Riemann Geometry

Chang-Mao, Yang

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Contents

1 Riemann Geometry

[1]:
```python
import sympy as sp
from IPython.display import Math, Markdown
```

[2]:
```python
%reload_ext autoreload
%autoreload 2
from riemann_geometry import RiemannGeometry
```

[3]:
```python
RiemannGeometry
```

[3]:
```
riemann_geometry.core.RiemannGeometry
```

1.1 Conversion

1.1.1 Einstein Conversion

Here using the Einstein summation convention,

> When an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.

1.1.2 Tensor denotion

On the manofold $\mathcal{M}$ with dimension $\dim \mathcal{M} = n$. The ***metric*** g_{ab} and the ***inverse of metric*** g^{ab} is denoted in matrix form, that is

$$g_{ab} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} g^{11} & g^{12} & \cdots & g^{1n} \\ g^{21} & g^{22} & \cdots & g^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g^{n1} & g^{n2} & \cdots & g^{nn} \end{pmatrix}.$$

For the ***Christoffel symbols*** $\Gamma^a{}_{bc}$, notice that it only has 3 indices, so here define it in the form

$$\Gamma^a{}_{bcd} = \left(\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \cdots & \Gamma^1_{1n} \\ \Gamma^1_{21} & \Gamma^1_{22} & \cdots & \Gamma^1_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^1_{n1} & \Gamma^1_{n2} & \cdots & \Gamma^1_{nn} \end{pmatrix} \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} & \cdots & \Gamma^2_{1n} \\ \Gamma^2_{21} & \Gamma^2_{22} & \cdots & \Gamma^2_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^2_{n1} & \Gamma^2_{n2} & \cdots & \Gamma^2_{nn} \end{pmatrix} \cdots \begin{pmatrix} \Gamma^n_{11} & \Gamma^n_{12} & \cdots & \Gamma^n_{1n} \\ \Gamma^n_{21} & \Gamma^n_{22} & \cdots & \Gamma^n_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^n_{n1} & \Gamma^n_{n2} & \cdots & \Gamma^n_{nn} \end{pmatrix} \right)$$

Then the ***Riemann tensor*** $R^a{}_{bcd}$ also denote as a matrix in a matrix form,

$$R^a{}_{bcd} = \begin{pmatrix} R^1{}_{1cd} & R^1{}_{1cd} & \cdots & R^1{}_{1cd} \\ R^2{}_{2cd} & R^2{}_{2cd} & \cdots & R^2{}_{2cd} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{ncd} & R^n{}_{ncd} & \cdots & R^n{}_{ncd} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R^1{}_{111} & R^1{}_{211} & \cdots & R^1{}_{n11} \\ R^2{}_{111} & R^2{}_{211} & \cdots & R^2{}_{n11} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{111} & R^n{}_{211} & \cdots & R^n{}_{n11} \end{pmatrix} & \begin{pmatrix} R^1{}_{112} & R^1{}_{212} & \cdots & R^1{}_{n12} \\ R^2{}_{112} & R^2{}_{212} & \cdots & R^2{}_{n12} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{112} & R^n{}_{212} & \cdots & R^n{}_{n12} \end{pmatrix} & \cdots & \begin{pmatrix} R^1{}_{11n} & R^1{}_{21n} & \cdots & R^1{}_{n1n} \\ R^2{}_{11n} & R^2{}_{21n} & \cdots & R^2{}_{n1n} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{11n} & R^n{}_{21n} & \cdots & R^n{}_{n1n} \end{pmatrix} \\ \begin{pmatrix} R^1{}_{121} & R^1{}_{221} & \cdots & R^1{}_{n21} \\ R^2{}_{121} & R^2{}_{221} & \cdots & R^2{}_{n21} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{121} & R^n{}_{221} & \cdots & R^n{}_{n21} \end{pmatrix} & \begin{pmatrix} R^1{}_{122} & R^1{}_{222} & \cdots & R^1{}_{n22} \\ R^2{}_{122} & R^2{}_{222} & \cdots & R^2{}_{n22} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{122} & R^n{}_{222} & \cdots & R^n{}_{n22} \end{pmatrix} & \cdots & \begin{pmatrix} R^1{}_{12n} & R^1{}_{22n} & \cdots & R^1{}_{n2n} \\ R^2{}_{12n} & R^2{}_{22n} & \cdots & R^2{}_{n2n} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{12n} & R^n{}_{22n} & \cdots & R^n{}_{n2n} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} R^1{}_{1n1} & R^1{}_{2n1} & \cdots & R^1{}_{nn1} \\ R^2{}_{1n1} & R^2{}_{2n1} & \cdots & R^2{}_{nn1} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{1n1} & R^n{}_{2n1} & \cdots & R^n{}_{nn1} \end{pmatrix} & \begin{pmatrix} R^1{}_{1n2} & R^1{}_{2n2} & \cdots & R^1{}_{nn2} \\ R^2{}_{1n2} & R^2{}_{2n2} & \cdots & R^2{}_{nn2} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{1n2} & R^n{}_{2n2} & \cdots & R^n{}_{nn2} \end{pmatrix} & \cdots & \begin{pmatrix} R^1{}_{1nn} & R^1{}_{2nn} & \cdots & R^1{}_{nnn} \\ R^2{}_{1nn} & R^2{}_{2nn} & \cdots & R^2{}_{nnn} \\ \vdots & \vdots & \ddots & \vdots \\ R^n{}_{1nn} & R^n{}_{2nn} & \cdots & R^n{}_{nnn} \end{pmatrix} \end{pmatrix}.$$

