Class Notes
Introduction to fluid mechanics

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Contents

1 The Equation of Motion

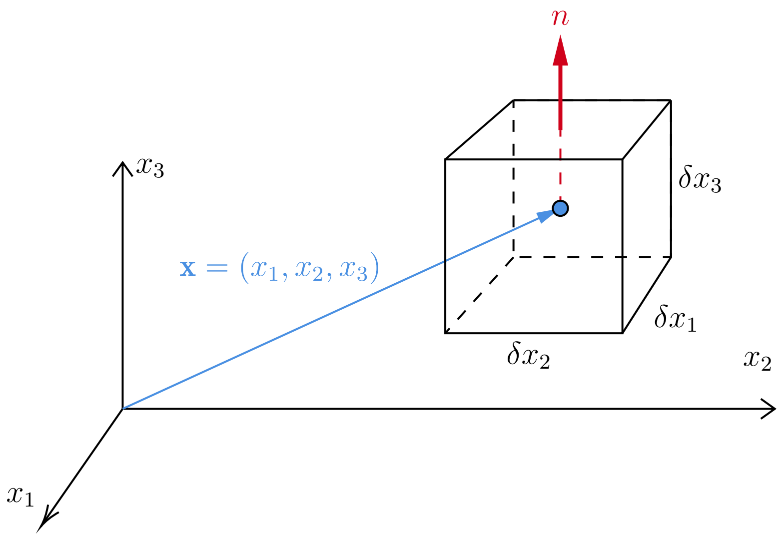
1.1 Introduction

1.1.1 Euler's equation:

Consider a fluid in a domain D in $\mathbb{R}^n$ ($n = 2$, $n = 3$).

Let $x \in D$, and $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ be the fluid density, velocity vector field and the pressure at the point x and time t.Consider an infinitesimal element of the fluid of volumn ∂V located at point x at time t with mass $\delta m = \rho(\mathbf{x}, t)$, which is moving $\mathbf{u}(\mathbf{x}, t)$ and momentum $\delta m \cdot \mathbf{u}(\mathbf{x}, t)$

The normal force directed into the indeinetesmal volumn across a face of area δa is $\mathbf{n} \cdot p(\mathbf{x}, t) \cdot \delta a$



Note that the pressure is the magnitade of the torce per unit area or normal stress, imposed on the fluid from neighboring fluid elements.

1.1.2 Convective derivative

convective derivative 對流導數 / material derivative 物質導數 / advective derivative 隨流導數 / convective derivative 對流導數 / derivative following the motion 隨體導數 / hydrodynamic derivative 水動力導數 / Lagrange derivative 拉格朗日導數 / substantial derivative 隨質導數 Couvder a fluid particle moving in flaid, whose position $\mathbf{x}$ at time t is $\mathbf{x}(t)$. Then

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t) \tag{1}$$

Hence, if $f(\mathbf{x},t)$ is a function on $D\times(0,T)$, then $f(\mathbf{x}(t),t)$ is the value if f at the moving fluid particle at $\mathbf{x}(t)$ at time t. We define the convective derivative of f:

$$\begin{aligned}\frac{Df(\mathbf{x},t)}{Dt} &= \frac{\partial f(\mathbf{x},t)}{\partial t} + \dot{\mathbf{x}}\cdot\nabla f(\mathbf{x},t)\\ &= f_t + \mathbf{u}\cdot\nabla f\end{aligned} \tag{2}$$

where $\nabla f = f(f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

Hence, if $f(\mathbf{x},t)$ is a function on $D\times(0,T)$, then $f(\mathbf{x}(t),t)$ is the vlaue of f at the moving fluid particle at $\mathbf{x}(t)$ at time t.

We define the convective derivative of f as:

$$\begin{aligned}\frac{Df(x,t)}{Dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{x}}(t)\cdot\nabla f\\ &= f_t + \mathbf{u}\cdot\nabla f\end{aligned}, \tag{3}$$

where $\nabla f = (f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.1 Def.

For any vector filed $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ on D, we let

$$\int_D \mathbf{F}dV = \left(\int_D F_1 dV, \int_D F_2 dV, \ldots, \int_D F_n dV\right). \tag{4}$$

1.1.2.2 Def.

We will assume that D is a smooth domain, i.e. for any $x_0\in\partial D$, $\mathbb{R}^n = (x', x_n), n = 2, 3$ $\exists\delta_0 > 0$ and a smooth function $\varphi : \mathbb{R}^{n-1}\to\mathbb{R}$, s.t.

$$\partial D\cap B(x_0,\delta_0) = \left\{(x',\varphi(x')) : \|x'\| < \delta_0, x'\in\mathbb{R}^{n-1}\right\}\cap B(x_0,\delta_0) \tag{5}$$

and

$$D\cap B(x_0,\delta_0) = \left\{(x', x_n) : x_n > \varphi(x'), x'\in\mathbb{R}^{n-1}, \|x'\| < \delta_0\right\}\cap B(x_0,\delta_0) \tag{6}$$

1.1.2.3 Claim

Conside the volume δV of an element of mass δm, which moves in the fluid by the fluid motion

$$\frac{d(\delta V)}{dt} = (\nabla\cdot\mathbf{u})(\mathbf{x},t)\cdot\delta V \quad \text{as} \quad \delta x_1, \delta x_2, \delta x_3 \to 0, \tag{7}$$

where $\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{u} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$, $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.4 proof

$$\begin{aligned}\frac{d(\delta V)}{dt} &= \frac{d}{dt}(\delta x_1, \delta x_2, \delta x_3)\\ &= \frac{d(\delta x_1)}{dt}\delta x_2\delta x_3 + \frac{d(\delta x_2)}{dt}\delta x_1\delta x_3 + \frac{d(\delta x_3)}{dt}\delta x_1\delta x_2\end{aligned} \tag{8}$$

For the first term

$$\begin{aligned}\frac{d(\delta x_1)}{dt} &\approx u_1\left(x_1+\frac{\delta x_1}{2}, x_2, x_3\right) - u_1\left(x_1-\frac{\delta x_1}{2}, x_2, x_3\right)\\ &= \frac{\partial u_1}{\partial x_1}(\xi_1, x_2, x_3)\delta x_1, \quad \text{fot some } \xi_1 \in \left(x_1-\frac{\delta x_1}{2}, x_1+\frac{\delta x_1}{2}\right)\end{aligned} \tag{9}$$

then

$$\frac{d(\delta x_1)}{dt}\delta x_2\delta x_3 \to \frac{\partial u_1}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \to 0 \tag{10}$$

Similarly

$$\frac{d(\delta x_2)}{dt}\delta x_2\delta x_3 \to \frac{\partial u_2}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \to 0 \tag{11}$$

