Rotation of The Fluid Flow

A brief introduction of rotation of fluid

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Final Project of *Introduction to Fluid Mechanics*

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Outline

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What is the rotation of a particle?

First we look at a moving particle in a space $\mathbb{R}^2$.

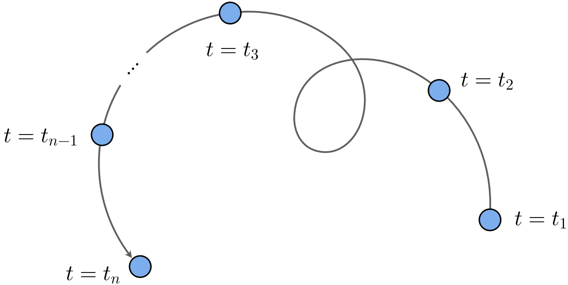
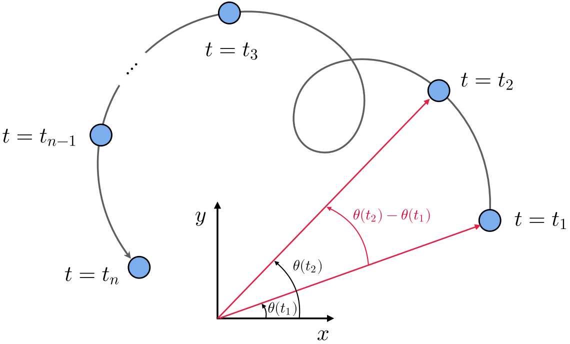


Figure: A moving particle in $\mathbb{R}^2$

However, if we add a coordinate



And define the magnitude angular velocity to be

$$\omega = \lim_{t_1 \to t_2} \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1} = \frac{d\theta}{dt} \tag{1}$$

Notice that, the velocity can be calculated in polar coordinate ($\mathbb{R}^2$) by position vector $\vec{r} = r\cos\theta\hat{e}_x + r\sin\theta\hat{e}_y$, that is

$$\begin{aligned}\vec{v} = \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(r\left(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y\right)\right)\\ &= \frac{dr}{dt}\left(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y\right) + r\left(-\sin\theta\hat{e}_x + \cos\theta\hat{e}_y\right)\frac{d\theta}{dt}\\ &= \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\end{aligned} \tag{2}$$

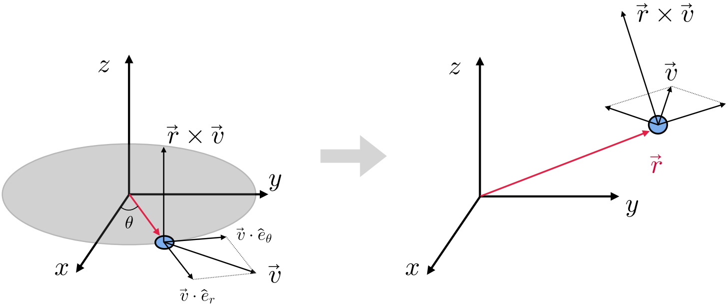
Extending the space to z component, we define the angular velocity for a particle on the xy-plane in $\mathbb{R}^3$

$$\vec{\omega} = \frac{\hat{e}_r \times \vec{v}}{r} = \frac{r(\hat{e}_r \times \hat{e}_\theta)}{r}\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{e}_z, \tag{3}$$

where $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_z$ are the orthonormal basis of cylinder coordinate.

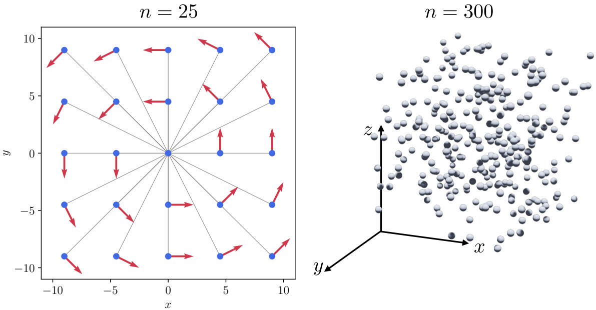
Now, consider the particle in $\mathbb{R}^3$ and we still using position vector $\vec{r}$ rather than the r-component vector, since $\hat{e}_r = \vec{r}/r$, we also can write

