Class Notes
Introduction to fluid mechanics

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Contents

1 Tensors

Tensors is *generalization of vector.*

- What;s tensors
- Tensors Algebra
- Tensor differentiation
- Tensor integration

1.1 Introduction

- **Newtonian mechanics** $\Rightarrow$ ODE
- **Quantum physics** $\Rightarrow$ Linear Algebra
- **Relativity** $\Rightarrow$ Tensor Calculus

1.1.1 Vector

Ex. Ch2 - Fig2 $\vec{A}$ is a vector

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 = A_1' \hat{e}_1' + A_2' \hat{e}_2' \tag{1}$$

and

$$\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = R(\theta) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{2}$$

Definition A vector $\vec{A}$ is an object that tranforms as the same as $\vec{r}$ under axis rotation.

In matrix notation:

$$A_i' = \sum_{j=1}^{n} R_{ij}(\theta) A_j, \quad \text{in } \mathbb{R}^n \tag{3}$$

We denote as

$$A_i' = R_{ij} A_j, \tag{4}$$

also called Einstein's convention (*Summing over repested indeces.*)

1.1.2 Tensor

Definition

A rank-2 tensor A_{ij} is an object such that each index transforms like a vector. i.e.

$$A_{ij}' = R_{ik} R_{g\ell} A_{k\ell} \tag{5}$$

In general, a rank-n tensor $A_{i_1,\ldots,i_n}$ transformation as

$$A_{i_1,\ldots,i_n}' = R_{i_1,j_1} R_{i_2,j_2} \cdots R_{i_n,j_n} \cdot A_{j_1,\ldots,j_n} \tag{6}$$

Rmk:

- In $\mathbb{R}^n$, $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ are constant basis.
- In polar coordinate system, basis are not constant
  Ch1 - Fig 2 polar

1.2 Manifold and coordinate

Manifold (mfld) 流形

What is the dimension of a manifold M ?
Fig3 - A manifold locally likes R^n
Definition
An n-dimension manifold M is something which locally look like $\mathbb{R}^n$.

$$\begin{aligned} x^i : & U \to \mathbb{R}^n \\ & p \mapsto x(p) \end{aligned} \tag{7}$$

Ex. $M = S^2$ (Two sphere) Fig4 - sphere

Rmk. If $(x^1, x^2, \ldots, x^n)$ be a set of n coordinates , which is non-degenerate if

$$p \longleftrightarrow (x^1, x^2, \ldots, x^n) \tag{8}$$

is 1-1 or *one-to-one mapping*. e.g. Fig4 - degenerate sphere

In overlapping region
Fig5 - overlapping

Ex. **stereo graphic projection** spherical coordinate θ, φ, notice that it is singular at $\theta = 0$ (north pole) and $\theta = \pi$ (south pole).

$$x^2 + y^2 + z^2 = 1 \tag{9}$$

Fig6 - projection

- U_N:

$$\begin{aligned} &(x, y, z-1) = \lambda(X, Y, -1) \\ &\Rightarrow \frac{x}{X} = \frac{y}{Y} = \frac{z-1}{-1} = \lambda \\ &\Rightarrow \begin{cases} X = \frac{x}{\lambda} = \frac{x}{1-z} \\ Y = \frac{y}{\lambda} = \frac{y}{1-z} \end{cases} \end{aligned} \tag{10}$$

- U_S:

$$\begin{aligned} &(x, y, z-1) = \lambda(X, Y, +1) \\ &\Rightarrow \frac{x}{X} = \frac{y}{Y} = \frac{z+1}{+1} = \lambda \\ &\Rightarrow \begin{cases} X = \frac{x}{\lambda} = \frac{x}{1+z} \\ Y = \frac{y}{\lambda} = \frac{y}{1+z} \end{cases} \end{aligned} \tag{11}$$

  Then, the manifold $M = U_N \cup U_S$, where $U_N \cap U_S = \phi$. And we called

- Two subset U_N and U_S to be a *patch* of covering (覆蓋).
- The set $\{U_N, U_S\}$ is the Altas.

  Claim For a point

$$p \in U_N \cap U_S \tag{12}$$

  we have

$$\begin{cases} X' = \frac{X}{X^2+Y^2} \\ Y' = \frac{Y}{X^2+Y^2} \end{cases} \tag{13}$$

  coordinate transformation between 2 system.

1.2.0.1 Tensors on M

The tensors on M are geometric quantities that obey coordinate transformation in overlapping region.

1.2.0.2 Curve on M

Fig7 - curves on M

In general, an m-dimension object in M can be parametrized by

$$x^a = x^a\left(u^1, u^2, \ldots, u^m\right), \quad a = 1, 2, \ldots, n \tag{14}$$

In particular, if $m = n - 1$, i.e.

$$x^a = x^a\left(u^1, u^2, \ldots, u^{n-1}\right), \quad a = 1, 2, \ldots, n \tag{15}$$

is a *hypersurface*. We also using a function $f(x^1, x^2, \ldots, x^n) = 0$.

1.2.0.3 Tangent space

For a mani fold M

Fig - tangent space

If M is locally the same as $\mathbb{R}^n$, introducting coordinate patchs

$$\{U_1, U_2, \ldots\} = \text{atlas}. \tag{16}$$

In the overlapping $U_\alpha \cap U_\beta \neq \phi$, $\alpha \neq \beta$ Fig - overlapping region we have $x^a = x^a(y^1, y^2, \ldots, y^n)$, $a = 1, 2, \ldots, n$.

1.2.1 ⊙ Tensor transformations of coordinates

Consider a change of coordinates

$$x^a \mapsto x'^a = f^a(x^1, x^2, \ldots, x^n) \equiv x'^a(x) \tag{17}$$

which means x' is a function of x. We define

Definition A matrix

$$J'^a{}_b \equiv \frac{\partial x'^a}{\partial x^b} \tag{18}$$

called *Jocobian matrix*, and

$$J' \equiv |J'^a_b| = \det(J^a_b). \tag{19}$$

Also, by implicit function theorem $x^a = x'^a(x')$, we have

$$J^a_b = \frac{\partial x'^a}{\partial x^b}. \tag{20}$$

In fact $J'^a_b J^b_c = \delta^a_c$, then $J' = \det(J^a_b) = 1/J$. (Notice δ^a_b is the Kronecker delta function.)

Rmk:

$$J'^a_b = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \frac{\partial x'^2}{\partial x'^1} & \frac{\partial x^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x'^n}{\partial x^1} & \frac{\partial x'^n}{\partial x^2} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{pmatrix} \tag{21}$$

Rmk: $\frac{\partial x^a}{\partial x'^b}$ can be viewd as coefficients of infinitesmall defferentials, then

$$\begin{aligned} dx'^a &= \frac{\partial x'^a}{\partial x^1}dx^1 + \frac{\partial x'^a}{\partial x^2}dx^2 + \cdots + \frac{\partial x'^a}{\partial x^n}dx^n \\ &= \sum_{b=1}^{n} \frac{\partial x^a}{\partial x'^b}dx^b \\ &= \frac{\partial x^a}{\partial x'^b}dx^b, \quad \text{Einstein convension} \end{aligned} \tag{22}$$

We shall classify geometric tensors quantities by transformation properties.

