Note: Levi-Civita symbol

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Contents

1 Definition

In three dimensions, the Levi-Civita symbol is defined by:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (2,1,3) \text{ or } (2,1,3) \\ 0 & \text{if } i = j,\ i = k \text{ or } k = i \end{cases} \tag{1}$$

That is, ε_{ijk} is 1 if (i,j,k) is an even permutation of $(1,2,3)$, -1 if it is an odd permutation, and 0 if any index is repeated.

2 Properties

2.1 Product

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right). \end{aligned} \tag{2}$$

2.2 Special cases

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{3}$$

2.2.1 Proof of special cases (method 1)

Using the properties of product

$$\begin{aligned}
\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{imn} &= \sum_{i=1}^{3} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \sum_{i=1}^{3} \Big(\delta_{ii} \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im} \left(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}\right) + \delta_{in} \left(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}\right) \Big) \\
&= \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) \sum_{i=1}^{3} \delta_{ii} + \sum_{i=1}^{3} \Big(- \delta_{im} \left(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}\right) + \delta_{in} \left(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}\right) \Big) \\
&= 3 \cdot \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) + \sum_{i=1}^{3} \left(-\delta_{im}\delta_{ji}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{in}\delta_{jm}\delta_{ki}\right) \\
&= 3 \cdot \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{jm}\delta_{kn} + \delta_{km}\delta_{jn} + \delta_{jn}\delta_{km} - \delta_{kn}\delta_{jm} \\
&= 3 \cdot \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - 2\delta_{jm}\delta_{kn} + 2\delta_{km}\delta_{jn} \\
&= \delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}
\end{aligned} \tag{4}$$

2.2.2 Proof of special cases (method 2)

Claim $\varepsilon_{i,j,k}\varepsilon_{i,m,n} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, where $i, j, k, m, n \in \{1, 2, 3\}$. WLOG, we may assume $\{i, j, k\}$ are all distinct and $\{i, m, n\}$ are all distinct.

1. If $j = m$ then $k = n$ and $j \neq k$, and we can rewrite
$$\varepsilon_{i,j,k}\varepsilon_{i,m,n} = \varepsilon_{i,j,k}\varepsilon_{i,j,k} = 1 = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{5}$$

2. If $j = n$ then $k = m$ and $j \neq k$, and we can rewrite
$$\varepsilon_{i,j,k}\varepsilon_{i,m,n} = \varepsilon_{i,j,k}\varepsilon_{i,k,j} = -\varepsilon_{i,j,k}\varepsilon_{i,j,k} = -1 = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{6}$$

2.2.3 Proof of special cases (method 3)

Consider $\varepsilon_{i,j,k}\varepsilon_{i,m,n}$ for $i = 1, 2, 3$, we have

1. When $i = 1$, the indices (j, k) and (m, n) can only be $(2, 3)$ or $(3, 2)$, otherwise this term will be zero.
$$\varepsilon_{1,j,k}\varepsilon_{1,m,n} : \begin{cases} \varepsilon_{1,2,3}\varepsilon_{1,2,3} & = 1, \\ \varepsilon_{1,2,3}\varepsilon_{1,3,2} & = -1, \\ \varepsilon_{1,3,2}\varepsilon_{1,2,3} & = -1, \\ \varepsilon_{1,3,2}\varepsilon_{1,3,2} & = 1. \end{cases} \tag{7}$$
Above result shows that: $\varepsilon_{1,j,k}\varepsilon_{1,m,n} = 1$ if $j = m$ and $k = n$; $\varepsilon_{1,j,k}\varepsilon_{1,m,n} = -1$ if $j = n$ and $k = m$. So we can write this term to be
$$\varepsilon_{1,j,k}\varepsilon_{1,m,n} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{8}$$