Also, for the ***Ricci tensor*** R_{ab}, it will also denote as a matrix form (since it only has two indices), that is

$$R_{ab} = R^c{}_{acb} = \begin{pmatrix} R_{11} & R_{11} & \cdots & R_{11} \\ R_{22} & R_{22} & \cdots & R_{22} \\ \vdots & \vdots & \ddots & \vdots \\ R_{nn} & R_{nn} & \cdots & R_{nn} \end{pmatrix}.$$

Last, since the ***Ricci scalar*** R is just a scalar, so it has no metrix definition.

1.2 Example for Manifolds in Euclidean $\mathbb{R}^n$

1.2.1 Flat Manifold in $\mathbb{R}^2$

Cartesian coordinate In 2-dimensional flat space $\mathbb{R}^2$, the line element in Cartesian coordinate is

$$ds^2 = dx^2 + dy^2 + dz^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[4]:
```python
x, y = sp.symbols('x y')
coords = sp.Matrix([x, y])
metric = sp.diag(1, 1)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|██████████████████████████████| 2/2 [00:00<00:00,
1008.00it/s]
Riemann Tensor      : 100%|██████████████████████████████| 2/2 [00:00<00:00,
1059.97it/s]
Ricci Tensor        : 100%|██████████████████████████████| 2/2 [00:00<00:00,
6172.63it/s]
Ricci Scalar        : 100%|█████████████████████████████| 2/2 [00:00<00:00,
50231.19it/s]
```

[4]:

$$x^a = \begin{pmatrix} x \\ y \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = 0.$$

Polar coordinate In 2-dimensional flat space $\mathbb{R}^2$, the line element in polar coordinate is

$$ds^2 = dr^2 + r^2 d\theta^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[5]:
```python
r, theta = sp.symbols('r theta')
coords = sp.Matrix([r, theta])
metric = sp.diag(1, r**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|███████████████████████████████| 2/2 [00:00<00:00,
308.02it/s]
Riemann Tensor      : 100%|███████████████████████████████| 2/2 [00:00<00:00,
613.34it/s]
Ricci Tensor        : 100%|██████████████████████████████| 2/2 [00:00<00:00,
6355.01it/s]
Ricci Scalar        : 100%|█████████████████████████████| 2/2 [00:00<00:00,
21959.71it/s]
```

[5]:

$$x^a = \begin{pmatrix} r \\ \theta \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{r} \\ \frac{1}{r} & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = 0.$$

1.2.2 Flat Manifold in $\mathbb{R}^3$

Cartesian coordinate In 3-dimensional flat space $\mathbb{R}^3$, the line element in Cartesian coordinate is

$$ds^2 = dx^2 + dy^2 + dz^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[6]:
```python
x, y, z = sp.symbols('x y z')
coords = sp.Matrix([x, y, z])
metric = sp.diag(1, 1, 1)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|██████████| 3/3 [00:00<00:00,576.67it/s]
Riemann Tensor      : 100%|██████████| 3/3 [00:00<00:00,361.92it/s]
Ricci Tensor        : 100%|██████████| 3/3 [00:00<00:00,5007.13it/s]
Ricci Scalar        : 100%|██████████| 3/3 [00:00<00:00,17050.02it/s]
```

[6]:

$$x^a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = 0.$$

Spherical coordinate In 3-dimensional flat space $\mathbb{R}^3$, the line element in spherical coordinate is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[7]:
```python
r, theta, phi = sp.symbols('r theta varphi')
coords = sp.Matrix([r, theta, phi])
metric = sp.diag(1, r**2, r**2 * (sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|██████████| 3/3[00:00<00:00, 45.22it/s]
Riemann Tensor      : 100%|██████████| 3/3[00:00<00:00, 95.38it/s]
Ricci Tensor        : 100%|██████████| 3/3 [00:00<00:00,
5094.30it/s]
Ricci Scalar        : 100%|██████████| 3/3 [00:00<00:00,
45425.68it/s]
```

[7]:

$$x^a = \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2(\theta) \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \frac{1}{\tan(\theta)} \\ \frac{1}{r} & \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = 0.$$

Cylindrical coordinate In 3-dimensional flat space $\mathbb{R}^3$, the line element in cylindrical coordinate is

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[8]:
```python
r, phi, z = sp.symbols('r theta z')
coords = sp.Matrix([r, phi, z])
metric = sp.diag(1, r**2, 1)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|██████████| 3/3 [00:00<00:00,369.50it/s]
Riemann Tensor      : 100%|██████████| 3/3 [00:00<00:00,360.67it/s]
Ricci Tensor        : 100%|██████████| 3/3 [00:00<00:00,3424.85it/s]
Ricci Scalar        : 100%|██████████| 3/3 [00:00<00:00,41120.63it/s]
```