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}. \tag{4}$$



Then, how about a group of particles? Average angular velocity?

$$\tilde{\Omega} = \frac{1}{n}\sum_{i=1}^{n}\vec{\omega}_i = \frac{1}{n}\sum_{i=1}^{n}\frac{\vec{r}_i \times \vec{v}}{\|\vec{r}_i\|^2} \quad \text{or} \quad \iint_{\Sigma} \vec{\omega}\cdot d\vec{A} \tag{5}$$

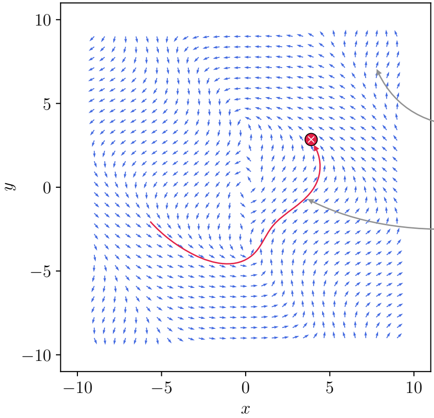


It seems to be some kind of approach method.

Stream line and Path line

There are two ways to describe how fluid flow in *Experimental Physics*.

$\vec{u} = \vec{u}(t_n)$



Stream line $\dfrac{dy}{dx} = \dfrac{u_y(x,y)}{u_x(x,y)}$

at specific time $t = t_n$

Path line

$$\frac{d\vec{r}(x(t), y(y))}{dt} = \vec{u}(x(t), y(t))$$

So, a more intuitive way to describe how a group of particles (fluid) rotate, is to place a marker is these particles.

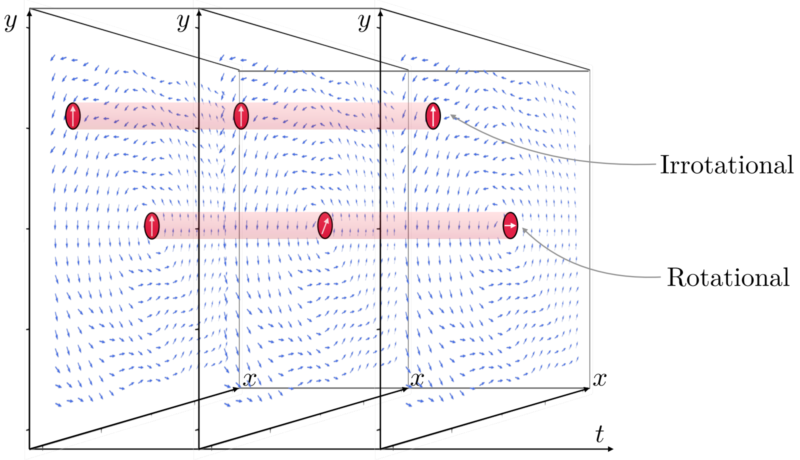
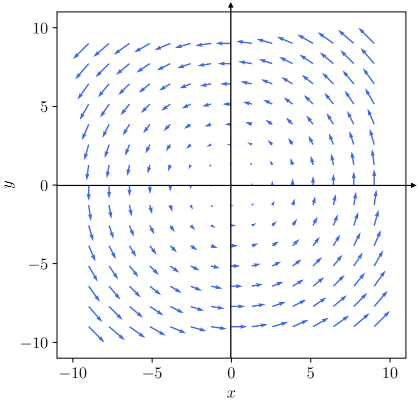


Figure: The *world line* of marker and vector field

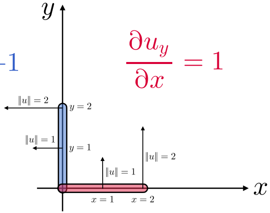
Example

Now, consider a fluid flow given by

$$\vec{u}(x,y) = u_x\hat{e}_x + u_y\hat{e}_y = -y\,\hat{e}_x + x\,\hat{e}_y \tag{6}$$



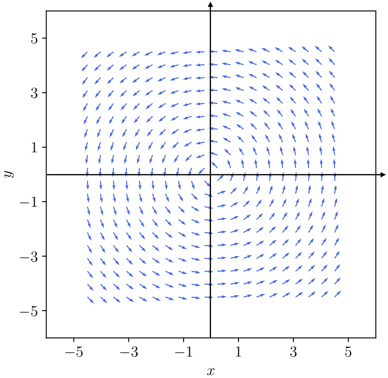
$$\frac{\partial u_x}{\partial y} = -1 \qquad \frac{\partial u_y}{\partial x} = 1$$