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2. When $i = 2$, the indices (j, k) and (m, n) can only be $(1, 3)$ or $(3, 1)$, otherwise this term will be zero.
$$\varepsilon_{2,j,k}\varepsilon_{2,m,n} : \begin{cases} \varepsilon_{2,1,3}\varepsilon_{2,1,3} & = 1, \\ \varepsilon_{2,1,3}\varepsilon_{2,3,1} & = -1, \\ \varepsilon_{2,3,1}\varepsilon_{2,1,3} & = -1, \\ \varepsilon_{2,3,1}\varepsilon_{2,3,1} & = 1. \end{cases} \tag{9}$$
Above result shows that: $\varepsilon_{2,j,k}\varepsilon_{2,m,n} = 1$ if $j = m$ and $k = n$; $\varepsilon_{2,j,k}\varepsilon_{2,m,n} = -1$ if $j = n$ and $k = m$. So we can write this term to be
$$\varepsilon_{2,j,k}\varepsilon_{2,m,n} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{10}$$

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3. When $i = 3$, the indices (j, k) and (m, n) can only be $(1, 2)$ or $(2, 1)$, otherwise this term will be zero.
$$\varepsilon_{3,j,k}\varepsilon_{3,m,n} : \begin{cases} \varepsilon_{3,1,2}\varepsilon_{3,1,2} & = 1, \\ \varepsilon_{3,1,2}\varepsilon_{3,2,1} & = -1, \\ \varepsilon_{3,2,1}\varepsilon_{3,1,2} & = -1, \\ \varepsilon_{3,2,1}\varepsilon_{3,2,1} & = 1. \end{cases} \tag{11}$$
Above result shows that: $\varepsilon_{3,j,k}\varepsilon_{3,m,n} = 1$ if $j = m$ and $k = n$; $\varepsilon_{3,j,k}\varepsilon_{3,m,n} = -1$ if $j = n$ and $k = m$. So we can write this term to be
$$\varepsilon_{3,j,k}\varepsilon_{3,m,n} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{12}$$

Then summing over i, we have

$$\sum_{i=1}^{3} \varepsilon_{i,j,k}\varepsilon_{i,m,n} = \varepsilon_{1,j,k}\varepsilon_{1,m,n} + \varepsilon_{2,j,k}\varepsilon_{2,m,n} + \varepsilon_{3,j,k}\varepsilon_{3,m,n}. \tag{13}$$

We can observe that, at most one of the three terms will be different from zero. Using the above conclusion, we can write

$$\sum_{i=1}^{3} \varepsilon_{i,j,k}\varepsilon_{i,m,n} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{14}$$

If we write down all the possibility if $\varepsilon_{i,j,k}\varepsilon_{i,m,n}$ for $i, j, k, m, n \in \{1, 2, 3\}$, we have

