Parameter Estimation of Coupled Oscillation System through Markov Chain Monte Carlo Method

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Introduction:

Parameter estimation is crucial in scientific research, especially in data analysis and modeling. Traditional optimization techniques, such as the least square method, often face challenges with increasing parameter dimensionality. The Markov Chain Monte Carlo (MCMC) method, with its Metropolis-Hastings algorithm, emerges as a powerful alternative, finding applications in areas like gravitational wave data analysis.

Using coupled oscillators as a model, which provides insights into lattice vibrations, we built an experimental system with springs of unknown constants. Through the MCMC method, we achieved greater efficiency in estimating these constants compared to traditional methods.

Experimental Equipment:

Our experimental equipment can be interpreted as shown in FIG. (1), consists of a standard pulley track and two carts.

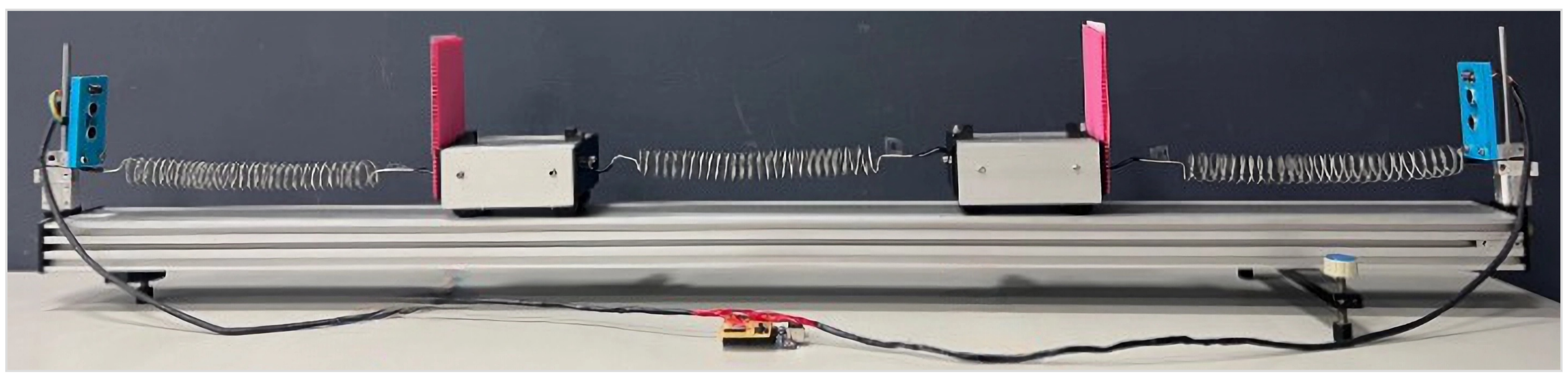


FIG. (1) Experimental setup of the Coupled Oscillators system

These carts are interconnected using three homemade springs made by white iron wire, forming a coupled oscillator system. To capture the oscillators' motion, we employed two ultrasonic distance sensors, housed in 3D-printed frames and connected to an Arduino circuit board. Furthermore, we developed a customized webpage hosted on GitHub for real-time visualization of the displacement data recorded by these sensors. This setup enabled us to measure and document the dynamics of the coupled oscillators effectively.

Analytic Model:

Our experimental equipment can be interpreted as shown in following FIG.(2).

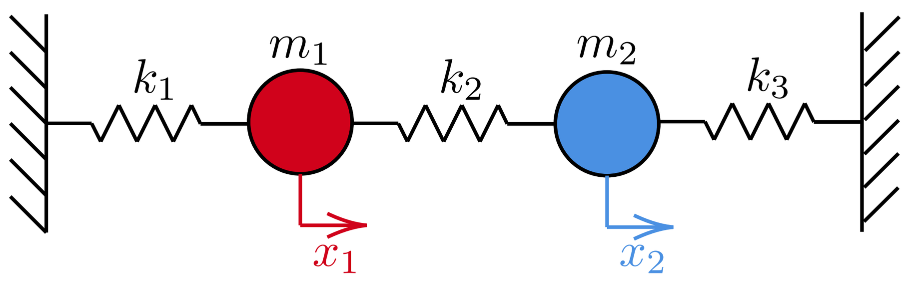


FIG. (2) Coupled Oscillation system with three springs and two masses

The system's Lagrangian is represented by

$$L = \frac{1}{2}\sum_{i=1}^{2} m_i \frac{d^2 x_i}{dt^2} - \frac{k_1 x_1^2 + k_2 (x_1 - x_2)^2 + k_3 x_2^2}{2},$$

where m_i is the mass of i^{th} oscillator and $(k_1,\ k_2,\ k_3)$ are the spring constants. Using Euler-Lagrange equation, we obtain a set of the second-order differential equations

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

After some algebra, the general expressions for the time-dependent displacement $x_1(t)$ and $x_2(t)$ can be exactly solved.

