Physics, Calculus, Quiz 10
December 26, 2023, Chang-Mao Yang

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1. (a) 推導公式：$\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$ 其中 $(n \neq 1)$。

   (b) 用此公式求 $\displaystyle\int \tan^3 x\,dx$。

2. 求積分 $\displaystyle\int x\sqrt{1-x^4}\,dx$。

3. 求積分 $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$。

4. 用 §8.1 的公式求曲線 $y=\sqrt{4-x^2}$，$0 \leq x \leq 2$，從 $(2,0)$ 到 $(0,2)$ 之弧長。

5. 求定積分 $\displaystyle\int \frac{x}{x^2+4x+13}\,dx$。

6. 求積分：$\displaystyle\int \frac{dx}{1+e^x}$。（hint：令 $u=e^x$。）

1. (a) 推導公式：$\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$ 其中 $(n \neq 1)$。

(b) 用此公式求 $\displaystyle\int \tan^3 x\,dx$。

(a)

$$\int \tan^n x dx = \int \tan^{n-2} x \cdot (\tan^2 x) dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \quad , \quad \text{let } u = \tan x \Rightarrow du = \sec^2 x dx$$

$$= \int u^{n-2} du - \int \tan^{n-2} x dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \quad \blacksquare$$

(b)

$$\int \tan^3 x dx = \frac{\tan^2 x}{2} - \int \tan x dx = \frac{\tan^2 x}{2} - \int \frac{\sin x}{\cos x} dx \quad , \quad \text{let } u = \cos x \Rightarrow du = -\sin x dx$$

$$= \frac{\tan^2 x}{2} - \int \frac{-1}{u} du = \frac{\tan^2 x}{2} + \ln|u| + C$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + C \quad \blacksquare$$

2. 求積分 $\int x\sqrt{1-x^4}\,dx$。

思路：1. 不是單純 $x^n, \cos x, \sin x, \ln x, e^x, \dots$ 或分式

2. 有根號：令 $u=1-x^4$，$x=(1-u)^{1/4}$，$dx=-\frac{1}{4}(1-u)^{-\frac{3}{4}}du$

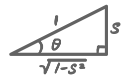
$$\int x\sqrt{1-x^4}\,dx=\int(1-u)^{1/4}\cdot\sqrt{u}\,(-1/4)(1-u)^{-\frac{3}{4}}du=-\frac{1}{4}\int(1-u)^{-\frac{2}{4}}\sqrt{u}\,du$$

$$=-\frac{1}{4}\int\sqrt{\frac{u}{1-u}}\,du$$

(通常到這裡就可以放棄這個方法了，但還是可以繼續算。令 $s=\sqrt{u}$ or $1-u$) let $s=\sqrt{u}$, $u=s^2$ $\Rightarrow du=2s\,ds$

$$=-\frac{1}{4}\int\sqrt{\frac{s^2}{1-s^2}}\,2s\,ds=-\frac{1}{2}\int\frac{s^2}{\sqrt{1-s^2}}\,ds$$

, let $s=\sin\theta \Rightarrow ds=\cos\theta\,d\theta$



$$=-\frac{1}{2}\int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta=-\frac{1}{2}\int\sin^2\theta\,d\theta=-\frac{1}{2}\int\frac{1-\cos2\theta}{2}\,d\theta=\frac{1}{4}\int\cos2\theta\,d\theta-\frac{\theta}{4}$$

$$=\frac{1}{4}\cdot\frac{1}{2}\sin2\theta-\frac{\theta}{4}+C=\frac{1}{4}(\sin\theta\cos\theta-\theta)+C$$

, since $\cos\theta=\sqrt{1-s^2}$, $\sin\theta=s$

$$=\frac{1}{4}\left(s\sqrt{1-s^2}-\sin^{-1}s\right)+C=\frac{1}{4}\left(\sqrt{u}\sqrt{1-u}-\sin^{-1}(\sqrt{u})\right)+C$$