$$
\begin{array}{llllll}
\varepsilon_{1,1,1}\varepsilon_{1,1,1}=0 & \varepsilon_{2,1,1}\varepsilon_{2,1,1}=0 & \varepsilon_{3,1,1}\varepsilon_{3,1,1}=0 & \varepsilon_{1,1,1}\varepsilon_{1,1,2}=0 & \varepsilon_{2,1,1}\varepsilon_{2,1,2}=0 & \varepsilon_{3,1,1}\varepsilon_{3,1,2}=0 \\
\varepsilon_{1,1,1}\varepsilon_{1,1,3}=0 & \varepsilon_{2,1,1}\varepsilon_{2,1,3}=0 & \varepsilon_{3,1,1}\varepsilon_{3,1,3}=0 & \varepsilon_{1,1,1}\varepsilon_{1,2,1}=0 & \varepsilon_{2,1,1}\varepsilon_{2,2,1}=0 & \varepsilon_{3,1,1}\varepsilon_{3,2,1}=0 \\
\varepsilon_{1,1,1}\varepsilon_{1,2,2}=0 & \varepsilon_{2,1,1}\varepsilon_{2,2,2}=0 & \varepsilon_{3,1,1}\varepsilon_{3,2,2}=0 & \varepsilon_{1,1,1}\varepsilon_{1,2,3}=0 & \varepsilon_{2,1,1}\varepsilon_{2,2,3}=0 & \varepsilon_{3,1,1}\varepsilon_{3,2,3}=0 \\
\varepsilon_{1,1,1}\varepsilon_{1,3,1}=0 & \varepsilon_{2,1,1}\varepsilon_{2,3,1}=0 & \varepsilon_{3,1,1}\varepsilon_{3,3,1}=0 & \varepsilon_{1,1,1}\varepsilon_{1,3,2}=0 & \varepsilon_{2,1,1}\varepsilon_{2,3,2}=0 & \varepsilon_{3,1,1}\varepsilon_{3,3,2}=0 \\
\varepsilon_{1,1,1}\varepsilon_{1,3,3}=0 & \varepsilon_{2,1,1}\varepsilon_{2,3,3}=0 & \varepsilon_{3,1,1}\varepsilon_{3,3,3}=0 & \varepsilon_{1,1,2}\varepsilon_{1,1,1}=0 & \varepsilon_{2,1,2}\varepsilon_{2,1,1}=0 & \varepsilon_{3,1,2}\varepsilon_{3,1,1}=0 \\
\varepsilon_{1,1,2}\varepsilon_{1,1,2}=0 & \varepsilon_{2,1,2}\varepsilon_{2,1,2}=0 & \varepsilon_{3,1,2}\varepsilon_{3,1,2}=1 & \varepsilon_{1,1,2}\varepsilon_{1,1,3}=0 & \varepsilon_{2,1,2}\varepsilon_{2,1,3}=0 & \varepsilon_{3,1,2}\varepsilon_{3,1,3}=0 \\
\varepsilon_{1,1,2}\varepsilon_{1,2,1}=0 & \varepsilon_{2,1,2}\varepsilon_{2,2,1}=0 & \varepsilon_{3,1,2}\varepsilon_{3,2,1}=-1 & \varepsilon_{1,1,2}\varepsilon_{1,2,2}=0 & \varepsilon_{2,1,2}\varepsilon_{2,2,2}=0 & \varepsilon_{3,1,2}\varepsilon_{3,2,2}=0 \\
\varepsilon_{1,1,2}\varepsilon_{1,2,3}=0 & \varepsilon_{2,1,2}\varepsilon_{2,2,3}=0 & \varepsilon_{3,1,2}\varepsilon_{3,2,3}=0 & \varepsilon_{1,1,2}\varepsilon_{1,3,1}=0 & \varepsilon_{2,1,2}\varepsilon_{2,3,1}=0 & \varepsilon_{3,1,2}\varepsilon_{3,3,1}=0 \\
\varepsilon_{1,1,2}\varepsilon_{1,3,2}=0 & \varepsilon_{2,1,2}\varepsilon_{2,3,2}=0 & \varepsilon_{3,1,2}\varepsilon_{3,3,2}=0 & \varepsilon_{1,1,2}\varepsilon_{1,3,3}=0 & \varepsilon_{2,1,2}\varepsilon_{2,3,3}=0 & \varepsilon_{3,1,2}\varepsilon_{3,3,3}=0 \\
\varepsilon_{1,1,3}\varepsilon_{1,1,1}=0 & \varepsilon_{2,1,3}\varepsilon_{2,1,1}=0 & \varepsilon_{3,1,3}\varepsilon_{3,1,1}=0 & \varepsilon_{1,1,3}\varepsilon_{1,1,2}=0 & \varepsilon_{2,1,3}\varepsilon_{2,1,2}=0 & \varepsilon_{3,1,3}\varepsilon_{3,1,2}=0 \\
