Physics, Classical Dynamics
Couple Oscillator #June 19, 2023

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1 System of Odinary Differential Equation

1.1 General Solution

For the System

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1}$$

in the form $\ddot{\vec{x}} = \mathcal{F}\vec{x}$, we solve the eigen problem for matrix $\mathcal{F}$

$$\left(\mathcal{F} - \lambda\mathcal{I}\right)\mu = 0, \tag{2}$$

we obtain the eigenvalues

$$\lambda_{1,2} = \frac{\operatorname{tr}(\mathcal{F}) \pm \sqrt{\operatorname{tr}(\mathcal{F})^2 - 4\det(\mathcal{F})}}{2}, \tag{3}$$

where the trace and determinant of matrix are

$$\begin{aligned} \operatorname{tr}(\mathcal{F}) &= -\frac{m_2k_1 + (m_1+m_2)\,k_2 + m_1k_3}{m_1m_2}, \\ \det(\mathcal{F}) &= \frac{k_1k_2 + k_2k_3 + k_3k_1}{m_1m_2}. \end{aligned} \tag{4}$$

Also, the corresponding eigenvectors are

$$\vec{\mu}_i = \begin{pmatrix} m_2\lambda_i + (k_2+k_3) \\ k_2 \end{pmatrix}, \quad i = 1, 2. \tag{5}$$

Since the general solution for the equation $\ddot{x} = \lambda x$ is $x\,(t) = C_1e^{\sqrt{\lambda}t} + C_2e^{-\sqrt{\lambda}t}$, we define

$$\omega_i = \pm\sqrt{\lambda_i}, \quad i = 1, 2 \tag{6}$$

the general solution for this system is

$$\vec{x}\,(t) = \sum_{i=1}^{2}\left(A_ie^{\omega_it} + B_ie^{-\omega_it}\right)\vec{\mu}_i. \tag{7}$$

Also, we define the velocity vector

$$\vec{v}\,(t) = \frac{d\vec{x}}{dt} = \sum_{i=1}^{2}\left(A_i\omega_ie^{\omega_it} - B_i\omega_ie^{-\omega_it}\right)\vec{\mu}_i. \tag{8}$$

1.2 Initial Value Problem

For initial state at $t = 0$, we have initial position vector $\vec{x}\,(0)$ and initial velocity vector $\vec{v}\,(0)$, which are equal to

$$\begin{aligned} \vec{x}\,(0) &= \sum_{i=1}^{2} (A_i + B_i)\,\vec{\mu}_i, \\ \vec{v}\,(0) &= \sum_{i=1}^{2} (A_i\omega_i - B_i\omega_i)\,\vec{\mu}_i. \end{aligned} \tag{9}$$

repectively. If we expand these 2 condition, we may have

$$\begin{aligned} \vec{x}\,(0) &= (A_1 + B_1)\,\vec{\mu}_1 + (A_2 + B_2)\,\vec{\mu}_2 \\ \vec{v}\,(0) &= (A_1\omega_1 - B_1\omega_1)\,\vec{\mu}_2 + (A_2\omega_2 - B_2\omega_2)\,\vec{\mu}_2 \end{aligned}, \tag{10}$$

which can be ewwriten in matrix form

$$\begin{aligned} \vec{x}\,(0) &= \begin{pmatrix} \vec{\mu}_1 & \vec{\mu}_2 \end{pmatrix} \begin{pmatrix} A_1 + B_1 \\ A_2 + B_2 \end{pmatrix} \\ \vec{v}\,(0) &= \begin{pmatrix} \vec{\mu}_1 & \vec{\mu}_2 \end{pmatrix} \begin{pmatrix} A_1\omega_1 - B_1\omega_1 \\ A_2\omega_2 - B_2\omega_2 \end{pmatrix} \end{aligned}. \tag{11}$$

Since we need to obtain the value of A_i and B_i, using the same matrix to reduce equations

$$\begin{aligned} \vec{x}\,(0) &= \begin{pmatrix} \vec{\mu}_1 & \vec{\mu}_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \vec{v}\,(0) &= \begin{pmatrix} \vec{\mu}_1 & \vec{\mu}_2 \end{pmatrix} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}. \tag{12}$$

Defining following 2 by 2 matrices in order to reduce the equation

$$\hat{\mu} = \begin{pmatrix} \vec{\mu}_1 & \vec{\mu}_2 \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \quad \text{and} \quad \hat{C} = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}. \tag{13}$$

Plug in, we have

$$\vec{x}\,(0) = \hat{\mu}\hat{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}\,(0) = \hat{\mu}\hat{\omega}\hat{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{14}$$

Applying the correponding inverse matrix for two equation, we obtain

$$\hat{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \hat{\mu}^{-1}\vec{x}\,(0), \quad \hat{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \hat{\omega}^{-1}\hat{\mu}^{-1}\vec{v}\,(0). \tag{15}$$

Also we could stack two equation into one single matrix equation

$$\hat{C} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \hat{\mu}^{-1}\vec{x}\,(0) & \hat{\omega}^{-1}\hat{\mu}^{-1}\vec{v}\,(0) \end{pmatrix}. \tag{16}$$

Applying the inverse matrix from right, solving that

$$\hat{C} = \frac{1}{2} \begin{pmatrix} \hat{\mu}^{-1}\vec{x}\,(0) & \hat{\omega}^{-1}\hat{\mu}^{-1}\vec{v}\,(0) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{17}$$

1.3 Further Calculation

From above result, given a initial condition $\vec{x}(0)$ and $\vec{v}(0)$, we colud obtain the general solution for this system, which is

$$\vec{x}(t) = \sum_{i=1}^{2} \left(A_i e^{\omega_i t} + B_i e^{-\omega_i t} \right) \vec{\mu}_i, \tag{18}$$

where the coefficients can be solved by

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \hat{\mu}^{-1}\vec{x}(0) & \hat{\omega}^{-1}\hat{\mu}^{-1}\vec{v}(0) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{19}$$

After a little bit calculation, matrices $\hat{\mu}^{-1}$ and $\hat{\omega}^{-1}$ can be obtained

$$\begin{aligned} \hat{\omega}^{-1} &= \begin{pmatrix} 1/\omega_1 & 0 \\ 0 & 1/\omega_2 \end{pmatrix} \\ \hat{\mu}^{-1} &= \frac{1}{k_2 m_2 (\lambda_1 - \lambda_2)} \begin{pmatrix} k_2 & -m_2\lambda_2 - (k_2 + k_3) \\ -k_2 & m_2\lambda_1 + (k_2 + k_3) \end{pmatrix} \end{aligned} \tag{20}$$