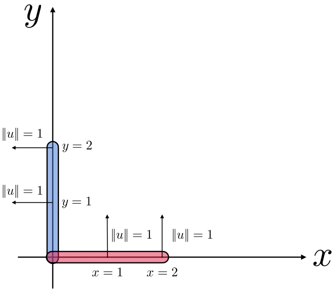


$$\operatorname{curl}\vec{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 2$$

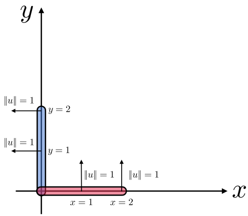
Now, normalize the preview example, we get

$$\vec{u}(x,y) = -\frac{y}{\sqrt{x^2+y^2}}\,\hat{e}_x + \frac{x}{\sqrt{x^2+y^2}}\,\hat{e}_y \tag{7}$$





Calculating the curl of $(-y\,\hat{e}_x + x\,\hat{e}_y)/\sqrt{x^2+y^2}$, we get



$$\frac{\partial u_y}{\partial x} = -\frac{y^2}{(x^2+y^2)^{3/2}} \xrightarrow{x=0} -\frac{1}{y}$$

$$\frac{\partial u_x}{\partial y} = \frac{x^2}{(x^2+y^2)^{3/2}} \xrightarrow{y=0} \frac{1}{x}$$

$$\begin{aligned}\operatorname{curl}\vec{u} &= \left(\frac{y^2}{(x^2+y^2)^{3/2}}\right) - \left(-\frac{x^2}{(x^2+y^2)^{3/2}}\right)\\ &= \frac{1}{\sqrt{x^2+y^2}}\end{aligned}$$

However, for $x, y \neq 0$, $\vec{u}(0,y) = -\hat{e}_x$ and $\vec{u}(x,0) = \hat{e}_y$, then $\partial_y \vec{u}(0,y) = \partial_x \vec{u}(x,0) = 0$.

By observing the result of

$$\vec{u}(x,y) = -\frac{y}{\sqrt{x^2+y^2}}\,\hat{e}_x + \frac{x}{\sqrt{x^2+y^2}}\,\hat{e}_y,$$

the two approach methods give

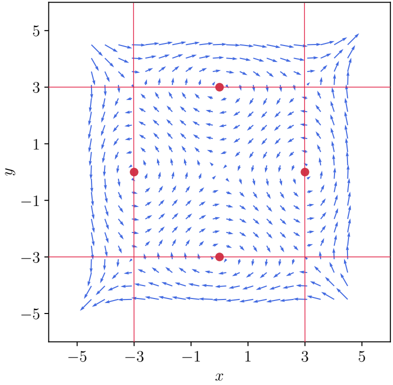
$$y\text{-axis:}\begin{cases}\text{curl}\left(\vec{u}\,\Big|_{x=0}\right) = 0 \\ \text{curl}\left(\vec{u}\,\right)\Big|_{x=0} = 1/y\end{cases}, \quad x\text{-axis:}\begin{cases}\text{curl}\left(\vec{u}\,\Big|_{y=0}\right) = 0 \\ \text{curl}\left(\vec{u}\,\right)\Big|_{y=0} = 1/x\end{cases} \tag{8}$$

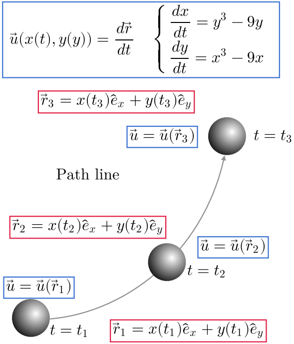
Here state a conclusion,

> *We cannot determine the rotational effect by looking at vectors at specific points. Instead, we should examine the changes in the vector field to understand its rotational effect.*

Using another example fluid flow given by

$$\vec{u}(x,y) = \left(y^3 - 9y\right)\hat{e}_x + \left(x^3 - 9x\right)\hat{e}_y \tag{9}$$





After we solving the system of non-linear differential equations, we have the path line and moving particles (click here to see video).

