# Simulation of the Orbit near Compact star


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Numerical simulations plays a important role in nowdays physics. In this project, we compare the gravitational potential in classical dynamics and general relativity with analytic calculation. And using the python to generate the orbit by force from analytic potential. Last, using vpython to script a javascript on web, in order to visualize the orbit by WebGL.


## I. ANALYTIC

In the Riemannian geometry of General Relativity, dot products are computed using a metric tensor $g_{\mu\nu}$ which depends on the stress-energy tensor $T_{\mu\nu}$ in Einstein's equation.

### A. Classical Dynamics

In spherical coordinates, the position vector is $\vec{r} = r\hat{r}$, where the spherical basis vectors denote as $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ which can be shown in figure (1). Then the velocity is

$$\frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\right)\hat{r} + r\left(\frac{d\phi}{dt}\right)\sin\theta\hat{\phi} + r\left(\frac{d\theta}{dt}\right)\hat{\theta}, \tag{1}$$

Denote $\dot{r} = \frac{dr}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$ and $\dot{\phi} = \frac{d\phi}{dt}$, and only consider the $xy$-plane, which is $\theta = \pi/2$. The velocity becomes

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \tag{2}$$

, and the inner product of velocity can be calculated by

$$\begin{aligned}\dot{\vec{r}}\cdot\dot{\vec{r}} &= \left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\cdot\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\\ &= \dot{r}^2\left(\hat{r}\cdot\hat{r}\right) + 2\dot{r}\dot{\theta}\left(\hat{r}\cdot\hat{\phi}\right) + r^2\dot{\theta}^2\left(\hat{\phi}\cdot\hat{\phi}\right)\end{aligned}$$

Solving that

$$\dot{\vec{r}}\cdot\dot{\vec{r}} = \dot{r}^2 + r^2\dot{\phi}^2 \tag{3}$$

Since, in classical dynamics, we have known that angular momentum $\vec{L}$ is conserved, we may calculate it by

$$\begin{aligned}\vec{L} &= \vec{r}\times\vec{p}\\ &= m\vec{r}\times\dot{\vec{r}}\\ &= mr\hat{r}\times\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)\\ &= mr^2\left(\hat{r}\times\hat{r}\right) + mr^2\dot{\phi}\left(\hat{r}\times\hat{\phi}\right).\end{aligned}$$

With the orthogonality in spherical coordinates, we obtain the angularmomentum

$$\vec{L} = mr^2\dot{\phi}\hat{\theta}. \tag{4}$$

Now, we may consider the total energy $E$ of Kepler problem, which is

$$E = T + V\left(\vec{r}\right). \tag{5}$$

The kinetic energy is $m\dot{\vec{r}}\cdot\dot{\vec{r}}/2$, and the gravitational potential is

$$V\left(\vec{r}\right) = -\frac{GMm}{|\vec{r}|}, \tag{6}$$

where $G$ is gravitational constant, $M$ and $m$ is the mass of two obejct. Using equation (3) and (4) plug in (5), we have

$$\begin{aligned}E &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r}\\ &= \frac{1}{2}m\dot{r}^2 + \frac{\left(mr^2\dot{\phi}\right)\left(mr^2\dot{\phi}\right)}{2mr^2} - \frac{GMm}{r}\\ &= \frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right)\end{aligned}$$

solving that

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + m\left(\frac{L^2}{2m^2r^2} - \frac{GM}{r}\right)} \tag{7}$$

and define the effective potential

$$V_{\text{eff}}\left(r\right) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{8}$$

### B. General Relativity

In Minkowski space, the inner product is given by the metric tensor $g_{\mu\nu}$. And the Lagrangian in space time is

$$L = \sqrt{-g_{\mu\nu}\left(x\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}, \tag{9}$$

where $\lambda$ is the parameterisation of the worldline.