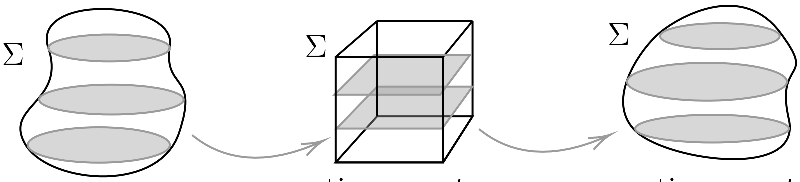
and

$$\frac{d(\delta x_3)}{dt}\delta x_2\delta x_3 \to \frac{\partial u_3}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \to 0 \tag{12}$$

so that

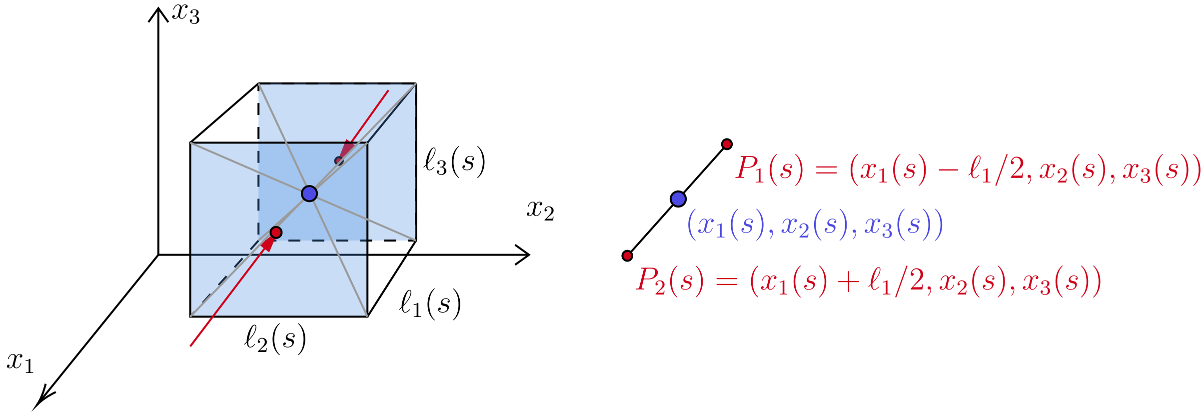
$$\frac{d(\delta V)}{dt} = \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\delta x_1\delta x_2\delta x_3 = (\nabla\cdot\mathbf{u})\delta V \tag{13}$$

1.1.2.5 Note



tagged porton of fluid particle

Consider a a tagged (marked) portion Σ of fluid with center of mass at $(x_1(s), x_2(s), x_3(s))$ at time s. Let $m(x)$ and $V(s)$ be the mass and volumn of this portion Σ of fluid at time s.The portion of fluid particle moves aling with fluid. see as (2/21 fig1)



For a time $t > 0$, suppose at time t, the tagged portion Σ of fluid particles is a cube centered at (x_1, x_2, x_3) with side lengh ℓ_1, ℓ_2, ℓ_3, see as (2/21 fig2 - textbook p.4),where

$$\begin{aligned} P_1(s) &= \left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \\ P_2(s) &= \left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \end{aligned} \tag{14}$$

We assume that Σ remain a cube for $s \approx t$ with side length, $\ell_1(s), \ell_2(s), \ell_3(s)$, then $V(s) = \ell_1(s) \cdot \ell_2(s) \cdot \ell_3(s)$

$$\left.\frac{dV(s)}{ds}\right|_{s=t} = \left.\frac{d\ell_1(s)}{ds}\right|_{s=t} \ell_2(s)\ell_3(s) + \left.\frac{d\ell_2(s)}{ds}\right|_{s=t} \ell_1(s)\ell_3(s) + \left.\frac{d\ell_3(s)}{ds}\right|_{s=t} \ell_1(s)\ell_2(s) \tag{15}$$

where

$$\begin{aligned} \frac{d\ell_1(s)}{ds} &= u_1(P_2(t), t) - u_1(P_1(t), t) \\ &= u_1 \left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t \right) - u_1 \left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t \right) \\ &\approx \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3, t) \cdot \ell_1 \end{aligned} \tag{16}$$

Similarly

$$\left.\frac{d\ell_i}{ds}\right|_{s=t} = \frac{\partial u_i}{\partial x_i}(x_1, x_2, x_3) \cdot \ell_i, \quad \forall i = 1, 2, 3. \tag{17}$$

Now we write $\left.\frac{d}{ds}\right|_{s=t} = \frac{d}{dt}$, combinded with equation

1.1.3 Continuity equation

Let $\rho(\mathbf{x}, t)$ be the density of fluid at time s.Since $M(s) = \text{const.}$, $\forall s > 0$ and $\frac{dM(s)}{ds} = 0$, $\forall s > 0$.Therefore, since it is similar to the cube, the density is

$$\rho(\mathbf{x}, s) \approx \frac{M(s)}{V(s)} \tag{18}$$

and the derevative is

$$\begin{aligned}
\frac{d}{ds}\rho(\mathbf{x}, s)\bigg|_{s=t} &\approx \frac{d}{ds}\frac{M(s)}{V(s)}\bigg|_{s=t} \\
&= \frac{M'(s)V(s) - M(s)V'(s)}{V^2(s)}\bigg|_{s=t} \\
&= \frac{0 - M(s)\frac{d}{ds}V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)(\operatorname{div}\mathbf{u})V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)}{V(s)}(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t} \\
&= -\rho(\mathbf{x}(s), s)(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t}
\end{aligned} \tag{19}$$

we get

$$-\frac{d}{dt}\rho(\mathbf{x}(t), t) = \rho \cdot (\nabla \cdot \mathbf{u}(t)) \tag{20}$$

On the other hand, by chain rule

$$\frac{d}{dt}\rho(\mathbf{x}(t), t) = \rho_t + (\nabla\rho) \cdot \mathbf{u}(t) \tag{21}$$

combining together we have

$$\begin{aligned}
&\Rightarrow \rho_t + (\nabla\rho) \cdot \mathbf{u} = \rho \cdot (\nabla \cdot \mathbf{u}) \\
&\Rightarrow \rho_t + (\nabla\rho) \cdot \mathbf{u} - \rho \cdot (\nabla \cdot \mathbf{u}) = 0 \\
&\Rightarrow \rho_t + \nabla \cdot (\rho\mathbf{u}) = 0
\end{aligned} \tag{22}$$

and the equation $\rho_t + \nabla \cdot (\rho\mathbf{u}) = 0$ is called the *contunuity equation.*

1.1.4 Heuristic proof of the Euler equation

In the ansense of an externally applied forces, the net force $\mathbf{F}$, acting on δV, is due to the pressure field.

Write $\mathbf{F} = (F_1, F_2, F_3)$, we get

$$\begin{aligned}
\mathbf{F}(x_1, x_2, x_3, t) &\approx \left(P\left(x_1 - \frac{\delta x_1}{2}, x_2, x_3, t\right) - P\left(x_1 + \frac{\delta x_1}{2}, x_2, x_3, t\right)\right)\delta x_2 \delta x_3 \\
&= -\frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t)\delta x_1 \delta x_2 \delta x_3, \quad \delta x_1, \delta x_2, \delta x_3 \to 0 \\
&= \frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t)\delta V
\end{aligned} \tag{23}$$

for some $\zeta_1 \in \left(x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2}\right)$.

