{H}(D_1, (\tau_1, \ldots, \tau_{2^r})) = \{D_k \mid k \geq 1\}$. For an arbitrary permutation $\varphi \in W_{\widetilde{C_1}}$ and the corresponding symmetric mapping $\Phi = (\varphi : \widetilde{C_1}) = (\varphi : \widetilde{D_1})$, the following statements are equivalent*:

(i) $\Phi \circ C_{k_0} = D_{k_0}$ *for some* $k_0 \geq 1$.
(ii) $\Phi \circ C_k = D_k$ *for all* $k \geq 1$.

Proof. (ii) ⇒ (i) is trivial. For (i) ⇒ (ii), we first show that statement (ii) is true for the generators C_1 and D_1: If $k_0 > 1$, we can divide the cubic grid of order k_0 into 2^r subgrids of order $k_0 - 1$. By the construction principle for CSSIs, the indexings C_{k_0} and D_{k_0} traverse these subgrids "along" the canonical corner-labelings $\widetilde{C_1}$ and $\widetilde{D_1}$, respectively. Since, by assumption, $\Phi \circ C_{k_0} = D_{k_0}$, the corresponding relation also holds true for the corner-labelings $\widetilde{C_1}$ and $\widetilde{D_1}$. Because of the isomorphisms $C_1 \simeq \widetilde{C_1}$ and $D_1 \simeq \widetilde{D_1}$, respectively, this finally yields the validity of the equation $\Phi \circ C_1 = D_1$.

We proceed with proving (ii) by induction on k. Assuming that $D_k = \Phi \circ C_k$, we show this relation for $k+1$ by applying Lemmas 1 and 2. Since $\{C_k \mid k \geq 1\}$ is a CSSI, we get

$$\begin{aligned} & C_k \;\;{}^{\widetilde{C_1}}_{(\tau_1,\ldots,\tau_{2^r})}\!\!\ll C_{k+1} \\ \overset{\text{Lemma 1}}{\Longrightarrow}\; & \underbrace{\Phi \circ C_k}_{=D_k} \;\;{}^{\widetilde{C_1}}_{(\tau_1\circ\varphi^{-1},\ldots,\tau_{2^r}\circ\varphi^{-1})}\!\!\ll C_{k+1} \\ \overset{\text{Lemma 2}}{\Longrightarrow}\; & D_k \;\;{}^{\widetilde{D_1}}_{(\tau_1,\ldots,\tau_{2^r})}\!\!\ll \Phi \circ C_{k+1}, \end{aligned}$$

where the last relation makes use of $\widetilde{D_1} = \varphi^{-1} \circ \widetilde{C_1}$, which we immediately obtain from the given equation $D_1 = \Phi \circ C_1$.[3] This implies that $D_{k+1} = \Phi \circ C_{k+1}$ because of the CSSI property of $\{D_k \mid k \geq 1\}$. □

[3] A disturbance by Φ implies a transformation of the corner-labelings by φ^{-1}, which can be easily checked.

In particular, the result of Theorem 3 implies that any questions concerning the structural similarity of two CSSIs can be reduced to the analysis of their generators. Any symmetric correspondence between two CSSIs in the large can be detected in the small, that is, in the structure of their generators. Thus, in order to give a classification of CSSIs where two families of curves that are equal modulo symmetry (rotation and reflection) are not distinguished, we need only distinguish between generators which differ modulo symmetry. We may, therefore, exclusively restrict our attention to the analysis of different types of generators and of suitable sequences of permutations. So, our result greatly simplifies the complete classification and the construction of all structurally different CSSIs. Moreover, it lays the foundations of a mechanized analysis of, for example, locality properties of multidimensional Hilbert curves (see [15]).

4. Applications: Computing and Analyzing CHPs

First in this section we attack a classification of all structurally different CHPs for higher dimensions. Whereas we can provide concrete combinatorial results for the 2D and 3D cases, the high-dimensional cases appear to be much more difficult. The basic tool for such an analysis, however, is given by Theorem 3. In the following subsections we sketch how to construct Hilbert curves in higher dimensions and, thus, verify the existence of such objects in arbitrary dimensions. Also in this section we discuss computational aspects of Hilbert curves and, finally, we conclude with locality properties of such curves. The general structural behavior of CHPs is sufficient to extend some results provided in previous work, such as [9].