Using the Euler-Lagrange equations, we may obtain the geodesic equation in space time

$$\frac{d^2x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{10}$$

where the Christoffel symbols are given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) \tag{11}$$

### 1. *Metric tensor in flat Minkowski space*

In flat space time, the metric tensor in Cartesian coordinate is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

which can also be written in spherical coordinate

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{13}$$

### 2. *Metric tensor and geodesic equation in bend Minkowski space*

For a non-rotating, spherically symmetric massive object with mass $M$, the solution $g_{\mu\nu}$ of Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{14}$$

in spherical coordinate is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{R_s}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{R_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{15}$$

also called Schwarzschild metric tensor with Schwarzschild radius $R_s$.

Here choosing the index to be $\mu, \nu = t, r, \theta, \phi,$. Then, plug in to the Christoffel symbols (11). After some calculation, we obtain

$$\begin{aligned} &\Gamma^t_{tr} = \Gamma^t_{rt} = -\Gamma^r_{rr} = \frac{R_s}{2r(r - R_s)}, \qquad \Gamma^r_{tt} = \frac{R_s (r - R_s)}{2r^3}, \\ &\Gamma^r_{\phi\phi} = (R_s - r)\sin^2\theta, \qquad \Gamma^r_{\theta\theta} = R_s - r, \\ &\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \\ &\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta, \end{aligned} \tag{16}$$

and we need to results substituted into the geodesic equations (10) for four index $\theta, t, \phi, r$

$$0 = \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{17}$$

$$0 = \frac{d^2 (ct)}{d\lambda^2} + \Gamma^t_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{18}$$

$$0 = \frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{19}$$

$$0 = \frac{d^2 r}{d\lambda^2} + \Gamma^r_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{20}$$

Here we must notice that, we are using the Einstein summation convention, so we need to summed over repeated indices.

1. geodesic equation ($\theta$ parts): Since the center mass $M$ has spherical symmetry, we can always choose our system to be $\theta = \pi/2$, which is $xy$-plane

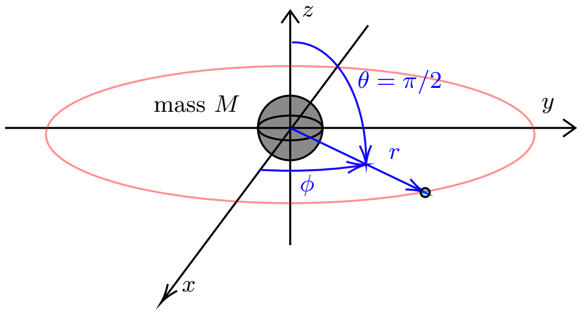


FIG. 1. Spherical coordinate label

equation (17) become

$$\begin{aligned} 0 &= \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\theta r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} + \Gamma^\theta_{r\theta}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= \left(\frac{d\theta}{d\lambda} + 2\Gamma^\theta_{\theta r}\frac{dr}{d\lambda}\right)\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= \left(\frac{d\theta}{d\lambda} + 2\Gamma^\theta_{\theta r}\frac{dr}{d\lambda}\right)\frac{d(\pi/2)}{d\lambda} - \sin\frac{\pi}{2}\cos\frac{\pi}{2}\left(\frac{d\phi}{d\lambda}\right)^2 = 0, \end{aligned}$$

so setting $\theta = \pi/2$ won't violate geodesic equation.

2. geodesic equation ($t$ parts): Due to the symmetry of $\Gamma^t_{tr} = \Gamma^t_{rt}$, equation (18) would become

$$\begin{aligned} 0 &= \frac{d^2(ct)}{d\lambda^2} + \Gamma^t_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \\ &= \frac{d^2(ct)}{d\lambda^2} + 2\Gamma^t_{tr}\frac{d(ct)}{d\lambda}\frac{dr}{d\lambda} \end{aligned}$$

plugging result of christoffel symbols (16)

$$\begin{aligned} 0 &= \frac{d^2(ct)}{d\lambda^2} + \frac{R_s}{r(r - R_s)}\frac{d(ct)}{d\lambda}\frac{dr}{d\lambda} \\ 0 &= \frac{d^2(ct)}{d\lambda^2}\left(\frac{r - R_s}{r}\right) + \frac{R_s}{r^2}\frac{d(ct)}{d\lambda}\frac{dr}{d\lambda} \\ &= \frac{d^2(ct)}{d\lambda^2}\left(1 - \frac{R_s}{r}\right) - \frac{d(ct)}{d\lambda}\frac{dr}{d\lambda}\frac{d}{dr}\left(1 - \frac{R_s}{r}\right) \\ &= \frac{d^2(ct)}{d\lambda^2}\left(1 - \frac{R_s}{r}\right) - \frac{d(ct)}{d\lambda}\frac{d}{d\lambda}\left(1 - \frac{R_s}{r}\right), \end{aligned}$$