, then by $u=1-x^4$

$$=\frac{1}{4}\left[\sqrt{1-x^4}\sqrt{1-(1-x^4)}-\sin^{-1}\left(\sqrt{1-x^4}\right)\right]+C$$

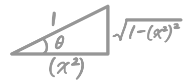
$$=\frac{1}{4}\left(x^2\sqrt{1-x^4}-\sin^{-1}\sqrt{1-x^4}\right)+C \quad ■$$

$$=\frac{1}{4}\left(x^2\sqrt{1-x^4}-\cos^{-1}(x^2)\right)+C \quad ■$$

$$=\frac{1}{4}\left(x^2\sqrt{1-x^4}+\sin^{-1}(x^2)\right)+C' \quad ■$$

(三個答案都可，積分常數不同而已。)

Here: $\sin^{-1}\sqrt{1-x^4}=\cos^{-1}x^2=\frac{\pi}{2}-\sin^{-1}(x^2)$



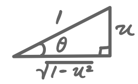
$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

3. 想辦法湊出 $(1-x^2)$ or $(1+x^2)$ 的形式（這樣就可以用三角函數換掉。）

① 法一：(令 $u=x^2 \Rightarrow \sqrt{1-x^4}=\sqrt{1-u^2}$) let $u=x^2 \Rightarrow du=2x\,dx$

$$\int x\sqrt{1-x^4}\,dx=\frac{1}{2}\int\sqrt{1-(x^2)^2}\,2x\,dx=\frac{1}{2}\int\sqrt{1-u^2}\,du$$

, let $u=\sin\theta \Rightarrow du=\cos\theta\,d\theta$



$$=\frac{1}{2}\int\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta=\frac{1}{2}\int\cos^2\theta\,d\theta=\frac{1}{2}\int\frac{1+\cos2\theta}{2}\,d\theta=\frac{1}{4}\left(\theta+\int\cos2\theta\,d\theta\right)$$

$$=\frac{1}{4}\left(\theta+\frac{1}{2}\sin2\theta\right)+C=\frac{1}{4}\theta+\frac{1}{4}\sin\theta\cos\theta+C=\frac{1}{4}\sin^{-1}u+\frac{1}{4}u\sqrt{1-u^2}+C$$

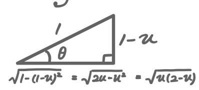
$$=\frac{1}{4}\left(x^2\sqrt{1-x^4}+\sin^{-1}x^2\right)+C \quad ■$$

② 法二 ($\sqrt{1-x^4}=\sqrt{1-x^2}\sqrt{1+x^2}$) let $u=1+x^2 \Rightarrow du=2x\,dx$, also $1-x^2=2-u$

$$\int x\sqrt{1-x^4}\,dx=\frac{1}{2}\int\sqrt{1-x^2}\sqrt{1+x^2}\,2x\,dx=\frac{1}{2}\int\sqrt{2-u}\sqrt{u}\,du=\frac{1}{2}\int\sqrt{2u-u^2}\,du=\frac{1}{2}\int\sqrt{1-(1-2u+u^2)}\,du$$

$$=\frac{1}{2}\int\sqrt{1-(1-u)^2}\,du$$

, let $1-u=\sin\theta$, $du=-\cos\theta\,d\theta$



$$=\frac{1}{2}\int\sqrt{1-\sin^2\theta}\,(-\cos\theta)\,d\theta=-\frac{1}{2}\int\cos^2\theta\,d\theta=-\frac{1}{2}\int\frac{1+\cos2\theta}{2}\,d\theta=-\frac{1}{4}\left(\theta+\frac{1}{2}\sin2\theta\right)+C$$

$$=-\frac{1}{4}\theta-\frac{1}{4}\sin\theta\cos\theta+C=-\frac{1}{4}\sin^{-1}(1-u)-\frac{1}{4}(1-u)\sqrt{u(2-u)}+C$$