\varepsilon_{1,1,3}\varepsilon_{1,1,3}=0 & \varepsilon_{2,1,3}\varepsilon_{2,1,3}=1 & \varepsilon_{3,1,3}\varepsilon_{3,1,3}=0 & \varepsilon_{1,1,3}\varepsilon_{1,2,1}=0 & \varepsilon_{2,1,3}\varepsilon_{2,2,1}=0 & \varepsilon_{3,1,3}\varepsilon_{3,2,1}=0 \\
\varepsilon_{1,1,3}\varepsilon_{1,2,2}=0 & \varepsilon_{2,1,3}\varepsilon_{2,2,2}=0 & \varepsilon_{3,1,3}\varepsilon_{3,2,2}=0 & \varepsilon_{1,1,3}\varepsilon_{1,2,3}=0 & \varepsilon_{2,1,3}\varepsilon_{2,2,3}=0 & \varepsilon_{3,1,3}\varepsilon_{3,2,3}=0 \\
\varepsilon_{1,1,3}\varepsilon_{1,3,1}=0 & \varepsilon_{2,1,3}\varepsilon_{2,3,1}=-1 & \varepsilon_{3,1,3}\varepsilon_{3,3,1}=0 & \varepsilon_{1,1,3}\varepsilon_{1,3,2}=0 & \varepsilon_{2,1,3}\varepsilon_{2,3,2}=0 & \varepsilon_{3,1,3}\varepsilon_{3,3,2}=0 \\
\varepsilon_{1,1,3}\varepsilon_{1,3,3}=0 & \varepsilon_{2,1,3}\varepsilon_{2,3,3}=0 & \varepsilon_{3,1,3}\varepsilon_{3,3,3}=0 & \varepsilon_{1,2,1}\varepsilon_{1,1,1}=0 & \varepsilon_{2,2,1}\varepsilon_{2,1,1}=0 & \varepsilon_{3,2,1}\varepsilon_{3,1,1}=0 \\
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\varepsilon_{1,2,3}\varepsilon_{1,3,2}=-1 & \varepsilon_{2,2,3}\varepsilon_{2,3,2}=0 & \varepsilon_{3,2,3}\varepsilon_{3,3,2}=0 & \varepsilon_{1,2,3}\varepsilon_{1,3,3}=0 & \varepsilon_{2,2,3}\varepsilon_{2,3,3}=0 & \varepsilon_{3,2,3}\varepsilon_{3,3,3}=0 \\
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\varepsilon_{1,3,2}\varepsilon_{1,2,3}=-1 & \varepsilon_{2,3,2}\varepsilon_{2,2,3}=0 & \varepsilon_{3,3,2}\varepsilon_{3,2,3}=0 & \varepsilon_{1,3,2}\varepsilon_{1,3,1}=0 & \varepsilon_{2,3,2}\varepsilon_{2,3,1}=0 & \varepsilon_{3,3,2}\varepsilon_{3,3,1}=0 \\
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\varepsilon_{1,3,3}\varepsilon_{1,1,3}=0 & \varepsilon_{2,3,3}\varepsilon_{2,1,3}=0 & \varepsilon_{3,3,3}\varepsilon_{3,1,3}=0 & \varepsilon_{1,3,3}\varepsilon_{1,2,1}=0 & \varepsilon_{2,3,3}\varepsilon_{2,2,1}=0 & \varepsilon_{3,3,3}\varepsilon_{3,2,1}=0 \\
\varepsilon_{1,3,3}\varepsilon_{1,2,2}=0 & \varepsilon_{2,3,3}\varepsilon_{2,2,2}=0 & \varepsilon_{3,3,3}\varepsilon_{3,2,2}=0 & \varepsilon_{1,3,3}\varepsilon_{1,2,3}=0 & \varepsilon_{2,3,3}\varepsilon_{2,2,3}=0 & \varepsilon_{3,3,3}\varepsilon_{3,2,3}=0 \\
\varepsilon_{1,3,3}\varepsilon_{1,3,1}=0 & \varepsilon_{2,3,3}\varepsilon_{2,3,1}=0 & \varepsilon_{3,3,3}\varepsilon_{3,3,1}=0 & \varepsilon_{1,3,3}\varepsilon_{1,3,2}=0 & \varepsilon_{2,3,3}\varepsilon_{2,3,2}=0 & \varepsilon_{3,3,3}\varepsilon_{3,3,2}=0 \\
\varepsilon_{1,3,3}\varepsilon_{1,3,3}=0 & \varepsilon_{2,3,3}\varepsilon_{2,3,3}=0 & \varepsilon_{3,3,3}\varepsilon_{3,3,3}=0 & & &
\end{array}
\tag{15}
$$