which can be reduced by chain rule of derivatives

$$\frac{d}{d\lambda}\left(\frac{d(ct)}{d\lambda}\left(1 - \frac{r_s}{r}\right)\right) = 0. \tag{21}$$

The equation (21) provide that $\frac{d(ct)}{d\lambda}\left(1 - \frac{r_s}{r}\right)$ is a conserved quantity. Due to its unit, we may define

$$\varepsilon \equiv \frac{E}{m} \equiv c\frac{d(ct)}{d\lambda}\left(1 - \frac{r_s}{r}\right) \tag{22}$$

3. geodesic equation ($\phi$ parts): Due to the symmetry of $\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r}$ and $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}$, equation (18) would become

$$\begin{aligned} 0 &= \frac{d^2\phi}{d\lambda^2} + 2\Gamma^{\phi}_{r\phi}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\Gamma^{\phi}_{\theta\phi}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} \\ &= \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} \\ 0 &= r^2\frac{d^2\phi}{d\lambda^2} + 2r\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = r^2\frac{d^2\phi}{d\lambda^2} + \frac{d\left(r^2\right)}{d\lambda}\frac{d\phi}{d\lambda} \end{aligned}$$

which can be reduced by chain rule of derivatives

$$0 = \frac{d}{d\lambda}\left(r^2\frac{d\phi}{d\lambda}\right). \tag{23}$$

The eqaition (23) provide that $r^2\frac{d\phi}{d\lambda}$ is a conserved quantity. Since its unit is anular momentum per mass, we may define

$$\ell \equiv \frac{L}{m} \equiv r^2\frac{d\phi}{d\lambda} \tag{24}$$

4. geodesic equation ($r$ parts): Substitude Christoffel symbols (16) in to the equation (20),

$$\begin{aligned} 0 &= \frac{d^2r}{d\lambda^2} + \Gamma^{r}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \\ &= \frac{d^2r}{d\lambda^2} + \Gamma^{r}_{tt}\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 + \Gamma^{r}_{rr}\left(\frac{dr}{d\lambda}\right)^2 \\ &\quad + \Gamma^{r}_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + \Gamma^{r}_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= \frac{d^2r}{d\lambda^2} + \frac{R_s\left(r-R_s\right)}{2r^3}\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 \\ &\quad + \frac{R_s}{2r\left(R_s-r\right)}\left(\frac{dr}{d\lambda}\right)^2 + \left(R_s-r\right)\left(\frac{d\theta}{d\lambda}\right)^2 \\ &\quad + \left(R_s-r\right)\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 \end{aligned}$$

Define a new function $A$ to reduce the calculation

$$A\left(r\right) = 1 - \frac{R_s}{r}, \tag{25}$$

with $\theta = \pi/2$ plug in

$$\begin{aligned} 0 &= \frac{d^2r}{d\lambda^2} + \frac{R_s}{2r^2}A\left(r\right)\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 \\ &\quad + \frac{R_s}{2r^2}A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \left(R_s-r\right)\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= \frac{d^2r}{d\lambda^2} + \frac{R_s}{2r^2}A\left(r\right)\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 \\ &\quad + \frac{R_s}{2r^2}A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r\left(\frac{r-R_s}{r}\right)\left(\frac{d\phi}{d\lambda}\right)^2 \end{aligned}$$

solving that

$$\begin{aligned} 0 &= \frac{d^2r}{d\lambda^2} + \frac{R_s}{2r^2}A\left(r\right)\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 \\ &\quad + \frac{R_s}{2r^2}A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - rA\left(r\right)\left(\frac{d\phi}{d\lambda}\right)^2 \end{aligned} \tag{26}$$