Definition A *contravariant* vector (rank-1) is a set of quantities X^a defined on $p \in M$, such that under coordinate transformation:

$$x^a \mapsto x'^a(x), \tag{23}$$

we have

$$X'^a = \frac{\partial x'^a}{\partial x^b}X^b \tag{24}$$

Fig - point p in overlapping patch

e.g. tangent vector at p in M. Fig - tangent vector at p on M For f is a function defined on the curve γ, where

$$f = f(x^a(u)), \tag{25}$$

and

$$\frac{df}{du} = \frac{\partial f}{\partial x^a}\frac{dx^a}{du} \tag{26}$$

holds for any f. Therefore

$$\frac{d}{du} = \left(\frac{dx^a}{du}\right)\frac{\partial}{\partial x^a}, \tag{27}$$

and here where $\partial/\partial x^a$ is like a basis. Notice that, for $X'^a = dx^a/du$

$$X'^a = \frac{dx'^a}{du} = \frac{dx'^a}{dx^b}\frac{dx^b}{du} = \frac{dx'^a}{dx^b}X^a, \tag{28}$$

therefore, dx^a/du is a contravariant vector.

How about higher rank tensors? e.g. rank-2

$$X'^{ab} = \frac{\partial x^a}{\partial x^c}\frac{\partial x^b}{\partial x^d}X^{cd}. \tag{29}$$

In general, for rank-n tensor

$$X'^{i_1,\dots,i_n} = \frac{\partial x^{i_1}}{\partial x^{j_1}}\frac{\partial x^{i_2}}{\partial x^{j_2}}\cdots\frac{\partial x^{i_3}}{\partial x^{j_3}}X'^{j_1,\dots,j_n} \tag{30}$$

Rmk: For a scalar ϕ (without indices or rank-0), we have

$$\phi'(x') = \phi(x) \tag{31}$$

Here, notice ϕ' is not the differentiation of function ϕ.

Definition A covarariant vector (rank-1) is a set of quantities X_a defined at $p \in M$, such that under coordinate transformation, we have

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b. \tag{32}$$

e.g. a gradient of ϕ

$$\frac{\partial \phi}{\partial x'^a} = \frac{\partial \phi}{\partial x^b}\frac{\partial x^b}{\partial x'^a}, \tag{33}$$

Also, for rank-2 covariant vector

$$X'_{ab} = \frac{\partial x^c}{\partial x^a}\frac{\partial x^d}{\partial x^b} X_{cd} \tag{34}$$

If a tensor has a form:

$$X^{a_1,\dots,a_p}_{b_1,\dots,b_q}, \tag{35}$$

we called a *mixed-type* tensor denoted by (p, q) type (up,down).

In summary:

tensor	Type
ϕ	$(0, 0)$
X^a	$(1, 0)$
X_a	$(0, 1)$
X^{ab}	$(2, 0)$
X_{ab}	$(0, 2)$

1.2.1.1 Coordinate-independent

Physical laws are described by tensor equatios which are coordinates-independent, e.g. Suppose a law is written in the form

$$X_{ab} = Y_{ab} \tag{36}$$

in x-system, X and Ymust the same type, since the physical laws must valid in any system. We rewrite the equation to $X_{ab} - Y_{ab} = 0$, and this equation must holds for $X'_{cd} - Y'_{cd} = 0$, and since the transformation is

$$\begin{aligned} X'_{cd} &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^d}{\partial x'^b} X_{cd} \\ Y'_{cd} &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^d}{\partial x'^b} Y_{cd} \end{aligned} \tag{37}$$

so that

$$(X'_{ab} - Y'_{ab}) = \frac{\partial x^a}{\partial x'^c}\frac{\partial x^d}{\partial x'^b}(X_{ab} - Y_{ab}) = 0 \tag{38}$$

example Maxwell's equations for $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$ is given by

$$F_{\mu\nu} = -F_{\mu\nu}, \tag{39}$$

where the tensor is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{40}$$

and for $j^\mu = \left(\rho c, \vec{J}\right)$, we have

$$\partial_\mu F^{\mu\nu} = j^{nu}, \tag{41}$$

rand-1 controvariant equation under Lorentz transformation $x'^\mu = x^\mu(x)$.

1.2.2 ⊙ Tensor fiels

> **Definition** Over M, we asign smoothly every point a tensor, which forms a tensor field.

e.g. a vector field
Fig - vector field
Now, we denote as

$$X^{ab}(x), \quad x \in M. \tag{42}$$

1.2.3 ⊙ Elementary operation of tensors

1. Addition For three $(1,2)$-type tensors X^a_{bc}, Y^a_{bc} and X^a_{bc}, we have

$$X^a_{bc} = Y^a_{bc} + Z^a_{bc} \tag{43}$$

2. scalar multiplication For $k \in \mathbb{R}$, we have kX^a_{bc}

3. symmetrization

$$\begin{aligned} X_{ab} &= \frac{1}{2}\left(X^{ab} + X^{ba}\right) + \frac{1}{2}\left(X^{ab} - X^{ba}\right) \\ &= X_{(a,b)} + X_{[a,b]} \end{aligned} \tag{44}$$

then

$$\begin{aligned} X_{(a,b)} &= +X_{(b,a)} \\ X_{[a,b]} &= -X_{[b,a]} \end{aligned} \tag{45}$$

and $X_{(a,b)}$ called symmetric tensor, $X_{[a,b]}$ called anti-symmetric tensor.

For rank-n, $n \geq 3$, e.x

$$X^{a_1,a_2,\ldots,a_n}, \quad a_1, a_2, \ldots, a_n = 1, 2, \ldots, n \tag{46}$$

we have n^3 elements. We define

$$\begin{aligned} X_{(a_1,a_2,\ldots,a_N)} &= \frac{1}{n!}\sum \text{all permutations} \\ X_{[a_1,a_2,\ldots,a_N]} &= \frac{1}{n!}\sum \text{alternating permutations} \end{aligned} \tag{47}$$

Ex: $n = 3$, for anti-symmetric tensor

$$X_{[a,b,c]} = \frac{1}{6}\left(X_{abc} + X_{bca} + X_{cab} - X_{bac} - X_{cba} - X_{acb}\right) \tag{48}$$

For aymmetric tensor

$$X_{(a,b,c)} = \frac{1}{6}(+ + + + ++) \tag{49}$$

Check: Also, the symmetrized tensor still remain the properties of original tensor, e.x.

$$X'_{(a,b)} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}X_{(c,d)} \tag{50}$$

4. tensor product Ex: For $(1,1)$-type tensor Y^a_b and $(0,2)$-type tensor Z_{cd}, we want to construct a $(1,3)$-type tensor.