, note $u=1+x^2$

$$=-\frac{1}{4}\sin^{-1}(-x^2)-\frac{1}{4}(-x^2)\sqrt{(1+x^2)(2-1-x^2)}+C=\frac{1}{4}\left(x^2\sqrt{(1+x^2)(1-x^2)}-\sin^{-1}(-x^2)\right)+C$$

$$=\frac{1}{4}\left(x^2\sqrt{1-x^2}+\sin^{-1}x^2\right)+C$$

3. 求積分 $\int_0^1 \sqrt{x^2+1}\,dx$。

1. 不是單純 x^n, $\cos x$, $\sin x$, $\ln x$, e^x, ... 或分式

2. 看到 $x^2+1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta$, let $x\Big|_0^1 = \tan\theta\Big|_0^{\pi/4} \Rightarrow dx = \sec^2\theta\, d\theta$

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4} \sqrt{\tan^2+1}\,\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^3\theta\, d\theta ,$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\sqrt{2}\cdot 1 - 0\right) - \frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{1+0}\right|$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \blacksquare$$

solve for $\int \sec^3\theta\, d\theta = \int \underbrace{\sec\theta}_{u} (\underbrace{\sec^2\theta\, d\theta}_{dv})$

let $\begin{cases} u = \sec\theta & \Rightarrow du = \sec\theta\tan\theta\, d\theta \\ dv = \sec^2\theta\, d\theta & \Rightarrow v = \tan\theta \end{cases}$

$$\int \sec^3\theta\, d\theta = \underbrace{\sec\theta}_{u}\ \underbrace{\tan\theta}_{v} - \int \underbrace{\tan\theta}_{v}(\underbrace{\sec\theta\tan\theta\, d\theta}_{du})$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$\Rightarrow \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

3. 有根號：令 $u\Big|_1^2 = x^2+1\Big|_0^1$ $x = \sqrt{u-1}$ $du = 2x\,dx \Rightarrow dx = \frac{1}{2\sqrt{u-1}}du$

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_1^2 \sqrt{u}\,\frac{1}{2\sqrt{u-1}}du = \frac{1}{2}\int_1^2 \sqrt{\frac{u}{u-1}}\,du ,$$

(通常到這裡就可以放棄這個方法了，但還是可以繼續算。令 $s = \sqrt{u}$ or $u-1$)

① 法一 (令 $s=\sqrt{u}$)：let $s\Big|_1^{\sqrt{2}} = \sqrt{u}\Big|_1^2 \Rightarrow u = s^2 \Rightarrow du = 2s\,ds$

$$\int_0^1 \sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^2 \sqrt{\frac{u}{u-1}}\,du = \frac{1}{2}\int_1^{\sqrt{2}} \frac{s}{\sqrt{s^2-1}}\,2s\,ds = \int_1^{\sqrt{2}} \frac{s^2}{\sqrt{s^2-1}}\,ds$$

看到 $s^2-1 \Rightarrow \sec^2\theta - 1 = \tan^2\theta$

let $s = \sec\theta$, $ds = \sec\theta\tan\theta\, d\theta$

($s: 1 \sim \sqrt{2} \longrightarrow \theta: 0 \sim \pi/4$)

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta - 1}}\tan\theta\sec\theta\, d\theta = \int_0^{\pi/4} \sec^3\theta\, d\theta$$

(跟上面結果一樣。)

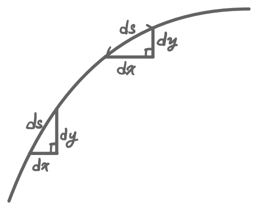
$$= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} = \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \blacksquare$$

② 令 $s = u-1 \Rightarrow ds = du$ $u: 1\sim 2 \Rightarrow s = 0 \sim 1$

$$\int_0^1 \sqrt{x^2+1}\,dx = \frac{1}{2}\int_1^2 \sqrt{\frac{u}{u-1}}\,du = \frac{1}{2}\int_0^1 \frac{\sqrt{s+1}}{\sqrt{s}}\,ds$$

