Introducting Relativity

Problem Set 1 (Due 2024/3/12)

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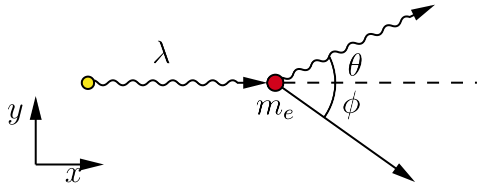
1. (Compton scattering 20%) A photon of wave length λ is scattered by a rest electron of mass m_e. If the scattered photon has wave length λ' with angle θ from the incident direction. Show that
$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta).$$
(Hint1. The 4-momentum $p^\mu = (E/c, \vec{p})$ satisfies $p^\mu p_\mu = (E/c)^2 - \vec{p}^2 = m_0 c^2$.)
(Hint2. The Energy and momentum for photon are given by $E = pc = h\nu$ and $c = \lambda\nu$)



Since, the momentum of the photon is $p = h\nu/c = h/\lambda$ The 4-momentum of the photon γ and electron e are

	Before	After
photon γ	$p^\mu_\gamma = \left(\frac{h}{\lambda}, \frac{h}{\lambda}, 0, 0\right)$	$p'^\mu_\gamma = \left(\frac{h}{\lambda'}, \frac{h}{\lambda'}\cos\theta, \frac{h}{\lambda'}\sin\theta, 0\right)$
electron e	$p^\mu_{m_e} = (m_e c, 0, 0, 0)$	$p'^\mu_{m_e}$ = unknown

Then using the conservation of 4-momentum, we have$p^\mu_\gamma + p^\mu_{m_e} = p'^\mu_\gamma + p'^\mu_{m_e}$, then rearranging the equation $p'^\mu_{m_e} = p^\mu_\gamma + p^\mu_{m_e} - p'^\mu_\gamma$. Using the properties of momentum for the $p'^\mu_{m_e}$, i.e. $(p'_{m_e})^\mu (p'_{m_e})_\mu = m_e^2 c^2$, so that

$$\begin{aligned}
(p'_{m_e})^\mu (p'_{m_e})_\mu &= \left[(p_\gamma)^\mu + (p_{m_e})^\mu - (p'_\gamma)^\mu\right]\left[(p_\gamma)_\mu + (p_{m_e})_\mu - (p'_\gamma)_\mu\right], &(1)\\
&= (p_\gamma)^\mu (p_\gamma)_\mu + (p_{m_e})^\mu (p_{m_e})_\mu + (p'_\gamma)^\mu (p'_\gamma)_\mu &(2)\\
&\quad + (p_\gamma)^\mu (p_{m_e})_\mu - (p_\gamma)^\mu (p'_\gamma)_\mu + (p_{m_e})^\mu (p_\gamma)_\mu - (p_{m_e})^\mu (p'_\gamma)_\mu &(3)\\
&\quad - (p'_\gamma)^\mu (p_\gamma)_\mu - (p'_\gamma)^\mu (p_{m_e})_\mu, &(4)\\
&= (p_\gamma)^\mu (p_\gamma)_\mu + (p_{m_e})^\mu (p_{m_e})_\mu + (p'_\gamma)^\mu (p'_\gamma)_\mu &(5)\\
&\quad + 2(p_\gamma)^\mu (p_{m_e})_\mu - 2(p_\gamma)^\mu (p'_\gamma)_\mu - 2(p_{m_e})^\mu (p'_\gamma)_\mu. &(6)
\end{aligned}$$

Plugin all the values of 4-momentum, we have

$$m_e^2 c^2 = 0 + m_e^2 c^2 + 0 + 2\frac{h}{\lambda} m_e c - 2\frac{h^2}{\lambda\lambda'}(1-\cos\theta) - 2 m_e c \frac{h}{\lambda'}. \quad (7)$$

Rearranging the equation

$$\begin{aligned}
0 &= \frac{h}{\lambda} m_e c - \frac{h^2}{\lambda\lambda'}(1-\cos\theta) - m_e c\frac{h}{\lambda'}, &(8)\\
\frac{\lambda\lambda'}{h m_e c}\cdot 0 = 0 &= \lambda' - \frac{h}{m_e c}(1-\cos\theta) - \lambda, &(9)
\end{aligned}$$

That is $\lambda - \lambda' = \frac{h}{m_e c}(1-\cos\theta)$.