Define

$$X^a_{bcd} = Y^a_b Z_{cd} \tag{51}$$

Check:

1. $Y'^a_b = \dfrac{\partial x'^a}{\partial x^e}\dfrac{\partial x'^f}{\partial x^b}Y^e_f$

2. $Z'_{cd} = \dfrac{\partial x^g}{\partial x'^c}\dfrac{\partial x^h}{\partial x'^d}Z^{gh}$

   then

$$\begin{aligned} X'^a_{bcd} &= Y'^a_b Z'_{cd} \\ &= \frac{\partial x'^a}{\partial x^e}\frac{\partial x'^f}{\partial x^b}\frac{\partial x^g}{\partial x'^c}\frac{\partial x^h}{\partial x'^d}Y^e_f Z^{gh} \end{aligned} \tag{52}$$

5. tensor contraction

contraction means summing over some indices

$$X^a_{bcd} \quad \longrightarrow \quad X^c_{bcd} \tag{53}$$

that is $(1,3)$-type to $(0,2)$-type. That is

$$X^c_{acd} = X^1_{a1d} + X^2_{a2d} + \cdots + X^n_{and} \tag{54}$$

Notice that repeated need to be sum, so it is not a $(1,3)$-type anymore.

Check:

$$\begin{aligned}
X'^{c}_{bcd} &= \frac{\partial x'^c}{\partial x^e}\frac{\partial x^f}{\partial x'^b}\frac{\partial x^g}{\partial x'^c}\frac{\partial x^h}{\partial x'^d}X^e_{fgh} \\
&= \left(\frac{\partial x'^c}{\partial x^e}\frac{\partial x^g}{\partial x'^c}\right)\frac{\partial x^f}{\partial x'^b}\frac{\partial x^h}{\partial x'^d}X^e_{fgh} \\
&= \delta^g_e \frac{\partial x^f}{\partial x'^b}\frac{\partial x^h}{\partial x'^d}X^e_{fgh} \\
&= \frac{\partial x^f}{\partial x'^b}\frac{\partial x^h}{\partial x'^d}X^g_{fgh}
\end{aligned} \tag{55}$$

1.2.4 Vector fields

Fig vector filed

Define A vector filed on M is an assignment (smoothly) of tangent at each point of M.

Recall:
Fig cirlce
In patch $\{x^a\}$, the tangent vector in this patch is expressed as

$$X = X^a \frac{\partial}{\partial x^a} = \underbrace{X^a}_{\text{component}} \underbrace{\frac{\partial}{\partial x^a}}_{\text{basis}} = X'^b \frac{\partial}{\partial x'^b} \tag{56}$$

Similar to $\vec{A} = \sum_{i=1}^{3} a_i \hat{e}_i = \sum_{j=1}^{3} a'_j \hat{e}'_j$

we then check

$$\frac{\partial}{\partial x^a} = \frac{\partial x'^b}{\partial x^a}\frac{\partial}{\partial x'^b} \tag{57}$$

so

$$X = X^a \frac{\partial}{\partial x^a} = X^a \frac{\partial x'^b}{\partial x^a}\frac{\partial}{\partial x'^b} = X'^b \frac{\partial}{\partial x'^b} \tag{58}$$

so the component is a controvariant.

We need to clearify the notation. $T_p(M)$ is the tangent space at p, then

$$T(M) = \bigcup_p T_p(M) \tag{59}$$

is a vector field, where $X \in T(M)$ and $X\Big|_p = T_p(M)$

Lie bracket Suppose X and Y are two vector field, i.e. locally

$$X = X^a \frac{\partial}{\partial x^a} \quad \text{and} \quad Y = Y^b \frac{\partial}{\partial x^b}. \tag{60}$$

Then the *Lie bracket* is still a vector field. **Check**: For a function f

$$\begin{aligned}
[X,Y]f &= (XY - YX)\, f \\
&= \left(XY^b \frac{\partial}{\partial x^b} - YX^a \frac{\partial}{\partial x^a} \right) f \\
&= \left(X^a \frac{\partial}{\partial x^a} \left(Y^b \frac{\partial f}{\partial x^b} \right) - Y^b \frac{\partial}{\partial x^b} \left(X^a \frac{\partial f}{\partial x^a} \right) \right) \\
&= X^a \left(\frac{\partial Y^b}{\partial x^a} \frac{\partial f}{\partial x^b} + Y^b \frac{\partial^2 f}{\partial x^a x^b} \right) - Y^b \left(\frac{\partial X^a}{\partial x^b} \frac{\partial f}{\partial x^a} + X^a \frac{\partial^2 f}{\partial x^b x^a} \right) \\
&= X^a \frac{\partial Y^b}{\partial x^a} \frac{\partial f}{\partial x^b} + X^a Y^b \frac{\partial^2 f}{\partial x^a x^b} - Y^b \frac{\partial X^a}{\partial x^b} \frac{\partial f}{\partial x^a} + Y^b X^a \frac{\partial^2 f}{\partial x^b x^a} \\
&= X^a \frac{\partial Y^b}{\partial x^a} \frac{\partial f}{\partial x^b} - Y^b \frac{\partial X^a}{\partial x^b} \frac{\partial f}{\partial x^a} \\
&= \left(X^a \frac{\partial Y^b}{\partial x^a} \frac{\partial}{\partial x^b} - Y^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial x^a} \right) f
\end{aligned} \tag{61}$$

Now we define the Lie bracket to be $Z = [X,Y] = Z^a \frac{\partial}{\partial x^a}$, since

$$Z = [X,Y] = X^b \frac{\partial Y^a}{\partial x^b} \frac{\partial}{\partial x^a} - Y^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial x^a} = \left(X^b \frac{\partial Y^a}{\partial x^b} - Y^b \frac{\partial X^a}{\partial x^b} \right) \frac{\partial}{\partial x^a} = Z^a \frac{\partial}{\partial x^a} \tag{62}$$

Let

$$Z^a = X^b \frac{\partial Y^a}{\partial x^b} - Y^b \frac{\partial X^a}{\partial x^b} \tag{63}$$

Check: Z^a is a contravariant vector.

Why $X = X^a \frac{\partial}{\partial x^a}$? For example a transformation between two 2-D system: FIg

$$\vec{r} = \sum x_i \hat{e}_i = \sum x_i' \hat{e}_i' \tag{64}$$

Since $x_i' = \vec{r} \cdot \hat{e}_i'$, so that

$$\begin{aligned}
x_1' &= \vec{r} \cdot \hat{e}_1' \\
&= \vec{r} \cdot (\cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_1) \\
&= \cos\theta\, x_1 + \sin\theta\, x_2 \\[1em]
x_2' &= \vec{r} \cdot \hat{e}_2' \\
&= \vec{r} \cdot (-\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_1) \\
&= -\sin\theta\, x_1 + \cos\theta\, x_2
\end{aligned} \tag{65}$$

That is

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \tag{66}$$

Last, we have

$$\begin{aligned} \frac{\partial}{\partial x'_1} &= \left(\frac{\partial x_1}{\partial x'_1}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial x_2}{\partial x'_1}\right)\frac{\partial}{\partial x_2} \\ &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x'_2} &= \left(\frac{\partial x_1}{\partial x'_2}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial x_2}{\partial x'_2}\right)\frac{\partial}{\partial x_2} \\ &= -\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2} \end{aligned} \tag{67}$$

Example: Tangent vector field on a curve $\mathcal{C}$:

Fig

(e.g. a curve in $\mathbb{R}^2$: $x^1 = \cos u$, $x^2 = \sin u$). We know that

$$X = X^a \frac{\partial}{\partial x^a} \tag{68}$$

then how to describe X^a depend on curve $\mathcal{C}$ (relation with $\partial/\partial u$)?