By Newton's second law, the equation of motion for the elemnet of fund mass δm, at point $\mathbf{x}(t)$ is

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \mathbf{F} = -(\nabla P)\delta V \tag{24}$$

also

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \delta m \frac{d}{dt}\mathbf{u}(\mathbf{x},t) = \delta m \left(\mathbf{u}_t + (\nabla \cdot \mathbf{u})\right) \mathbf{u} \tag{25}$$

then

$$\begin{aligned} \delta m \left(\mathbf{u}_t + (\nabla \cdot \mathbf{u})\right) \mathbf{u} &= -(\nabla P)\delta V \\ \mathbf{u}_t + (\nabla \cdot \mathbf{u})\, \mathbf{u} &= -(\nabla P)\frac{\delta V}{\delta m} = -(\nabla P)\frac{1}{\delta m/\delta V} \end{aligned} \tag{26}$$

we get a equation

$$\mathbf{u}_t + (\nabla \cdot \mathbf{u})\, \mathbf{u} = -\frac{\nabla P}{\rho} \tag{27}$$

called *Euler's equation*.

Notice that

$$\begin{aligned} (\nabla \cdot \mathbf{u})\, \mathbf{u} &= \left(\sum_{i=0}^{3} u_i \frac{\partial}{\partial x_i}\right) \mathbf{u} \\ &= \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}\right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &= \begin{pmatrix} \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}\right) u_1 \\ \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}\right) u_2 \\ \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}\right) u_3 \end{pmatrix} \end{aligned} \tag{28}$$

1.1.5 Lemma

Let D be a bounded domain and $F : \bar{D} \times [0, a_0] \to \mathbb{R}$ be a smooth function (or C^∞), then

$$\frac{d}{dt}\int_D F(x,t)dx = \int_D \frac{dF(x,t)}{dt}dx \tag{29}$$

1.1.5.1 proof

we have

$$\begin{aligned} \frac{d}{dt}\int_D F(x,t)dx &= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\int_D F(x,t+\Delta t)dx - \frac{d}{dt}\int_D F(x,t)dx\right] \\ &= \lim_{\Delta t \to 0} \frac{d}{dt}\int_D \frac{F(x,t+\Delta t) - F(x,t)}{\Delta t}dx \\ &= \text{By M.V.T.} \\ &= \lim_{\Delta t \to 0}\int_D \frac{\frac{\partial}{\partial t}F(x,\xi)\Delta t}{\Delta t}dx, \quad \text{for some } \xi, \text{ where} t < \xi < t + \Delta \\ &= \lim_{\Delta t \to 0}\int_D \frac{\partial}{\partial t}F(x,\xi)dx \end{aligned} \tag{30}$$

Denote, $\frac{\partial}{\partial t}F(x,t) = F_t(x,t)$ and $\frac{\partial^2}{\partial t^2}F(x,t) = F_{tt}(x,t)$, so

$$\begin{aligned}
&\left|\frac{1}{\Delta t}\int_D [F(x,t+\Delta t) - F(x,t)] - \int_D \frac{\partial}{\partial t}F(x,t)dx\right| \\
&= \left|\int_D F_t(x,\xi)dx - \int_D F_t(x,\xi)dx\right| \\
&= \text{By MVT} \\
&= \left|\int_D [F_t(x,\xi) - F_t(x,t)]\right|dx \\
&= \text{By MVT} \\
&= \left|\int_D F_{tt}(x,z)(t-\xi)dz\right|, \quad z \text{ between } t \text{ and } \xi \\
&\leq M|t-\xi||D| \to 0, \quad \text{where } |D| \text{ is vlolumn of } D
\end{aligned} \tag{31}$$

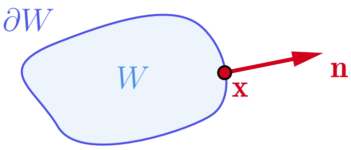
where $M = \sup_{(x,t)\in D\times(0,a)} F_{tt}(x,t)$.

1.1.6 The Continuity Equation

Recall that D is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ filled with fluid and $x = x(x_1, x_2, x_3) \in D$ is a particle of fluid moving through x at time t with velocity $\mathbf{u}(x,t)$. If W is a ant subregion of D, then the mass of fluid in W at time t is

$$m(W,t) = \int_W \rho(x,t)dV, \tag{32}$$

where $\rho(x,t)$ is the density of fluid at (x,t). Then $\frac{d}{dt}m(W,t) = \int_W \rho_t(x,t)dV$.



Let ∂W be the boundary of W. Suppose ∂W is smooth, let $\mathbf{n}(x)$ be the normal vector to ∂W at $x \in \partial W$, Let dA denote the

Then the volumn of fluid flow rate across ∂W per unit time, Since $\mathbf{u}\cdot\mathbf{n}\,\Delta A \Rightarrow$ the mass of fluid flow per unit time is $\rho\mathbf{u}\cdot\mathbf{n}\,\Delta A$

Since, by the conservation of mass, the rate of increase of mass in W is equal to the rate that mass is incoming ∂W

$$\int_W \rho_t dV = \frac{d}{dt}\int_W \rho dV = -\int_{\partial W} \rho\,(\mathbf{n}\cdot\mathbf{u})dV = \text{by divergence theorem} = -\int_W \operatorname{div}(\rho\mathbf{u})dV \tag{33}$$

By divergence theorem, we have

$$\int_W (\rho_t + \operatorname{div}(\rho\mathbf{u}))\,dV = 0, \quad \forall W \subset D \tag{34}$$

We now choose $W = B(x,r)$, and let $H(y,t) = \rho_t + \mathrm{div}(\rho\mathbf{u}) = H$, by above equation, we have

$$\int_{B(x,r)} H(y,t)dV_x = 0, \quad \forall x \in D, B(x,r) \subset D \tag{35}$$

Notice $H(y,t) = \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} H(y,t)dV = H(y,t)\dfrac{\int_{B(x,r)} dV}{|B(x,r)|}$, where $|B(x,r)|$ is the volumn of $B(x,r)$. Now

$$\begin{aligned}
\left|\frac{1}{B(x,r)}\int_{B(x,r)} H(x,t)dV - H(x,t)\right| &= \frac{1}{|B(x,r)|}\left|\int_{B(x,r)} [H(y,t) - H(x,t)]dV\right| \\
&\leq \frac{1}{|B(x,r)|}\int_{B(x,r)} |H(y,t) - H(x,t)|dV \\
&\leq \frac{1}{|B(x,r)|}\max_{y\in B(x,r)} |H(y,t) - H(x,t)| \cdot |B(x,r)| \\
&\to 0, \quad \text{as } r \to 0
\end{aligned} \tag{36}$$

so that

$$\lim_{r\to 0}\left|\frac{1}{B(x,r)}\int_{B(x,r)} H(x,t)dV - H(x,t)\right| = 0 \tag{37}$$

By equation (35) and (37), we have

$$H(x,t) = 0 \quad \Rightarrow \quad \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \tag{38}$$

which is called the continuity equation in $D \times (0,T)$.