4.1. *Classification Theorems for the* 2*D and* 3*D Cases*

According to the discussion after Theorem 3 we have to proceed in two steps in order to determine the number of structurally different CHPs of dimension r. Firstly, we need to list all structurally different generators, i.e., all continuous indexing schemes of the r-dimensional grid of order 1 "modulo symmetry." Secondly, for each such generator we combinatorially have to check all suitable sequences of permutations such that 2^r copies of the generator can be arranged in a grid of order 2 along the canonical corner-labeling in a continuous way. Proceeding in this manner, we obtain a list of *all* CHPs of dimension r, which cannot be transformed into each other via symmetry mappings (i.e., roatations or reflections). Note that, for a fixed generator, any two different sets of permutations yield CHPs which are not identical "modulo symmetry." This observation is due to the fact that, in the definition of a CHP, we require an arrangement along the canonical corner-labeling, which remains fixed for a given generator.

Our first theorem investigates the 2D setting. The result given below justifies the name "class with Hilbert property" (CHP). Also, note that the subsequent proofs make decisive use of the geometric clearness provided by our formalism.

Theorem 4. *The classical* 2*D Hilbert curve* $\mathcal{H}\big(\mathrm{Hil}_1^2, ((2\,4), \mathrm{id}, \mathrm{id}, (1\,3))\big)$ *is the only CHP of dimension* 2 *modulo symmetry.*

Proof. According to the discussion above, we first need to show that Hil_1^2 is the only continuous 2D generator, which is obvious. In addition, a simple combinatorial consideration shows that no other sequence of permutations yields a continuous indexing of order 2 whose start- and endpoints are located at corners of the grid. However, any constructor for a curve of higher order must have the property that both start- and endpoints are corner-points of the grid. □

What about the 3D case? Are there any differences concerning the amount of possible CHPs? The analysis of the "Simple Indexing Schemes" (which are related to our CHPs) in [5] already shows that the number of CHPs in the 3D case grows drastically compared with the 2D setting. Many "Simple Indexing Schemes" in [5] now, by our analysis, turn out to be identical modulo symmetry. Our goal is to specify all structurally different CHPs, that is, all CHPs that are not identical modulo symmetry (rotation and reflection). We restrict ourselves to CHPs which, like the 2D Hilbert curve, have the property that start- and endpoints are corner-points of the grid. In the following, such CHPs are called *simple*. Note that, in our setting, the meaning of "simple" is different from [5].

For the generator of a simple CHP it is easy to see that the start- and endpoints cannot lie diagonally opposite to each other: If this were the case, there would be no possibility to arrange 2^r copies of such a generator to obtain a simple continuous curve of order 2. Starting at the lower left corner, due to the diagonality of the start- and endpoints, the endpoint of the second copy of the generator would always be an outer corner-point of the grid of order 2. Hence, there is no way to arrange a third copy of the generator in a continuous way. This observation yields the following useful remark.

Remark 5. For a generator C of a simple r-dimensional CHP and its canonical corner-labeling $\widetilde{C}$ it holds that $d_{\mathrm{h}}(\widetilde{C}^{-1}(1), \widetilde{C}^{-1}(2^r)) < r$, where d_{h} denotes the Hamming-distance of two binary numbers.

Since, by Theorem 3, we find all symmetric similarities of two CHPs in the structure of their generating elements, we may restrict our attention to the investigation of the generators and all suitable sequences of permutations. In addition, Lemmas 1 and 2 can be seen as helpful tools to describe symmetrically disturbed CHPs in a very constructive way. They at least provide formulas of how to calculate the sequence of permutations for a disturbed CHP out of the given sequence of the original CHP. The following theorem also generalizes and gives answers to the work of Sagan [18].

Theorem 6. *For the* 3*D case, there are* $6 \cdot 2^8 = 1536$ *structurally different* (*i.e., not identical modulo reflection and rotation*) *simple CHPs. These types are listed in Table* 1.

Proof. Theorem 3 says that we can restrict our attention to checking all curves of order 1 that are different modulo symmetry. Given such a continuous generator C, the total amount of CHPs which can be constructed by C is given by all possibilities of piecing together eight (rotated or reflected) versions of C ("subcurves") along its canonical corner-labeling $\widetilde{C}$.