Consider a function on $\mathcal{C}$: $f(x^a(u))$, rate of change of f is

$$\frac{df}{du} = \frac{\partial}{\partial x^a}\left(\frac{\partial x^a}{\partial u}\right). \tag{69}$$

That is the derevative

$$\frac{d}{du} = \left(\frac{\partial x^a}{\partial u}\right)\frac{\partial}{\partial x^a} = X^a\frac{\partial}{\partial x^a} \tag{70}$$

1.2.4.1 Two vector field

For a vector field $X = X^a\dfrac{\partial}{\partial x^a}$

FIg

where $X\Big|_p = X^a(p)\dfrac{\partial}{\partial x^a}\Big|_p$. Then consider a new vector field $Y = Y^a\dfrac{\partial}{\partial x^a}$

Claim: (How to generate a vector field by this two vector fields)

1. $X + Y = Z$ is a vector field: $X + Y = (X^a + Y^a)\dfrac{\partial}{\partial x^a}$

1. Lie bracket $[X, Y] = Z$ is a vector field, where

$$Z^a = X^b\frac{\partial Y^a}{\partial x^b} - Y^b\frac{\partial X^a}{\partial x^b}, \tag{71}$$

then we can check $X'^a = \dfrac{\partial x'^a}{\partial x^b}Z^b$ holds.

Also, the Lie bracket $[X, Y]$ satisfies

1. $[X, Y] = -[Y, X]$ (skew-symmetric)
2. $[XY, Z] = X[Y, Z] + [X, Z]Y$ (Leibnitz rule)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [X, [X, Y]] = 0$

1.3 Tensor Calculus

How to define a partial derivative of a tensor field? Recall the definition of derivative in calculus:

Definition:

$$\frac{df}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \tag{72}$$

The question is: Is the subtraction meaniful? In $\mathbb{R}^2$, $x + \epsilon$ and x are using the same coordinate. However, the coordinate in manifold using the different coordinate system.

Fig

In other words,

Problem 1 f_p and f_a refer to different coordinate system.

and

Problem 2 If $\partial_b X^a$ is a $(1, 1)$-type tensor,

$$\partial'_b X'^a = \left(\frac{\partial x'^a}{\partial x^c}\right)\left(\frac{\partial x^d}{\partial x'^b}\right)\partial_d X^c \tag{73}$$

must be the $(1, 1)$-type tensor as well.

Consider ordinary derivative

$$\begin{aligned}
\partial'_c X'^a &= \partial'_c \left(\frac{\partial x'^a}{\partial x^b} X^b\right) \\
&= \frac{\partial x'^a}{\partial x^b}\left(\partial'_c X^b\right) + \left(\partial'_c \frac{\partial x'^a}{\partial x^b}\right) X^b \\
&= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\left(\partial_d X^b\right) + \left(\frac{\partial^2 x'^a}{\partial x'^c x^b}\right) X^b \\
&= \underbrace{\frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\left(\partial_d X^b\right)}_{\text{This is what we want.}} + \underbrace{\left(\frac{\partial^2 x'^a}{\partial x'^c x^b}\right) X^b}_{\text{This is what we unwant term.}}
\end{aligned} \tag{74}$$

There are two approachs

1. Lie derivative
2. introducting a *connetion* to defone derivative.

1.3.1 ⊙ Lie derivative

- integral curve

FIg family of curve

Consider a differentiation of a tensor T^a along the curve $x^a(u)$.

key point: move/drag $T^a\big|_p$ to point Q.

Lie first define express the T'^a by

$$T^a \to T'^a(x') = \left(\frac{\partial x'^a}{\partial x^b}\right) T^b \tag{75}$$

Notice $x'^a \simeq x^a + \delta u X^a$, then $\dfrac{\partial x'^a}{\partial x^b} = \delta^a_b + \delta u \partial_b x^a$, so that

Lie define the dragged of $T^a(x)$ by

$$\begin{aligned} T^a \to T'^a(x') &= \left(\frac{\partial x'^a}{\partial x^b}\right) T^b \\ &= \left(\delta^a_b + \delta u \partial_b X^a\right) T^b(x) \\ &= T^a(x) + \delta u \partial_b X^a T^b(x), \end{aligned} \tag{76}$$

and the *Lie derivative* is

$$\begin{aligned} L_x T^a &= \lim_{\delta u \to 0} \frac{T^a(x') - T'^a(x')}{\delta u}, \text{point at } Q\text{−dragged point} \\ &= \lim_{\delta u \to 0} \frac{T^a(x + \delta X^a) - \left(T^a(x) + \delta u \partial_b X^a\right) T^b(x)}{\delta u} \\ &= \lim_{\delta u \to 0} \frac{\left(T^a(x) + \delta u \partial_b T^a X^b\right) - \left(T^a(x) + \delta u \partial_b X^a T^b\right)}{\delta u} \\ &= X^b \partial_b T^a - \partial_b X^a T^b \end{aligned} \tag{77}$$

Check $L_x T^a = x^b \partial_b T^a - \partial_b X^a T^b$ is a $(1,0)$-type tensor.

Example:

$$L_x T^{ab} = X^c \partial_c T^{ab} - \partial_c X^a T^{cb} - \partial_c X^b T^{ac} \tag{78}$$

Try $L_x T^{abc}$ and $L_x X_a$

Example (scalar)

$$\lim_{\delta u \to 0} \frac{\phi(x'^a) - \phi^{\text{dragged}}(x'^a)}{\delta u} = \lim_{\delta u \to 0} \frac{\phi(x^a + \delta u X^a) - \phi(x^a)}{\delta u} = X^a \partial_a \phi \tag{79}$$

Review: Lie dedrivative

$$\lim_{\delta u \to 0} \frac{T(Q) - T(P)}{\delta u} \tag{80}$$

Fig Drag $T(P) \to T'(Q)$ then define $L_X T = \lim_{\delta u \to 0} \dfrac{T(Q) - T'(Q)}{\delta u}$, where the drag

$$T'^a(x + \delta u\, X) = \left(\frac{\partial x'^a}{\partial x^b}\right) T^b(x) \tag{81}$$

So that the Lie dedrivative $L_X = X^b \partial_b T^a - T^b \partial_b X^a$, which is a contravariant vector. In general, for $T^{a_1, a_2, \cdots, a_n}$, the Lie dedrivative is defined to be

$$L_X T^{a_1, a_2, \ldots, a_n} = X^b \partial_b T^{a_1, a_2, \ldots, a_n} - () \tag{82}$$

