系級............... 學號 .... 姓名..........................

微積分 (I) Quiz #0 (35 minutes) 2024/02/26

1. (10 points) 若 $\int_0^\infty x^k e^{-x}\, dx = k!$ (k 為正整數), 試證明 $\int_0^\infty x^{k+1} e^{-x}\, dx = (k+1)!$.

$\int x^{k+1} e^{-x} dx$ 用 integration by parts: $\begin{cases} u = x^{k+1}, & dv = e^{-x}dx \\ du = (k+1)x^k dx, & v = -e^{-x} \end{cases}$

$= -x^{k+1}e^{-x} + (k+1)\int x^k e^{-x} dx$

$\int_0^\infty x^{k+1}e^{-x}dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx = \lim_{t\to\infty}\left(-x^{k+1}e^{-x}\right)\Big|_0^t + \lim_{t\to\infty}\int_0^t x^k e^{-x}dx$

$= \lim_{t\to\infty}\left(-t^{k+1}e^{-t}\right) + \int_0^\infty x^k e^{-t} dt$

其中 $\lim_{t\to\infty}\frac{t^{k+1}}{e^t}\left(\frac{\infty}{\infty}\right) = \lim_{t\to\infty}\frac{(k+1)t^k}{e^t} = \cdots = \lim_{t\to\infty}\frac{(k+1)!}{e^t} = 0$

$\therefore \int_0^\infty x^{k+1}e^{-x}dx = 0 + (k+1)\int_0^\infty x^k e^{-t}dt = (k+1)\,k! = (k+1)!$

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2. (10 points) 判斷數字 a 必須有多大才能滿足不等式:

$$\int_a^\infty \frac{1}{1+x^2}\, dx < 0.001$$

$= \lim_{t\to\infty}\int_a^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\tan^{-1}(x)\Big|_a^t = \lim_{t\to\infty}\tan^{-1}(t) - \tan^{-1}(a)$

$= \frac{\pi}{2} - \tan^{-1}(a) < 0.001$

$\Rightarrow \tan^{-1}(a) > \left(\frac{\pi}{2} - 0.001\right)$, ($y = \tan^{-1}(x)$ 為嚴格遞增函數)

$\Rightarrow a > \tan\left(\frac{\pi}{2} - 0.001\right)$

3. (10+10=20 points) 計算以下積分: (a) $\int_0^\infty \sin\theta \cdot e^{\cos\theta}\, d\theta$

$$= \lim_{t\to\infty} \int_0^t \sin\theta\, e^{\cos\theta}\, d\theta$$

$$= \lim_{t\to\infty} \left(-e^{\cos\theta}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-e^{\cos t} + e^{\cos 0}\right)$$

$$= e - \lim_{t\to\infty} e^{\cos t}$$, $\cos t$ 在 $-1, 1$ 之間振盪

$\Rightarrow \lim_{t\to\infty} e^{\cos t}$ 在 e^{-1}, e^{1} 之間振盪 故原式發散

, $\int \sin\theta\, e^{\cos\theta}\, d\theta$: 令 $u = \cos\theta$

$\Rightarrow du = -\sin\theta\, d\theta$

$$= \int -e^u\, du = -e^u + C$$

$$= -e^{\cos\theta} + C$$

(b) $\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\, dx$

$u = \ln x$, $dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx$, $v = -x^{-1}$

$$\int_1^t \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx$$

$$= \frac{-\ln x}{x} - x^{-1} + C$$

$$= \lim_{t\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-\ln t}{t} - \frac{1}{t} + 0 + 1\right)$$

$$= \lim_{t\to\infty} \frac{-\ln t}{t} \underset{(\text{羅})}{} - \lim_{t\to\infty} \frac{1}{t} + 1$$

$$= \lim_{t\to\infty} \frac{-1/t}{1} - 0 + 1$$

$$= 0 - 0 + 1 = 1$$