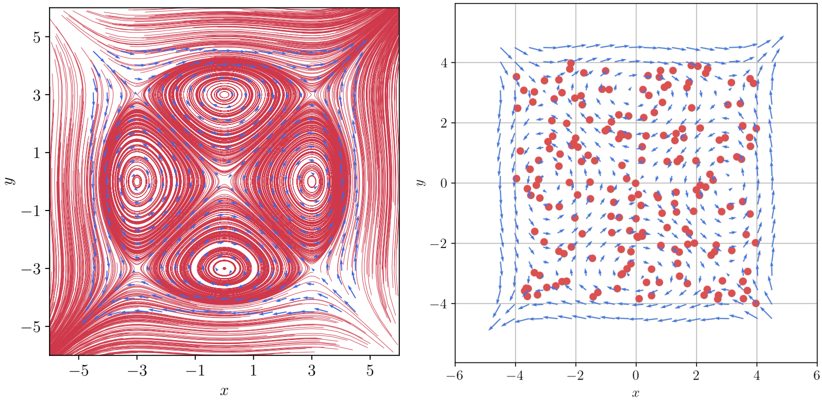
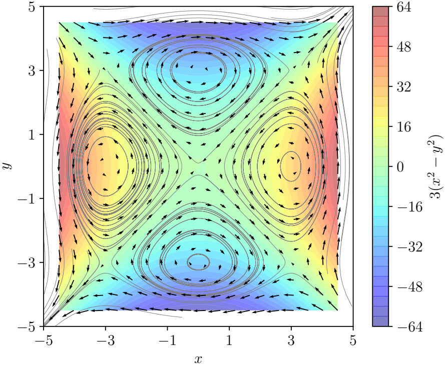


Figure: Left figure is the path line of vector field, reft figure is a snapshot of the simulation video

Also calculate the curl of vector field

$$\begin{aligned}\operatorname{curl}\vec{u} &= \partial_x(x^3 - 9x) - \partial_y(y^3 - 9y)\\ &= 3(x^3 - y^2),\end{aligned} \tag{10}$$

and make a contour:



Here state a conclusion,

The curl is to measure the rotation at specific point.

Outline

Definition of Circulation

Now, consider a simple closed oriented curve C , the *circulation* Γ_C of vector field $\vec{u}$ on C is given by

$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell}, \tag{11}$$

where $d\vec{\ell}$ is the infinitesimal displacement vector on C. More precisely,

Definition of circulation in $\mathbb{R}^2$

For a vector field $\vec{u} : S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, given a piecewise smooth oriented curve C parametrized by $\vec{\ell}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$, where $a \leq t \leq b$ and $\vec{\ell}(a) = \vec{\ell}(b)$, the circulation Γ of $\vec{u}$ on C is given by

$$\Gamma_C = \oint_C \vec{u} \cdot \frac{d\vec{\ell}(t)}{dt} dt = \int_a^b \vec{u}\left(x(t), y(t)\right) \cdot \frac{d\vec{\ell}(t)}{dt} dt. \tag{12}$$

But why are we using circulation ? By Stokes' theorem we have

$$\oint_{\partial\Sigma} \vec{u} \cdot d\vec{\ell} = \int_{\Sigma} (\operatorname{curl} \vec{u}) \cdot d\vec{\Sigma}, \tag{13}$$

where $\ell = \partial\Sigma$ is the boundary of Σ, and Σ is smooth oriented surface. If we only consider $\mathbb{R}^2$, we have the circulation

$$\Gamma_{\partial\Sigma} = \oint_{\partial\Sigma} \vec{u} \cdot d\vec{\ell} = \iint_{\Sigma} (\operatorname{curl} \vec{u})\ dx\, dy. \tag{14}$$

Definition of Circulation

Let the area of Σ to be $A = \iint_\Sigma dxdy$, we have

$$\Gamma_{\partial\Sigma} = A \cdot \frac{\iint_\Sigma (\operatorname{curl} \vec{u}) \ dx\, dy}{\iint_\Sigma dxdy} \approx \text{Area} \times \text{mean of } \operatorname{curl} \vec{u} \text{ in } \Sigma. \tag{15}$$

Here state a conclusion,

The circulation is to measure the rotation in a region.