1.2 Proof of Euler's Equation

1.2.1 Balance of Momentum 1 (BM1)

The force per unit area on a point $x \in \partial W$ is $-p \cdot \mathbf{n}$, since the total force on W due force the pressure on ∂W is

$$\begin{aligned}
\mathbf{f} &= -\int_{\partial W} p \cdot \mathbf{n}dA \\
&= \left(-\int_{\partial W} pn_1 dA, -\int_{\partial W} pn_2 dA, -\int_{\partial W} pn_3 dA\right) \\
&= \left(-\int_{\partial W} (p,0,0)\cdot\mathbf{n}dA, -\int_{\partial W} (0,p,0)\cdot\mathbf{n}dA, -\int_{\partial W} (0,0,p)\cdot\mathbf{n}dA\right) \\
&= \text{by divergence theorem} \\
&= \left(-\int_W \mathrm{div}(p,0,0)dV, -\int_W \mathrm{div}(0,p,0)dV, -\int_W \mathrm{div}(0,0,p)dV\right) \\
&= -\int_W (\nabla p)dV
\end{aligned} \tag{39}$$

Now, the total force on the fluid due to the pressure

$$\partial W = -\int_W \nabla p dV \tag{40}$$

If $\mathbf{b}(\mathbf{x}, t)$ denotes the given body force per unit mass (ex: gravity), then the toal body force is

$$F_B = \int_W \rho \cdot \mathbf{b} \cdot dV. \tag{41}$$

By equations (40) and (41), the force per unit volume is $-\nabla p + \rho \mathbf{b}$

By the Newton's second law,

$$\frac{D}{Dt}(\delta m\, \mathbf{u}) = \frac{D}{Dt}(\delta m\, \mathbf{u}(\mathbf{x}, t)) = (-p + \rho \mathbf{b})\delta V \tag{42}$$

we then have

$$\Rightarrow \rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b} \quad \text{BM1 (Euler equation)} \tag{43}$$

1.2.2 Balance of Momentum 2 (BM2)

WLOG. we write $\mathbf{u}$ for $\mathbf{u} = (u_1, u_2, u_3)$ and b for $\mathbf{b}$, Integral from of balance of momentum

By (BM1)

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \rho \mathbf{b} \tag{44}$$

then by 44 and continuity equation

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = \rho_t \mathbf{u} + \rho \mathbf{u}_t = -\operatorname{div}(\rho \mathbf{u})\mathbf{u} - \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \rho \mathbf{b} \tag{45}$$

Let $\mathbf{e}$ be a fixed vector in space, then

$$\begin{aligned} \mathbf{e} \cdot \frac{\partial}{\partial t}(\rho \mathbf{u}) &= -\operatorname{div}(\rho \mathbf{u})\mathbf{u} \cdot \mathbf{e} - \rho(\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{e} - (\nabla p) \cdot \mathbf{e} + \rho \mathbf{b} \cdot \mathbf{e} \\ &= -\operatorname{div}\left(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e})\right) - \operatorname{div}\left(p\mathbf{e}\right) + \rho \mathbf{b} \cdot \mathbf{e} \\ &= -\operatorname{div}\left(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e}) + p\mathbf{e}\right) + \rho \mathbf{b} \cdot \mathbf{e} \end{aligned} \tag{46}$$

Since, we have

1. the divergence of $p\mathbf{e}$

$$\begin{aligned} \operatorname{div}(p\mathbf{e}) &= \operatorname{div}\left((pe_1, pe_2, pe_3)\right) \\ &= \frac{\partial p}{\partial x_1} e_1 + \frac{\partial p}{\partial x_2} e_2 + \frac{\partial p}{\partial x_3} e_3 \\ &= \sum_{i=1}^{3} \frac{\partial p}{\partial x_i} e_i = \nabla p \cdot \mathbf{e} \end{aligned} \tag{47}$$

1. the divergence of $\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e})$

$$\begin{aligned} \operatorname{div}(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e})) &= \sum_{i=0}^{3} \frac{\partial}{\partial x_i} \left[\rho u_i (\mathbf{u} \cdot \mathbf{e})\right] \\ &= \sum_{i=0}^{3} \left(\frac{\partial}{\partial x_i}(\rho u_i)\right)(\mathbf{u} \cdot \mathbf{e}) + \sum_{i=0}^{3} (\rho u_i)\left(\frac{\partial}{\partial x_i}(\mathbf{u} \cdot \mathbf{e})\right) \\ &= \operatorname{div}(\rho \mathbf{u})(\mathbf{u} \cdot \mathbf{e}) + \rho(\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \mathbf{e}) \end{aligned} \tag{48}$$

Hence, if W is a fixed region in space in the fluid

$$\begin{aligned}
\mathbf{e}\cdot\frac{d}{dt}\int_W \rho\mathbf{u}dV &= \int_W e\cdot\frac{d}{dt}(\rho\mathbf{u})dV \\
&= -\int_W \operatorname{div}(p\mathbf{e}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e}))dV + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV \\
&= \text{By divergence theorem.} \\
&= -\int_{\partial W}(p\mathbf{e}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e}))\cdot\mathbf{n}dA + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV \\
&= -\int_{\partial W} p\mathbf{e}\cdot\mathbf{n}dA - \int_{\partial W}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})\cdot\mathbf{n}dA + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV, \quad \forall\mathbf{e}\in\mathbb{R}^n, n=2\text{ or }3
\end{aligned} \tag{49}$$

then

$$\frac{d}{dt}\int_W \mathbf{u}dV = -\int_{\partial W} p\mathbf{n}dA - \int_{\partial W}\rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}dA + \int_{\partial W}\rho\mathbf{b}\cdot\mathbf{e}dV \tag{50}$$

or

$$\frac{d}{dt}\int_W \mathbf{u}dV = -\int_{\partial W}(p\mathbf{n}+\rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u})\,dA + \int_{\partial W}\rho\mathbf{b}\cdot\mathbf{e}dV, \quad \text{BM2} \tag{51}$$

and BM2 is also the Integral form of balance of momentum.

Note: The quantity $p\mathbf{n}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ is the momentum per unit area crossing ∂W when $\mathbf{n}$ is unit vector outer normal to ∂W.

1.2.3 Balance of Momentum 3 (BM3)

Let D be a region that the fluid is moving and $x\in D$

Let $\varphi(\mathbf{x},t)$ be the trajectory of the partiacle that is at point x, i.e. φ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi(\mathbf{x},t) = \mathbf{u}\left(\varphi(\mathbf{x},t),t\right) && \forall t>0 \text{ at time } t \\
&\varphi(\mathbf{x},0) = x && \varphi(\mathbf{x},t) = \varphi_t(\mathbf{x})
\end{aligned} \tag{52}$$

We will assume that φ is smooth and for fixed t, $\varphi_t : t \to \varphi(\mathbf{x},t)$ is invertible.

φ_t doesn't mean $\partial/\partial t$ here!

We called φ is the fluid flow map.

If W is the a region in D, then $W_t := \varphi_t(W)$ is the region of the fluid at time t whose initial position is in W at time t.

Then by the balance of momentum

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = \mathbf{F}_{\partial W_t} + \int_{W_t}\rho\mathbf{b}dV, \tag{53}$$

where $\mathbf{F}_{\partial W_t}$ is the force on ∂W_t due to perssure, i.e.

$$\mathbf{F}_{\partial W_t} = -\int_{\partial W_t} p\mathbf{n}dA = -\int_{W_t}\nabla p dV \tag{54}$$

so that

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = -\int_{W_t}\nabla p dV + \int_{W_t}\rho\mathbf{b}dV, \quad \text{BM3} \tag{55}$$

Recall

$$\frac{d}{dt}(\delta V) = (\operatorname{div}\mathbf{u})\delta V \tag{56}$$

for an infitesmal volume δV of fluid moving in te fluid with the fluid velocity, We will now given a rigorous proof of the result.

