系級............ 組別： A / B 學號 .... 姓名......................

微積分 (I) Quiz #5~~(期中考練習)~~ (45 minutes) 2023/11/13

除了選擇，填充和簡答題之外，你的答案必須提供完整說明，如果只有答案沒有任何說明得零分！

1. (4+6=10 points) 計算以下數值：(a) $\arcsin(\sin(5\pi/4))$.

$$= \arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \quad \left(\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

(b) $\cos\left(2\sin^{-1}(5/13)\right)$ 令 $\theta = \sin^{-1}(5/13) \Leftrightarrow \sin\theta = 5/13$,

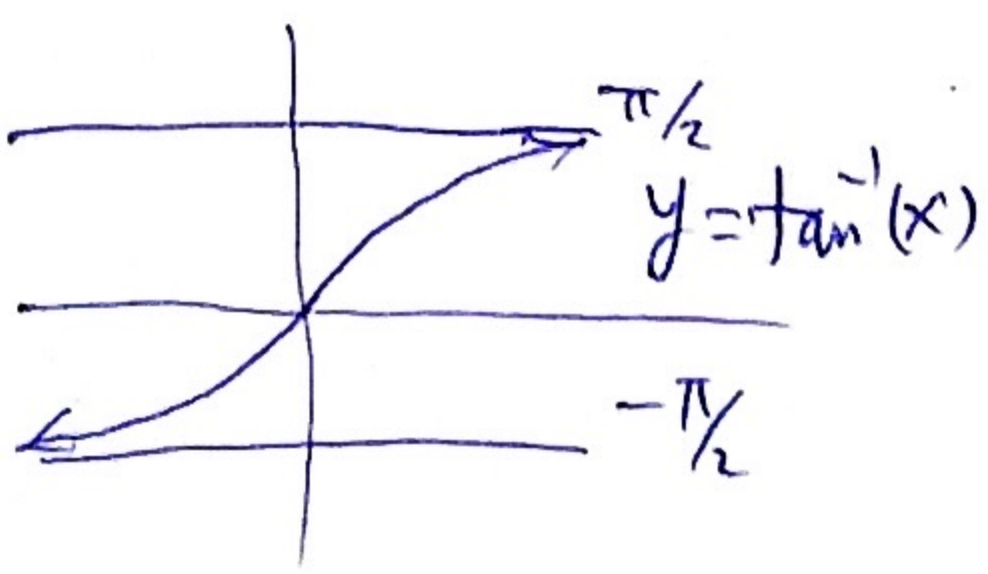
(13, 5, 12, θ)

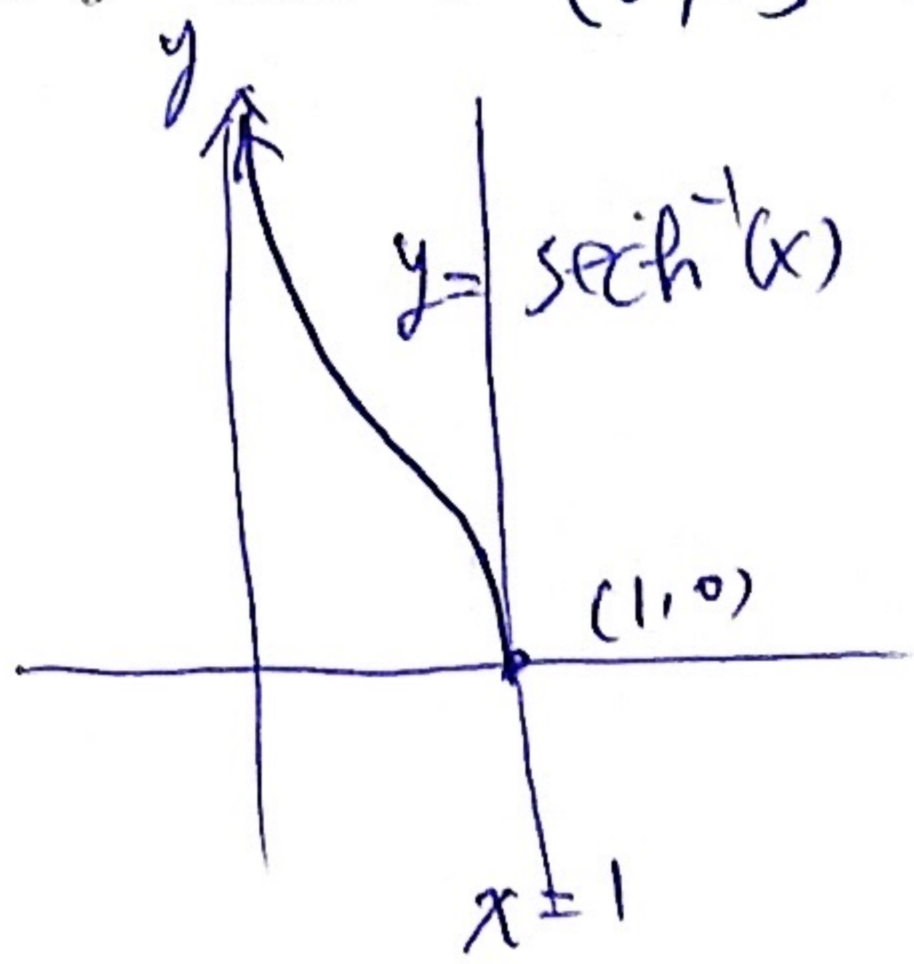
$$= \cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{144-25}{169}$$