By an exhaustive search, we get that there are three different (modulo symmetry) types of continuous generators, namely, Hil_1^3.A, Hil_1^3.B, and Hil_1^3.C (see Figure 3). As

Table 1. Description of all simple 3D CHPs.

Generator	Version	τ_1	τ_2	τ_3	τ_4
Hil_1^3.A	(a)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	(3 7)(4 8) / (2 8 4)(3 7 5)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)
	(b)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	id / (2 4)(5 7)	(1 7 3)(4 6 8) / (1 7 5 3)(2 8 6 4)
	(c)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	id / (2 4)(5 7)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(d)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	(3 7)(4 8) / (2 8 4)(3 7 5)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)
Hil_1^3.B	(a)	(2 8)(5 7) / (2 6 8)(3 5 7)	id / (2 6)(3 7)	(3 5)(6 8) / (2 8 6)(3 7 5)	(2 8)(5 7) / (2 6 8)(3 5 7)
	(b)	(2 8)(5 7) / (2 6 8)(3 5 7)	id / (2 6)(3 7)	(3 5)(6 8) / (2 8 6)(3 7 5)	(3 5)(6 8) / (2 8 6)(3 7 5)

Generator	Version	τ_5	τ_6	τ_7	τ_8
Hil_1^3.A	(a)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(b)	(1 3 5)(2 6 8) / (1 3 5 7)(2 4 6 8)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(c)	(2 8)(3 5) / (2 4 8)(3 5 7)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(d)	(1 3 5)(2 6 8) / (1 3 5 7)(2 4 6 8)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
Hil_1^3.B	(a)	(1 3)(4 6) / (1 3 7)(2 4 6)	(1 3)(4 6) / (1 3 7)(2 4 6)	id / (2 6)(3 7)	(1 7)(2 4) / (1 7 3)(2 6 4)
	(b)	(1 7)(2 4) / (1 7 3)(2 6 4)	(1 3)(4 6) / (1 3 7)(2 4 6)	id / (2 6)(3 7)	(1 7)(2 4) / (1 7 3)(2 6 4)

described above, we now have to check whether there are continuous arrangements of these generators along their canonical corner-labelings. Beginning with type A, an exhaustive combinatorial search yields that there are four possible continuous formations of Hil_1^3.A along $\widetilde{\mathrm{Hil}_1^3}$.A. All possibilities are shown in Figure 4, where the orientation of each subcube is given by the position of an edge (drawn, in Figure 4, in bold lines). For each subcube, there are two symmetry mappings which yield possible arrangements for the generator within such a subgrid. The permutations expressing these mappings are listed in Table 1.

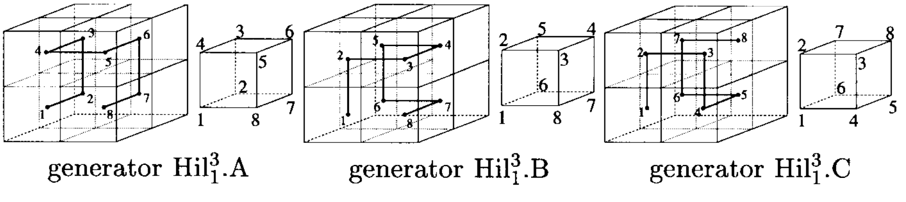


Fig. 3. Continuous 3D generators Hil_1^3.x and their canonical corner-labelings $\widetilde{\mathrm{Hil}_1^3}$.x.