Now that

$$\begin{aligned} \text{volume}(W_t) &= \int_{W_t} 1dV \\ &= \int_{W_t} 1dy, \quad \text{put } y \text{ to be } \varphi_t(\mathbf{x}) \\ &= \int_W J(\mathbf{x},t)d\mathbf{x}, \end{aligned} \tag{57}$$

where $J(\mathbf{x},t)$ is the Jocobian determinant of the map φ_t , so that

$$\frac{d}{dt}\text{volume}(W_t) = \int_W \frac{\partial}{\partial t}J(\mathbf{x},t)d\mathbf{x} \tag{58}$$

1.3 Equivalence between BM1, BM2 and BM3

Quick Summary of *Balance of Momentum*

1. BM1: $\rho\dfrac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}$
2. BM2: $\dfrac{d}{dt}\displaystyle\int_W \rho\mathbf{u}dV = \int_{\partial W}(p\mathbf{n} + \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u})\,dA + \int_W \rho\mathbf{b}dV$
3. BM3: $\dfrac{d}{dt}\displaystyle\int_{W_t} \rho\mathbf{u}dV = -\int_{W_t}\nabla pdV + \int_{W_t}\rho\mathbf{b}dV$

1.3.1 Lemma

$$\frac{\partial J(\mathbf{x},t)}{\partial t} = \operatorname{div}(\mathbf{u}(y,t))\cdot J(\mathbf{x},t) \tag{59}$$

1.3.1.1 proof:

$\mathbf{y} = \varphi(\mathbf{x},t) = (y_1,y_2,y_3)$, and $\mathbf{x} = (x_1,x_2,x_3)$. Observe that

$$\frac{\partial\varphi}{\partial t} = \mathbf{u}(\varphi(\mathbf{x},t),t), \quad \text{or} \quad \frac{\partial y_i}{\partial t} = u_i(y,t)\,, \forall i = 1,2,3 \tag{60}$$

then

$$\begin{aligned} J(\mathbf{x},t) &= \operatorname{div}\left(\frac{\partial y_i}{\partial x_j}\right)_{1\le i\le 3} \\ &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_1}{\partial x_{\sigma(1)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \end{aligned} \tag{61}$$

here S is the family of the permutations $\{1,2,3\}$, and

$$\operatorname{sign}\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation;} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases} \tag{62}$$

then

$$\begin{aligned} \frac{\partial J(\mathbf{x},t)}{\partial t} &= \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial^2 y_2}{\partial t \partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial^2 y_3}{\partial t \partial x_{\sigma(3)}} \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{63}$$

then calculate I_1 first

$$\begin{aligned} I_1 &= \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S} (\operatorname{sign}\sigma) \left(\frac{\partial}{\partial x_{\sigma(2)}} \left(\frac{\partial y_1}{\partial t} \right) \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S} (\operatorname{sign}\sigma) \left(\frac{\partial u_1}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S} (\operatorname{sign}\sigma) \left(\sum_{k=1}^{3} \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S} \sum_{k=1}^{3} (\operatorname{sign}\sigma) \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{k=1}^{3} \frac{\partial u_1}{\partial y_k} \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \frac{\partial u_1}{\partial y_1} \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \frac{\partial u_1}{\partial y_2} \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \frac{\partial u_1}{\partial y_3} \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_3}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \frac{\partial u_1}{\partial y_1} \sum_{\sigma\in S} (\operatorname{sign}\sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} + 0 + 0 \\ &= \frac{\partial u_1}{\partial y_1} J(\mathbf{x},t) \end{aligned} \tag{64}$$

using the same we can get

$$\begin{aligned}\frac{\partial J(\mathbf{x},t)}{\partial t} &= I_1 + I_2 + I_3 \\ &= \frac{\partial u_1}{\partial y_1} J(\mathbf{x},t) + \frac{\partial u_2}{\partial y_2} J(\mathbf{x},t) + \frac{\partial u_3}{\partial y_3} J(\mathbf{x},t) \\ &= \left(\sum_{i=1}^{3} \frac{\partial u_i}{\partial y_i}\right) J(\mathbf{x},t) \\ &= \operatorname{div}_y(\mathbf{u})\, J(\mathbf{x},t)\end{aligned} \tag{65}$$

1.3.2 Transport Theorem

For any smooth function $f : D \times [0,T] \to \mathbb{R}$, we have

$$\frac{d}{dt}\int_{W_t} \rho f \, dV_y = \int_{W_t} \rho \frac{Df}{dt} \, dV_y \tag{66}$$

1.3.2.1 proof

Change $W_t \to W$

$$\int_{W_t} \rho f dV_y = \int_W \rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\, dx \tag{67}$$

we have

$$\begin{aligned}\frac{d}{dt}\int_{W_t} \rho f dV &= \frac{d}{dt}\int_W \rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\, dx \\ &= \int_W \frac{d}{dt}\Big(\rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\Big)\, dx \\ &= \int_W \frac{d}{dt}\Big(\rho(y,t) f(y,t) J(\mathbf{x},t)\Big)\, dx \\ &= \int_W \frac{d}{dt}\Big(\rho(y,t) f(y,t)\Big) J(\mathbf{x},t)\, dx + \int_W \rho(y,t) f(y,t) \frac{d}{dt}\Big(J(\mathbf{x},t)\Big)\, dx \\ &= \int_W \frac{d}{dt}(\rho f)\, J(\mathbf{x},t)\, dx + \int_W (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x},t)\, dx \\ &= \int_W \left(\frac{d}{dt}(\rho f) + (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x},t)\right) dx \\ &= \int_W \left(\frac{\partial}{\partial t}(\rho f) + \nabla_y(\rho f) \cdot \left(\frac{dy}{dt}\right) + (\rho f) \operatorname{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\ &= \int_W \left(\frac{\partial}{\partial t}(\rho f) + \nabla_y(\rho f) \cdot \mathbf{u} + (\rho f) \operatorname{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\ &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\ &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u})\right) dV_y\end{aligned} \tag{68}$$

and consider the first term in the integral

$$\begin{aligned}
\frac{D(\rho f)}{Dt} &= \frac{\partial(\rho f)}{\partial t} + \mathbf{u}\cdot\nabla_y(\rho f) \\
&= f\frac{\partial \rho}{\partial t} + \rho\frac{\partial f}{\partial t} + \mathbf{u}\cdot(f\,\nabla_y\rho + \rho\,\nabla_y f) \\
&= f\left(\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla_y\rho\right) + \rho\left(\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla_y f\right) \\
&= f\Big(\rho_t + \mathbf{u}\cdot\nabla_y\rho\Big) + \rho\Big(f_t + \mathbf{u}\cdot\nabla_y f\Big) \\
&= f\Big(\rho_t + \operatorname{div}(\rho\mathbf{u}) - \rho\operatorname{div}(\mathbf{u})\Big) + \rho\Big(f_t + \mathbf{u}\cdot\nabla_y f\Big) \\
&= f\underbrace{\Big(\rho_t + \operatorname{div}(\rho\mathbf{u})\Big)}_{\text{continuity equation=0}} - f\rho\operatorname{div}(\mathbf{u}) + \rho\Big(f_t + \mathbf{u}\cdot\nabla_y f\Big) \\
&= -f\rho\operatorname{div}(\mathbf{u}) + \rho\Big(f_t + \mathbf{u}\cdot\nabla_y f\Big) \\
&= \rho\Big(f_t + \mathbf{u}\cdot\nabla_y f\Big) - \rho f\operatorname{div}(\mathbf{u}) \\
&= \rho\frac{Df}{Dt} - (\rho f)\operatorname{div}(\mathbf{u})
\end{aligned} \tag{69}$$