Rmk:

$$\begin{aligned} L_X Y^a &= X^b \partial_b Y^a - Y^b \partial_b X^a \\ &= [X, Y]^a \end{aligned} \tag{83}$$

Rmk: For scalar ϕ, its drag is $\phi'(x') = \phi(x)$, so

$$L_X \phi = \lim_{\delta u \to 0} \frac{\phi(X + \delta u) - \phi(X)}{\delta u} = X^c \partial_c \phi \tag{84}$$

How about covariant vector $L_X Y_a$? We write

$$L_X Y_a = \lim_{\delta u \to 0} \frac{Y_a(X + \delta u X) - Y'_a(X')}{\delta u} \tag{85}$$

Its drag:

1.3.1.1 Summary

Contravariant	Covariant
$L_X Y^a = X^b \partial_b Y^a - Y^b \partial_b X^a$	$L_X Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b$

Ex: $(1,1)$-type

$$L_X T^a_b = X^c \partial_c T^a_b - T^c_b \partial_c X^a + T^a_c \partial_b X^c \tag{86}$$

Check for $(2,1)$-type

$$L_X(Y^a Y_a) = \tag{87}$$

1.3.2 ⊙ Covariant differentiation

協變微分（引入了 connection）

If we replace the dragged vector by a ”*parallel vector*”.

$$X^a_{\parallel}(x+\delta x) = X^a(x) - \delta \bar{X}^a \tag{88}$$

Notice $\delta \bar{X}^a$ very small, so it is propotion to (linear to) X^a and δx, i.e.

$$\delta \bar{X}^a = -\Gamma^a_{bc} X^b \delta x^c, \tag{89}$$

where Γ^a_{bc} is a 3 indices object connect the ... and ... called ”*connection coefficient*”.

Notice the repeating indices

$$\delta \bar{X}^a = -\sum_{b,c=1}^{n} \Gamma^a_{bc} X^b \delta x^c, \tag{90}$$

Fig

Then we have

$$\begin{aligned}\nabla_c X^a &\equiv \lim_{\delta x^a \to 0} \frac{X^a(x+\delta x^a) - X^a_{\parallel}}{\delta x^a} \\ &= \lim_{\delta x^c \to 0} \frac{(X^a(x) + \delta x^c \partial_c X^a) - (X^a(x) - \Gamma^a_{bc} X^v \delta x^c)}{\delta x^c}\end{aligned} \tag{91}$$

So we define the covariant derivative

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b \tag{92}$$

We hope $\nabla_c X^a$ is a $(1,1)$-type tensor, so we will obtain the transformation rule for connection foefficients Γ^a_{bc}

Check: Is it a $(1,1)$-type tensor?

$$\nabla'_c X^a = \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x^c}\right)\nabla_b X^d \tag{93}$$

Then the left hand side is

$$\begin{aligned} \text{LHS} &= \partial'_c X'^a + \Gamma'^a_{bc} X'^b \\ &= \partial'_c\left(\frac{\partial x'^a}{\partial x^b}X^b\right) + \Gamma'^a_{bc}\left(\frac{\partial x'^b}{\partial x^d}X^d\right) \\ &= \partial'_c\left(\frac{\partial x'^a}{\partial x^b}\right)X^b + \frac{\partial x'^a}{\partial x^b}\partial'_c X^b + \Gamma'^a_{bc}\left(\frac{\partial x'^b}{\partial x^d}X^d\right) \end{aligned} \tag{94}$$

and the right hand side

$$\begin{aligned} \text{RHS} &= \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\left(\partial_b X^d + \Gamma^d_{eb}X^e\right) \\ &= \left(\frac{\partial x'^a}{\partial x^d}\right)\frac{\partial x^b}{\partial x'^c}\frac{\partial}{\partial x^b}X^d + \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\Gamma^d_{eb}X^e \\ &= \left(\frac{\partial x'^a}{\partial x^d}\right)\frac{\partial}{\partial x'^c}X^d + \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\Gamma^d_{eb}X^e \\ &= \left(\frac{\partial x'^a}{\partial x^d}\right)\partial'_c X^d + \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\Gamma^d_{eb}X^e \end{aligned} \tag{95}$$

Then solving that

$$\begin{aligned} \text{LHS} &= \partial'_c\left(\frac{\partial x'^a}{\partial x^b}\right)X^b + \frac{\partial x'^a}{\partial x^b}\partial'_c X^b + \Gamma'^a_{bc}\frac{\partial x'^b}{\partial x^d}X^d \\ \text{RHS} &= \frac{\partial x'^a}{\partial x^d}\partial'_c X^d + \left(\frac{\partial x'^a}{\partial x^d}\right)\frac{\partial x^b}{\partial x'^c}\Gamma^d_{eb}X^e \\ \Rightarrow \partial'_c&\left(\frac{\partial x'^a}{\partial x^b}\right)X^b + \Gamma'^a_{bc}\frac{\partial x'^b}{\partial x^d}X^d = \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\Gamma^d_{eb}X^e \\ \Rightarrow &\left[\partial'_c\left(\frac{\partial x'^a}{\partial x^e}\right) + \Gamma'^a_{bc}\frac{\partial x'^b}{\partial x^e}\right]X^e = \left(\frac{\partial x'^a}{\partial x^d}\right)\left(\frac{\partial x^b}{\partial x'^c}\right)\Gamma^d_{eb}X^e \end{aligned} \tag{96}$$

Deriving that

$$\partial'_c\left(\frac{\partial x'^a}{\partial x^e}\right) + \Gamma'^a_{bc}\frac{\partial x'^b}{\partial x^e} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^b}{\partial x'^c}\Gamma^d_{eb} \tag{97}$$

Rmk: A manifold M with prescribed connection on it is called an "*Affined manifold*" denoted as (M, Γ).

Ex: For the covariant tensor $\nabla_c X_a$. Since $\nabla_c(X_a X^a) = \partial_c(X_a X^a)$, we can expand

$$\nabla_c\left(X_a X^a\right) = \left(\nabla_c X_a\right)X^a + X_a \nabla_c X^a = \partial_c(X_a X^a) \tag{98}$$

so that

$$\nabla_c X_a = \partial_c X_a - \Gamma^a_{bc}X_b \tag{99}$$

In general, for the tensor $\nabla_c T^a_{bc}$

Recall: Lie derivative $L_x T^{a\cdots}_{b\ldots}$ is a tensor Recall: Covariant derivative $\nabla_c T^{a\cdots}_{b\ldots}$ is defined by using a affine connection Γ^a_{bc} e.g. $\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b$ is a $(1,1)$-tensor, the price we need to pay is that Γ^a_{bc} is NOT a tensor.

$$\nabla_c T = \lim_{\delta x_c \to 0} \frac{T(x') - T(x)}{\delta x_c} \tag{100}$$

Also, if we define

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} \tag{101}$$

is a *torsion tensor* which is a $(1,2)$-type tensor.