[8]:

$$x^a = \begin{pmatrix} r \\ \theta \\ z \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = 0.$$

1.2.3 Curved Manifolds

Sphere S^2 On a sphere, the line element is given by

$$ds^2 = dr^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\varphi,$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[9]:
```python
theta, phi = sp.symbols('theta varphi')
R = sp.symbols('R')
coords = sp.Matrix([theta, phi])
metric = sp.diag(R**2, R**2 * (sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|██████████| 2/2 [00:00<00:00,105.26it/s]
Riemann Tensor      : 100%|██████████| 2/2[00:00<00:00, 62.22it/s]
Ricci Tensor        : 100%|██████████| 2/2 [00:00<00:00,336.35it/s]
Ricci Scalar        : 100%|██████████| 2/2 [00:00<00:00,13421.77it/s]
```

[9]:

$$x^a = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 \\ 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\tan(\theta)} \\ \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \sin^2(\theta) \\ -\sin^2(\theta) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}, \quad R = \frac{2}{R^2}.$$

Torus T^2 On a sphere, the line element is given by

$$ds^2 = (R + r\cos v)^2\, du^2 + r^2 dv^2,$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

[10]:
```
u, v = sp.symbols('u v')
R, r = sp.symbols('R, r')
coords = sp.Matrix([u, v])
metric = sp.diag((R + r*sp.cos(v))**2, r**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|██████████| 2/2[00:00<00:00, 12.76it/s]
Riemann Tensor      : 100%|██████████| 2/2[00:00<00:00, 13.72it/s]
Ricci Tensor        : 100%|██████████| 2/2[00:00<00:00, 33.62it/s]
Ricci Scalar        : 100%|██████████| 2/2 [00:00<00:00,
8971.77it/s]
```

[10]:

$$x^a = \begin{pmatrix} u \\ v \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} (R + r\cos(v))^2 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{1}{R^2 + 2Rr\cos(v) + r^2\cos^2(v)} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & -\frac{r\sin(v)}{R+r\cos(v)} \\ -\frac{r\sin(v)}{R+r\cos(v)} & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{(R+r\cos(v))\sin(v)}{r} & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{r\cos(v)}{R+r\cos(v)} \\ -\frac{r\cos(v)}{R+r\cos(v)} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{(R+r\cos(v))\cos(v)}{r} \\ \frac{(R+r\cos(v))\cos(v)}{r} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} \frac{(R+r\cos(v))\cos(v)}{r} & 0 \\ 0 & \frac{r\cos(v)}{R+r\cos(v)} \end{pmatrix}, \quad R = \frac{2\cos(v)}{r(R + r\cos(v))}.$$

1.3 Example for Manifolds in Spacetime $\mathbb{R}^{n+1}$

1.3.1 Flat Manifolds in $\mathbb{R}^{3+1}$

Cartesian coordinate Minkowski metric is the metric in flat spacetime, the line element in Cartesian coordinate is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[15]:
```
t, x, y, z = sp.symbols('t x y z')
coords = sp.Matrix([t, x, y, z])
metric = sp.diag(-1, 1, 1, 1)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|██████████| 4/4 [00:00<00:00,234.91it/s]
Riemann Tensor      : 100%|██████████| 4/4 [00:00<00:00,132.96it/s]
Ricci Tensor        : 100%|██████████| 4/4 [00:00<00:00,2240.85it/s]
Ricci Scalar        : 100%|██████████| 4/4 [00:00<00:00,45964.98it/s]
```

[15]:

$$x^a = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad R = 0.$$

Spheical coordinate Minkowski metric is the metric in flat spacetime, the line element in spherical coordinate is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[16]:
```
t, r, theta, phi = sp.symbols('t r theta varphi')
coords = sp.Matrix([t, r, theta, phi])
metric = sp.diag(1, -1, -r**2, -r**2*(sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|██████████| 4/4[00:00<00:00, 65.89it/s]
Riemann Tensor      : 100%|██████████| 4/4[00:00<00:00, 67.29it/s]
Ricci Tensor        : 100%|██████████| 4/4 [00:00<00:00,3629.86it/s]
Ricci Scalar        : 100%|██████████| 4/4 [00:00<00:00,39475.80it/s]
```

[16]:

$$x^a = \begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&-r&0\\0&0&0&-r\sin^2(\theta) \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&\frac{1}{r}&0\\0&\frac{1}{r}&0&0\\0&0&0&-\sin(\theta)\cos(\theta) \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&\frac{1}{r}\\0&0&0&\frac{1}{\tan(\theta)}\\0&\frac{1}{r}&\frac{1}{\tan(\theta)}&0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad R = 0.$$

