Introducting Relativity

Problem Set 1 (Due 2024/3/12)

Chang-Mao Yang, 楊長茂 (409220055), March 12, 2024

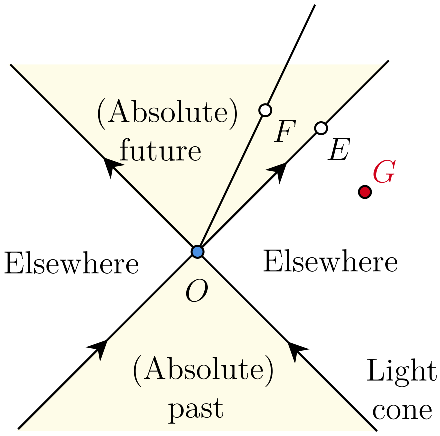


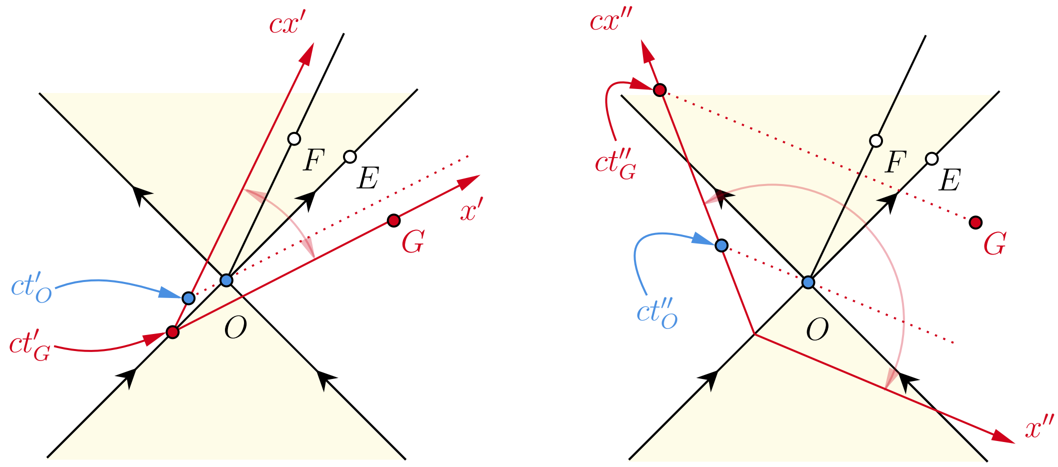
Figure 1: Light cone and causality of the frame S

1. (Causality) Let E be the event on the light-cone and G be an event outside light-cone in the inertia frame S.

   (a) Draw the axis of the frame S' in S so that G occurs earlier than O.

   (b) Draw the axis of the frame S'' in S so that G occurs later than O.

The 2 drawing figure see as Figure 2.



(a) G occurs earlier than O in S'
$t'_O > t'_G$

(b) G occurs later than O in S''
$t''_O < t''_G$

Figure 2: Axes of the frame S' and S'' in S

2. (Lorentz transformation) Given the Lorentz transformation in x-direction

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \equiv L(v) \begin{pmatrix} ct \\ x \end{pmatrix}.$$

Show that from the combination of two Lorentz transformations $L(v_2)L(v_1) = L(v)$ one has

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$

For two different velocities v_1 and v_2, we define $\gamma_i = \dfrac{1}{\sqrt{1-\beta_i^2}}$ where $\beta_i = \dfrac{v_i}{c}$, $i = 1, 2$. Also, same for the addition velocity v we define $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, where $\beta = \dfrac{v}{c}$.

For convenience, we first compute

$$\begin{aligned}
\gamma_1\gamma_2 &= \frac{1}{\sqrt{1-\beta_1^2}}\frac{1}{\sqrt{1-\beta_2^2}}, && (1)\\
&= \frac{1}{\sqrt{1+\beta_1^2\beta_2^2 - (\beta_1^2+\beta_2^2)}}, && (2)\\
&= \frac{1}{\sqrt{(1+2\beta_1\beta_2+\beta_1^2\beta_2^2) - (\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2)}}, && (3)\\
&= \frac{1}{\sqrt{(1+\beta_1\beta_2)^2 - (\beta_1+\beta_2)^2}}, && (4)\\
&= \frac{1}{1+\beta_1\beta_2}\frac{1}{\sqrt{1-(\beta_1+\beta_2)^2/(1+\beta_1\beta_2)^2}}. && (5)
\end{aligned}$$

Now, we the product of two matrix $L(v_1)$ and $L(v_2)$ is

$$\begin{aligned}
L(v_1)L(v_2) &= \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} && (6)\\
&= \begin{pmatrix} \gamma_1\gamma_2 + \beta_1\gamma_1\beta_2\gamma_2 & -\beta_2\gamma_2\gamma_1 - \beta_1\gamma_1\gamma_2 \\ -\beta_2\gamma_2\gamma_1 - \beta_1\gamma_1\gamma_2 & \gamma_1\gamma_2 + \beta_1\gamma_1\beta_2\gamma_2 \end{pmatrix} && (7)\\
&= \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_1 & -(\beta_1+\beta_2) \\ -(\beta_1+\beta_2) & \beta_1\beta_2 + 1 \end{pmatrix} && (8)\\
&= \frac{1}{1+\beta_1\beta_2}\frac{1}{\sqrt{1-(\beta_1+\beta_2)^2/(1+\beta_1\beta_2)^2}} \begin{pmatrix} 1+\beta_1\beta_1 & -(\beta_1+\beta_2) \\ -(\beta_1+\beta_2) & \beta_1\beta_2 + 1 \end{pmatrix} && (9)\\
&= \frac{1}{\sqrt{1-(\beta_1+\beta_2)^2/(1+\beta_1\beta_2)^2}} \begin{pmatrix} 1 & -\dfrac{\beta_1+\beta_2}{1+\beta_1\beta_2} \\ -\dfrac{\beta_1+\beta_2}{1+\beta_1\beta_2} & 1. \end{pmatrix} && (10)
\end{aligned}$$

Ones we define $L(v) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = L(v_1)L(v_2)$, we compare the result, yield that

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \text{where } \beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}. \tag{11}$$

This result shows that the addition velocity is $v = c\beta = \dfrac{c\beta_1 + c\beta_2}{1+\beta_1\beta_2} = \dfrac{v_1+v_2}{1+v_1v_2/c^2}$.