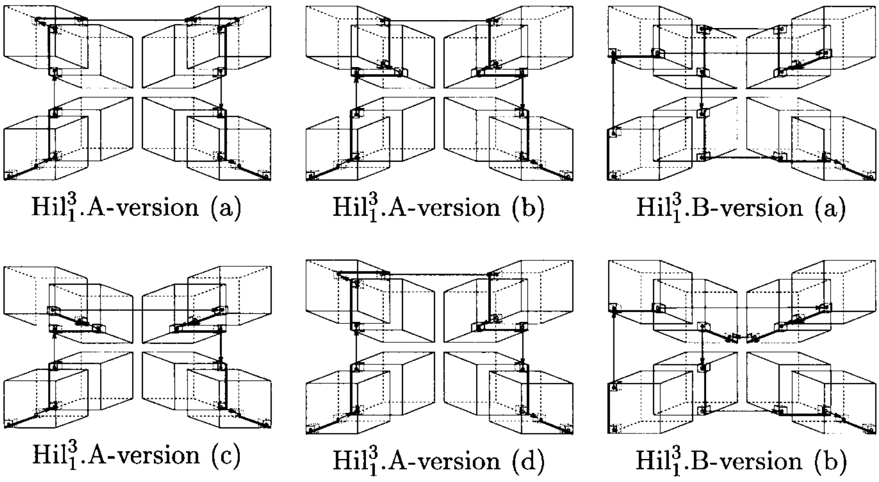


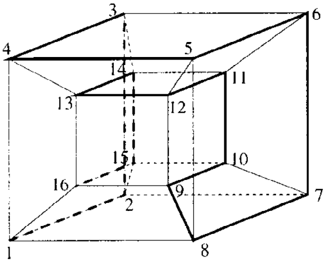
Fig. 4. Construction principles for CHPs with generators Hil_1^3.A and Hil_1^3.B.

Analogously, we find out the possible arrangements for generator type B. Note that there are no more than two different continuous arrangements of this generator along its canonical corner-labeling. Finally, by Remark 5, Hil_1^3.C cannot be the constructor of a simple CHP. Table 1 thus yields that there are exactly $4 \cdot 2^8 + 2 \cdot 2^8 = 6 \cdot 2^8$ structurally different CHPs. □

A complete classification of the high-dimensional cases and of nonsimple CHPs appears to be much more difficult and remains open. We end this section by sketching several further results based on our characterization of curves with Hilbert property.

4.2. *Construction of an r-Dimensional Hilbert Curve*

As already mentioned, CHPs seem to outperform many other space-filling curves concerning their properties important for applications like data structures or parallel processing (e.g., computational effort, locality, etc.). Since such qualities might depend only weakly on the inside structure of a CHP, it, however, seems to be important to have at least one easily constructible CHP for each dimension. Without giving an explicit proof here, we just indicate how the construction of a high-dimensional CHP can be done inductively in an easy way: A continuous generator of dimension r can be derived inductively simply by "joining together" two continuous generators of dimension $r - 1$. A similar consideration finally helps specify the suitable permutations in order to obtain indexings of higher order. As an example we give a CHP of dimension 4, whose generator Hil_1^4 is constructed by joining together two generators Hil_1^3.A (see Figure 3). The generator Hil_1^4 and a suitable sequence of permutations are shown in Figure 5. Note that this construction principle can be extended to obtain Hilbert curves in arbitrary dimensions in an easy and constructive way: Following the construction principle of Hil_1^3.A, first pass through an $(r - 1)$-dimensional structure, then in "two steps" do a change of dimension in the rth dimension, and, finally, again pass through an $(r - 1)$-dimensional structure. This



$$\begin{aligned}
\tau_1 &= (2\ 16)(3\ 13)(6\ 12)(7\ 9)\\
\tau_2 &= (3\ 15)(4\ 16)(5\ 9)(6\ 10)\\
\tau_3 &= \tau_2\\
\tau_4 &= (1\ 3\ 13\ 11\ 9\ 7)(2\ 4\ 14\ 12\ 10\ 8)(5\ 15)(6\ 16)\\
\tau_5 &= \tau_4\\
\tau_6 &= (1\ 5\ 13\ 9)(2\ 6\ 14\ 10)(3\ 11\ 15\ 7)(4\ 12\ 16\ 8)\\
\tau_7 &= \tau_6\\
\tau_8 &= (1\ 7)(4\ 6)(10\ 16)(11\ 13)\\
\tau_9 &= \tau_8\\
\tau_{10} &= (1\ 9\ 13\ 5)(2\ 10\ 14\ 6)(3\ 7\ 15\ 11)(4\ 8\ 16\ 12)\\
\tau_{11} &= \tau_{10}\\
\tau_{12} &= (1\ 11)(2\ 12)(3\ 5\ 7\ 9\ 15\ 13)(4\ 6\ 8\ 10\ 16\ 14)\\
\tau_{13} &= \tau_{12}\\
\tau_{14} &= (1\ 13)(2\ 14)(7\ 11)(8\ 12)\\
\tau_{15} &= \tau_{14}\\
\tau_{16} &= (1\ 15)(4\ 14)(5\ 11)(8\ 10)
\end{aligned}$$

Fig. 5. Constructing elements for a 4D CHP (generator Hil_1^4 and permutations).

method applies to finding the generators as well as to finding the permutations. Thus the construction principle of $\mathrm{Hil}_1^3.A$ in a sense is iterated $r-2$ times in order to generate an indexing of dimension r.