(通常到這裡就可以放棄這個方法了，但還是可以繼續算。太複雜，我放棄。)

4. 用 §8.1 的公式求曲線 $y=\sqrt{4-x^2}$，$0\le x\le 2$，從 $(2,0)$ 到 $(0,2)$ 之弧長。



$$(ds)^2 = (dx)^2+(dy)^2$$
$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx \quad \Rightarrow S = \int ds = \int \sqrt{1+(y')^2}\,dx$$

$$y=\sqrt{4-x^2}\ ,\quad y' = \frac{-2x}{2\sqrt{4-x^2}}\ ,\quad 0\le x\le 2$$

$$S = \int_0^2 \sqrt{1+(y')^2}\,dx = \int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = \int_0^2 \sqrt{\frac{1}{1+(\frac{x}{2})^2}}\,dx\ ,\quad \text{let } \begin{array}{l} u=\frac{x}{2} \Rightarrow 2du = dx \\ x: 0\sim 2 \Rightarrow u: 0\sim 1 \end{array}$$

$$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx\ ,\quad \text{let } x=\sin\theta\ ,\ dx = \cos\theta\,d\theta\ ,\quad x: 0\sim 1\ ,\ \theta = 0\sim \pi/2$$

$$= 2\int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = 2\int_0^{\pi/2} \frac{\cos\theta}{\cos\theta}\,d\theta = 2\int_0^{\pi/2} d\theta = 2\cdot\theta\Big|_0^{\pi/2} = \pi \ \blacksquare$$

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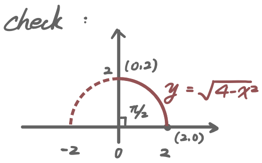
check :



弧長 : $r\theta = 2\cdot\frac{\pi}{2} = \pi$ ✓

calculate the integral : $\displaystyle\int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx$

1. 看到 $a^2-x^2 \Rightarrow$ 三角代換 $(x = a\sin\theta)$

let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta \Rightarrow \begin{array}{l} x: 0\sim 2 \\ \theta: 0\sim \pi/2 \end{array}$

$$\int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = \int_0^{\pi/2} \sqrt{\frac{4}{4-4\sin^2\theta}}\,2\cos\theta\,d\theta = 2\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = 2\cdot\theta\Big|_0^{\pi/2} = \pi$$

2. 看到 $a^2-x^2 \longrightarrow a^2\{1-(\frac{x}{a})^2\} \xrightarrow{u=x/a} a^2(1-u^2)$ ，此法跟上面一樣

$$\int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = 2\cdot\int_0^{\pi/2} \frac{\cos\theta}{\cos\theta}\,d\theta = 2\cdot\theta\Big|_0^{\pi/2} = \pi$$

3. 看到根號 let $u = 4-x^2 \Rightarrow x=\sqrt{4-u}$, $du = -2x\,dx = -2\sqrt{4-u}\,dx$, $\begin{array}{l} x: 0\sim 2 \\ u: 4\sim 0 \end{array}$

$$\int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = \int_4^0 \sqrt{\frac{4}{u}}\,\frac{du}{-2\sqrt{4-u}} = \int_0^4 \frac{1}{\sqrt{4u-u^2}}\,du = \int_0^4 \frac{1}{\sqrt{4-(4-4u+u^2)}}\,du$$

$$= \int_0^4 \frac{1}{\sqrt{2^2-(4-u)^2}}\,du$$ ⋯ 會發現最後還是需要用三角代換。

5. 求定積分 $\displaystyle\int \frac{x}{x^2+4x+13}\,dx$。

思路：

1. 不是單純 $x^n, \cos x, \sin x, \ln x, e^x, \dots$
2. 沒有 x^2-a^2 , x^2+a^2 , a^2-x^2 之形式
3. 此為分式之形式：