then plugin to the integral

$$\begin{aligned}
\frac{d}{dt}\int_{W_t}\rho f dV &= \int_W\left(\frac{D(\rho f)}{Dt} + (\rho f)\operatorname{div}_y(\mathbf{u})\right)dV \\
&= \int_W\left(\rho\frac{Df}{Dt} - (\rho f)\operatorname{div}(\mathbf{u}) + (\rho f)\operatorname{div}_y(\mathbf{u})\right)dV \\
&= \int_W\left(\rho\frac{Df}{Dt}\right)dV
\end{aligned} \tag{70}$$

we get

$$\frac{d}{dt}\int_{W_t}\rho f dV = \int_W\left(\rho\frac{Df}{Dt}\right)dV, \quad \forall f \text{ is smooth function} \tag{71}$$

Notice that:

$$\frac{d}{dt}\int_{W_t} \rho f dV = \int_W \left(\rho \frac{Df}{Dt}\right) dV \tag{72}$$

so when we consider a vector fucntion $\mathbf{u}$, we can write

$$\begin{aligned}
\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV &= \frac{d}{dt}\int_{W_t} \rho(u_1, u_2, u_3) dV \\
&= \left(\frac{d}{dt}\int_{W_t} \rho u_1 dV, \frac{d}{dt}\int_{W_t} \rho u_2 dV, \frac{d}{dt}\int_{W_t} \rho u_3 dV\right) \\
&= \left(\int_W \left(\rho\frac{Du_1}{Dt}\right) dV, \int_W \left(\rho\frac{Du_2}{Dt}\right) dV, \int_W \left(\rho\frac{Du_3}{Dt}\right) dV\right) \\
&= \int_W \left(\rho\frac{Du_1}{Dt}, \rho\frac{Du_2}{Dt}\rho\frac{Du_3}{Dt}\right) dV \\
&= \int_W \rho\frac{D}{Dt}(u_1, u_2, u_3)\ dV \\
&= \int_W \rho\frac{D\mathbf{u}}{Dt} dV
\end{aligned} \tag{73}$$

Now we rewrite BM3

$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV = -\int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV \tag{74}$$

to be

$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV = 0 \tag{75}$$

then using the result from above

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\
0 &= \int_{W_t} \rho\frac{D\mathbf{u}}{Dt} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\
0 &= \int_{W_t} \left(\rho\frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b}\right) dV, \quad \forall W_t
\end{aligned} \tag{76}$$

imply

$$\rho\frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b} = 0 \tag{77}$$

which is just BM1, this means BM3 is equivalent to BM1.

Similarly,

$$\frac{d}{dt}\int_W \rho \mathbf{u} dV = \int_{\partial W_t} p\mathbf{n} + \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u} dA + \int_W \rho \mathbf{b} dV \tag{78}$$

so that

$$\text{BM1} \Leftrightarrow \text{BM2} \Leftrightarrow \text{BM3} \tag{79}$$

By an arrgument similart ot the proof of the transport theorem, for any smooth function $f : D \times [0, T] \to \mathbb{R}$, we have

$$\frac{d}{dt}\left(\int_{W_t} f dV\right) = \int_{W_t} \left(\frac{Df}{Dt} + f(\operatorname{div}\mathbf{u})\right) dV \tag{80}$$

or in the other form as the textbook

$$\frac{d}{dt}\int_{W_t} f dV = \int_{W_t} \left(\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u})\right) dV, \tag{81}$$

since $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f$, and these 2 equations (80) and (81) is called the *Transport theorem without mass density.*

Rmk: Here we consider the *Transport theorem* (with mass density):

$$\frac{d}{dt}\int_{W_t} \rho f dV = \int_{W_t} \rho \frac{Df}{Dt} dV, \tag{82}$$

if we take ρ to be a constant (ex: $\rho = 1$), we have

$$\frac{d}{dt}\int_{W_t} f dV = \int_{W_t} \frac{Df}{Dt} dV, \tag{83}$$

however compare to the equation without no mass density (80),

$$\frac{d}{dt}\left(\int_{W_t} f dV\right) = \int_{W_t} \frac{Df}{Dt} dV + \int_{W_t} f(\mathrm{div}\,\mathbf{u}) dV \tag{84}$$

we have an extra term contain $f(\mathrm{div}\,\mathbf{u})$. **BUT**, if we consider more carefully, the mass density ρ here must satisfy the continuity equation, i.e.

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \tag{85}$$

so that if mass density is a constant (ex: $\rho = 1$), we have

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 + \mathrm{div}(\mathbf{u}) = 0 \tag{86}$$

so that $\mathrm{div}\,\mathbf{u} = 0$, which means the extra term vanishing, and the *transport theorem with mass density* and *transport theorem without mass density* are equivalent.

1.3.3 Def:

And ideal fluid is one in whuch there are no shear stresses.Hence Euler's equation

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{b} \tag{87}$$

holds for ideal fluid.

1.4 Incompressible

We call a flow incompressible if for any subregion $W \subset D$

$$\mathrm{volumn}(W_t) = \int_{W_t} dV = \mathrm{constant}. \tag{88}$$

int time t, $W_t = \varphi_t(W)$, where φ_t is flow map.

Then a flow is incompressible if and only if

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{W_t} dV_y \\ &= \frac{d}{dt}\int_{W} J(\mathbf{x},t)dV_x \\ &= \int_{W} \frac{\partial}{\partial t} J(\mathbf{x},t)dV_x \\ &= \int_{W} (\operatorname{div}\mathbf{u}) J(\mathbf{x},t)dV_x \\ &= \int_{W} (\operatorname{div}\mathbf{u}) dV_y \end{aligned} \tag{89}$$

imply

$$\operatorname{div}\mathbf{u} = 0 \Leftrightarrow \frac{\partial J}{\partial t} = (\operatorname{div}\mathbf{u})J = 0 \tag{90}$$

so that $J(\mathbf{x},t)$ is a constant, notice that $J(\mathbf{x},0) = 0$, we have

$$J(\mathbf{x},t) = 1, \quad \forall x \in D, t > 0 \tag{91}$$

Rmk: Since

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0 \Rightarrow \frac{D\rho}{Dt} = \rho_t + \mathbf{u}\cdot\nabla\rho = -\rho\operatorname{div}\mathbf{u} = 0 \tag{92}$$

so that $D\rho/Dt = 0$. Hence, the mass density is constant following the fluid for incompressible fluid.