### 3. *Time-like and Light-like geodesic*

Consider the space time interval

$$ds^2 = g_{\mu\nu}x^\mu x^\nu \tag{27}$$

and derivative with respect to the parameterisation of the worldline, we have

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{28}$$

and using the Schwarzschild metric (15), function $A\left(r\right)$ (25) and $\theta = \pi/2$ plug in

$$\begin{aligned} \left(\frac{ds}{d\lambda}\right)^2 &= g_{tt}\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 \\ &\quad + g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= -A\left(r\right)\left(\frac{d\left(ct\right)}{d\lambda}\right)^2 + A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 \end{aligned}$$

Using the conserved quantity by the defenetion (22) and (24), we have

$$\begin{aligned} \left(\frac{d\left(ct\right)}{d\lambda}\right)^2 &= \frac{\varepsilon^2}{A\left(t\right)^2} \\ \left(\frac{d\phi}{d\lambda}\right)^2 &= \frac{\ell^2}{r^4}. \end{aligned}$$

Plug in $\left(ds/d\lambda\right)^2$,

$$\begin{aligned} \left(\frac{ds}{d\lambda}\right)^2 &= -A\left(r\right)\varepsilon^2\left(A\left(t\right)^{-1}\right)^2 + A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 \\ &\quad + r^2\sin^2\theta\frac{\ell^2}{r^4} \\ &= -A\left(r\right)^{-1}\varepsilon^2 + A\left(r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} \end{aligned}$$

deriving that

$$0 = -\varepsilon^2 + \left(\frac{dr}{d\lambda}\right)^2 - A\left(r\right)\left(\frac{\ell^2}{r^2} - \left(\frac{ds}{d\lambda}\right)^2\right) \tag{29}$$

or simpify

$$\begin{aligned} 0 &= \varepsilon^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1 - \frac{R_s}{r}\right)\left(\frac{\ell^2}{r^2} - \left(\frac{ds}{d\lambda}\right)^2\right) \\ &= \varepsilon^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{\ell^2}{r^2} + \frac{R_s\ell^2}{r^3} - \frac{G_s}{r}\left(\frac{ds}{d\lambda}\right)^2 + \left(\frac{ds}{d\lambda}\right)^2 \end{aligned}$$

and plug in $R_s$ and choosing $\lambda = \tau$, which is proper time

$$0 = \varepsilon^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{\ell^2}{r^2} + \frac{2GM\ell^2}{c^2 r^3} - \frac{2GM}{c^2 r}\left(\frac{ds}{d\tau}\right)^2 + \left(\frac{ds}{d\tau}\right)^2 \tag{30}$$

Now, we consider two cases, one is light-light geodesic and the other is time-like geodesic. In our defination of metric tensor in Cartesian (12): When

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad \Rightarrow \quad \left(\frac{ds}{d\tau}\right)^2 = 0$$

is light-like interval, and when

$$c^2dt^2 - dx^2 - dy^2 - dz^2 > 0 \quad \Rightarrow \quad \left(\frac{ds}{d\tau}\right)^2 > 0$$

is time-like interval.

1. Light-like interval $(ds/d\tau)^2 = 0$: Eqaution (30) becomes

$$\boxed{\varepsilon^2 = \left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} - \frac{2GM\ell^2}{c^2 r^3}} \tag{31}$$

Notice, for the circular orbit, $r$ is a constant, and effetive potential is a constant with respect to $r$, so

$$\frac{dr}{d\tau} = 0, \quad \text{and} \quad \frac{d}{dr}V_{\text{eff}}(r), \tag{32}$$

where the effective potential is given by

$$V_{\text{eff}}(r) = \frac{\ell^2}{r^2} - \frac{2GM\ell^2}{c^2 r^3}. \tag{33}$$

Then we may derive

$$\begin{aligned} 0 &= \frac{d}{dr}\left(\frac{\ell^2}{r^2} - \frac{2GM\ell^2}{c^2 r^3}\right) \\ &= -2\frac{\ell^2}{r^3} + \frac{6GM\ell^2}{c^2 r^4} \\ 0 &= -2r + 3\frac{2GM}{c^2} \end{aligned}$$

solving that

$$r = \frac{3}{2}\frac{2GM}{c^2} = \frac{3}{2}R_s, \tag{34}$$

called "photon sphere".