① check 分母次方 > 分子次方 ⇒ ✓

② check 分母能不能因式分解

• 三次方：一定會有一個解，因為其圖形為：  or 

• 二次方：沒有解 or 兩個解，因為其圖形為：  or 

$x^2+4x+13$: $b^2-4ac = 16-4\cdot1\cdot13 = -36 < 0$ ⇒ 沒有解

○ 有解 ⇒ 拆成 $(x-x_1)(x-x_2)$ 之形式 (再分式拆解) ✗

○ 沒有解 ⇒ 配方 $(x-x_0)^2+y_0$ 之形式 (再三角代換) ✓

$$\int \frac{x}{x^2+4x+13}dx = \int \frac{x}{x^2+4x+4+9}dx = \int \frac{x}{(x+2)^2+3^2}dx$$

(看到 u^2+3^2 之形式 $\Rightarrow \tan^2\theta+1 = \sec^2\theta$)

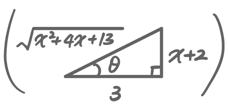
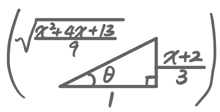
let $x+2 = 3\tan\theta \Rightarrow x = 3\tan\theta - 2$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{3\tan\theta-2}{9\tan^2\theta+9}(3\sec^2\theta d\theta) = \int \frac{3}{9}\,\frac{3\tan\theta-2}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int 3\tan\theta - 2\, d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta - \frac{2}{3}\int d\theta$$

let $u = \cos\theta$

() or ()

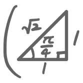
$$= -\int \frac{1}{u}du - \frac{2}{3}\theta = -\ln|u| - \frac{2}{3}\theta + C = \ln\left|\frac{1}{\cos\theta}\right| - \frac{2}{3}\theta + C$$

$$= \ln\left|\frac{\sqrt{x^2+4x+13}}{3}\right| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C = \ln\sqrt{x^2+4x+13} - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + (C-\ln 3)$$

$$= \frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C'$$

$$\Rightarrow \int_0^1 \frac{x}{x^2+4x+13}dx = \frac{1}{2}\ln(x^2+4x+13)\Big|_0^1 - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\Big|_0^1 = \frac{1}{2}\ln\left(\frac{1+4+13}{13}\right) - \frac{2}{3}\left(\tan^{-1}(1) - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \frac{1}{2}\ln\left(\frac{18}{13}\right) - \frac{2}{3}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{2}{3}\right)\right) \blacksquare$$

, ( $\tan\frac{\pi}{4} = 1 \Rightarrow \frac{\pi}{4} = \tan^{-1}(1)$)

6. 求積分：$\int \frac{dx}{1+e^x}$。(hint：令 $u = e^x$。)

1. let $u = e^x \Rightarrow du = e^x dx$, $x = \ln u$, $dx = \frac{1}{e^x} du = \frac{1}{u} du$

$$\int \frac{dx}{1+e^x} = \int \frac{1}{1+u} \left(\frac{1}{u} du\right) = \int \frac{1}{u(u+1)} du$$

① 看到分式（沒有根號，跟上一題思路相同）⇒ 分母次方 > 分子次方 ⇒ 分母二次方 ⇒ 分母有根 ⇒ 拆分式

$$\int \frac{dx}{1+e^x} \overset{u=e^x}{=\!=} \int \frac{1}{u(u+1)} du = \int \frac{1}{u} + \frac{-1}{u+1} du$$

$$= \int \frac{1}{u} du - \int \frac{1}{u+1} du$$

$$= \ln|u| - \ln|u+1| + C \quad \text{since } u = e^x > 0$$

$$= \ln|e^x| - \ln|e^x+1| + C$$

$$= x - \ln(1+e^x) + C \quad \blacksquare$$

$$\boxed{\begin{aligned} &\text{Note}: \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} \\ &\Rightarrow 1 = A(u+1) + Bu \\ &u=0: \ 1 = A(1) + B(0) \Rightarrow A = +1 \\ &u=-1: \ 1 = A(0) + B(-1) \Rightarrow B = -1 \end{aligned}}$$