If $\Gamma^a_{bc} = \Gamma^a_{cb}$, the the connection is *torsion free*.

1.3.3 $\odot$ Affine Geodesics

A curve induced by connection

$\nabla_c T^{a\cdots}_{b\ldots}$ covariant derivative along δx_c. Let X^a be a tangent vector od $x^a(u)$, that is

$$X^a = \frac{dx^a}{du} \tag{102}$$

Fig

then $X^c \nabla_c T^{a\cdots}_{b\ldots}$ is the covariant dericative along the curve, denoted by

$$\frac{DT^{a\ldots}_{b\ldots}}{Du} \tag{103}$$

Rmk: X^a is a $(1,0)$-type tensor, $\nabla_c T^{a\cdots}_{b\ldots}$ is a $(p, q+1)$-type tensor, $X^c \nabla_c T^{a\cdots}_{b\ldots}$ is a (p,q)-type tensor

Parallel transported tensor field Fig i.e.

$$\frac{DT^{a\ldots}_{b\ldots}}{Du} = 0 \tag{104}$$

1.3.3.1 Affien geodesic

If a cure $x^c(u)$ is constructed such that the tangent vector X^a at x' is just the parallel transported vector from x. The curve is called *affine geodesic*.

$$X^c \nabla_c X^a = \frac{DX^a}{Du} = 0 \tag{105}$$

Fig

$$X^c \left(\partial_c X^a + \Gamma^a_{bc} X^b\right) = 0 \tag{106}$$

then

$$\frac{dx^c}{du}\frac{\partial X^a}{\partial x^c} + \frac{dx^c}{du}\Gamma^a_{bc}\frac{dx^b}{du} = 0 \tag{107}$$

Plugin $X^a = dx^a/du^2$, we have

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0. \tag{108}$$

Rmk:

- Given a Γ^a_{bc}, then $x^a(u)$ is determined.
- After reparametrized $u \to \alpha u + \beta$, the geodesic still the same.
- In equation $\dfrac{d^2x^a}{du^2} + \Gamma^a_{bc}\dfrac{dx^b}{du}\dfrac{dx^c}{du} = 0$,only the symmetric part Γ^a_{bc} involved

1.4 Riemann tensor

In Euclilean space, the derevative

$$\partial_a\partial_c T = \partial_c\partial_a T \tag{109}$$

is commutative. However, in non-Euclilean space, is that

$$\nabla_a\nabla_c T \stackrel{?}{=} \nabla_c\nabla_a T. \tag{110}$$

In general, $\nabla_a\nabla_c T \neq \nabla_c\nabla_a T$, which is non-commutative.

1.4.0.1 Second derivative

Consider

$$\begin{cases} \nabla_d X^a = \partial_d X^a + \Gamma^a_{bd}X^b \\ \nabla_c Y^a_d = \partial_c Y^a_d + \Gamma^a_{bc}Y^b_d - \Gamma^b_{dc}Y^a_b \end{cases} \tag{111}$$

we can calculate the second derivative

$$\begin{aligned} \nabla_c\nabla_d X^a &= \nabla_c\left(\partial_d X^a + \Gamma^a_{bd}X^b\right) \\ &= \partial_c\left(\partial_d X^a + \Gamma^a_{bd}X^b\right) + \Gamma^a_{bc}\left(\partial_d X^b + \Gamma^b_{ed}X^e\right) - \Gamma^b_{dc}\left(\nabla_b X^a\right) \\ &= \partial_c\partial_d X^a + \partial_c\left(\Gamma^a_{bd}X^b\right) + \Gamma^a_{bc}\partial_d X^b + \Gamma^a_{bc}\Gamma^b_{ed}X^e - \Gamma^b_{dc}\left(\nabla_b X^a\right) \\ &= \partial_c\partial_d X^a + X^b\partial_c\Gamma^a_{bd} + \Gamma^a_{bd}\partial_c X^b + \Gamma^a_{bc}\partial_d X^b + \Gamma^a_{bc}\Gamma^b_{ed}X^e - \Gamma^b_{dc}\left(\nabla_b X^a\right) \end{aligned} \tag{112}$$

the the defference between second derivative, we have

$$\begin{aligned} \nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a &= \left(\partial_c\partial_d X^a + X^b\partial_c\Gamma^a_{bd} + \Gamma^a_{bd}\partial_c X^b + \Gamma^a_{bc}\partial_d X^b + \Gamma^a_{bc}\Gamma^b_{ed}X^e - \Gamma^b_{dc}\nabla_b X^a\right) \\ &\quad - \left(\partial_d\partial_c X^a + X^b\partial_d\Gamma^a_{bc} + \Gamma^a_{bc}\partial_d X^b + \Gamma^a_{bd}\partial_c X^b + \Gamma^a_{bd}\Gamma^b_{ec}X^e - \Gamma^b_{cd}\nabla_b X^a\right) \\ &= \left(\partial_c\partial_d - \partial_d\partial_c\right)X^a + X^b\left(\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}\right) \\ &\quad + \left(\Gamma^a_{bd}\partial_c X^b - \Gamma^a_{bc}\partial_d X^b\right) + \left(\Gamma^a_{bc}\partial_d X^b - \Gamma^a_{bd}\partial_c X^b\right) \\ &\quad + \left(\Gamma^a_{bc}\Gamma^b_{ed} - \Gamma^a_{bd}\Gamma^b_{ec}\right)X^e - \left(\Gamma^b_{dc} - \Gamma^b_{cd}\right)\nabla_b X^a \\ &= \left(\partial_c\partial_d - \partial_d\partial_c\right)X^a + X^b\left(\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}\right) \\ &\quad + \left(\Gamma^a_{bc}\Gamma^b_{ed} - \Gamma^a_{bd}\Gamma^b_{ec}\right)X^e - \left(\Gamma^b_{dc} - \Gamma^b_{cd}\right)\nabla_b X^a \end{aligned} \tag{113}$$

If Γ^a_{bc} is torsion free $\Gamma^a_{bc} = \Gamma^a_{cb}$, the between second derivative is given by

$$\begin{aligned} \nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a &= \left(\partial_c\partial_d - \partial_d\partial_c\right)X^a + X^b\left(\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}\right) \\ &\quad + \left(\Gamma^a_{bc}\Gamma^b_{ed} - \Gamma^a_{bd}\Gamma^b_{ec}\right)X^e - \left(\Gamma^b_{dc} - \Gamma^b_{cd}\right)\nabla_b X^a \\ &= \left(\partial_c\partial_d - \partial_d\partial_c\right)X^a + \left(\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}\right)X^b + \left(\Gamma^a_{bc}\Gamma^b_{ed} - \Gamma^a_{bd}\Gamma^b_{ec}\right)X^e - 0 \end{aligned} \tag{114}$$

so that

$$\begin{aligned}(\nabla_c\nabla_d - \nabla_d\nabla_c)\,X^a &= (\partial_c\partial_d - \partial_d\partial_c)\,X^a + (\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc})\,X^b + \left(\Gamma^a_{bc}\Gamma^b_{ed} - \Gamma^a_{bd}\Gamma^b_{ec}\right)X^e \\ &= (\partial_c\partial_d - \partial_d\partial_c)\,X^a + (\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc})\,X^b + (\Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc})\,X^b \\ &= (\partial_c\partial_d - \partial_d\partial_c)\,X^a + (\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc})\,X^b\end{aligned} \tag{115}$$