3 Application

3.1 Cross Product

For $\hat{e}_1 = (1,0,0)$, $\hat{e}_2 = (0,1,0)$ and $\hat{e}_3 = (0,0,1)$, we have

$$\hat{e}_j \times \hat{e}_k = \sum_{i=1}^{3} \varepsilon_{ijk}\hat{e}_i \tag{16}$$

proof:

$$\begin{aligned}
\hat{e}_1 \times \hat{e}_1 &= \sum_{i=1}^{3} \varepsilon_{i,1,1}\hat{e}_i = \varepsilon_{1,1,1}\hat{e}_1 + \varepsilon_{2,1,1}\hat{e}_2 + \varepsilon_{3,1,1}\hat{e}_3 = 0 \\
\hat{e}_1 \times \hat{e}_2 &= \sum_{i=1}^{3} \varepsilon_{i,1,2}\hat{e}_i = \varepsilon_{1,1,2}\hat{e}_1 + \varepsilon_{2,1,2}\hat{e}_2 + \varepsilon_{3,1,2}\hat{e}_3 = \hat{e}_3 \\
\hat{e}_1 \times \hat{e}_3 &= \sum_{i=1}^{3} \varepsilon_{i,1,3}\hat{e}_i = \varepsilon_{1,1,3}\hat{e}_1 + \varepsilon_{2,1,3}\hat{e}_2 + \varepsilon_{3,1,3}\hat{e}_3 = -\hat{e}_2 \\
\hat{e}_2 \times \hat{e}_1 &= \sum_{i=1}^{3} \varepsilon_{i,2,1}\hat{e}_i = \varepsilon_{1,2,1}\hat{e}_1 + \varepsilon_{2,2,1}\hat{e}_2 + \varepsilon_{3,2,1}\hat{e}_3 = -\hat{e}_3 \\
\hat{e}_2 \times \hat{e}_2 &= \sum_{i=1}^{3} \varepsilon_{i,2,2}\hat{e}_i = \varepsilon_{1,2,2}\hat{e}_1 + \varepsilon_{2,2,2}\hat{e}_2 + \varepsilon_{3,2,2}\hat{e}_3 = 0 \\
\hat{e}_2 \times \hat{e}_3 &= \sum_{i=1}^{3} \varepsilon_{i,2,3}\hat{e}_i = \varepsilon_{1,2,3}\hat{e}_1 + \varepsilon_{2,2,3}\hat{e}_2 + \varepsilon_{3,2,3}\hat{e}_3 = \hat{e}_1 \\
\hat{e}_3 \times \hat{e}_1 &= \sum_{i=1}^{3} \varepsilon_{i,3,1}\hat{e}_i = \varepsilon_{1,3,1}\hat{e}_1 + \varepsilon_{2,3,1}\hat{e}_2 + \varepsilon_{3,3,1}\hat{e}_3 = \hat{e}_2 \\
\hat{e}_3 \times \hat{e}_2 &= \sum_{i=1}^{3} \varepsilon_{i,3,2}\hat{e}_i = \varepsilon_{1,3,2}\hat{e}_1 + \varepsilon_{2,3,2}\hat{e}_2 + \varepsilon_{3,3,2}\hat{e}_3 = -\hat{e}_1 \\
\hat{e}_3 \times \hat{e}_3 &= \sum_{i=1}^{3} \varepsilon_{i,3,3}\hat{e}_i = \varepsilon_{1,3,3}\hat{e}_1 + \varepsilon_{2,3,3}\hat{e}_2 + \varepsilon_{3,3,3}\hat{e}_3 = 0
\end{aligned} \tag{17}$$