2. Time-like interval $(ds/d\tau)^2 < 0$: Notice that

$$\left(\frac{ds}{d\tau}\right)^2 = \frac{ds}{d\tau}\frac{ds}{d\tau} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

which is the inner product of the four velocity, so

$$\left(\frac{ds}{d\tau}\right)^2 = g_{\mu\nu}U^\mu U^\nu = -c^2.$$

Also, plugging $\varepsilon = E/m$ and $\ell = L/m$, equation (30) becomes

$$0 = \frac{E^2}{m^2c^2} - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{m^2r^2} + \frac{2GML^2}{m^2c^2r^3} + \frac{2GM}{c^2 r}c^2 - c^2$$

Solving that

$$\frac{E^2}{m^2c^2} - c^2 = \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{m^2r^2} - \frac{2GML^2}{m^2c^2r^3} - \frac{2GM}{r}$$

or rearange the equation

$$\boxed{\begin{aligned} \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) &= \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 \\ &\quad + m\left(\frac{L^2}{2m^2r^2} - \frac{GM}{r} - \frac{GML^2}{m^2c^2r^3}\right) \end{aligned}} \tag{35}$$

and define the effective potential

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GML^2}{mc^2r^3} \tag{36}$$

### C. Comparision

Here, compare the potential in Classical Dynamics (C.D.) and General Relativity (G.R.), which is equation (33) and eqaution (36)

$$V_{\text{eff}} = \begin{cases} \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r} & ,\text{(C.D.)} \\ \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r} - \dfrac{GML^2}{mc^2r^3} & ,\text{(G.R.)} \end{cases}. \tag{37}$$

Taking

$$G = M = m = c = 1 \tag{38}$$

the potential becomes

$$V_{\text{eff}} = \begin{cases} \dfrac{L^2}{2r^2} - \dfrac{1}{r} & ,\text{(C.D.)} \\ \dfrac{L^2}{2r^2} - \dfrac{1}{r} - \dfrac{L^2}{r^3} & ,\text{(G.R.)} \end{cases}. \tag{39}$$

## II. NUMERICAL

### A. The potential perturbation

There are multiple ways [1] to derive the potential for this problem. The previous deviation is using the General Relativity where the correcting term is propotion to $1/r^3$. And the other method is setting the correction term to be propotion to $1/r^2$ where the solution has been derived, let us called it "Bob's Theory". Therefore, we have three different potential in spherical coordinate

$$V_{\text{eff}}(r) = \begin{cases} \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r} & ,\text{(C.D.)} \\ \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r}\left(1 + \dfrac{3GM}{c^2 r}\right) & ,\text{(Bob)} \\ \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r}\left(1 + \dfrac{L^2}{m^2c^2r^2}\right) & ,\text{(G.R.)} \end{cases} . \tag{40}$$

If we substitude the first term of effective potential $V_{\text{eff}}$ back to the kinetic energy term, the energy in Cartesian coordinate is given by

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} + V(\vec{r}), \tag{41}$$

and we obtain the potential in the Cartesian coordinate, which is

$$V(r) = \begin{cases} -\dfrac{GMm}{r} & ,\text{(C.D.)} \\ -\dfrac{GMm}{r}\left(1 + \dfrac{3GM}{c^2 r}\right) & ,\text{(Bob)} \\ -\dfrac{GMm}{r}\left(1 + \dfrac{L^2}{m^2c^2r^2}\right) & ,\text{(G.R.)} \end{cases} . \tag{42}$$

and the force can be derive from $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$ appling the Newton's second law $\vec{F} = m\ddot{\vec{r}}$, we have

$$\boxed{\vec{F}(\vec{r}) = \begin{cases} \dfrac{GMm}{r^2}\hat{r} & ,\text{(C.D.)} \\ \dfrac{GMm}{r^2}\hat{r} + \dfrac{6G^2M^2m}{c^2r^3}\hat{r} & ,\text{(Bob)} \\ \dfrac{GMm}{r^2}\hat{r} + \dfrac{6GL^2M}{c^2mr^4}\hat{r} & ,\text{(G.R.)} \end{cases} .} \tag{43}$$

Now, we may using the numerical nethod to iterate the solution for the solution of $\vec{F}(\vec{r}) = m\ddot{\vec{r}}$.