So, now the method to estimate the rotation of the fluid flow with fluid velocity $\vec{u}$ are

1. Curl $\nabla \times \vec{u}(\vec{r}) = \operatorname{curl} \vec{u}(\vec{r})$ is to measure the rotation at point $\vec{r}$.
2. Circulation $\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell}$ is to measure the rotation in a region C.

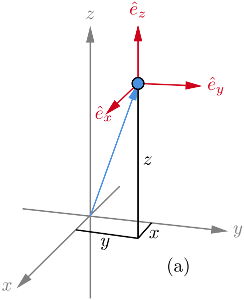
So, now let us consider some special case for $\nabla \times \vec{u} = 0$.

Outline

Vector calculus

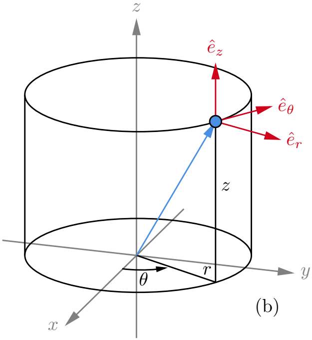
First, let us specify our coordinates, that is

Cartesian coordiante



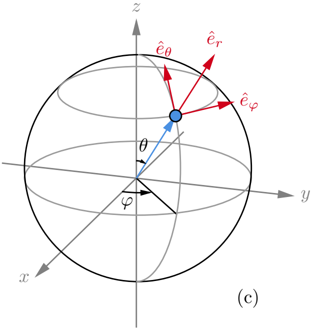
(a)

Cylindrical coordiante



(b)

Spherical coordiante



(c)

The curl in three different coordinate are

1. For a vector field $\vec{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \tag{16}$$

2. For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix} \tag{17}$$

3. For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\varphi \hat{e}_\varphi$

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\varphi \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & r\sin\theta\, A_z \end{vmatrix} \tag{18}$$

Or, expanding ...

1. For a vector field $\vec{A} = A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z$

$$\nabla\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{e}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{e}_z \quad (19)$$

2. For a vector field $\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_z\hat{e}_z$

$$\nabla\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z \quad (20)$$

3. For a vector field $\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\varphi\hat{e}_\varphi$

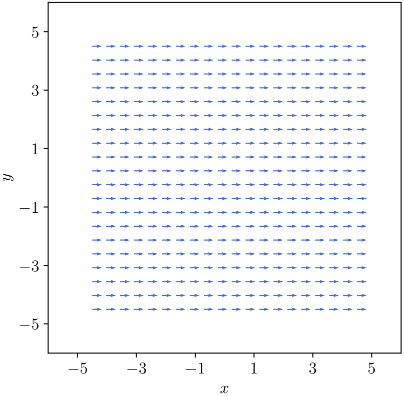
$$\begin{aligned}\nabla\times\vec{A} = &\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_z\sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{e}_\varphi\end{aligned} \quad (21)$$

Example for Cartesian

Here, for simplicity, we only consider the flow on the xy-plane and using the Cartesian coordinate to describe. We consider a flow velocity

$$\vec{u}(x, y) = 1 \cdot \hat{e}_x. \tag{22}$$

Obviously, the curl is zero, $\nabla \times \vec{u} = 1$, since it is a uniform flow.



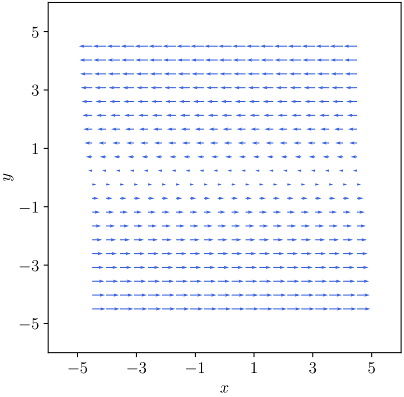
Another example is to consider the flow velocity

$$\vec{u}(x, y) = -y \cdot \hat{e}_x, \tag{23}$$

so that the velocity grows when y becomes larger. Now, the curl is

$$\nabla \times \vec{u} = 1 \cdot \hat{e}_z, \tag{24}$$

Although, the curl is not zero, we still can not see any rotation in the arrow graph of vector field.



Even the stream and path line, we still cannot see any rotation.