Desired Distribution:

Given the system parameters and initial conditions, the displacements $x_1(t)$ and $x_2(t)$ of the coupled oscillators can be determined. After measuring the masses and initial positions of the oscillators, the real-time data can be used to estimate the unknown spring constants k_1, k_2 and k_3. The fitting error is defined as:

$$\text{err}(k_1, k_2, k_3) = \sum_{i=1}^{2}\sum_{j=0}^{N-1} \frac{(x_i(t_j) - x_{i,j})^2}{N},$$

where N is the total time steps. The desired distribution for MCMC is

$$p(k_1, k_2, k_3) = \exp(-\text{err}(k_1, k_2, k_3)).$$

Optimized values for k_1, k_2 and k_3 correspond to the distribution's maximum. To enhance optimization accuracy, we applied a Gamma correction to improve the distribution's contrast. The corrected distribution is given by:

$$P = p^{\gamma}, \quad p \in (0, 1]$$

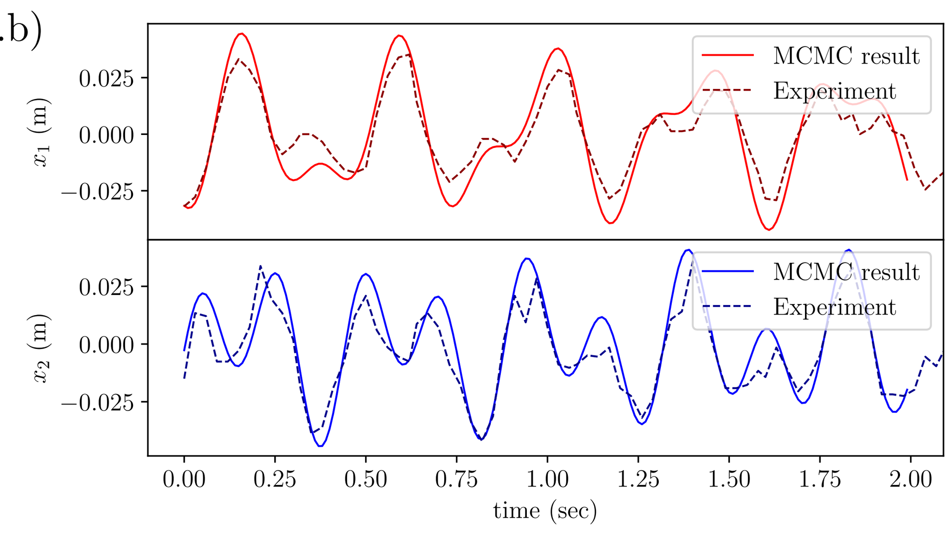
This Gamma correction ensures a clearer distinction between peak and valley values in the distribution, aiding in the optimization process.

MCMC method and Result:

Using the defined distribution function P, we applied the Metropolis-Hastings algorithm within the MCMC method to explore the parameter space of $(k_1,\ k_2,\ k_3)$. After sampling 100,000 combinations, we obtained a simulated posterior distribution, visualized using the "`Corner.py`" package, as shown in FIG. (3.a). The distributions for the spring constants k_1, k_2 and k_3 resemble normal distributions, ensuring accurate estimations.

The comparison between experimental measurements and MCMC simulation result is presented in FIG. (3.b).

