BSC6882: Game Theory

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1 Game theory payoff matrices

Game theory is the field of study which analyzes interactions between various "players" which implement "strategies." The strategy chosen by each player (e.g. 1, 2, 3, ...) is associated with costs and/or benefits that depend on the strategy of competing players. If player 1 chooses strategy A, the payoff (the net outcome of all benefits minus all costs) will depend on whether player 2 chooses strategy 1, 2, 3, etc.

The game is thus structured as a matrix of payoffs. The row is the strategy chosen by player 1, and the column is the strategy chosen by our opponent, player 2. The following matrix describes the payoffs associated with a game of two strategies (1,2). If player 1 chooses strategy i and player 2 chooses strategy j, then player 1 receives the payoff a_{ij} while player 2 receives payoff a_{ji}.

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{1}$$

2 Example payoff matrix

$$M = \begin{bmatrix} 3 & 6 & 5 & 3 \\ 5 & 9 & 1 & 2 \\ 3 & 11 & 5 & 1 \\ 2 & 7 & 6 & 4 \end{bmatrix} \tag{2}$$

Classical game theory analyzes the behavior of rational individuals attempting to maximize their own payoff in a group of other rational individuals.

This matrix in particular is interesting because there are rational explanations for why an individual may choose any of the 4 different strategies.

2.0.1 Strategy 1: Min-max strategy

For example, strategy 1 (row 1) is known as a min-max strategy: an individual should choose 1 if they desire to minimize their maximum loss scenario. The worst case payoff is 3 (if player 2 chooses 1 or 4). This is the best worst-case strategy of the 4 (e.g. strategy 2's worth case is payoff of 1; strategy 3's worst case is payoff of 1; strategy 4's worst case is 2).

2.0.2 Strategy 2: Maximize total payoff

Two players may wish to cooperate, and maximize their total, collective payoff ($P = a_{ij} + a_{ji}$). Here, we scan the payoff matrix to find the row-column pair which maximizes P. This is strategy 2, which nets each player 9 ($P = 9 + 9$).

2.0.3 Strategy 3: Take advantage of cooperators

The previous strategy requires cooperation by the opponent, and is subject to cheaters! For example, if I suspect my opponent to be a "sucker" who will attempt to cooperate, I can refuse to cooperate and instead maximize my own selfish gain. Opponent chooses strategy 2, (column 2), and I find that I should selfishly choose strategy 3, resulting in a payoff of 11 ($11 > 9$). My opponent is a sucker for choosing to cooperate and is left with payoff of 1 (row 2, column 3).

2.0.4 Strategy 4: Nash equilibrium (no regrets)

Knowing the strategies of the other players, and treating the strategies of the other players as set in stone, can I benefit by changing my strategy?

If I know my opponent will choose 1, I choose 2 (max of column 1). If I know my opponent will choose 2, I choose 3. If I know my opponent will choose 3, I choose 4. If I know my opponent will choose 4, I choose 4. Only the fourth scenario represents a "no regrets" policy where I will not wish to change strategy after play, in any circumstance.

3 Replicator equation

In the previous section, we have reviewed classical game theory, where individuals choose a given strategy amongst competing individuals. Here, we transition to talking about "evolutionary game theory" wherein individuals (i.e. cancer cells) adhere to a fixed strategy, and replicate in proportion to their strategy's payoff. Rather than two players competing with different strategies of choice (1, 2, 3, ...), we have N players each with a fixed strategy. We monitor this population of N players to predict the proportion of individuals that adhere to each strategy, i, over time: $x_i(t)$.

The most common equation in evolutionary game theory (EGT) is the replicator equation. For two strategies ($i \in [1,2]$), the governing equations are:

$$\dot{x}_1 = x_1(f_1 - \phi) \tag{3}$$

$$\dot{x}_2 = x_2(f_2 - \phi) \tag{4}$$

Intuitively, the growth rate of each population is positive if the population's fitness f_i is above the average fitness of all cell types, *phi*. Fitness is given by:

$$f_1 = (Ax)_1 = a_{11}x_1 + a_{12}x_2 \tag{5}$$

$$f_2 = (Ax)_2 = a_{21}x_1 + a_{22}x_2 \tag{6}$$

and,

$$\phi = x_1 f_1 + x_2 f_2 \tag{7}$$

4 Availability of code

All the Matlab code used to generate the plots in these lecture notes can be found online[1].

References

1. West, J. BSC-6882 and BSC-6883 lecture notes. https://github.com/jeffreywest/IMO-lecture-notes (2023).