3. (Length contraction) Let S and S' are inertial frames relative with $v = \alpha c$ where $0 < \alpha < 1$. If a rod at rest in S' makes an angle of $\pi/6$ with Ox' in S' and $\pi/4$ with Ox in S. Find the parameter α.

Define the length of rod to be L in S and L' in S'. The length projection on x, y-axis in S and x, y-axis in S' are given by

$$\begin{cases} L_x = L\cos(\pi/4) = L\sqrt{2}/2 \\ L_y = L\sin(\pi/4) = L\sqrt{2}/2 \end{cases} \quad \text{and} \quad \begin{cases} L'_x = L'\cos(\pi/6) = L'\sqrt{3}/2 \\ L'_y = L'\sin(\pi/6) = L'/2 \end{cases}. \tag{12}$$

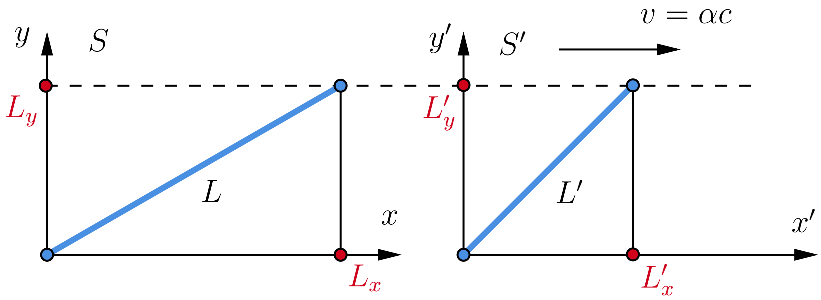


Figure 3: Length contraction

Also, notice that the projection on y are the same $L_y = L'_y$(see Figure 4), so we may get

$$L_y = L'_y \quad \Rightarrow \quad L\sqrt{2}/2 = L'/2 \quad \Rightarrow \quad \sqrt{2}L = L'. \tag{13}$$

According to the length contraction, we have the relation of length projection between two frames S and S', that is

$$L'_x = \gamma(v)\, L_x = \frac{1}{\sqrt{1 - v^2/c^2}}\, L_x = \frac{1}{\sqrt{1-\alpha^2}}\, L_x. \tag{14}$$

Plugin the length in two frame and using the the relation of two length (13), we have

$$\frac{1}{\sqrt{1-\alpha^2}} = \frac{L'_x}{L_x} = \frac{L'\sqrt{3}/2}{L\sqrt{2}/2} = \frac{L\sqrt{2}\sqrt{3}/2}{L\sqrt{2}/2} = \sqrt{3} \tag{15}$$

solving the value of α,

$$\sqrt{1-\alpha^2} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad 1 - \alpha^2 = \pm\frac{1}{3} \quad \Rightarrow \quad \alpha^2 = 1 \pm \frac{1}{3} \quad \Rightarrow \quad \alpha = \pm\sqrt{1 \pm \frac{1}{3}}, \tag{16}$$

we have the all possible values for α, that is $\alpha = \pm\sqrt{1/2}, \pm\sqrt{3/2}$. However, we must have $0 < \alpha < 1$, so $\alpha = \sqrt{2/3}$.

4. (Aberration)A light ray from a star to a telescope observer has an inclination θ' to the horizontal in S' and θ in S, where S and S' are related by speed v. Show that

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta + v/c)},$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

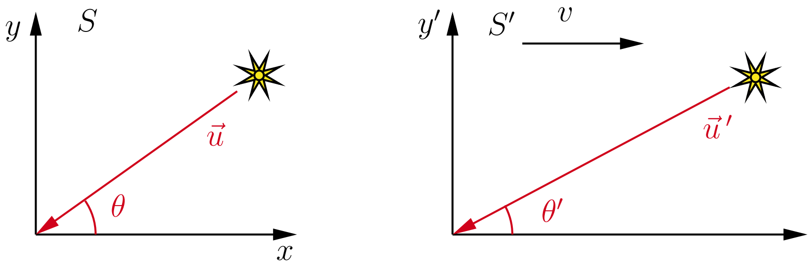


Figure 4: Rest frame and telescope frame

Using the addition formulae of velocity, we have

$$u'_x = \frac{u_x + v}{1 + u_x v/c^2} \quad \text{and} \quad u'_y = \frac{u_y}{\gamma(1 + u_x v/c^2)}. \tag{17}$$

So the tangent value angle θ' of the beam in S' is

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{\dfrac{u_y}{\gamma(1 + u_x v/c^2)}}{\dfrac{u_x + v}{1 + u_x v/c^2}} = \frac{u_y}{\gamma(u_x + v)} = \frac{c\sin\theta}{\gamma(c\cos\theta + v)} = \frac{\sin\theta}{\gamma(\cos\theta + v/c)}. \tag{18}$$