If we write

$$(\nabla_c\nabla_d - \nabla_d\nabla_c)\,X^a = (\partial_c\partial_d - \partial_d\partial_c)\,X^a + R^a_{bcd}X^b, \tag{116}$$

where we define the *Reimann tensor* is given by

$$R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}. \tag{117}$$

Note: If it is not torsion-free

$$(\nabla_c\nabla_d - \nabla_d\nabla_c)\,X^a = R^a_{bcd}X^b\left(\partial_c\partial_d + \left(\Gamma^b_{cd} - \Gamma^b_{dc}\right)\nabla_b X^a - \partial_d\partial_c\right)X^a, \tag{118}$$

1.4.1 Geodesic coordiante

Locally, we may choose $\{x^a\}$ s.t. $\Gamma^a_{bc} \overset{*}{=} 0$, $[x^a]_p \overset{*}{=} 0$

Suppose,

$$x^a \mapsto x'^a + \frac{1}{2}Q^a_{bc}x^bx^c, \tag{119}$$

where we require $Q^a_{bc} = Q^a_{bc}$ (constant), then we calculte

$$\left.\frac{\partial x'^a}{\partial x^d}\right|_p = \left.\delta^a_d\right|_p + \left.Q^a_{bc}x^b\right|_p = \delta^a_d \tag{120}$$

then

$$\left.\frac{\partial^2 x'^a}{\partial x^d\partial x^c}\right|_p = Q^a_{cd} \tag{121}$$

Recall:

$$\begin{aligned}\Gamma'^a_{bc} &= \left.\left(\frac{\partial x'^a}{\partial x^d}\frac{\partial x^f}{\partial x'^c}\frac{\partial x^e}{\partial x'^b}\right)\right|_p \left.\Gamma^d_{ef}\right|_p - \left.\frac{\partial^2 x'^a}{\partial x^d\partial x^c}\frac{\partial x^d}{\partial x'^c}\frac{\partial x^e}{\partial x'^b}\right|_p \\ &= \left.\delta^a_b\,\delta^f_c\,\delta^e_b\,\Gamma^d_{ef}\right|_p - Q^a_{ed}\delta^d_c\delta^e_b \\ &= \left.\Gamma^a_{bc}\right|_p - Q^a_{bc}\end{aligned} \tag{122}$$

So, if we choose

$$Q^a_{bc} = \left.\Gamma^a_{bc}\right|_p \tag{123}$$

then

$$\Gamma'^a_{bc} \overset{*}{=} 0 \tag{124}$$

This "trick" for computation set $\Gamma^a_{bc}\Big|_p = 0$. After computation, restore Γ by $\partial_a \to \nabla_a$.

Rmk: In general, it is impossible to find a coordinate transforamtion, s.t. $\Gamma^a_{bc} = 0$ globally. If it does, then the manifold is affine flat manifold.

1.5 Affine flatness

1.5.1 ⊙ Integrable connetion

1.5.1.1 Definition:

If a parallel transport of a vector from P to Q is *independent* pf path then the connection is *integrable.*

1.5.1.2 Lemma 1.

The connection is *integrable* or *torsion free*t $\iff R^a_{bcd} = 0$.

proof $\Rightarrow$ (necessary) If Γ^a_{bc} is integrable $\Rightarrow \dfrac{dc^c}{du}\nabla_c X^a = 0$ is path integrable $\Rightarrow$ Since, $\dfrac{dc^c}{du}$ is arbitrary. $\nabla_c X^a = 0$, that is

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b = 0 \tag{125}$$

which is 1st order P.D.E. the existence of solution: $\partial_d \partial_c X^a = \partial_c \partial_d X^a$,

$$\begin{aligned}\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a &= R^a_{bcd} X^b + \left(\Gamma^e_{cd} - \Gamma^e_{dc}\right) \nabla_e X^a + \left(\partial_d \partial_c X^a - \partial_c \partial_d\right) X^a \\ &= R^a_{bcd} X^b = 0\end{aligned} \tag{126}$$

Since X^b is arbitrary, $R^a_{bcd} = 0$.

proof $\Leftarrow$ (sufficient) Consider an infinitesmal loop:
FIg
compute parallel transport along two paths.

1. For the path C_1:

- $x^a \to x^a + \delta x^a$

$$\begin{aligned}X^a(x+\delta x) &= X^a(x) + \bar{\delta} X^a(x) \\ &= X^a(x) - \Gamma^a_{bc} X^b \delta x^c\end{aligned} \tag{127}$$

- $x^a + \delta x^a \to (x^a + \delta x^a) + dx^a$

$$\begin{aligned}X^a(x+\delta x+dx) &= X^a(x+\delta x) + \bar{\delta} X^a(x+\delta x) \\ &= \left(X^a(x) - \Gamma^a_{bc} X^b \delta x^c\right) - \left(\Gamma^a_{bc} + \partial_d \Gamma^a_{bc} \delta x^d\right)\left(x^b - \Gamma^b_{ef} X^e \delta x^f\right) dx^c\end{aligned} \tag{128}$$

1. For the path C_2:

$$X^a(x+dx+\delta x) = X^a - \Gamma^a_{bc}X^b dx^c - \Gamma^a_{bc}x^b\delta x^c \\ = \tag{129}$$

Last we have the difference

$$\begin{aligned} \Delta X &= X^a_{C_1}(x+\delta x+dx) - X^a_{C_2}(x+\delta x+dx) \\ &= (\partial\Gamma^a_{bd} - \partial)\, X^b\delta x^d\delta x^c \\ &= R^a_{cbd} \end{aligned} \tag{130}$$

1.5.1.3 Lemma 2

A manifold M is affine flat ($\Gamma^a_{bc} = 0$ globally) $\Longleftrightarrow$ The connection symmetric and integrable.

proof $\Rightarrow$ (necessary) If M is affoine flat, then $\Gamma^a_{bc} = 0$ everywhere, then parallel transport is path independent, trivially.

proof $\Leftarrow$ (sufficient) If Γ^a_{bc} is integrable, around P choose L.T. vector $\{X^a_1, \ldots, X^a_n\}$, where $\dim M = n$. Now, using Γ^a_{bc} to parallel transport $\{X^a_i\}$ everywhere.