2. (Relativistic energy 20%) Using the relativistic force defined by

$$\vec{F} = \frac{d}{dt}\left(\gamma m_0 \vec{u}\right), \quad \gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

and the work done by $\vec{F}$ as

$$\frac{dE}{dt} = \vec{F}\cdot\vec{u}$$

to show that

$$E = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}} + \text{constant}.$$

Using the work done by $\vec{F}$, we can solve the energy by integrating the equation

$$\begin{aligned}
E &= \int \frac{dE}{dt}dt = \int \vec{F}\cdot\vec{u}dt && (10)\\
&= \int \frac{d}{dt}\left(\gamma m_0\vec{u}\right)\cdot\vec{u}dt = \int \vec{u}\cdot\frac{d\left(\gamma m_0\vec{u}\right)}{dt}dt && (11)\\
&= \int \vec{u}\cdot\frac{d\left(\gamma m_0\vec{u}\right)}{dt}dt = \int \vec{u}\cdot d\left(\gamma m_0\vec{u}\right) && (12)\\
&= \int \vec{u}\cdot\left(\gamma m_0 d\vec{u} + m_0\vec{u}d\gamma\right) && (13)\\
&= \int \gamma m_0\vec{u}\cdot d\vec{u} + \int m_0\vec{u}\cdot\vec{u}\,d\gamma && (14)\\
&= \int \gamma m_0\vec{u}\cdot d\vec{u} + \int m_0 u^2 d\left(1-u^2/c^2\right)^{-1/2} && (15)\\
&= \int \gamma m_0\vec{u}\cdot d\vec{u} + \int m_0 u^2\frac{-1}{2}\left(1-u^2/c^2\right)^{-3/2}d\left(-u^2/c^2\right) && (16)\\
&= \int \gamma m_0\vec{u}\cdot d\vec{u} + \int \frac{1}{2}m_0u^2\gamma^3 d\left(u^2/c^2\right) && (17)\\
&= \int \gamma m_0\vec{u}\cdot d\vec{u} + \int \frac{1}{2}m_0u^2\gamma^3\frac{2\vec{u}}{c^2}\cdot d\vec{u} && (18)\\
&= m_0\int\left(\gamma+\gamma^3\frac{u^2}{c^2}\right)\vec{u}\cdot d\vec{u} = m_0\int\gamma^3\left(\gamma^{-2}+\frac{u^2}{c^2}\right)\vec{u}\cdot d\vec{u} && (19)\\
&= m_0\int\gamma^3\left(\left(1-\frac{u^2}{c^2}\right)+\frac{u^2}{c^2}\right)\vec{u}\cdot d\vec{u} = m_0\int\gamma^3\vec{u}\cdot d\vec{u} && (20)\\
&= m_0\int\left(1-\frac{u^2}{c^2}\right)^{-3/2}du^2/2 = \frac{-m_0c^2}{2}\int\left(1-\frac{u^2}{c^2}\right)^{-3/2}d(-u^2/c^2) && (21)\\
&= -\frac{m_0c^2}{2}\cdot\left(1-\frac{u^2}{c^2}\right)^{-1/2}\cdot(-2) + \text{constant}. && (22)\\
&= \frac{m_0c^2}{\sqrt{1-u^2/c^2}} + \text{constant}. && (23)
\end{aligned}$$

3. (Vector fields 60%) In $\mathbb{R}^2$ a point can be expressed in Cartesian coordinate $(x^a) = (x, y)$ or polar coordinate $(x'^a) = (r, \theta)$.

   (1a) Find the transformation matrix $J' = (\partial x'^a / \partial x^b)$ and its inverse $J = (\partial x^a / \partial x'^b)$ in terms of x'^a.

   (1b) Let $f(x, y)$ be a function on the circle $x^2 + y^2 = a^2$. Obtain the vector field $X = X^a \partial_a$ such that $df/d\theta = Xf$.

   (1c) Using J' to obtain the corresponding X' a for the vector field in (1b).

(1a) Notice that

$$x^a(x'^a) = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \tag{24}$$

so

$$dx^a = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dr\cos\theta - r\sin\theta d\theta \\ dr\sin\theta + r\cos\theta d\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \frac{dx^a}{dx'^b} dx'^b, \tag{25}$$

which means the Jacobian $J = (\partial x^a / \partial x'^b)$ is

$$J = \frac{dx^a}{dx'^b} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}. \tag{26}$$

And its determinant is $\det(J) = r$, then we have the inverse transformation

$$J' = \frac{dx'^a}{dx^b} = \frac{1}{\det(J)} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}. \tag{27}$$

(1b) For the function $f(x, y)$ on the circle $x^2 + y^2 = a^2$, we can using a new coordinate $x'^a = (r, \theta)$ as same as (1a), i.e. $x = r\cos\theta$ and $y = r\sin\theta$, where $r = a$. So that

$$\frac{df}{d\theta} = \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial f}{\partial y} = \left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}\right) f = \left(-a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y}\right) f \tag{28}$$

denote as

$$\frac{df}{d\theta} = Xf = X^a \partial_a f, \tag{29}$$

where X is the vector field given by

$$X = -a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y} \tag{30}$$

(1c) For a vector field in (1b), we can using the transformation to obtain

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b \tag{31}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/a & \cos\theta/a \end{pmatrix} \begin{pmatrix} -a\sin\theta \\ a\cos\theta \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} -a\sin\theta\cos\theta + a\cos\theta\sin\theta \\ \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{33}$$