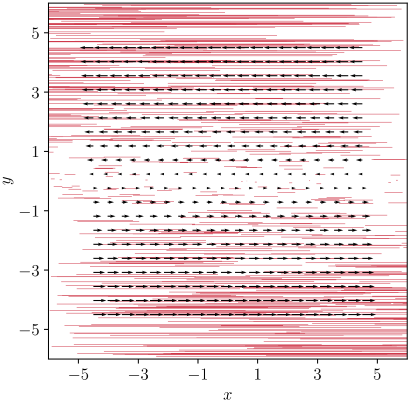


Figure: Given random initial position condition to solve *path line*.

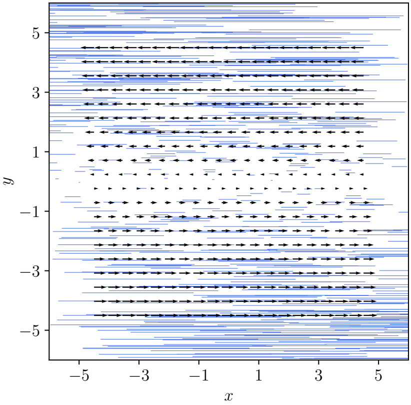
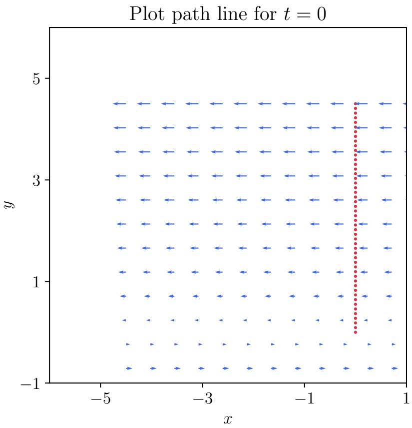
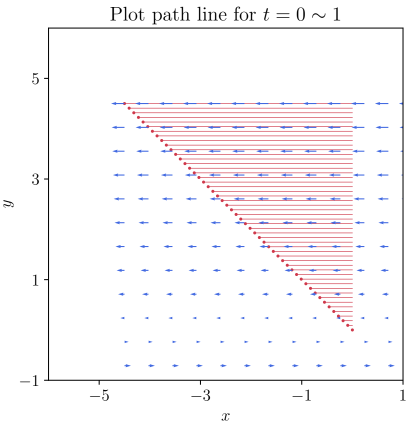


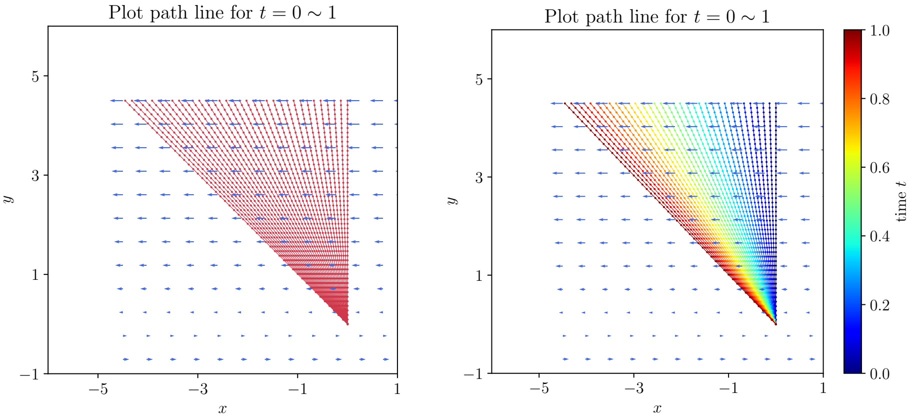
Figure: Given random initial position condition to solve *stream line*.

But if we look at some specific particle in space:





And link the particles in each time, and color them in different colors:



Here, for simplicity, we only consider the flow on the xy-plane and using the cylindrical coordinate to describe. So, the fluid velocity now becomes

$$\vec{u}(r,\theta) = u_r(r,\theta)\hat{e}_r + u_\theta(r,\theta)\hat{e}_\theta, \tag{25}$$

and then the curl

$$\nabla\times\vec{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{e}_z \tag{26}$$

becomes

$$\nabla\times\vec{u} = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{e}_z, \tag{27}$$

also notice that the differential displacement in this case is

$$d\vec{\ell} = dr\,\hat{e}_r + rd\theta\,\hat{e}_\theta. \tag{28}$$

Example for Cylindrical

Now, let $\vec{u}(r,\theta) = u_\theta(r,\theta)\hat{e}_\theta$, and solve for $0 = \nabla \times \vec{u}$, we have