$$= 119/169$$

2. (5+5=10 points) 說明以下函數的定義域和值域並且描繪其圖形(正確的凹凸形狀與漸進線)：

(a) $y = \tan^{-1}x$: $\mathbb{R} \to (-\pi/2, \pi/2)$ (b) $y = \operatorname{sech}^{-1}x$ $(0,1] \to [0,\infty)$





3. (10 points) 推導以下公式：

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

令 $y = \tanh^{-1}(x)$

$\Leftrightarrow \tanh(y) = x$.

$\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$

$\Rightarrow xe^{2y} + x = e^{2y} - 1$

$\Rightarrow (1+x) = (1-x)e^{2y}$

$\Rightarrow e^{2y} = \frac{1+x}{1-x}$, $(-1<x<1)$

$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

4. (5+5=10 points) 求以下積分：(a) $\int e^{-5x}\,dx$,

$$\boxed{\begin{aligned}&令\ y=-5x \Rightarrow dy=-5dx\\ &\qquad\Rightarrow dx=\frac{dy}{(-5)}\end{aligned}}$$

$$=\int e^{y}\frac{dy}{-5}=\frac{-1}{5}e^{y}+C$$
$$=\frac{-1}{5}e^{-5x}+C$$

(b) $\int \sec^2 x\tan^3 x\,dx=\int u^3du=\frac{u^4}{4}+C$

令 $u=\tan x$

$\Rightarrow du=\sec^2 x\,dx$

$$=\frac{\tan^4 x}{4}+C$$

5. (10 points) 求定積分：$\int_1^2 \frac{e^x}{(1-e^x)^2}\,dx$ 令 $v=1-e^x \Rightarrow dv=-e^x dx$

$$=\int_{1-e}^{1-e^2}\frac{-dv}{v^2}=-\int_{1-e}^{1-e^2}v^{-2}dv=v^{-1}\Big|_{1-e}^{1-e^2}\qquad \therefore\ e^x dx=-dv$$

$$=\frac{1}{1-e^2}-\frac{1}{1-e}=\frac{1-(1+e)}{1-e^2}=\frac{-e}{1-e^2}=\frac{e}{e^2-1}$$

另解：先求不定積分

$$\int\frac{e^x dx}{(1-e^x)^2}=-\int v^{-2}dv=v^{-1}+C=\frac{1}{(1-e^x)}$$

$$\therefore\int_1^2\frac{e^x}{(1-e^x)^2}dx=\frac{1}{1-e^x}\Big|_1^2=\left(\frac{1}{1-e^2}-\frac{1}{1-e}\right)$$

6. (10 points) 求定積分：$\int_1^2 \frac{\sqrt{1+1/x}}{x^2}\,dx$

令 $u=1+\frac{1}{x}\Rightarrow du=-1(x^{-2})dx\Rightarrow \frac{dx}{x^2}=-du$

$$\int_1^2\frac{\sqrt{1+1/x}}{x^2}dx=\int_2^{\frac{3}{2}}\sqrt{u}\cdot(-du)=\int_{3/2}^2 u^{1/2}du=\frac{2}{3}u^{\frac{3}{2}}\Big|_{3/2}^2=\frac{2}{3}\left(2^{\frac{3}{2}}-\left(\frac{3}{2}\right)^{\frac{3}{2}}\right)$$

另解 先求不定積分：$\int\frac{\sqrt{1+\frac{1}{x}}}{x^2}dx=\int\sqrt{u}\,(-du)=-\frac{2}{3}u^{\frac{3}{2}}+C=-\frac{2}{3}\left(1+\frac{1}{x}\right)^{\frac{3}{2}}+C$

$$\therefore\ 原式=-\frac{2}{3}\left(1+\frac{1}{x}\right)^{\frac{3}{2}}\Big|_1^2=\frac{2}{3}\left((1+1)^{\frac{3}{2}}-\left(1+\frac{1}{2}\right)^{3/2}\right)=\frac{2}{3}\left(2^{3/2}-\left(\frac{3}{2}\right)^{3/2}\right)$$