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微積分（II）Quiz #3 (50 minutes) 2024/03/18

1. (10 points) 求以下函數的Maclaurin級數： $f(x) = 2^x$, 與其收斂半徑R.

(解法1) $2^x = e^{\ln 2^x} = e^{x(\ln 2)} = \sum_{n=0}^{\infty} \frac{[(\ln 2)x]^n}{n!}$, $R = \infty$

$= \sum_{n=0}^{\infty} \frac{(\ln 2)^n x^n}{n!}$

(解法2) $f(x) = 2^x,\ f(0) = 2^0 = 1$

$f' = 2^x(\ln 2),\ f'(0) = (\ln 2)$

$f'' = 2^x(\ln 2)^2,\ f''(0) = (\ln 2)^2$

$\vdots$

$f^{(n)} = 2^x(\ln 2)^n,\ f^{(n)}(0) = (\ln 2)^n$

$\Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n}$ 求收斂半徑R:

$\lim_{n\to\infty} \frac{\frac{(\ln 2)^{n+1}}{(n+1)!}|x^{n+1}|}{\frac{(\ln 2)^n}{n!}|x^n|} = (\ln 2)|x| \lim_{n\to\infty} \frac{1}{(n+1)}$

$= 0 \cdot |x| < 1,\ \forall x \in \mathbb{R} \Rightarrow R = \infty$

2. (10 points) 用級數計算不定積分： $\int x^2 \sin(x^2)dx$ 並且求其收斂半徑R.

使用書上公式: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $R = \infty$

$\Rightarrow x^2 \sin(x^2) = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

$\therefore \int x^2 \sin(x^2) = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \left(\int \frac{(-1)^n x^{4n+4}}{(2n+1)!} dx \right)$

$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(2n+1)!(4n+5)}$

3. (10 points) 用級數計算極限： $\lim_{x\to 0} \frac{\sqrt[3]{8+x} - 2 - x/12}{x^2}$

$\sqrt[3]{8+x} = \sqrt[3]{8(1+\frac{x}{8})} = \sqrt[3]{8}(1+\frac{x}{8})^{1/3} = 2\sum_{n=0}^{\infty} \binom{1/3}{n} \left(\frac{x}{8}\right)^n$ binomial 級數

$= 2\left[1 + \binom{1/3}{1}\left(\frac{x}{8}\right) + \binom{1/3}{2}\left(\frac{x}{8}\right)^2 + \binom{1/3}{3}\left(\frac{x}{8}\right)^3 + \cdots\right]$

$= 2 + 2\cdot\left(\frac{1}{3}\right)\left(\frac{x}{8}\right) + 2\cdot\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}\left(\frac{x}{8}\right)^2 + \left(A_3x^3 + A_4x^4 + \cdots\right)$

$= 2 + \frac{x}{12} + 2\cdot\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}\frac{x^2}{64} + O(x^3)$

$\therefore \lim_{x\to 0} \frac{\left(2 + \frac{x}{12} + \frac{-x^2}{9\cdot 32} + A_3x^3 + A_4x^4 + \cdots\right) - 2 - \frac{x}{12}}{x^2}$

$= \lim_{x\to 0} \left(\frac{-1}{9\cdot 32} + A_3x + A_4x^2 + A_5x^3 + \cdots\right) = \frac{-1}{9\times 32} = \frac{-1}{288}$

4. (10 points) 求 $f(x)=\ln(1+2x)$ 在 $a=1$ 處的2次Taylor多項式，如果使用此多項式在區間 $[0.5,1.5]$上做近似，試估計此近似的最大誤差。

$f(x)=\ln(1+2x)$, $f(1)=\ln(3)$

$f'=\frac{2}{1+2x}$, $f'(1)=\frac{2}{3}$

$f''=\frac{(-1)2\cdot 2}{(1+2x)^2} \Rightarrow f''(1)=\frac{-4}{9}$

$f'''=(-1)(-2)\cdot 4\cdot(1+2x)^{-3}\cdot 2=\frac{16}{(1+2x)^3}$

$T_2(x)=\ln(3)+\frac{2}{3}(x-1)+\frac{\frac{-4}{9}}{2!}(x-1)^2=\ln 3+\frac{2}{3}(x-1)-\frac{2}{9}(x-1)^2$

其中，$g(x)=(1+2x)^3 \Rightarrow g'=3(1+2x)^2\cdot 2>0\ \forall x$

為 increasing 函數

$(1+2x)^3$在 $x=0.5$有最小值 $\Rightarrow |f'''(x)|$在 $x=0.5$有最大值 $M=\frac{16}{(1+2\times0.5)^3}=\frac{16}{2^3}=2$

$\therefore$最大誤差: $\frac{2}{3!}|0.5|^3$

或 $\frac{2\cdot(\frac{1}{2})^3}{6}=\frac{1}{3\cdot 8}=\frac{1}{24}$

9 **Taylor's Inequality** If $|f^{(n+1)}(x)|\leqslant M$ for $|x-a|\leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)|\leqslant\frac{M}{(n+1)!}|x-a|^{n+1}\qquad\text{for } |x-a|\leqslant d$$

5. (10 points) 求函數 $f(x)=\sec x$ 的Maclaurin級數的前三項(非零項).

$\sec(x)=\frac{1}{\cos x}$, $\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$
$=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$

$$\begin{array}{r l}
 & 1+\frac{x^2}{2}+\frac{5}{24}x^4+\cdots \\
1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots \big) & 1 \\
 & 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720} \\
\hline
 & \frac{x^2}{2}-\frac{x^4}{24}+\frac{x^6}{720} \\
 & \frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{48} \\
\hline
 & \frac{5}{24}x^4+A_6x^6+\cdots \\
 & \frac{5}{24}x^4\ \cdots
\end{array}$$

$\left(\frac{-1}{24}-\left(\frac{-1}{4}\right)=\frac{1}{4}-\frac{1}{24}=\frac{5}{24}\right)$

$\therefore$前三項為: $1+\frac{x^2}{2}+\frac{5x^4}{24}$

6. (10 points) 求函數 $f(x)=e^{-x^2}\cos x$ 的Maclaurin級數的前三項(非零項).

$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=1-\frac{x^2}{1}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots=1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\cdots$

$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots$

$$\begin{array}{r l}
 & 1-x^2+\frac{x^4}{2}+\cdots \\
\times) & 1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots \\
\hline
 & 1-x^2+\frac{x^4}{2}+\cdots \\
 & -\frac{x^2}{2}+\frac{x^4}{2}+\cdots \\
 & \frac{x^4}{24}+\cdots \\
\hline
 & 1-\frac{3x^2}{2}+\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{24}\right)x^4
\end{array}$$

$\Rightarrow$ 前三項為

$1-\frac{3x^2}{2}+\frac{25}{24}x^4$