4.3. *Recursive Computation of CSSIs*

Note that whenever a CSSI $\mathcal{C} = \{C_k \mid k \geq 1\}$ is explicitly given by its generator and the sequence of permutations, we may use the recursive formula (1) of Section 3.1 to compute the indexings C_k. In other words, the defining formula (1) itself provides a computation scheme for CSSI, which is parameterized by the generating elements (generator and sequence of permutations). This formula might, for example, also be used to investigate locality properties of CSSIs by mechanical methods. The first steps toward an automated technique to analyze locality of r-dimensional curves can be found in [15]. There, mechanisms are provided for analyzing locality on condition that the given curves have self-similar character. This underlines the usefulness of the simple structure of CSSIs in particular with respect to aspects of computation.

4.4. *Aspects of Locality*

The locality properties of Hilbert curves have already been studied in great detail. As an example for such investigations, we briefly note a result of Gotsman and Lindenbaum [9] for multidimensional Hilbert curves. In [9] they investigate a curve $C\colon \{1,\ldots,n^r\} \to \{1,\ldots,n\}^r$ with the help of their locality measure

$$L_2(C) := \max_{i,j\in\{1,\ldots,n^r\}, i\neq j} \frac{d_2(C(i),C(j))^r}{|i-j|}, \tag{2}$$

where d_2 denotes the Euclidean metric. In their Theorem 3 they claim the upper bound $L_2(H_k^r) \leq (r+3)^{r/2}2^r$ for any r-dimensional Hilbert curve of order k, without precisely specifying what an r-dimensional Hilbert curve shall be. Since the proof of their result does not utilize the special Hilbert structure of the curve, this result can even be extended to arbitrary CSSIs.

When making use of the special CHP-property of a class of curves one can get even more precise results. For the 2D case (see Theorem 4), Gotsman and Lindenbaum present (see Theorem 4 of [9]) a result for the Euclidean metric.[4] Moreover, apart from the given locality measure L_2, we can consider measures L_p (with $p = 1$ or $p = \infty$), replacing the Euclidean distance d_2 in definition (2) by the Manhattan metric d_1, and the Maximum metric d_∞, respectively. With this notation the result of Gotsman and Lindenbaum can be improved and generalized to the following theorem.

Theorem 7. *For the 2D Hilbert indexing* $\mathcal{H}^2 = \{\mathrm{Hil}_k^2 \mid k \geq 1\}$, *we have*

$$\begin{aligned} L_1(\mathrm{Hil}_k^2) &\leq 9\tfrac{3}{5}, \\ L_2(\mathrm{Hil}_k^2) &\leq 6\tfrac{1}{2}, \\ L_\infty(\mathrm{Hil}_k^2) &\leq 6\tfrac{2}{5}, \end{aligned}$$

for all $\mathrm{Hil}_k^2 \in \mathcal{H}^2$ *with* $\mathrm{ord}(\mathrm{Hil}_k^2) = k$.

Proof. The key idea for determining the upper bounds relies on refining and extending the method given by Gotsman and Lindenbaum [9, Theorem 4]: We fix $|i-j| =: s$, where $i \leq j$. Then we choose $m \in \mathbb{N}$ such that $4^{m-1} < s \leq 4^m$. If we divide our 2D grid of order k into subgrids of sidelength 2^{m-1}, then, for each subcase $s \in](\ell_1 - 1)\cdot 4^{m-1}, \ell_1 \cdot 4^{m-1}]$, $\ell_1 \in \{2, 3, 4\}$, the subpath $S := \{\mathrm{Hil}_k^2(i), \ldots, \mathrm{Hil}_k^2(j)\}$ has to lie either in a sequence of ℓ_1 or $\ell_1 + 1$ of these subgrids. At this point, we make decisive use of the recursive CHP structure of our curve and use the fact that we can list all possible arrangements of such "subgrid-sequences" that occur in Hil_k^2. These possibilities are listed in Figure 6 up to rotations or reflexions. In order to arrive at upper bounds for $L_p(\mathrm{Hil}_k^2)$, we do an even more refined analysis and consider all subcases $s \in](\ell_2 - 1)\cdot 4^{m-2}, \ell_2 \cdot 4^{m-2}]$, $\ell_2 \in \{5, \ldots, 16\}$, i.e., we consider all subgrids of sidelength $E := 2^{m-2}$ of our grid (see also Figure 6). For each $\ell_2 \in \{5, \ldots, 16\}$, the third column in Table 4.4 shows the arrangement which behaves worst in terms of locality among all possible arrangements that are listed in Figure 6. This finally determines an upper bound for