1.3.2 Curved manifolds in $\mathbb{R}^{3+1}$

Schwarzschild One of the solution for Einstein field equations, is Schwarzschild metric, where the line element is given by

$$ds^2 = \left(1-\frac{r_s}{r}\right)dt^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\varphi$$

then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[11]:
```python
t, r, theta, phi = sp.symbols('t r theta varphi')
rs = sp.symbols('r_s')
coords = sp.Matrix([t, r, theta, phi])
metric = sp.diag((1-rs/r), -1/(1-rs/r), -r**2, -r**2*(sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|| 4/4[00:00<00:00, 40.57it/s]
Riemann Tensor      : 100%|| 4/4[00:00<00:00, 12.59it/s]
Ricci Tensor        : 100%|| 4/4 [00:00<00:00,1763.79it/s]
Ricci Scalar        : 100%|| 4/4 [00:00<00:00,22610.80it/s]
```

[11]:

$$x^a = \begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1-\frac{r_s}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{1}{1-\frac{r_s}{r}} & 0 & 0 & 0 \\ 0 & -1+\frac{r_s}{r} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \begin{pmatrix} \begin{pmatrix} 0 & \frac{r_s}{2r(r-r_s)} & 0 & 0 \\ \frac{r_s}{2r(r-r_s)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{r_s(r-r_s)}{2r^3} & 0 & 0 & 0 \\ 0 & -\frac{r_s}{2r(r-r_s)} & 0 & 0 \\ 0 & 0 & -r+r_s & 0 \\ 0 & 0 & 0 & (-r+r_s)\sin^2(\theta) \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{1}{\tan(\theta)} \\ 0 & \frac{1}{r} & \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \end{pmatrix},$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{r_s}{r^2(r-r_s)} & 0 & 0 \\ -\frac{r_s}{r^2(r-r_s)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -\frac{r_s}{2r} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s}{2r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & -\frac{r_s\sin^2(\theta)}{2r} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s\sin^2(\theta)}{2r} & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{r_s(r-r_s)}{r^4} & 0 & 0 \\ \frac{r_s(-r+r_s)}{r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r_s}{2r} & 0 \\ 0 & \frac{r_s}{2r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_s\sin^2(\theta)}{2r} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{r_s\sin^2(\theta)}{2r} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{r_s(-r+r_s)}{2r^4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s(r-r_s)}{2r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r_s}{2r^2(r-r_s)} & 0 \\ 0 & -\frac{r_s}{2r^2(r-r_s)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r_s\sin^2(\theta)}{r} \\ 0 & 0 & -\frac{r_s\sin^2(\theta)}{r} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & \frac{r_s(-r+r_s)}{2r^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s(r-r_s)}{2r^4} & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r_s}{2r^2(r-r_s)} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{r_s}{2r^2(r-r_s)} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_s}{r} \\ 0 & 0 & \frac{r_s}{r} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R = 0.$$

Kerr One of the solution for Einstein field equations, which corresponds to the gravitational field of a uncharged, rotating, spherically symmetric body, is Kerr metric, and the line element is given by

$$ds^2 = -\left(1-\frac{r_s r}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2+a^2+\frac{r_s r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\ d\phi^2 - \frac{2r_s r a\sin^2\theta}{\Sigma}\,dt\,d\phi$$

which is in *Boyer-Lindquist coordinates*, then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[24]:
```python
t, r, theta, phi = sp.symbols('t r theta varphi')
rs, J, M = sp.symbols('r_s J, M')
coords = sp.Matrix([t, r, theta, phi])
a = J/M
mu = r**2 + a**2
Sigma = r**2 + a**2 * sp.cos(theta)**2
Delta = r**2 + a**2 - rs*r

metric = sp.Matrix([
    [-(1-rs*r/Sigma),             0,           0,     -rs*r*a*sp.sin(theta)**2/Sigma],
    [0,                           Sigma/Delta, 0,     0],
    [0,                           0,           Sigma, 0],
    [-rs*r*a*sp.sin(theta)**2/Sigma, 0,           0,      (mu+rs*r*a**2*sp.sin(theta)**2/Sigma)*sp.sin(theta)**2]
])
# WARM: !!! It would take many time to calcuate!!!!!
# geo = RiemannGeometry(metric, coords)
# geo.calculate()
# geo
```

Reissner-Nordström One of the solution for Einstein field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body, is Weak-Field metric, and the line element is given by

$$ds^2 = \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)dt^2 - \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)dr^2 - d\theta^2 - r^2\sin^2\theta d\varphi,$$

where Φ is a function of r (usual Newtonian gravitational potential $\Phi = -GM/r$), then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[12]:
```python
t, r, theta, phi = sp.symbols('t r theta varphi')
rs, rQ = sp.symbols('r_s r_Q')
coords = sp.Matrix([t, r, theta, phi])
metric = sp.diag((1-rs/r + rQ**2/r**2), -1/(1-rs/r+ rQ**2/r**2), -r**2, -r**2*(sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|| 4/4[00:00<00:00, 22.81it/s]
Riemann Tensor      : 100%|| 4/4[00:00<00:00,  4.55it/s]
Ricci Tensor        : 100%|| 4/4[00:00<00:00, 59.79it/s]
Ricci Scalar        : 100%|| 4/4 [00:00<00:00,
11732.32it/s]
```

