Note: Rotational Invariance of Cross Product

Chang-Mao Yang 楊長茂

April 13, 2024

Contents

1 Definitions

1.1 Levi-Civita symbol

In three dimensions, the Levi-Civita symbol is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),\ (2,1,3) \text{ or } (2,1,3), \\ 0 & \text{if } i=j,\ i=k \text{ or } k=i. \end{cases} \tag{1}$$

1.2 Cross Product

Given two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, define the cross product is given by

$$\vec{a} \times \vec{b} = \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} a_i b_j \hat{e}_k \tag{2}$$

1.3 Rotation

Let $\mathbf{R}$ be a rotational matrix in $\mathbb{R}^3$, where

$$\mathbf{R} = \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \end{pmatrix}. \tag{3}$$

For a vector $\vec{a}$ in $\mathbb{R}^3$, define the rotation of vector is given by

$$\mathbf{R}(\vec{a}) = \mathbf{R}\vec{a} = \sum_{i,j=1}^{3} (\mathbf{R}\vec{a})_i \,\hat{e}_i = \sum_{i,j=1}^{3} R_{i,j} a_j \hat{e}_i \tag{4}$$

Notice, for the rotation matrix, we have the following properties

- $\mathbf{R}^T\mathbf{R} = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, where $\mathbf{I}$ is identity matrix. (That is $\mathbf{R}^T = \mathbf{R}^{-1}$, where $\mathbf{R}^{-1}$ is the inverse of $\mathbf{R}$)
- The determinant of $\mathbf{R}$ is 1, $\det(\mathbf{R}) = 1$.

1.4 Determinant

For a matrix $\mathbf{A}$ in $\mathbb{R}^3$, where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}. \tag{5}$$

The determinant of $\mathbf{A}$ can be defined by

$$\det(\mathbf{A}) = \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} a_{1,i} a_{2,j} a_{3,k}, \tag{6}$$

also known as *Leibniz formula* for determinants in $\mathbb{R}^3$.

This can be easily shown by triple product of vectors. Given three vectors in $\mathbb{R}^3$,

$$\vec{a} = (a_1, a_2, a_3), \quad \vec{b} = (b_1, b_2, b_3) \quad \text{and} \quad \vec{c} = (c_1, c_2, c_3), \tag{7}$$

the triple product is given by

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} a_i b_j c_k. \tag{8}$$

After changing the symbols $a_i \to a_{1,i}$, $b_j \to b_{2,j}$ and $c_k \to c_{3,k}$, that is

$$\vec{a} = (a_{1,1}, a_{1,2}, a_{1,3}), \quad \vec{b} = (a_{2,1}, a_{2,2}, a_{2,3}) \quad \text{and} \quad \vec{c} = (a_{3,1}, a_{3,2}, a_{3,3}), \tag{9}$$

the triple product then corresponding to the determinant of $\mathbf{A}$,

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \det(\mathbf{A}) = \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} a_{1,i} a_{2,j} a_{3,k}. \tag{10}$$

2 Rotational Invariance of Cross Product

Let $\mathbf{R}$ be a rotational matrix in $\mathbb{R}^3$, where

$$\mathbf{R} = \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \end{pmatrix}. \tag{11}$$

For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbf{R}^3$, we have

$$\mathbf{R}\left(\vec{u} \times \vec{v}\right) = \mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right), \tag{12}$$

where

$$\vec{u} = (u_1, u_2, u_3) \quad \text{and} \quad \vec{v} = (v_1, v_2, v_3). \tag{13}$$

2.1 Proof 1

Proof. First, we consider

$$\mathbf{R}^{-1}\left(\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right)\right) = \sum_{i,j=1}^{3} (\mathbf{R}^{-1})_{i,j} \left(\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right)\right)_j \hat{e}_i \tag{14}$$

Since, $\mathbf{R}^{-1} = \mathbf{R}^T \Rightarrow (\mathbf{R}^{-1})_{i,j} = (\mathbf{R}^T)_{i,j} = (\mathbf{R})_{j,i} = R_{j,i}$, then, pluggin to the equation and expand it by the definition (4) of rotational matrix in component form

