0.0.1 Def:

And ideal fluid is one in whuch there are no shear stresses.Hence Euler's equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{b} \tag{1}$$

holds for ideal fluid.

0.1 Incompressible

We call a flow incompressible if for any subregion $W \subset D$

$$\text{volumn}(W_t) = \int_{W_t} dV = \text{constant.} \tag{2}$$

int time t, $W_t = \varphi_t(W)$, where φ_t is flow map.

Then a flow is incompressible if and only if

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{W_t} dV_y \\ &= \frac{d}{dt}\int_W J(\mathbf{x}, t) dV_x \\ &= \int_W \frac{\partial}{\partial t} J(\mathbf{x}, t) dV_x \\ &= \int_W (\operatorname{div}\mathbf{u}) J(\mathbf{x}, t) dV_x \\ &= \int_W (\operatorname{div}\mathbf{u}) dV_y \end{aligned} \tag{3}$$

imply

$$\operatorname{div}\mathbf{u} = 0 \Leftrightarrow \frac{\partial J}{\partial t} = (\operatorname{div}\mathbf{u})J = 0 \tag{4}$$

so that $J(\mathbf{x}, t)$ is a constant, notice that $J(\mathbf{x}, 0) = 0$, we have

$$J(\mathbf{x}, t) = 1, \quad \forall x \in D, t > 0 \tag{5}$$

Rmk: Since

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0 \Rightarrow \frac{D\rho}{Dt} = \rho_t + \mathbf{u} \cdot \nabla\rho = -\rho \operatorname{div}\mathbf{u} = 0 \tag{6}$$

so that $D\rho/Dt = 0$. Hence, the mass density is constant following the fluid for incompressible fluid.

0.2 Homogeneous

0.2.1 Def:

A fluid is said to be homogeneous, if $\rho(\mathbf{x}, t) = \rho(t)$,$\forall x \in D$

Rmk: For incompressible homogeneous fluid,

$$\rho(\mathbf{x}, t) = \rho(t) \quad \text{and} \quad \frac{D\rho}{Dt} = 0 \Rightarrow \rho_t = 0 \tag{7}$$

Then $\rho(t) = \text{constant} = \rho(0)$, $\forall t > 0$