[12]:

$$x^a = \begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1-\frac{r_s}{r}+\frac{r_Q^2}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{r^2}{r^2-rr_s+r_Q^2} & 0 & 0 & 0 \\ 0 & \frac{-r^2+rr_s-r_Q^2}{r^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \begin{pmatrix} \begin{pmatrix} 0 & \frac{\frac{rr_s}{2}-r_Q^2}{r\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \\ \frac{\frac{rr_s}{2}-r_Q^2}{r\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{\left(rr_s-2r_Q^2\right)\left(r^2-rr_s+r_Q^2\right)}{2r^5} & 0 & 0 & 0 \\ 0 & \frac{-\frac{rr_s}{2}+r_Q^2}{r\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \\ 0 & 0 & -r+r_s-\frac{r_Q^2}{r} & 0 \\ 0 & 0 & 0 & \frac{\left(-r^2+rr_s-r_Q^2\right)\sin^2(\theta)}{r} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{1}{\tan(\theta)} \\ 0 & \frac{1}{r} & \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \end{pmatrix},$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{rr_s-3r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \\ \frac{-rr_s+3r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \frac{-\frac{rr_s}{2}+r_Q^2}{r^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\frac{rr_s}{2}-r_Q^2}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & \frac{\left(-\frac{rr_s}{2}+r_Q^2\right)\sin^2(\theta)}{r^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\left(\frac{rr_s}{2}-r_Q^2\right)\sin^2(\theta)}{r^2} & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{r^3r_s-3r^2r_Q^2-r^2r_s^2+4rr_Q^2r_s-3r_Q^4}{r^6} & 0 & 0 \\ \frac{-r^3r_s+3r^2r_Q^2+r^2r_s^2-4rr_Q^2r_s+3r_Q^4}{r^6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\frac{rr_s}{2}+r_Q^2}{r^2} & 0 \\ 0 & \frac{\frac{rr_s}{2}-r_Q^2}{r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\left(-\frac{rr_s}{2}+r_Q^2\right)\sin^2(\theta)}{r^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\left(\frac{rr_s}{2}-r_Q^2\right)\sin^2(\theta)}{r^2} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -\frac{\left(rr_s-2r_Q^2\right)\left(r^2-rr_s+r_Q^2\right)}{2r^6} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\left(rr_s-2r_Q^2\right)\left(r^2-rr_s+r_Q^2\right)}{2r^6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\frac{rr_s}{2}-r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 \\ 0 & \frac{-\frac{rr_s}{2}+r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\left(rr_s-r_Q^2\right)\sin^2(\theta)}{r^2} \\ 0 & 0 & \frac{\left(-rr_s+r_Q^2\right)\sin^2(\theta)}{r^2} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & -\frac{\left(rr_s-2r_Q^2\right)\left(r^2-rr_s+r_Q^2\right)}{2r^6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\left(rr_s-2r_Q^2\right)\left(r^2-rr_s+r_Q^2\right)}{2r^6} & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\frac{rr_s}{2}-r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-\frac{rr_s}{2}+r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-rr_s+r_Q^2}{r^2} \\ 0 & 0 & \frac{rr_s-r_Q^2}{r^2} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} \frac{r_Q^2\left(r^2-rr_s+r_Q^2\right)}{r^6} & 0 & 0 & 0 \\ 0 & -\frac{r_Q^2}{r^2\left(r^2-rr_s+r_Q^2\right)} & 0 & 0 \\ 0 & 0 & \frac{r_Q^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{r_Q^2\sin^2(\theta)}{r^2} \end{pmatrix}, \quad R = 0.$$

Weak-Field One of the solution for Einstein field equations in the limit of weak gravity, is Weak-Field metric, where the line element is given by

$$ds^2 = (1+2\Phi^2(r))\,dt^2 - (1+2\Phi^2(r))\,dr^2 - (1+2\Phi^2(r))\,d\theta^2 - (1+2\Phi^2(r))\,r^2\sin^2\theta d\varphi,$$

where Φ is a function of r (usual Newtonian gravitational potential $\Phi = -GM/r$), then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[25]:
```
# coordinates
t, r, theta, phi = sp.symbols('t r theta varphi')
# dependent function
Phi = sp.Function('Phi')(r)

# M = sp.symbols('M')
# Phi = -M/r
# ------------------------------
coords = sp.Matrix([t, r, theta, phi])
metrix = sp.diag(-(1+2*Phi), +(1-2*Phi), +(1-2*Phi)*r**2, +(1-2*Phi)*r**2*(sp.sin(theta))**2)

# WARM: !!! It would take many time to calcuate!!!!!
geo = RiemannGeometry(metrix, coords)
geo.calculate()
geo
```

```
Calculating …
Christoffel Symbols : 100%|██████████| 4/4[00:00<00:00, 13.07it/s]
Riemann Tensor      : 100%|██████████| 4/4[00:01<00:00,  3.22it/s]
Ricci Tensor        : 100%|██████████| 4/4[00:00<00:00,  9.54it/s]
Ricci Scalar        : 100%|█████████| 4/4 [00:00<00:00,
12087.33it/s]
```