(3.b)



(3.a)

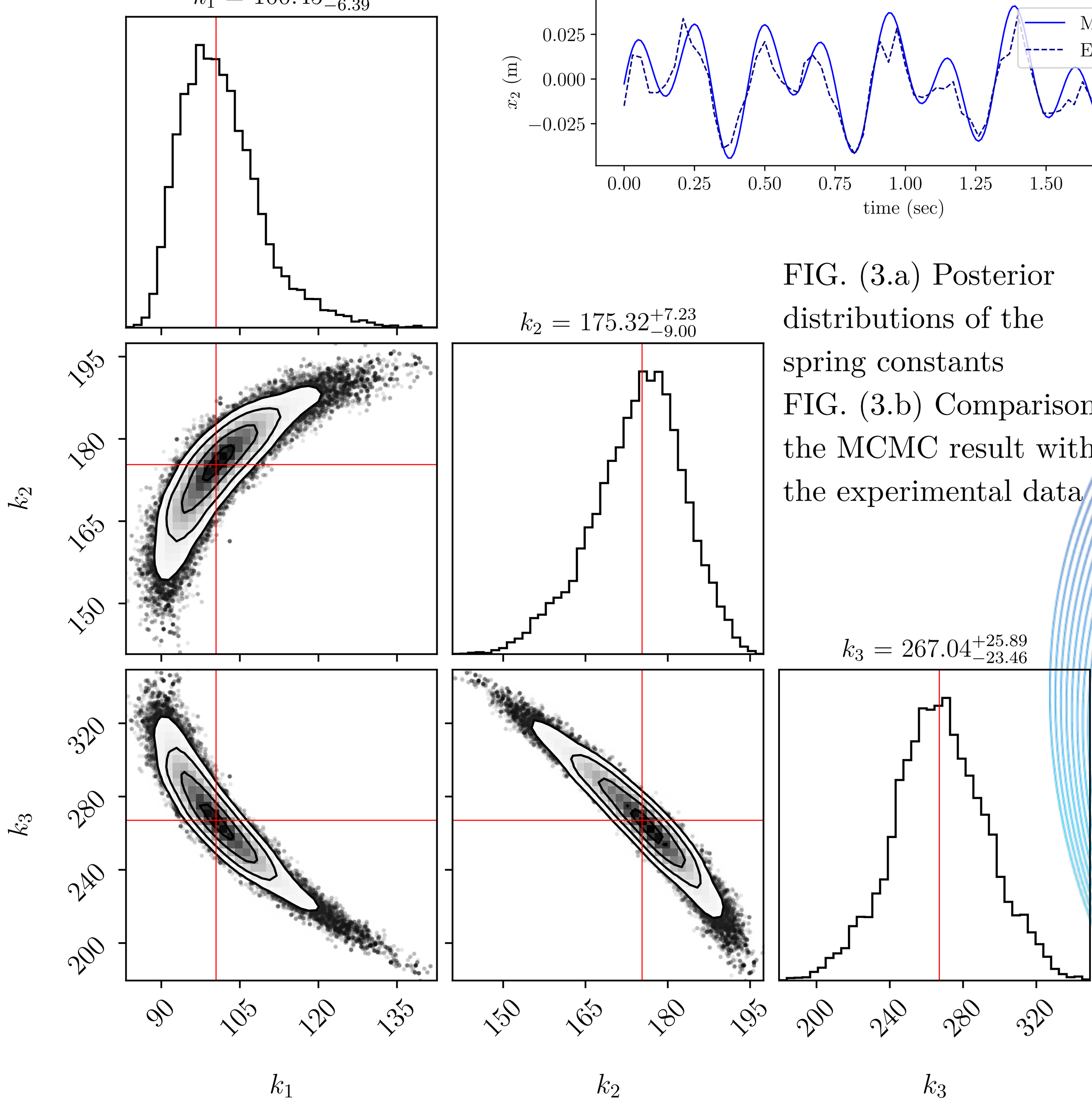


FIG. (3.a) Posterior distributions of the spring constants
FIG. (3.b) Comparison of the MCMC result with the experimental data

Conclusion:

We applied the MCMC method, particularly the Metropolis-Hastings algorithm, to the coupled oscillators system. Through this method, we explored the parameter space of the spring constants k_1, k_2 and k_3, achieving a posterior distribution resembling a normal distribution.

Despite some experimental discrepancies, the MCMC results aligned well with the experimental data. Our findings highlight the MCMC method's superiority over traditional techniques in parameter estimation for complex systems.

Future endeavors will aim to refine the experimental process and delve deeper into the system's dynamics by considering springs with varied elastic constants. This will further our understanding of coupled oscillators and pave the way for enhanced models and analytical methods.

Reference:

[1] W.K. Hastings, "Monte Carlo sampling methods using Markov chains and their applications," Biometrika, vol. 57, no. 1, pp. 97–109, 1970.
[2] Zenodo Archive. 2016. "Corner.py: Scatterplot Matrices in Python." http://dx.doi.org/ 10.5281/zenodo.53155. doi:10.5281/zenodo.53155.

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