Then, for vectors $\vec{a}$ and $\vec{b}$, if we write $\vec{a} = \sum_{i=1}^{3} a_i\hat{e}_i$ and $\vec{b} = \sum_{i=1}^{3} b_i\hat{e}_i$. The cross product can be writen as

$$\begin{aligned}
\vec{a} \times \vec{b} &= \left(\sum_{i=1}^{3} a_i\hat{e}_i\right) \times \left(\sum_{i=1}^{3} b_j\hat{e}_j\right) \\
&= \sum_{i,j=1}^{3} a_i b_j \left(\hat{e}_i \times \hat{e}_j\right) \\
&= \sum_{i,j=1}^{3} a_i b_j \sum_{k=1}^{3} \varepsilon_{i,j,k}\hat{e}_k,
\end{aligned} \tag{18}$$

denoted as

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^{3} a_i b_j \varepsilon_{i,j,k}\hat{e}_k \quad \text{or} \quad \left(\vec{a} \times \vec{b}\right)_k = \sum_{i,j=1}^{3} a_i b_j \varepsilon_{i,j,k}. \tag{19}$$

3.2 Vector Triple Product

For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, if we write $\vec{a} = \sum_{i=1}^{3} a_i \hat{e}_i$, $\vec{b} = \sum_{i=1}^{3} b_i \hat{e}_i$ and $\vec{c} = \sum_{i=1}^{3} c_i \hat{e}_i$, the product

$$\begin{aligned}
\vec{a} \times \left(\vec{b} \times \vec{c}\right) &= \sum_{ijk=1}^{3} \hat{e}_i \left(\vec{a} \times \left(\vec{b} \times \vec{c}\right)\right)_i && (20)\\
&= \sum_{i,j,k=1}^{3} \hat{e}_i \varepsilon_{i,j,k} a_j \left(\vec{b} \times \vec{c}\right)_k && (21)\\
&= \sum_{i,j,k=1}^{3} \hat{e}_i \varepsilon_{i,j,k} a_j \sum_{\ell m=1}^{3} \varepsilon_{k,\ell,m} b_\ell c_m && (22)\\
&= \sum_{i,j,k,\ell,m=1}^{3} \hat{e}_i \varepsilon_{i,j,k} \varepsilon_{k,\ell,m} a_j b_\ell c_m && (23)\\
&= \sum_{i,j,k,\ell,m=1}^{3} \hat{e}_i \varepsilon_{k,i,j} \varepsilon_{k,\ell,m} a_j b_\ell c_m && (24)\\
&= \sum_{i,j,\ell,m=1}^{3} \hat{e}_i \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right) a_j b_\ell c_m && (25)\\
&= \sum_{i,j=1}^{3} \left(a_j b_i c_j - a_j b_j c_i\right) \hat{e}_i && (26)\\
&= \left(\sum_{j=1}^{3} a_j c_j\right)\left(\sum_{i=1}^{3} \hat{e}_i b_j\right) - \left(\sum_{j=1}^{3} a_j b_j\right)\left(\sum_{i=1}^{3} \hat{e}_i c_j\right) && (27)\\
&= \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c} && (28)
\end{aligned}$$

Using above result, the proof in class can be reduced to

$$\begin{aligned}
\mathbf{u} \times (\nabla \times \mathbf{u}) &= \sum_{i,j=1}^{3} \left(u_j \partial_i u_j - u_j \partial_j u_i\right) \hat{e}_i && (29)\\
&= \sum_{i,j=1}^{3} \partial_i \hat{e}_i \left(\frac{1}{2} u_j^2\right) - \sum_{i,j=1}^{3} u_j \partial_j u_i \hat{e}_i && (30)\\
&= \sum_{i=1}^{3} \partial_i \hat{e}_i \left(\sum_{j=1}^{3} \frac{1}{2} u_j^2\right) - \sum_{j=1}^{3} u_j \partial_j \left(\sum_{i=1}^{3} u_i \hat{e}_i\right) && (31)\\
&= \frac{1}{2}\nabla\|\mathbf{u}\| - \sum_{j=1}^{3} u_j \partial_j \mathbf{u} && (32)\\
&= \frac{1}{2}\nabla\|\mathbf{u}\| - (\mathbf{u} \cdot \nabla)\,\mathbf{u}. && (33)
\end{aligned}$$

Last, rearranging the equation, we get $(\mathbf{u} \cdot \nabla)\,\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla\|\mathbf{u}\|$.