### B. Runge Kutta Method

#### 1. *Reduce the problem in vector form*

In this problem, we want to solve the problem

$$\ddot{\vec{r}} = \frac{1}{m}\vec{F}(\vec{r}) \tag{44}$$

Since there are 6 parameters in for this differential equation, which is position $\vec{r} = (x, y, z)$ and velocity $\vec{v} = (v_x, v_y, v_z)$, we define a $6 \times 1$ vector to using the Runge Kurra method, rather than using a lot of computation formula. Define

$$\vec{Y} = \left(\vec{r} \;\; \dot{\vec{r}}\right)^T = \left(x \;\; y \;\; z \;\; v_x \;\; v_y \;\; v_z\right)^T \tag{45}$$

and its differential is given by

$$\begin{aligned} \dot{\vec{Y}} &= \left(\dot{x} \;\; \dot{y} \;\; \dot{z} \;\; \dot{v}_x \;\; \dot{v}_y \;\; \dot{v}_z\right)^T \\ &= \left(v_x \;\; v_y \;\; v_z \;\; a_x \;\; a_y \;\; a_z\right)^T \\ &= \left(\dot{\vec{r}} \;\; \ddot{\vec{r}}\right)^T \\ &= \left(\dot{\vec{r}} \;\; \vec{F}(\vec{r})/m\right)^T \end{aligned}$$

Therefore, this problem becomes a very simple equation

$$\dot{\vec{Y}} = \vec{f}\left(\vec{r}, \dot{\vec{r}}\right), \tag{46}$$

where the vector function is given by

$$\vec{f}\left(\vec{r}, \dot{\vec{r}}\right) = \begin{pmatrix} \dot{\vec{r}} \\ \vec{F}(\vec{r})/m \end{pmatrix}, \tag{47}$$

and $\vec{F}$ is the function of the force (43).

#### 2. *General Explicit Runge Kutta method*

The family of explicit Runge-Kutta methods is given by

$$\vec{Y}_{n+1} = \vec{Y}_n + h\sum_{i=1}^{s} b_i \vec{f}_i, \tag{48}$$

where $f_i$ is given by

$$\vec{f}_i = \vec{f}\left(x_n + c_i h \;,\; \vec{Y}_n + h\sum_{j=1}^{i-1} a_{i,j}\vec{f}_j\right) \tag{49}$$

and $h$ is the step, $b_i, c_i, a_{i,j}$ are the parameter in Runge Kutta method, which can be obtain from the Runge Kutta error term in $s$ order.

#### 3. *General Embedded Runge Kutta method*

Since the orbit problem may easily diverge, in order to ensure the error in every step to be very small. Here, introduce another method called embedded Runge Kutta method, which need to calculated by

$$\begin{aligned} \hat{\vec{Y}}_{n+1} &= \vec{Y}_n + h\sum_{i=1}^{s} \hat{b}_i \vec{f}_i, \\ \vec{Y}_{n+1} &= \vec{Y}_n + h\sum_{i=1}^{s} b_i \vec{f}_i, \end{aligned} \tag{50}$$

and form $\vec{f}_i$ are the same as explicit Runge Kutta method

$$\vec{f}_i = \vec{f}\left(x_n + c_i h \;,\; \vec{Y}_n + h\sum_{j=1}^{i-1} a_{i,j}\vec{f}_j\right) \tag{51}$$

and the error in every step is given by

$$\text{error} = \left\| \hat{\vec{Y}}_n - \vec{Y}_n \right\|, \tag{52}$$

and $h$ is the step, $\hat{b}_i, b_i, c_i, a_{i,j}$ are the parameter in Runge Kutta method, which can be obtain from the Runge Kutta error term in $s$ order.