[25]:

$$x^a = \begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} -2\Phi(r)-1 & 0 & 0 & 0 \\ 0 & 1-2\Phi(r) & 0 & 0 \\ 0 & 0 & r^2\cdot(1-2\Phi(r)) & 0 \\ 0 & 0 & 0 & r^2\cdot(1-2\Phi(r))\sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{1}{-2\Phi(r)-1} & 0 & 0 & 0 \\ 0 & \frac{1}{1-2\Phi(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{2r^2\Phi(r)-r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2r^2\Phi(r)\sin^2(\theta)-r^2\sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & \frac{\frac{d}{dr}\Phi(r)}{2\Phi(r)+1} & 0 & 0 \\ \frac{\frac{d}{dr}\Phi(r)}{2\Phi(r)+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\frac{d}{dr}\Phi(r)}{2\Phi(r)-1} & 0 & 0 & 0 \\ 0 & \frac{\frac{d}{dr}\Phi(r)}{2\Phi(r)-1} & 0 & 0 \\ 0 & 0 & \frac{r\left(-r\frac{d}{dr}\Phi(r)-2\Phi(r)+1\right)}{2\Phi(r)-1} & 0 \\ 0 & 0 & 0 & \frac{r\left(-r\frac{d}{dr}\Phi(r)-2\Phi(r)+1\right)\sin^2(\theta)}{2\Phi(r)-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r\frac{d}{dr}\Phi(r)+2\Phi(r)-1}{r(2\Phi(r)-1)} & 0 \\ 0 & \frac{r\frac{d}{dr}\Phi(r)+2\Phi(r)-1}{r(2\Phi(r)-1)} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r\frac{d}{dr}\Phi(r)+2\Phi(r)-1}{r(2\Phi(r)-1)} \\ 0 & 0 & 0 & \frac{1}{\tan(\theta)} \\ 0 & \frac{r\frac{d}{dr}\Phi(r)+2\Phi(r)-1}{r(2\Phi(r)-1)} & \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \frac{-4\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)+4\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2+\frac{d^2}{dr^2}\Phi(r)}{8\Phi^3(r)+4\Phi^2(r)-2\Phi(r)-1} & 0 & 0 \\ \frac{4\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)-4\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2-\frac{d^2}{dr^2}\Phi(r)}{8\Phi^3(r)+4\Phi^2(r)-2\Phi(r)-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \frac{r\left(-r\frac{d}{dr}\Phi(r)-2\Phi(r)+1\right)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-1} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & \frac{r\left(-r\frac{d}{dr}\Phi(r)-2\Phi(r)+1\right)\sin^2(\theta)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\sin^2(\theta)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-1} & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{4\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)-4\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2-\frac{d^2}{dr^2}\Phi(r)}{8\Phi^3(r)-4\Phi^2(r)-2\Phi(r)+1} & 0 & 0 \\ \frac{-4\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)+4\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2+\frac{d^2}{dr^2}\Phi(r)}{8\Phi^3(r)-4\Phi^2(r)-2\Phi(r)+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r\left(-2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-2\Phi(r)\frac{d}{dr}\Phi(r)+\frac{d}{dr}\Phi(r)\right)}{4\Phi^2(r)-4\Phi(r)+1} & 0 \\ 0 & \frac{r\left(2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)-2r\left(\frac{d}{dr}\Phi(r)\right)^2-r\frac{d^2}{dr^2}\Phi(r)+2\Phi(r)\frac{d}{dr}\Phi(r)-\frac{d}{dr}\Phi(r)\right)}{4\Phi^2(r)-4\Phi(r)+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r\left(-2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-2\Phi(r)\frac{d}{dr}\Phi(r)+\frac{d}{dr}\Phi(r)\right)\sin^2(\theta)}{4\Phi^2(r)-4\Phi(r)+1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{r\left(2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)-2r\left(\frac{d}{dr}\Phi(r)\right)^2-r\frac{d^2}{dr^2}\Phi(r)+2\Phi(r)\frac{d}{dr}\Phi(r)-\frac{d}{dr}\Phi(r)\right)\sin^2(\theta)}{4\Phi^2(r)-4\Phi(r)+1} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & \frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\frac{d}{dr}\Phi(r)}{r(2\Phi(r)-1)^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\frac{d}{dr}\Phi(r)}{r(2\Phi(r)-1)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)-2r\left(\frac{d}{dr}\Phi(r)\right)^2-r\frac{d^2}{dr^2}\Phi(r)+2\Phi(r)\frac{d}{dr}\Phi(r)-\frac{d}{dr}\Phi(r)}{r(4\Phi^2(r)-4\Phi(r)+1)} & 0 \\ 0 & \frac{-2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-2\Phi(r)\frac{d}{dr}\Phi(r)+\frac{d}{dr}\Phi(r)}{r(4\Phi^2(r)-4\Phi(r)+1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^2(\theta)-\frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)^2\sin^2(\theta)}{(2\Phi(r)-1)^2} \\ 0 & 0 & -\sin^2(\theta)+\frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)^2\sin^2(\theta)}{(2\Phi(r)-1)^2} & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & \frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\frac{d}{dr}\Phi(r)}{r(2\Phi(r)-1)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\left(r\frac{d}{dr}\Phi(r)+2\Phi(r)-1\right)\frac{d}{dr}\Phi(r)}{r(2\Phi(r)-1)^2} & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)-2r\left(\frac{d}{dr}\Phi(r)\right)^2-r\frac{d^2}{dr^2}\Phi(r)+2\Phi(r)\frac{d}{dr}\Phi(r)-\frac{d}{dr}\Phi(r)}{r(4\Phi^2(r)-4\Phi(r)+1)} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-2r\Phi(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-2\Phi(r)\frac{d}{dr}\Phi(r)+\frac{d}{dr}\Phi(r)}{r(4\Phi^2(r)-4\Phi(r)+1)} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r\left(r\frac{d}{dr}\Phi(r)+4\Phi(r)-2\right)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-4\Phi(r)+1} \\ 0 & 0 & \frac{r\left(-r\frac{d}{dr}\Phi(r)-4\Phi(r)+2\right)\frac{d}{dr}\Phi(r)}{4\Phi^2(r)-4\Phi(r)+1} & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix},$$