Hence, for any $x \in M$, $\{X^a_i(x)\}$ is L.T., then $|X^a_i| \neq 0$ (by L.T.), so $\exists!$ inverse X^i_b s.t.

$$X^a_i X^i_a = \delta^a_b. \tag{131}$$

Since

$$0 = \nabla_b X^a_i = \partial_b X^a_i + \Gamma^a_{cb}X^c_i \quad \Rightarrow \quad \Gamma^a_{cb} = -X^i_c\partial X^a_i. \tag{132}$$

Thus

$$o = \Gamma^a_{bc} - \Gamma^a_{cb} = X^i_c\partial_b X^a_i - X^i_b\partial_c X^a_i = \left(X^i_c\partial_b - X^i_b\partial_c\right) X^a_i \tag{133}$$

Because, (X^a_i) is non-degenerate $\Rightarrow$ $\partial_c X^i_b = \partial_b X^i_c$, then $\forall x \in M, \exists$ functions $f^i(x)$ s.t. $X^i_b = \partial_b f^i$

1.5.1.4 Recall

- (M, Γ) affine manifold
- Γ^a_{bc} affine connection
- e.g. for a $(1,1)$-type tensor): $\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc}X^b$
- trosion free: $\Gamma^a_{bc} = \Gamma^a_{cb}$
- Integrable: parrellel transport is path independent
- affine geodesic $x^a(u)$

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^b}{du} = 0 \tag{134}$$

- Riemannian tensor ($(1,3)$-type tensor)

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{135}$$

- How flat?
- M is affine flat $\Leftrightarrow$ Γ^a_{bc} is tegrable symmetry
- $\Gamma^a_{bcd} = 0 \Leftrightarrow \Gamma^a_{bc}$ is tegrable symmetry
- M is affine flat $\Leftrightarrow \Gamma^a_{bcd} = 0$
- geodesic coordinate $\{x^a\}$: $\Gamma^a_{dc} = 0$

1.6 Metric

度規：$(M, g_{\mu\nu})$

If we add an additional structure called mretric $g_{ab}(x)$, a $(0,2)$-type symmetric tensor.

1.6.0.1 Def: (Riemannian manifold)

A Riemannian manifold (M, g) where g_{ab} is the metric defoned by

$$(ds)^2 = g_{ab}dx^a dx^b \tag{136}$$

also called *1st fundamental form.*

Fig

e.g. in $\mathbb{R}^3$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{137}$$

where

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{138}$$

e.g. in $\mathbb{R}^2$

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 = (dx)^2 + (dy)^2 \tag{139}$$

where

$$g_{ab}\Big|_{\text{polar}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g_{ab}\Big|_{\text{Cartisian}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{140}$$

For $x^a \in T_p(M)$, we define

$$\cos(X, Y) = \frac{g_{ab}X^a Y^b}{\sqrt{|X|^2|Y|^2}} \tag{141}$$

where $|X|^2 = g_{ab}X^a X^b$. Notice that

$$\begin{cases} |X|^2 > 0 & \text{positive} \\ |X|^2 < 0 & \text{negative} \end{cases} \tag{142}$$

Rmk:

1. $X^a \perp Y^b$, if $g_{ab}X^aY^b = 0$
2. g_{ab} is non-singular, if $\det(g_{ab}) \neq 0$
3. $(g_{ab})^{-1} = g^{ab}$, that is $g_{ab}g^{ab} = \delta^a_b$
4. g_{ab} and g^{ab} can be used to lowering and rasing tensorial indices. e.g.

$$\begin{aligned} X^a &= g^{ab}X_b \\ X_a &= g_{ab}X^b \end{aligned} \tag{143}$$

Notice that

$$g_{ab}T^{bc} = T_a{}^c \tag{144}$$

In general,

$$X_b{}^a = g^{ac}X_{bc} \quad \neq \quad X^a{}_b = g^{ac}X_{cb} \tag{145}$$

1.6.0.2 Geodesic equation

For the Riemannian manifold (M, g),

$$(ds)^2 = g_{ab}dx^adx^b \tag{146}$$

Then the path $x^a(u)$ from $u = P$ to $u = Q$, can be interpreted as

$$S = \int_P^Q ds = \int_P^Q \sqrt{g_{ab}dx^adx^b} = \int_P^Q \sqrt{g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}}du \tag{147}$$

then if we want to minimize the path to find the shortest path from P to Q , we can define the Lagrangian to be

$$L(x^a, x^b, u) = \sqrt{g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}} = \sqrt{g_{ab}\dot{x}^a\dot{x}^b} \tag{148}$$

the the path reduced to

$$S = \int_P^Q L(x^a, x^b, u)du \tag{149}$$

Using the *Least Action Principle* the shortest path x^a corresponding to the solution for the *Euler-Lagrangian equation*

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{150}$$

the EL equation becomes

$$g^{ab}\ddot{x}^b + \left(\partial_c g_{ab} - \frac{1}{2}\partial_c g_{db}\right).. \tag{151}$$

choosing a linear parameter $u = \alpha s + \beta$, the equation becomes

$$\frac{d^2x^a}{ds^2} + \begin{Bmatrix} a \\ bc \end{Bmatrix}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{152}$$

Recall: The affine geodesic

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{153}$$

Comparing to the affine geodesic, define

$$\begin{Bmatrix} a \\ bc \end{Bmatrix} = \frac{1}{2}g^{ad}\{bc,d\} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right) \tag{154}$$

That is we can deine a *metric connection* Γ by the metric g by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right) \tag{155}$$

called *Christoffel* symbols.

HW: prove Γ^a_{bc} is metric connection $\Leftrightarrow \nabla_c g_{ab} = 0$

1.6.1 $\odot$ Affine flatness

1.6.1.1 Def: (metric flatness)

$\exists$ a special coordinate, s.t. g_{ab} is constant everywhere, e.g.

$$g_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{156}$$

1.6.1.2 Theorem: (metrix flatness)

A metric is flat $\Leftrightarrow R^a_{bcd} = 0$

That is, once we identify

$$\Gamma^a_{bc} = \left\{\begin{pmatrix} a \\ bc \end{pmatrix}\right\} \tag{157}$$

then

$$\text{affine flatness } (\Gamma^a_{bc} = 0) = \text{metrix flateness } (g_{ab} = \text{constant}) \tag{158}$$

1.7 Riemannian tensor

1.7.0.1 symmetry

- R^a_{bcd} has symmeties.
- So that $g_{ae}R^e_{bcd} = R_{abcd}$ has $4^4 = 256$ components?

In fact,

$$\begin{aligned} R_{abcd} &= -R_{bacd} \quad (a \leftrightarrow b) \\ R_{abcd} &= -R_{abdc} \quad (c \leftrightarrow d) \\ R_{abcd} &= -R_{cdab} \quad (ab \leftrightarrow cd) \end{aligned} \tag{159}$$

1.7.0.2 summation

$$R_{abcd} + R_{adbc} + R_{acbd} = 0 \tag{160}$$

we can check that in the geodesic equation.

Ricci tensor

$$R_{ab} = R^{c}_{acb} \tag{161}$$

Ricci scalar

$$g^{ab}R_{ab} = R^{a}{}_{a} = R \tag{162}$$

Einstein equations

- vacuum equation

$$R_{ab} - \frac{1}{2}g_{ab}R = 0 \tag{163}$$

- with matter

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{c^4}T_{ab} \tag{164}$$

where T_{ab} is energy-momentum tensor.