### C. Simulation force

We may consider all the case for Classical Dynamics, Bob's theory and General Relativity by taking the force in the form

$$\vec{F}(\vec{r}) = \frac{GMm}{r^2}\left(1 + \frac{\beta}{r} + \frac{\gamma}{r^2}\right)\frac{\vec{r}}{|r|}, \tag{53}$$

where

$$\beta = \frac{6GM}{c^2}, \tag{54}$$

$$\gamma = \frac{6L^2}{c^2 m^2}. \tag{55}$$

When we want to calculate the orbit for the Bob's theory, we may set $\gamma = 1$. In other case, when we want to calculate the orbit for the General Relativity, we may set $\beta = 0$.

### D. Numerical analysis and parameter Table

In order to reduce our code, we transfer the constant and orbit parameter into Astronomical system of units. Also, we reduce the iteration function by changing the units and conserved quantities.

#### 1. *iteration function*

By general relativity, the gravitational force near black hole is

$$\vec{F} = \frac{GMm}{r^2}\left(1 + \frac{\gamma}{r^2}\right), \quad \gamma = \frac{6L^2}{m^2c^2}.$$

With the conserved quantity, angular momentum, which can be evaluated by the intial condition $r_0, v_0$ plug in to the $\gamma$, obtaining

$$\gamma = \frac{6\left(mr_0v_0\right)^2}{m^2c^2} = \frac{6r_0^2v_0^2}{c^2}$$

Therefore the ODE for the orbit is reduced to

$$\ddot{\vec{r}} = \frac{GM}{r^2}\left(1 + \frac{6r_0^2v_0^2}{r^2c^2}\right).$$

#### 2. *Useful Table*


[1] James D. Wells. (2022) When effective theories predict: The Inevitability of Mercury' s anomalous perihelion precession.

[2] Wells, James D. (2011) When effective theories predict: the inevitability of Mercury's anomalous perihelion precession. doI: 10.48550/ARXIV.1106.1568

[3] Dr. David R. Williams. Mercury fact sheet. `https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html`. Accessed: 2017-12-07.

[4] eigenchris (2021) Relativity 108c: Schwarzschild Metric - Geodesics (Mercury perihelion advance, photon sphere) `https://youtu.be/2pPvUx-EUqE`

| constant | notation(math) | value | S.I. unit |
|---|---|---|---|
| solar mass | $(M_\odot)$ | $1.9885 \times 10^{30}$ | (kg) |
| Earth mass | $(M_\oplus)$ | $3.3010 \times 10^{23}$ | (kg) |
| astronomical unit | $(AU)$ | $1.4960 \times 10^{11}$ | (m) |
| day | (day) | $8.6400 \times 10^{4}$ | (m) |
| arc second | $(1^\circ/\text{arcsec})$ | 1/3600 | (degree) |
| arc second | (1/arcsec) | $\pi/648000$ | (radius) |

TABLE I. Astronomical system of units

| constant | value (S.I.) | value (A.U.) |
|---|---|---|
| gravitational constant $G$ | $6.6743 \times 10^{-11}(\text{m}^3/\text{kg}/\text{s}^2)$ | $2.9593 \times 10^{-4}(\text{AU}^3/M_\odot/\text{day}^2)$ |
| speed of light $c$ | $2.9979 \times 10^{8}(\text{m/s})$ | $1.7314 \times 10^{2}(\text{AU/day})$ |

TABLE II. Useful physics constant

| constant | value (A.U.) | value (S.I.) |
|---|---|---|
| solar mass | $1.9885 \times 10^{30}(\text{kg})$ | $1(M_\odot)$ |
| planet mass | $3.3010 \times 10^{23}(\text{kg})$ | $0.0553(M_\oplus)$ |
| period | $7.6006 \times 10^{6}(\text{s})$ | 87.97(day) |
| Perihelion radius | $4.6000 \cdot 10^{10}(\text{m})$ | 0.3074(AU) |
| Aphelion radius | $6.9818 \cdot 10^{10}(\text{m})$ | 0.4667(AU) |
| Perihelion velocity | $5.8980 \cdot 10^{4}(\text{m})$ | 0.0340(AU/day) |
| Aphelion velocity | $3.8860 \cdot 10^{4}(\text{m})$ | 0.0224(AU/day) |

TABLE III. parameters for the Mercury orbit system