$$R_{ab} = \begin{pmatrix}
\frac{-4r\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)-2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-8\Phi^2(r)\frac{d}{dr}\Phi(r)+2\frac{d}{dr}\Phi(r)}{r(8\Phi^3(r)-4\Phi^2(r)-2\Phi(r)+1)} & 0 & 0 & 0 \\
0 & \frac{-24r\Phi^3(r)\frac{d^2}{dr^2}\Phi(r)+24r\Phi^2(r)\left(\frac{d}{dr}\Phi(r)\right)^2-4r\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)+12r\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2+6r\Phi(r)\frac{d^2}{dr^2}\Phi(r)+4r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-16\Phi^3(r)\frac{d}{dr}\Phi(r)-8\Phi^2(r)\frac{d}{dr}\Phi(r)+4\Phi(r)\frac{d}{dr}\Phi(r)+2\frac{d}{dr}\Phi(r)}{r(16\Phi^4(r)-8\Phi^2(r)+1)} & 0 & 0 \\
0 & 0 & \frac{r\left(-4r\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-16\Phi^2(r)\frac{d}{dr}\Phi(r)+4\Phi(r)\frac{d}{dr}\Phi(r)+2\frac{d}{dr}\Phi(r)\right)}{8\Phi^3(r)-4\Phi^2(r)-2\Phi(r)+1} & 0 \\
0 & 0 & 0 & \frac{r\left(-4r\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)+2r\left(\frac{d}{dr}\Phi(r)\right)^2+r\frac{d^2}{dr^2}\Phi(r)-16\Phi^2(r)\frac{d}{dr}\Phi(r)+4\Phi(r)\frac{d}{dr}\Phi(r)+2\frac{d}{dr}\Phi(r)\right)\sin^2(\theta)}{8\Phi^3(r)-4\Phi^2(r)-2\Phi(r)+1}
\end{pmatrix},$$

$$R = \frac{2\cdot\left(24r\Phi^3(r)\frac{d^2}{dr^2}\Phi(r)-12r\Phi^2(r)\left(\frac{d}{dr}\Phi(r)\right)^2+4r\Phi^2(r)\frac{d^2}{dr^2}\Phi(r)-8r\Phi(r)\left(\frac{d}{dr}\Phi(r)\right)^2-6r\Phi(r)\frac{d^2}{dr^2}\Phi(r)-5r\left(\frac{d}{dr}\Phi(r)\right)^2-r\frac{d^2}{dr^2}\Phi(r)+48\Phi^3(r)\frac{d}{dr}\Phi(r)+8\Phi^2(r)\frac{d}{dr}\Phi(r)-12\Phi(r)\frac{d}{dr}\Phi(r)-2\frac{d}{dr}\Phi(r)\right)}{r\left(32\Phi^5(r)-16\Phi^4(r)-16\Phi^3(r)+8\Phi^2(r)+2\Phi(r)-1\right)}.$$

Friedmann–Lemaître–Robertson–Walker One of the solution for Einstein field equations, is Friedmann–Lemaître–Robertson–Walker metric, where the line element is given by

$$ds^2 = dt^2 - \frac{a^2(t)}{1-kr^2}dr^2 - r^2a^2(t)d\theta^2 - r^2a^2(t)\sin^2\theta d\varphi,$$

where a is a function of time $a(t)$ (called scale factor), then we can calculate the ***Christoffel symbols***, ***Riemann tensor***, ***Ricci tensor*** and ***Ricci scalar***.

Here using Geometrized unit system $G = c = 1$.

