系級........... 組別: A / B 學號 .... 姓名.....................

微積分 (I) Quiz #4 (45 minutes) 2023/10/23

除了選擇,填充和簡答題之外，你的答案必須提供完整說明，如果只有答案沒有任何說明得零分!

1. (10 points) Find $x \in R$ such that $\tanh(x) = 12/13$.

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{12}{13} \Leftrightarrow 13e^x - 13e^{-x} = 12e^x + 12e^{-x}$$

$$\Leftrightarrow e^x - 25e^{-x} = 0$$

$$\Rightarrow e^{-x}(e^{2x} - 25) = 0$$

$$\Rightarrow e^{2x} = 25$$

$$\Rightarrow 2x = \ln 25 = \ln 5^2 = 2\ln 5,$$

$$\Rightarrow \boxed{x = \ln 5}$$

2. (5+5=10 points) (a) 推導以下公式: $d(uv) = udv + vdu$

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$\Rightarrow d(uv) = \frac{d(uv)}{dx} \cdot dx = (u'v + uv')dx$$

$$= v(u'dx) + u(v'dx) = vdu + udv$$

(b) 給定一個球~~[illegible]~~体，其半徑經過測量為 21 公分(測量值最大誤差為0 .05公分)，如果使用上述測量值估算此球體積時，最大的可能誤差為何?

$$V(r) = \frac{4}{3}\pi r^3, \quad dV = 4\pi r^2 dr, \quad dr = 0.05, \quad r = 21$$

$$\Rightarrow dV = 4\pi \cdot (21)^2 \underbrace{dr}_{0.05} = 88.2 \text{ cm}^3$$

$$(21)^2 = 441$$

3. (10 points) Use a linear approximaton (or differentials) to estimate $\cos(29°)$.

$$\cos(29°) = \cos(30° - 1°) = \cos\left(\frac{\pi}{6} - \frac{\pi}{180}\right); \quad f(x) = \cos x, \quad a = \frac{\pi}{6}, \quad \Delta x = \frac{-\pi}{180}$$

$$1° = \frac{\pi}{180}$$

$$f\left(\frac{\pi}{6} - \frac{\pi}{180}\right) = f\left(\frac{\pi}{6} + \frac{-\pi}{180}\right)$$

$$\simeq f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(\frac{-\pi}{180}\right)$$

$$f' = -\sin(x)$$

$$f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = \frac{-1}{2}$$

$$= \cos\left(\frac{\pi}{6}\right) + \left(\frac{-1}{2}\right)\left(\frac{-\pi}{180}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{360}$$

4. (5+5=10 points) Find linearization $L(x)$ of (a) $f(x)=e^{-2x}$ at $a=0$.

$f(a)=e^{0}=1,\quad f'(x)=e^{-2x}\cdot(-2),\quad f'(0)=e^{0}\cdot(-2)=-2$

$\therefore L(x)=1+(-2)\cdot x=1-2x$

(b) $g(x)=\dfrac{1}{1+x^2}$ at $a=1$.

$g(1)=\frac{1}{2}\qquad g'(x)=\dfrac{(-1)\,2x}{(1+x^2)^2},\quad g'(1)=\dfrac{-2}{(1+1)^2}=\dfrac{-2}{4}=\dfrac{-1}{2}$

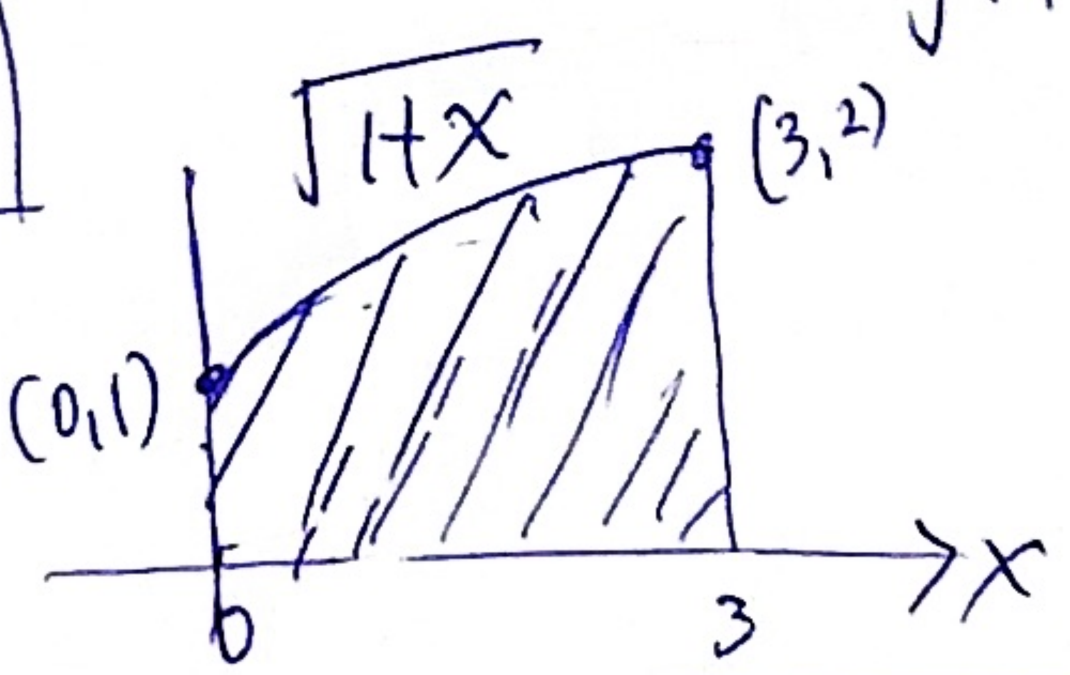
$L(x)=\frac{1}{2}+\frac{-1}{2}(x-1)$

5. (3+3+4=10 points) 簡答題. 極限 $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$ 是函數 $f(x)$ 在區間 $[0,b]$ 的面積，求 $f(x)$ 與 b. 描繪此圖形。

$\Delta x=\left(\frac{3}{n}\right)=\frac{b-0}{n}=\frac{b}{n},\ \therefore b=3,\quad x_i=i\left(\frac{3}{n}\right)$

$\sqrt{1+\frac{3i}{n}}=\sqrt{1+x_i}=f(x_i)$

$\boxed{f(x)=\sqrt{1+x}}$,



$\therefore f(x)=\sqrt{1+x}$

6. (10 points) 簡答題. 極限 $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{3}{n}\left[\left(2+\frac{3i}{n}\right)+\frac{1}{\left(2+\frac{3i}{n}\right)^2}\right]=\int_2^b f(x)dx$,

求 $f(x)$ 與 b.

$\boxed{\Delta x=\frac{b-2}{n}=\frac{3}{n}\Rightarrow b=5}$, $x_i=2+\left(\frac{3}{n}\right)i\Rightarrow\left[\left(2+\frac{3}{n}i\right)+\frac{1}{\left(2+\frac{3i}{n}\right)^2}\right]=x_i+\frac{1}{x_i^2}$

$=f(x_i)$

$\therefore f(x)=x+\frac{1}{x^2},\quad b=5$

$\Rightarrow f(x)=x+\frac{1}{x^2}$