系級........... 組別： A / B 學號 .... 姓名..................

微積分 (I) Quiz #10 (45 minutes) 2023/12/18

除了選擇，填充和簡答題之外，你的答案必須提供完整說明，如果只有答案沒有任何說明得零分！

1. (6+4=10 points) (a) 推導公式：$\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$.

(b) 用此公式求 $\displaystyle\int \tan^3 x\,dx$.

$$\int \tan^n x\,dx = \int \tan^{n-2}x \tan^2 x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}(x)\sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

其中 $\int \tan^{n-2}x \sec^2 x\,dx$：令 $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$= \int u^{n-2}\,du = \frac{u^{n-1}}{n-1} + C = \frac{\tan^{n-1}x}{n-1} + C$$

$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

利用上述公式：$\displaystyle\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \int \tan x\,dx = \frac{\tan^2 x}{2} - \int \frac{\sin x}{\cos x}\,dx$

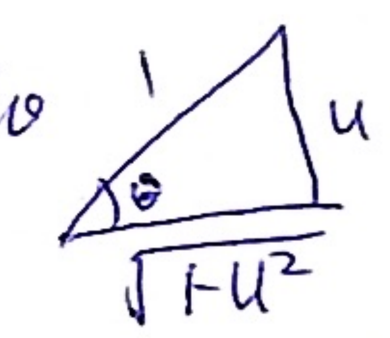
令 $y = \cos x$, $dy = -\sin x\,dx$

$$= \frac{\tan^2 x}{2} + \int \frac{dy}{y} = \frac{\tan^2 x}{2} + \ln|y| + C = \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

2. (10 points) 求積分：$\displaystyle\int x\sqrt{1-x^4}\,dx$.

令 $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

原式 $= \displaystyle\int \sqrt{1-u^2}\,\frac{du}{2}$，令 $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$



$$= \frac{1}{2}\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{2}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{4}\theta + \frac{1}{4}\cdot\frac{\overbrace{\sin 2\theta}^{2\sin\theta\cdot\cos\theta}}{2} + C = \frac{1}{4}\theta + \frac{\sin\theta\cos\theta}{4} + C$$

$$= \frac{1}{4}\sin^{-1}(u) + \frac{1}{4}\cdot u\cdot\sqrt{1-u^2} + C = \frac{1}{4}\sin^{-1}(x^2) + \frac{x^2\sqrt{1-x^4}}{4} + C$$

3. (10 points) 求定積分：$\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$.

令 $x = \tan\theta$, $\Rightarrow dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$.

$$= \int_0^{\pi/4} \sec(\theta)\cdot\sec^2(\theta)\,d\theta = \int_0^{\pi/4} \sec^3\theta\,d\theta$$

令 $u = \sec\theta$, $dv = \sec^2\theta\,d\theta$

$\Rightarrow du = \sec\theta\cdot\tan\theta\,d\theta$, $v = \tan\theta$

$$\int \sec^3\theta\,d\theta = \sec\theta\cdot\tan\theta - \int \sec\theta\cdot\underbrace{\tan^2\theta}_{(\sec^2\theta-1)\text{代入}}\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$\Rightarrow 2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$

$$\Rightarrow \int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}\sec\theta\tan\theta\Big|_0^{\pi/4} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\cdot(\sqrt{2}\cdot 1) + \frac{1}{2}\ln|\sqrt{2}+1|$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)$$

4. (10 points) 用用§8.1的公式求曲線 $y=\sqrt{4-x^2}$, $0 \le x \le 2$ 從(2,0)到(0,2)之弧長.

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{4-x^2} = \frac{4}{4-x^2}$$

$$\text{arc-length} = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = \boxed{2\int_0^2 \frac{dx}{\sqrt{4-x^2}}}$$

令 $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$\boxed{\theta = \sin^{-1}\left(\frac{x}{2}\right)}$

$$2\int \frac{2\cos\theta\, d\theta}{2\cos\theta} = 2\int d\theta$$

$$= 2\theta = 2\cdot\sin^{-1}\left(\frac{x}{2}\right)$$

$$\therefore \text{弧長} = 2\sin^{-1}\left(\frac{x}{2}\right)\Big|_0^2 = 2\left(\sin^{-1}(1) - \sin^{-1}(0)\right)$$

$$= 2\cdot\left(\frac{\pi}{2} - 0\right) = \pi$$

5. (10 points) 求定積分: $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx.$

$$\Rightarrow \int \frac{x}{(x+2)^2+9}\,dx \quad (令\ u = x+2) = \int \frac{u-2}{u^2+9}\,du = \int \frac{u}{u^2+9}\,du - \int \frac{2\,du}{u^2+3^2}$$

令 $y = u^2+9$

$dy = 2u\,du$

$u = 3\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{u}{3}\right)$

$du = 3\sec^2\theta\, d\theta$

$u^2+3^2 = 9(1+\tan^2\theta)$

$= 9\sec^2\theta$

$$= \frac{1}{2}\int \frac{2u\,du}{u^2+9} - 2\int \frac{3\sec^2\theta\, d\theta}{9\sec^2\theta}$$

$$= \frac{1}{2}\int \frac{dy}{y} - \frac{2}{3}\int d\theta = \frac{1}{2}\ln|y| - \frac{2}{3}\theta + C = \frac{1}{2}\ln(u^2+9) - \frac{2}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{2}\ln\left|(x+2)^2+9\right| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\therefore \text{原式} = \frac{1}{2}\ln\left|x^2+4x+13\right|\Big|_0^1 - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\Big|_0^1 = \frac{1}{2}(\ln 18 - \ln 13) - \frac{2}{3}\left(\tan^{-1}1 - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \frac{1}{2}\ln(18/13) - \frac{2}{3}\left(\pi/4 - \tan^{-1}(2/3)\right)$$

6. (10 points) 求積分: $\displaystyle\int \frac{dx}{1+e^x}.$

hint: 令$u = e^x$

令 $u = e^x \Rightarrow du = e^x dx = u\,dx \quad \therefore dx = \frac{1}{u}du$

$$\text{原式} = \int \frac{\frac{1}{u}du}{1+u} = \int \frac{du}{u(u+1)} = \int\left(\frac{1}{u} - \frac{1}{u+1}\right)du$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$\Rightarrow 1 = A(u+1) + Bu$

$1 = (A+B)u + A$

$\boxed{\therefore A = 1,\ A+B=0 \Rightarrow B=-1}$

$$= \ln|u| - \ln|u+1| + C$$

$$= \ln e^x - \ln(1+e^x) + C$$

$$= x - \ln(1+e^x) + C$$