$$M_p(\ell_2) := \max_{\substack{i,j \in \{1,\ldots,2^{kr}\}, \\ (\ell_2-1)\cdot 4^{m-2} < |i-j| \leq \ell_2 \cdot 4^{m-2}}} \frac{d_p(\mathrm{Hil}_k^2(i), \mathrm{Hil}_k^2(j))^2}{|i-j|}.$$

The result follows since $L_p(\mathrm{Hil}_k^2) \leq \max_{\ell_2 \in \{5,\ldots,16\}} M_p(\ell_2)$. $\square$

In addition to the theorem above, Gotsman and Lindenbaum [9, Theorem 4] establish the lower bound $6(1 - O(2^{-k})) \leq L_2(\mathrm{Hil}_k^2)$ by analyzing a suitable subcurve inside Hil_k^2. Similar subcurves can be used to derive the analogous result for the Manhattan metric and the Maximum metric, namely, $9(1 - O(2^{-k})) \leq L_1(\mathrm{Hil}_k^2)$ and $6(1 - O(2^{-k})) \leq$

[4] Gotsman and Lindenbaum claimed the upper bound $L_2(H_k^2) \leq 6\frac{2}{3}$. Checking their very sketchy proof, only an upper bound of $8\frac{1}{2}$ seems to follow directly.

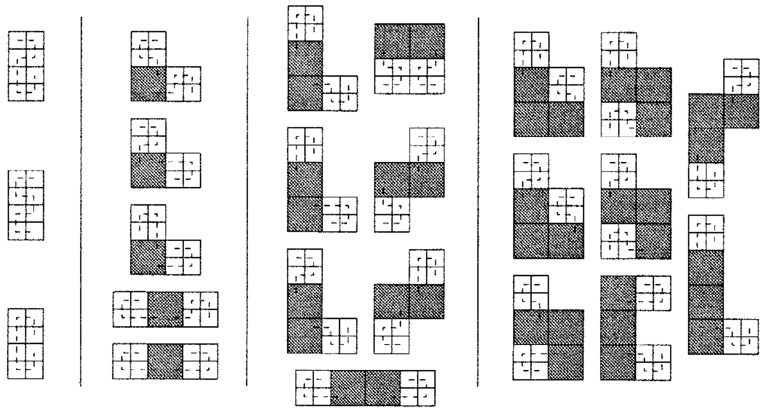


Fig. 6. For $s := |i-j| \in \,](\ell_1 - 1) \cdot 4^{m-1}, \ell_1 \cdot 4^{m-1}], \ell_1 \in \{2, 3, 4\}$, the subpath $S := \{\mathrm{Hil}_k^2(i), \ldots, \mathrm{Hil}_k^2(j)\}$ lies in a sequence of ℓ_1 or $\ell_1 + 1$ subgrids of sidelength 2^{m-1}. This list shows all possible arrangements of such "subgrid-sequences" in Hil_k^2 of lengths 2 to 5. (The shaded squares in this figure have sidelength 2^{m-1}.) For further analysis we also need to trace the exact course of the subpath in the more refined grid of sidelength 2^{m-2}. For that purpose, the outermost subgrids of each sequence are again subdivided and the dotted line illustrates the way Hil_k^2 traverses them.

$L_\infty(\mathrm{Hil}_k^2)$. Details can be found in the long version [1]. For further results with respect to the Manhattan metric refer to [6] and [15].

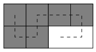
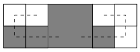
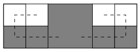
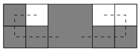
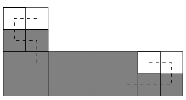
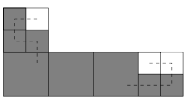
Similar techniques can be applied to all higher dimensional CHPs whenever the corresponding construction principle (i.e., generator and sequence of permutations) is given.