[14]:
```
# coordinates
t, r, theta, phi = sp.symbols('t r theta varphi')
# constants
k, c = sp.symbols('k c')
# dependent function
a = sp.Function('a')(t)
# ------------------------------
coords = sp.Matrix([t, r, theta, phi])
metric = sp.diag(1, -a**2/(1-k*r**2), -r**2*a**2, -r**2*a**2*(sp.sin(theta))**2)
geo = RiemannGeometry(metric, coords)
geo.calculate()
geo
```

```
Calculating …

Christoffel Symbols : 100%|████████████████████| 4/4[00:00<00:00, 31.66it/s]
Riemann Tensor      : 100%|████████████████████| 4/4[00:00<00:00,  9.94it/s]
Ricci Tensor        : 100%|████████████████████| 4/4[00:00<00:00, 51.95it/s]
Ricci Scalar        : 100%|██████████████████| 4/4 [00:00<00:00,
13740.55it/s]
```

[14]:

$$x^a = \begin{pmatrix} t \\ r \\ \theta \\ \varphi \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{-kr^2+1} & 0 & 0 \\ 0 & 0 & -r^2a^2(t) & 0 \\ 0 & 0 & 0 & -r^2a^2(t)\sin^2(\theta) \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{kr^2-1}{a^2(t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2a^2(t)} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2a^2(t)\sin^2(\theta)} \end{pmatrix},$$

$$\Gamma^a{}_{bc} = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{a(t)\frac{d}{dt}a(t)}{kr^2-1} & 0 & 0 \\ 0 & 0 & r^2a(t)\frac{d}{dt}a(t) & 0 \\ 0 & 0 & 0 & r^2a(t)\sin^2(\theta)\frac{d}{dt}a(t) \end{pmatrix} \begin{pmatrix} 0 & \frac{\frac{d}{dt}a(t)}{a(t)} & 0 & 0 \\ \frac{\frac{d}{dt}a(t)}{a(t)} & -\frac{kr}{kr^2-1} & 0 & 0 \\ 0 & 0 & kr^3-r & 0 \\ 0 & 0 & 0 & r(kr^2-1)\sin^2(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{\frac{d}{dt}a(t)}{a(t)} & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \frac{\frac{d}{dt}a(t)}{a(t)} & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \frac{\frac{d}{dt}a(t)}{a(t)} \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{1}{\tan(\theta)} \\ \frac{\frac{d}{dt}a(t)}{a(t)} & \frac{1}{r} & \frac{1}{\tan(\theta)} & 0 \end{pmatrix} \right),$$

$$R^a{}_{bcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -\frac{a(t)\frac{d^2}{dt^2}a(t)}{kr^2-1} & 0 & 0 \\ \frac{a(t)\frac{d^2}{dt^2}a(t)}{kr^2-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & r^2a(t)\frac{d^2}{dt^2}a(t) & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)\frac{d^2}{dt^2}a(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & r^2a(t)\sin^2(\theta)\frac{d^2}{dt^2}a(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)\sin^2(\theta)\frac{d^2}{dt^2}a(t) & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{\frac{d^2}{dt^2}a(t)}{a(t)} & 0 & 0 \\ -\frac{\frac{d^2}{dt^2}a(t)}{a(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r^2\left(k+\left(\frac{d}{dt}a(t)\right)^2\right) & 0 \\ 0 & r^2\left(-k-\left(\frac{d}{dt}a(t)\right)^2\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\left(k+\left(\frac{d}{dt}a(t)\right)^2\right)\sin^2(\theta) \\ 0 & 0 & 0 & 0 \\ 0 & r^2\left(-k-\left(\frac{d}{dt}a(t)\right)^2\right)\sin^2(\theta) & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \frac{\frac{d^2}{dt^2}a(t)}{a(t)} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\frac{d^2}{dt^2}a(t)}{a(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k+\left(\frac{d}{dt}a(t)\right)^2}{kr^2-1} & 0 \\ 0 & \frac{-k-\left(\frac{d}{dt}a(t)\right)^2}{kr^2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\left(k+\left(\frac{d}{dt}a(t)\right)^2\right)\sin^2(\theta) \\ 0 & 0 & r^2\left(-k-\left(\frac{d}{dt}a(t)\right)^2\right)\sin^2(\theta) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & \frac{\frac{d^2}{dt^2}a(t)}{a(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\frac{d^2}{dt^2}a(t)}{a(t)} & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k+\left(\frac{d}{dt}a(t)\right)^2}{kr^2-1} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-k-\left(\frac{d}{dt}a(t)\right)^2}{kr^2-1} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\left(-k-\left(\frac{d}{dt}a(t)\right)^2\right) \\ 0 & 0 & r^2\left(k+\left(\frac{d}{dt}a(t)\right)^2\right) & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$R_{ab} = \begin{pmatrix} -\frac{3\frac{d^2}{dt^2}a(t)}{a(t)} & 0 & 0 & 0 \\ 0 & \frac{-2k-a(t)\frac{d^2}{dt^2}a(t)-2\left(\frac{d}{dt}a(t)\right)^2}{kr^2-1} & 0 & 0 \\ 0 & 0 & r^2\cdot\left(2k+a(t)\frac{d^2}{dt^2}a(t)+2\left(\frac{d}{dt}a(t)\right)^2\right) & 0 \\ 0 & 0 & 0 & r^2\cdot\left(2k+a(t)\frac{d^2}{dt^2}a(t)+2\left(\frac{d}{dt}a(t)\right)^2\right)\sin^2(\theta) \end{pmatrix}, \quad R = \frac{6\left(-k-a(t)\frac{d^2}{dt^2}a(t)-\left(\frac{d}{dt}a(t)\right)^2\right)}{a^2(t)}.$$