1.5 Homogeneous

1.5.0.1 Def:

A fluid is said to be homogeneous, if $\rho(\mathbf{x},t) = \rho(t)$,$\forall x \in D$

Rmk: For incompressible homogeneous fluid,

$$\rho(\mathbf{x},t) = \rho(t) \quad \text{and} \quad \frac{D\rho}{Dt} = 0 \tag{93}$$

so that

$$\begin{aligned} 0 = \frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) \\ &= \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\operatorname{div}\mathbf{u} \\ &= \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\mathbf{0} + \rho\cdot 0 = \rho_t \end{aligned} \tag{94}$$

that is $\rho_t = 0$. Then $\rho(t) = \text{constant} = \rho(0)$, $\forall t > 0$

Rmk: For any subregion $W \subset D$, Let $W_t = \varphi_t(W)$, where φ_t is the flow map.If the fluid is incompressible,

$$\frac{d}{dt}\int_{W_t} \rho dV = \int_{W_t} \frac{D\rho}{Dt} dV = 0, \tag{95}$$

then

$$\int_{W_t} \rho(\mathbf{x}, t) dV_y = \int_W \rho(\varphi, t) J(\mathbf{x}, t) dV_x = \int_W \rho(\mathbf{x}, 0) dV_x \tag{96}$$

Now, we consider

$$0 = \frac{1}{\text{Volumn}(W)} \int_W \Big(\rho\left(\varphi(\mathbf{x}, t), t\right) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0)\Big) dV_x, \quad \forall W \subset D \tag{97}$$

Let $W = B(\mathbf{x}, r)$ and letting $r \to 0$, we have

$$\lim_{r\to 0} \frac{1}{\text{Volumn}(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} \Big(\rho\left(\varphi(\mathbf{x}, t), t\right) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0)\Big) dV_x. \tag{98}$$

Yield that

$$\rho\left(\varphi(\mathbf{x}, t), t\right) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0) = 0. \tag{99}$$

Hence

$$\rho\left(\varphi(\mathbf{x}, t), t\right) = \rho(\mathbf{x}, 0), \quad \forall \mathbf{x} \in D, t > 0 \tag{100}$$

if the fluid is incompressible.

Rmk:

- Incompressible: $\rho\left(\varphi(\mathbf{x}, t), t\right) = \rho(\mathbf{x}, 0), \quad \forall \mathbf{x} \in D, t > 0$
- Homogeneous: $\rho(\mathbf{x}, t) = \rho(t), \quad \forall x \in D$

e.g. For $\varphi(\mathbf{x}, t) = \varphi\left((x_1, x_2, x_3), t\right) = ((1+t)x_1, x_2, x_3)$, so the Jocobian

$$J(\mathbf{x}, t) = \begin{vmatrix} 1+t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 + t \tag{101}$$

We can choose $\rho(\mathbf{x}, t) = \dfrac{1}{1+t}$, then the fluid is compressible but homogeneous.

1.5.0.2 Example

Consider an imcompressible homogeneous fluid in a regionthen the density ρ is a constant.

Then $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p/\rho = 0$ (Euler's equation) in D, since it is incompressible, $\operatorname{div} \mathbf{u} = 0$ in D. That is

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{\nabla p}{\rho} = 0 \\ \operatorname{div} \mathbf{u} = 0 \end{cases}, \quad \text{in } D. \tag{102}$$

Taking the derevitive of Euler's equation

$$\begin{aligned}
\operatorname{div}\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\nabla p}{\rho}\right) &= 0 \\
\operatorname{div}(\mathbf{u}_t) + \operatorname{div}\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \operatorname{div}\left(\frac{\nabla p}{\rho}\right) &= 0 \\
\frac{\partial}{\partial t}\underbrace{\operatorname{div}(\mathbf{u})}_{=0} + \operatorname{div}\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \operatorname{div}\left(\frac{\nabla p}{\rho}\right) &= 0 \\
\operatorname{div}\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \operatorname{div}\left(\frac{\nabla p}{\rho}\right) &= 0
\end{aligned} \tag{103}$$

Now, we first calculate

$$\begin{aligned}
\operatorname{div}\left((\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) &= \sum_i \frac{\partial}{\partial x_i}\left(\mathbf{u}\cdot\nabla\right) u_i \\
&= \sum_i \frac{\partial}{\partial x_i}\left(\sum_j u_j \frac{\partial}{\partial x_j}\right) u_i \\
&= \sum_i \sum_j \frac{\partial}{\partial x_i}\left(\left(u_j \frac{\partial}{\partial x_j}\right) u_i\right) \\
&= \sum_i \sum_j \left(\frac{\partial}{\partial x_i}\left(u_j \frac{\partial}{\partial x_j}\right)\right) u_i + \left(u_j \frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial x_i} u_i\right) \\
&= \sum_i \sum_j \left(\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial}{\partial x_j}\right)\right) u_i + \left(u_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right) u_i + \left(u_j \frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial x_i} u_i\right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + 2\left(u_j \frac{\partial}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_i}\right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + 2\sum_i \sum_j \left(u_j \frac{\partial}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_i}\right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + 2\sum_j \left(u_j \frac{\partial}{\partial x_j}\right)\operatorname{div}\mathbf{u} \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + 2\left(\mathbf{u}\cdot\nabla\right)\operatorname{div}\mathbf{u}
\end{aligned} \tag{104}$$

notice $\operatorname{div}\mathbf{u} = 0$, we then have

$$\operatorname{div}\left((\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) + \frac{\nabla p}{\rho} = \sum_{i,j}\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\nabla p}{\rho} = 0 \tag{105}$$

So the Euler's equation becomes

$$\nabla p = -\rho \sum_{i,j}\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right). \tag{106}$$

Since the fluid is confined in fixed region of space by D, therefore the fluid cannot move into $\mathbb{R}^3 \setminus D$ (out side), so the normal component of the fluid satisfied

$$\mathbf{u}\cdot\mathbf{n}\Big|_{\partial D} = 0 \quad \Rightarrow \quad \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{n}\Big|_{\partial D} = 0, \tag{107}$$

then by Euler's equation, we have

$$\frac{\partial p}{\partial \mathbf{n}}\Big|_{\partial D} = \nabla p\cdot\mathbf{n} = -\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{n}\Big|_{\partial D}. \tag{108}$$

For the flat boundary surfaces, e.g. the wall of fixed rectangular box, $\mathbf{n} = (n_1, n_2, n_3)$ and

$$\begin{aligned}\mathbf{n}\cdot((\mathbf{u}\cdot\nabla)\,\mathbf{u}) &= \sum_{i,j} n_i\left(u_i\frac{\partial}{\partial x_j}u_i\right)\\ &= \sum_{i,j} u_j\frac{\partial}{\partial x_j}(u_i n_i)\\ &= \sum_{j} u_j\frac{\partial}{\partial x_j}(\mathbf{u}\cdot\mathbf{n}) = 0, \quad \text{Since } \mathbf{u}\cdot\mathbf{n}\Big|_{\partial D} = 0\end{aligned} \tag{109}$$

on each flat surface. Since

$$\frac{\partial p}{\partial n}\Big|_{\partial D} = 0 \tag{110}$$

on each flat surface. Then one can solve

$$\begin{cases}\nabla p = -\rho\displaystyle\sum_{i,j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j},\\ \dfrac{\partial p}{\partial \mathbf{n}}\Big|_{\partial D} = 0, \quad \text{(If the surface is flat.)}\end{cases} \tag{111}$$

using standard PDE method.