5. Conclusion

There is no denying the fact that dealing with dimensions greater than three makes the study of multidimensional structures quite hard due to the loss of geometric intuition. In this paper we tried to provide a mathematical mechanism to describe and analyze space-filling Hilbert curves in arbitrary dimensions that was as simple as possible. Using a formalism based on generating elements and permutations, which completely describe whole families of Hilbert curves, we were able to discover some nice combinatorial properties of Hilbert curves in arbitrary dimensions.

Our formalism still leaves a lot of freedom we have not made use of. So giving up the restriction to "pure" Hilbert curves, it would be fairly straightforward to study also generators with side-length b instead of 2 (see [1]). However, in this case, the formalism would become a little more complicated because there is no longer such a simple isomorphism between corner-labelings and generators. Note that, for example, Butz [3] studied locality in multidimensional curves with $b = 3$, paying less attention to a combinatorial study and structural issues of the curves as we did. From an application point of view, it may also be important to study noncubic grids and the corresponding indexings. Here our formalism in principle also works, but one has to take care of the

Table 2. Establishing upper bounds for $L_1(Hil_k^2)$, $L_2(Hil_k^2)$ und $L_\infty(Hil_k^2)$. Here we use $E := 2^{m-2}$, i.e., $E^2 = 4^{m-2}$.

$< s$	$s \leq$	Worst-case	$d_2^2 \leq$	$d_2^2/s \leq$	$d_1 \leq$	$d_1^2/s \leq$	$d_\infty \leq$	$d_\infty^2/s \leq$
$4E^2$	$5E^2$		$20E^2$	5	$6E$	9	$4E$	4
$5E^2$	$6E^2$		$26E^2$	$5\frac{1}{5}$	$6E$	$7\frac{1}{5}$	$5E$	5
$6E^2$	$7E^2$		$37E^2$	$6\frac{1}{6}$	$7E$	$8\frac{1}{6}$	$6E$	6
$7E^2$	$8E^2$		$40E^2$	$5\frac{5}{7}$	$8E$	$9\frac{1}{7}$	$6E$	$5\frac{1}{7}$
$8E^2$	$9E^2$		$40E^2$	5	$8E$	8	$6E$	$4\frac{1}{2}$
$9E^2$	$10E^2$		$50E^2$	$5\frac{5}{9}$	$8E$	$7\frac{1}{9}$	$7E$	$5\frac{4}{9}$
$10E^2$	$11E^2$		$65E^2$	$6\frac{1}{2}$	$9E$	$8\frac{1}{10}$	$8E$	$6\frac{2}{5}$
$11E^2$	$12E^2$		$68E^2$	$6\frac{2}{11}$	$10E$	$9\frac{1}{11}$	$8E$	$5\frac{9}{11}$
$12E^2$	$13E^2$		$68E^2$	$5\frac{2}{3}$	$10E$	$8\frac{1}{3}$	$8E$	$5\frac{1}{3}$
$13E^2$	$14E^2$		$68E^2$	$5\frac{3}{13}$	$10E$	$7\frac{9}{13}$	$8E$	$4\frac{12}{13}$
$14E^2$	$15E^2$		$73E^2$	$5\frac{3}{14}$	$11E$	$8\frac{9}{14}$	$8E$	$4\frac{4}{7}$
$15E^2$	$16E^2$		$80E^2$	6	$12E$	$9\frac{3}{5}$	$8E$	$4\frac{4}{15}$

fact that, in this case, only a more restricted form of permutations applies. It would also be possible to make use of more than one generator as we do in the Hilbert case, thus, also gaining curves with somewhat better locality properties than Hilbert ones (see [2] and [5] for 2D and 3D cases). However, this probably would extremely complicate the combinatorial analysis while only obtaining a modest improvement in locality properties. Our paper lays the basis for several further research directions. So it could be tempting to determine the number of structurally different r-dimensional curves with Hilbert property for $r > 3$. Moreover, a (mechanized) analysis of locality properties of r-dimensional ($r > 3$) Hilbert curves is still to be done (see [15]). An analysis of the construction of more complicated curves using more generators or different permutations for different levels remains open.

Acknowledgment

We thank two anonymous referees of *Theory of Computing Systems* whose comments helped to improve the presentation of this paper significantly.

References

[1] J. Alber. Locality properties of discrete space-filling curves: results with relevance for computer science. Studienarbeit, Universität Tübingen, 1997 (in German). Available through http://www-fs.informatik.uni-tuebingen.de/∼alber/publications.html.