$$\begin{aligned}
\mathbf{R}^{-1}\left(\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right)\right) &= \sum_{i,j=1}^{3} R_{j,i} \left(\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right)\right)_j \hat{e}_i & (15)\\
&= \sum_{i,j=1}^{3} R_{j,i} \left(\sum_{n,m=1}^{3} \varepsilon_{j,n,m} \left(\mathbf{R}\vec{u}\right)_n \left(\mathbf{R}\vec{v}\right)_m\right) \hat{e}_i & (16)\\
&= \sum_{i,j=1}^{3} R_{j,i} \left(\sum_{n,m=1}^{3} \varepsilon_{j,n,m} \left(\sum_{k=1}^{3} R_{n,k} u_k\right) \left(\sum_{\ell=1}^{3} R_{m,\ell} v_\ell\right)\right) \hat{e}_i & (17)\\
&= \sum_{i,j,n,m,k,\ell=1}^{3} R_{j,i} \left(\varepsilon_{j,n,m} \left(R_{n,k} u_k\right) \left(R_{m,\ell} v_\ell\right)\right) \hat{e}_i & (18)\\
&= \sum_{i,j,n,m,k,\ell=1}^{3} R_{j,i} \varepsilon_{n,m,j} R_{n,k} u_k R_{m,\ell} v_\ell \hat{e}_i & (19)\\
&= \sum_{i,k,\ell=1}^{3} \left(\sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,\ell}\right) u_k v_\ell \hat{e}_i & (20)
\end{aligned}$$

In order to simplify the above formula, I provide a specific equation detailed in Appendix (3), which shows that the summation inside is just a Levi-Civita symbol. Therefore, it follows that

$$\mathbf{R}^{-1}\left(\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right)\right) = \sum_{i,k,\ell=1}^{3} \varepsilon_{i,k,\ell} u_k v_\ell \hat{e}_i = \vec{u} \times \vec{v}. \tag{21}$$

Last, acting a rotation on both side, one may derive

$$\mathbf{R}\left(\vec{u}\right) \times \mathbf{R}\left(\vec{v}\right) = \mathbf{R}\left(\vec{u} \times \vec{v}\right), \tag{22}$$

as claimed. □

2.2 Proof 2

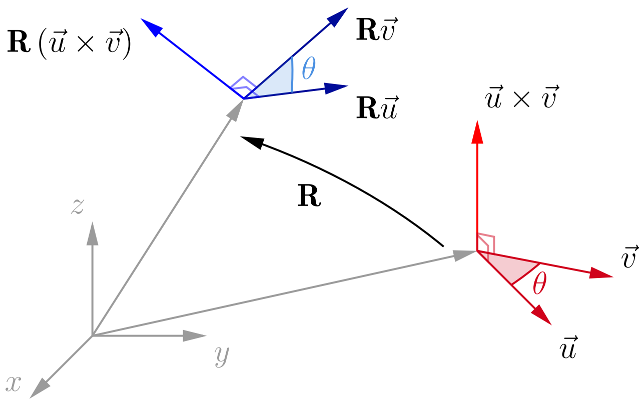

Figure 1: Rotation the cross product $\vec{u} \times \vec{v}$ by rotational matrix $\mathbf{R}$.

For the equation

$$\mathbf{R}(\vec{u}) \times \mathbf{R}(\vec{v}) = \mathbf{R}(\vec{u} \times \vec{v}) . \tag{23}$$

Considering the following discussions:

(a) The inner product of $\mathbf{R}(\vec{u})$ and $\mathbf{R}(\vec{u} \times \vec{v})$

$$[\mathbf{R}(\vec{u})] \cdot [\mathbf{R}(\vec{u} \times \vec{v})] = [\mathbf{R}(\vec{u})]^T [\mathbf{R}(\vec{u} \times \vec{v})] = (\vec{u})^T \mathbf{R}^T [\mathbf{R}(\vec{u} \times \vec{v})] \tag{24}$$

$$= (\vec{u})^T \mathbf{R}^{-1}\mathbf{R}(\vec{u} \times \vec{v}) = (\vec{u})^T (\vec{u} \times \vec{v}) = 0 \tag{25}$$

(b) The inner product of $\mathbf{R}(\vec{v})$ and $\mathbf{R}(\vec{u} \times \vec{v})$

$$[\mathbf{R}(\vec{v})] \cdot [\mathbf{R}(\vec{u} \times \vec{v})] = [\mathbf{R}(\vec{v})]^T [\mathbf{R}(\vec{u} \times \vec{v})] = (\vec{v})^T \mathbf{R}^T [\mathbf{R}(\vec{u} \times \vec{v})] \tag{26}$$

$$= (\vec{v})^T \mathbf{R}^{-1}\mathbf{R}(\vec{u} \times \vec{v}) = (\vec{v})^T (\vec{u} \times \vec{v}) = 0 \tag{27}$$

(c) If vectors $\vec{u}$ and $\vec{v}$ are parallel, that is $\vec{u} = \alpha\vec{v}$, the left-hand side of equation (23) becomes

$$\mathbf{R}(\vec{u}) \times \mathbf{R}(\vec{v}) = \mathbf{R}(\alpha\vec{v}) \times \mathbf{R}(\vec{v}) = \alpha\Big(\mathbf{R}(\vec{v}) \times \mathbf{R}(\vec{v})\Big) = 0 \tag{28}$$

and the right-hand side of equation (23) becomes

$$\mathbf{R}(\vec{u} \times \vec{v}) = \mathbf{R}(\alpha\vec{v} \times \vec{v}) = 0. \tag{29}$$

This means $0 = 0$, the equation is valid.