$$0 = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r u_\theta(r,\theta)\right) - \frac{\partial 0}{\partial \theta}\right)\hat{e}_z \tag{29}$$

$$\Rightarrow \quad 0 = u_\theta(r,\theta) + r\frac{\partial u_\theta(r,\theta)}{\partial r}, \quad \text{for } r \neq 0 \tag{30}$$

$$\Rightarrow \quad \frac{\partial u_\theta(r,\theta)}{\partial r} = -\frac{u_\theta(r,\theta)}{r}. \tag{31}$$

By experience, the general solution is $u_\theta(r,\theta) = r^{-1}f(\theta)$, where f is a function of θ. We can check by plug into the equation

$$\frac{\partial u_\theta(r,\theta)}{\partial r} = -r^{-2}f(\theta) = -\frac{r^{-1}f(\theta)}{r} = -\frac{u_\theta(r,\theta)}{r}.$$

Then we can calculate the circulation for $\vec{u}(r,\theta) = u_\theta(r,\theta)\hat{e}_\theta$ such that $\nabla \times \vec{u} = 0$, the general solution is $u_\theta(r,\theta) = r^{-1} f(\theta)$, then the circulation in cylindrical coordinate (polar coordinate) is

$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell} = \oint_C u_\theta(r,\theta)\, r d\theta = \oint_C f(\theta) d\theta. \tag{32}$$

Now, we can observing that, although the curl is zero, if $f(\theta) > 0$ on C, then $\Gamma_C > 0$, on the other hand, if $f(\theta) < 0$ on C, then $\Gamma_C < 0$.

Now, consider a more general case, let $\vec{u}(r,\theta) = u_r(r,\theta)\hat{e}_r + u_\theta(r,\theta)\hat{e}_\theta$, and solve for $\nabla \times \vec{u} = 0$, we have

$$0 = \frac{\partial}{\partial r}\left(r u_\theta(r,\theta)\right) - \frac{\partial u_r(r,\theta)}{\partial \theta}, \tag{33}$$

Also, by experience, let $u_r(r,\theta) = r^{-1} f(r,\theta)$, we have

$$\frac{\partial f(r,\theta)}{\partial r} = \frac{\partial u_r(r,\theta)}{\partial \theta}, \tag{34}$$

which is the first equation in *Cauchy–Riemann equations*.

In order to solve $\partial f(r,\theta)/\partial r = \partial u_r(r,\theta)/\partial\theta$, we can suppose there exists a potential function $\phi(r,\theta)$ such that

$$f(r,\theta) = \frac{\partial\phi(r,\theta)}{\partial\theta} \quad \text{and} \quad u_r(r,\theta) = \frac{\partial\phi(r,\theta)}{\partial r}. \tag{35}$$

With this assumption, the equation is automatically satisfied:

$$\frac{\partial f(r,\theta)}{\partial r} = \frac{\partial^2\phi(r,\theta)}{\partial r\partial\theta} = \frac{\partial^2\phi(r,\theta)}{\partial\theta\partial r} = \frac{\partial u_r(r,\theta)}{\partial\theta}.$$

Now, the solution is

$$\vec{u} = \frac{\partial\phi(r,\theta)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}\hat{e}_r = \nabla\phi \tag{36}$$

where ϕ is any function of r and θ.

Then we can calculate the circulation for $\vec{u} = u_r(r,\theta)\hat{e}_r + u_\theta(r,\theta)\hat{e}_\theta$ such that $\nabla \times \vec{u} = 0$. The general solution is $u_\theta(r,\theta) = \partial_r\phi\,\hat{e}_r + r^{-1}\partial_\theta\phi\,\hat{e}_r$, where ϕ is any function of r and θ. The circulation in cylindrical coordinate (polar coordinate) is

$$\Gamma_C = \oint_C \vec{u}\cdot d\vec{\ell} = \oint_C \Big(u_r(r,\theta)dr + u_\theta(r,\theta)\,rd\theta\Big) \tag{37}$$

$$= \oint_C \frac{\partial\phi(r,\theta)}{\partial r}dr + \frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}rd\theta \tag{38}$$

$$= \oint_C \frac{\partial\phi(r,\theta)}{\partial r}dr + \frac{\partial\phi(r,\theta)}{\partial\theta}d\theta \tag{39}$$

$$= \oint_C \nabla\phi\cdot d\vec{\ell} = \oint_C d\phi, \tag{40}$$

where $d\phi$ is the total differential.

