系級........... 組別: A / B 學號 .... 姓名......................

微積分 (I) Quiz #3 (45 minutes) 2023/10/16

除了選擇, 填充和簡答題之外，你的答案必須提供完整說明，如果只有答案沒有任何說明得零分!

1. (10 points) Find an equation of the tangent line to the curve defined by $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point(3,1).

$4(x^2+y^2)\cdot(2x+2yy') = 50x - 50y\cdot y'$

$x=3, y=1$ $\quad 4\cdot 10\cdot(6+2y') = 150 - 50y'$

$80y' + 240 = 150 - 50y'$

$130y' = -90$

$y' = \frac{-9}{13}$

$\Rightarrow \boxed{(y-1) = \frac{-9}{13}(x-3)}$

或 $\boxed{y = \frac{-9}{13}x + \frac{40}{13}}$

2. (10 points) Use implicit differentiation to find the slope of the tangent line at the point $(\pi/8, \pi/8)$: $\tan(x+y) + \sec(x-y) = 2$.

$\sec^2(x+y)\cdot(1+y') + \sec(x-y)\tan(x-y)\cdot(1-y') = 0$

$\left(\begin{matrix} x = \pi/8 \\ y = \pi/8 \end{matrix} \text{ 代入上式}\right) \Rightarrow$ $\sec^2(\frac{\pi}{4})(1+y') + \sec(0)\tan(0)(1-y') = 0$

$(\sqrt{2})^2(1+y') = 0$

$1+y' = 0$

$\Rightarrow y' = -1$

3. (5+5=10 points) 簡答題. (a) If $f(x) = \ln(\sin^2 x)$, find $f'(\pi/4)$.

$f' = \frac{1}{\sin^2 x}\cdot(2\sin x)\cos x = \frac{2\cos x}{\sin x} \Rightarrow f'(\pi/4) = \frac{2\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 2$

(b) If $g(t) = \sqrt{1+\ln t}$, find $g'(2)$

$g'(t) = \frac{1}{2}\frac{1}{\sqrt{1+\ln t}}\cdot\frac{d}{dt}(\ln t) = \frac{1}{2}\frac{1}{\sqrt{1+\ln t}}\cdot\frac{1}{t}$

$\therefore g'(2) = \frac{1}{2}\frac{1}{\sqrt{1+\ln 2}}\cdot\frac{1}{2} = \frac{1}{4\sqrt{1+\ln 2}}$

4. (10 points) Find y' if $x^y = y^x$. 兩边取 $\ln \Rightarrow \ln x^y = \ln y^x$

$\Leftrightarrow y\ln x = x\ln y$，兩邊對 x 微分：

$\frac{d}{dx}(y\ln x) = \frac{d}{dx}(x\ln y)$

$\Leftrightarrow y'\ln x + y\cdot\frac{1}{x} = \ln y + x\frac{1}{y}y'$

$\Rightarrow \left(\ln x - \frac{x}{y}\right)y' = \ln y - \frac{y}{x}$

$\Rightarrow \boxed{y' = \dfrac{\ln y - y/x}{\ln x - x/y}}$

5. (5+5=10 points) 簡答題. (a) Differentiate fhe function $f(x) = 2^x \cdot \log_2 x$

$f'(x) = \frac{d}{dx}(2^x)\cdot\log_2 x + 2^x\cdot\frac{d}{dx}(\log_2 x) = \underbrace{2^x\ln 2\cdot\log_2 x} + \underbrace{2^x\cdot\frac{1}{x}\cdot\frac{1}{\ln 2}}$

(b) $y = \ln\left(\frac{x^3}{10^x}\right)$, find $y'(1)$

$y = \ln\left(\frac{x^3}{10^x}\right) = \ln(x^3) - \ln(10^x)$

$= 3\ln(x) - x\ln(10)$

$\therefore y' = \frac{3}{x} - \ln(10) \Rightarrow y'(1) = \frac{3}{1} - \ln 10 = 3 - \ln(10)$

6. (10 points) 推導以下公式: $\frac{d}{dx}\left(\ln\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \csc x$

左式 $= \frac{d}{dx}\left(\ln\sqrt{1-\cos x} - \ln\sqrt{1+\cos x}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x)\right)$

$= \frac{1}{2}\cdot\frac{\frac{d}{dx}(1-\cos x)}{1-\cos x} - \frac{1}{2}\frac{\frac{d}{dx}(1+\cos x)}{1+\cos x}$

$= \frac{1}{2}\left(\frac{\sin x}{1-\cos x} - \frac{-\sin x}{1+\cos x}\right) = \frac{1}{2}\left(\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right)$

$= \frac{1}{2}\frac{\sin x + \cancel{\sin x\cos x} + \sin x - \cancel{\sin x\cos x}}{(1-\cos x)(1+\cos x)} = \frac{1}{2}\frac{2\sin x}{1-\cos^2 x}$

$= \frac{\sin x}{\sin^2 x} = \frac{1}{\sin x} = \csc x$