These three discussions shows that

$$\mathbf{R}(\vec{u}) \times \mathbf{R}(\vec{v}) = k\,\mathbf{R}(\vec{u} \times \vec{v}) . \tag{30}$$

If we define the angle between $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ is θ (see Figure 1), and taking the norm of both side of equation (30), one may derive

$$\|\mathbf{R}(\vec{u}) \times \mathbf{R}(\vec{v})\| = k\|\mathbf{R}(\vec{u} \times \vec{v})\| \tag{31}$$

$$\Rightarrow \|\mathbf{R}(\vec{u})\|\|\mathbf{R}(\vec{v})\| \sin\theta = k \det(\mathbf{R})\|\vec{u} \times \vec{v}\| \tag{32}$$

$$\Rightarrow \det(\mathbf{R})^2\|\vec{u}\|\|\vec{v}\| \sin\theta = k \det(\mathbf{R})\|\vec{u}\|\|\vec{v}\| \sin\theta \tag{33}$$

$$\Rightarrow \det(\mathbf{R}) = k. \tag{34}$$

Also, since the determinant of rotational matrix is 1, i.e. $\det(\mathbf{R}) = k = 1$, one may obtain

$$\mathbf{R}(\vec{u}) \times \mathbf{R}(\vec{v}) = \mathbf{R}(\vec{u} \times \vec{v}) , \tag{35}$$

as claimed. □

3 Appendix

Notice that the this term has free indices i, k, ℓ, we denote it as $f_{i,k,\ell}$, where $i, k, \ell \in \{1,2,3\}$ and

$$f_{i,k,\ell} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,\ell} \tag{36}$$

Now, considering the following three cases:

(a) If $(i,k,\ell) = (1,2,3)$, according to the equation (6), it just the determinant of $\mathbf{R}^T$, which equal to 1, that is

$$f_{1,2,3} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,1} R_{n,2} R_{m,3} = \det\left(\mathbf{R}^T\right) = \det(\mathbf{R}) = 1. \tag{37}$$

(b) If we exchange the indices $(i,k,\ell) \to (i,\ell,k)$, we have

$$f_{i,k,\ell} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,\ell} \quad \text{(original equation)}$$

$$f_{i,\ell,k} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,\ell} R_{m,k} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{m,k} R_{n,\ell} \tag{38}$$

$$= -\sum_{j,n,m=1}^{3} \varepsilon_{j,m,n} R_{j,i} R_{m,k} R_{n,\ell} \quad (\text{changing } m \to n,\ n \to m) \tag{39}$$

$$= -\sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,\ell} R_{m,k} = -f_{i,k,\ell}. \tag{40}$$

Using similar methods, one can derive

$$\begin{aligned} f_{i,\ell,k} &= -f_{i,k,\ell} \quad (k \leftrightarrow \ell) \\ f_{k,i,\ell} &= -f_{i,k,\ell} \quad (i \leftrightarrow k) \end{aligned} \tag{41}$$

This implies that, exchange of two indices changes the sign.

(c) Last, if $k = \ell$ $(\ell \to k)$, we have

$$f_{i,k,\ell} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,\ell} \quad \text{(original equation)}$$

$$f_{i,k,k} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,k} \quad (\text{changing } m \to n,\ n \to m) \tag{42}$$

$$= \sum_{j,n,m=1}^{3} \varepsilon_{j,m,n} R_{j,i} R_{m,k} R_{n,k} = \sum_{j,n,m=1}^{3} \varepsilon_{j,m,n} R_{j,i} R_{n,k} R_{m,k} \tag{43}$$

$$= -\sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,k} = -f_{i,k,k}. \tag{44}$$

Then we have $f_{i,k,k} = -f_{i,k,k}$, and the only possible value for $f_{i,k,k}$ is 0. Using similar methods, one can derive

$$\begin{aligned} f_{i,k,k} &= 0 \quad (\ell \to k) \\ f_{k,k,\ell} &= 0 \quad (i \to k) \\ f_{i,k,i} &= 0 \quad (\ell \to i) \end{aligned} \tag{45}$$

This implies that, the symbols $f_{i,k,\ell} = 0$, if there is any repeated indices.

These three cases (a), (b) and (c) leads to the conclusion that the symbols $f_{i,k,\ell}$ is just corresponding to the Levi-Civita symbol

$$f_{i,k,\ell} = \sum_{j,n,m=1}^{3} \varepsilon_{j,n,m} R_{j,i} R_{n,k} R_{m,\ell} = \varepsilon_{i,k,\ell}. \tag{46}$$