Usually, ϕ is a continuous function; however, not always. If a piecewise smooth oriented path C can be parameterized by

$$\vec{\ell}(t) = x(r(t), \theta(t))\hat{e}_x + y(r(t), \theta(t))\hat{e}_y, \tag{41}$$

where $t \in [a, b]$ and $\vec{\ell}(a) = \vec{\ell}(b)$, then the circulation becomes

$$\Gamma_C = \phi(\vec{r}(b)) - \phi(\vec{r}(a)). \tag{42}$$

However, if $\phi(r, \theta) = \theta$, the vector field is $\vec{u} = \nabla\phi = 1/r\hat{e}_\theta$. If the path C surrounds the origin, which is the pole $r = 0$, the circulation becomes $\Gamma_C = 2\pi - 0 = 2\pi > 0$, which means the circulation still can be measurement of rotation.

This is because function θ is *multivalued function.* Here we know that for all potential flow, the curl is zero (irrotational/vorticity-free)

$$\nabla \times \vec{u} = 0,$$

However,

$$\begin{cases} \text{if } \phi \text{ is single-valued} & \Rightarrow \quad \Gamma_C = 0, \quad \text{for any closed curve } C, \\ \text{if } \phi \text{ is multivalued} & \Rightarrow \quad \Gamma_C \neq 0, \quad \text{for some closed curves } C. \end{cases}$$

Here, I list some potential flow in cylindrical coordinate:

$\phi(r,\theta)$	$\vec{u} = \nabla\phi$
θ	$(1/r)\,\hat{e}_\theta$
$c_0 + c_1 \cdot \theta$	$(c_1/r)\,\hat{e}_\theta$
$\sum_{s=0}^{n} c_s \cdot \theta^s$	$\left(\sum_{s=1}^{n} s c_s \cdot \theta^{s-1}/r\right)\hat{e}_\theta$
$f(\theta)$	$(f'(\theta)/r)\,\hat{e}_\theta$
$\cos\theta$	$(-\sin\theta/r)\,\hat{e}_\theta$
$\sin\theta$	$(\cos\theta/r)\,\hat{e}_\theta$
$g(r)\cdot f(\theta)$	$(g'(r)f(\theta))\,\hat{e}_r + (g(r)f'(\theta)/r)\,\hat{e}_\theta$
$\vdots$	$\vdots$

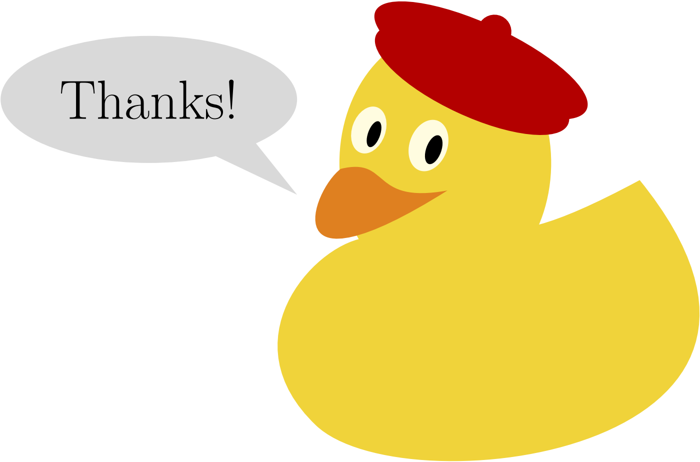
where $c_0, c_1, \ldots, c_n$ are constants, and f is any function of θ, g is any function of r.
Note that if the function depends on θ, $\hat{e}_\theta$ component of $\vec{u}$ is not zero.

Outline

Conclusion

In conclude,

1. The ***curl*** is to measure the rotation at specific point.
2. The ***circulation*** is to measure the rotation in a region.
3. If the potential function depends on the variable θ, the fluid velocity may have an $\hat{e}_\theta$ component.
4. If the potential function is discontinuous on the closed path, even if curl is zero, the circulation can still be nonzero.


Thanks!