[2] T. Asano, D. Ranjan, T. Roos, E. Welzl, and P. Widmayer. Space-filling curves and their use in the design of geometric data structures. *Theoretical Computer Science*, 181:3–15, 1997.

[3] A. R. Butz. Space filling curves and mathematical programming. *Information and Control*, 12:314–330, 1968.

[4] A. R. Butz. Convergence with Hilbert's space filling curve. *Journal of Computer and System Sciences*, 3:128–146, 1969.

[5] G. Chochia and M. Cole. Recursive 3D mesh indexing with improved locality. Technical report, University of Edinburgh, 1997. Short version appeared in the *Proceedings of High-Performance Computing and Networking* 1997 (*HPCN* '97), Lecture Notes in Computer Science 1225, Springer-Verlag, Berlin.

[6] G. Chochia, M. Cole, and T. Heywood. Implementing the Hierarchical PRAM on the 2D mesh: analyses and experiments. In *Proceedings of the IEEE Symposium on Parallel and Distributed Processing*, pages 587–595, 1995.

[7] C. Faloutsos and S. Roseman. Fractals for secondary key retrieval. In *Proceedings of the* 8*th ACM Symposium on Principles of Database Systems*, pages 27–252, 1989.

[8] U. Feige and R. Krauthgamer. Stereoscopic families of permutations, and their applications. In *Proceedings of the* 5*th IEEE Israel Symposium on the Theory of Computing and Systems*, pages 85–95, 1997.

[9] C. Gotsman and M. Lindenbaum. On the metric properties of discrete space-filling curves. *IEEE Transactions on Image Processing*, 5(5):79–797, May 1996.

[10] D. Hilbert. Über die stetige Abbildung einer Linie auf ein Flächenstück. *Mathematische Annalen*, 38:59–60, 1891.

[11] P. Indyk, R. Motwani, P. Raghavan, and S. Vempala. Locality-preserving hashing in multidimensional spaces. In *Proceedings of the* 29*th ACM Symposium on Theory of Computing*, pages 618–625, 1997.

[12] H. V. Jagadish. Linear clustering of objects with multiple attributes. *ACM SIGMOD Record*, 19(2):332–32, June 1990.

[13] H. V. Jagadish. Analysis of the Hilbert curve for representing two-dimensional space. *Information Processing Letters*, 62:17–22, 1997.

[14] I. Kamel and C. Faloutsos. Hilbert R-tree: an improved R-tree using fractals. In *Proceedings of the* 20*th International Conference on Very Large Data Bases*, pages 500–509, 1994.

[15] R. Niedermeier, K. Reinhardt, and P. Sanders. Towards optimal locality in mesh-indexings. In B. S. Chlebus and L. Czaja, editors, *Proceedings of the* 11*th Conference on Fundamentals of Computation Theory*, Lecture Notes in Computer Science 1279, pages 36–375, Kraków, Poland, Sept. 1997. Springer-Verlag, Berlin. A long version is available as technical report IB 12/97, Universität Karlsruhe. Submitted to *Discrete Applied Mathematics*.

[16] J.-C. Pérez and E. Vidal. An approximate nearest neighbours search algorithm based on the extended general spacefilling curves heuristic. In *Advances in Pattern Recognition* (*SSPR* '98 *and SPR* '98), pages 697–706, Sydney, Australia, Aug. 1998. Lecture Notes in Computer Science 1451, Springer-Verlag, Berlin.

[17] L. K. Platzman and J. J. Bartholdi. Spacefilling curves and the planar travelling salesman problem. *Journal of the ACM*, 36(4):719–737, 1989.

[18] H. Sagan. A three-dimensional Hilbert curve. *International Journal of Mathematics, Education, Science, and Technology*, 2(4):541–545, 1993.

[19] H. Sagan. *Space-Filling Curves*. Universitext. Springer-Verlag, Berlin, 1994.

[20] P. Sanders and T. Hansch. Efficient massively parallel quicksort. In *Proceedings of the* 4*th International Symposium on Solving Irregularly Structured Problems in Parallel*, pages 13–24. Lecture Notes in Computer Science 1253, Springer-Verlag, Berlin, June 1997.

Received April 14, 1999, *and in final form March* 7, 2000. *Online publication June* 16, 2000.