Rmk: If the surface are not flat, the equations become

$$\begin{cases}\nabla p = -\rho\displaystyle\sum_{i,j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j},\\ \dfrac{\partial p}{\partial \mathbf{n}}\Big|_{\partial D} = -\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\cdot\mathbf{n}\Big|_{\partial D}\end{cases} \tag{112}$$

1.6 Consevation of Energy

For fluid moving in a domain D with velocity $\mathbf{u}$, the kinetic energy of the fluid in W is

$$E_{\text{kinetic}} = \frac{1}{2}\int_W \rho\cdot\|\mathbf{u}\|^2 dV, \tag{113}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $\|\mathbf{u}\|^2 = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

Rmk: When $W = D$ and the fluid is incompressible with no external force acting on fluid, we have

$$\begin{aligned}
\frac{d}{dt}E_{\text{kinetic}} &= \frac{1}{2}\frac{d}{dt}\int_W \rho \cdot \|\mathbf{u}\|^2 dV \\
&= \frac{1}{2}\int_D \frac{\partial}{\partial t}\left(\rho\|\mathbf{u}\|^2\right) dV \\
&= \frac{1}{2}\int_D \rho_t\|\mathbf{u}\|^2 + \rho\frac{\partial}{\partial t}\|\mathbf{u}\|^2 dV \\
&= \frac{1}{2}\int_D \Big(- \operatorname{div}(\rho\mathbf{u})\|\mathbf{u}\|^2 + 2\rho\mathbf{u}\cdot\mathbf{u}_t \Big) dV \\
&= \frac{1}{2}\int_D \Big(- \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) + (\rho\mathbf{u})\nabla\|\mathbf{u}\|^2 + 2\rho\mathbf{u}\cdot\mathbf{u}_t \Big) dV
\end{aligned} \tag{114}$$

also notice that the euler equation $\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p/\rho = 0$, pluin

$$\begin{aligned}
\frac{d}{dt}E_{\text{kinetic}} &= \frac{1}{2}\int_D \Big(- \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) + (\rho\mathbf{u})\cdot\nabla\|\mathbf{u}\|^2 + 2\rho\mathbf{u}\cdot\mathbf{u}_t \Big) dV \\
&= \frac{1}{2}\int_D \left(- \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) + (\rho\mathbf{u})\cdot 2\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right) + 2\rho\mathbf{u}\cdot\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\nabla p}{\rho}\right)\right) dV \\
&= \frac{1}{2}\int_D \left(- \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) + 2\rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} - 2\rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} - 2\rho\mathbf{u}\cdot\frac{\nabla p}{\rho}\right) dV \\
&= \frac{1}{2}\int_D \left(- \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) - 2\rho\mathbf{u}\cdot\frac{\nabla p}{\rho}\right) dV \\
&= -\frac{1}{2}\int_D \operatorname{div}\left(\rho\mathbf{u}\|\mathbf{u}\|^2\right) dV - \int_D (\mathbf{u}\cdot\nabla p)\, dV \\
&= -\frac{1}{2}\int_{\partial D} \rho\|\mathbf{u}\|^2\mathbf{u}\cdot\mathbf{n}d\sigma - \int_D (\mathbf{u}\cdot\nabla p)\, dV
\end{aligned} \tag{115}$$

Since, $\mathbf{u}\cdot\mathbf{n} = 0$,

Rmk: When E_{kinetic}

$$\begin{aligned}
\frac{\partial}{\partial t}E_{\text{kinetic}} &= \cdots \\
&= \int_{W_t} \rho\mathbf{u}\cdot\left(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\cdot\mathbf{u}\right) dV \\
&= \int_{W_t} \rho\mathbf{u}\cdot\left(-\frac{\nabla p}{\rho} + \mathbf{b}\right) dV, \quad \text{By BM1} \\
&= -\int_{W_t} \left(\mathbf{u}\cdot\nabla\cdot p - \rho\mathbf{u}\cdot\mathbf{b}\right) dV,
\end{aligned} \tag{116}$$

On the other hand, if we alsp assume that all energy is kinetic, then the rate of change of kinetic energy of fluid os equalto the rate of work done on the fluid in W_t by the pressure and the body force, i.e.

$$\begin{aligned}\frac{\partial}{\partial t}E_{\text{kinetic}}(W_t) &= \int_{\partial W_t} p\mathbf{u}\cdot\mathbf{n}d\sigma + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}dV \\ &= -\int_{W_t}\operatorname{div}(\rho\mathbf{u})dV + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}dV \\ &= -\int_{W_t}(\mathbf{u}\cdot\nabla p + p\,(\operatorname{div}\mathbf{u})) + \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}dV \\ &= -\int_{W_t}(\mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\mathbf{b}) + \int_{W_t} p\,(\operatorname{div}\mathbf{u})dV,\end{aligned} \tag{117}$$

then by equations (116) and (117), we have

$$\int_{W_t} p\,(\operatorname{div}\mathbf{u})dV, \quad \forall W\subset D. \tag{118}$$

Let $W = \varphi_t^{-1}(B(x_0,r))$, where $B(x_0,r)\subset D$, then $B(x_0,r)=\varphi_t(W)=W_t$, we get

$$\frac{1}{B(x_0,r)}\int_{B(x_0,r)} p\ \operatorname{div}\mathbf{u}\,dV = 0, \quad \forall x_0\in D, r>0, B(x_0,r)\subset D. \tag{119}$$

Letting $r\to 0$, $p\operatorname{div}\mathbf{u}=0$ in D, we get $\operatorname{div}\mathbf{u}$ if $p\neq 0$ in D. So that the fluid must be incrompressible.

Summary In summary, the Euler equation are

$$\begin{cases}\rho\dfrac{D\mathbf{u}}{Dt} & = -\nabla p + \rho\mathbf{b} \\ \dfrac{D\rho}{Dt} = 0 & \\ \operatorname{div}\mathbf{u} = 0 & \end{cases}, \quad \text{in } D, \tag{120}$$

where $\mathbf{u}\cdot\mathbf{n}\Big|_{\partial D} = 0$

1.7 Isentropic fluid

A flow is said to be isentropic, if there exists function W (called enthalpy), s.t.

$$\nabla W = \frac{\nabla p}{\rho}. \tag{121}$$

Recall that in theremodynamics, one has the following basic quantitties which are the function of $\mathbf{x}$ and t:

- p : pressure
- ρ : density
- T : temperature
- s : entropy

- w : enthalpy (per unit mass)
- $\epsilon = W - p/\rho$: inertial energy (per unit mass)

These quantities are related by the first law of Thermodynamics, which we accept as a basic principle:

1. **TD1** :

$$dw = Tds + \frac{dp}{\rho} \tag{122}$$

1. **TD2** :

$$d\epsilon = dw - \frac{dp}{\rho} + \frac{p}{\rho^2}d\rho = Tds + \frac{p}{\rho^2}d\rho \tag{123}$$

If the pressure p is a function of ρ only, and s is constant, then

$$dw = Tds + \frac{dp}{\rho} = \frac{dp/d\rho}{\rho}d\rho \quad \Rightarrow \quad w = \int^{\rho} \frac{p'(\bar{\rho})}{\bar{\rho}}d\bar{\rho}. \tag{124}$$

Then the divergence

$$\nabla w = \sum_{i=1}^{3}\left(\frac{\partial W}{\partial \rho}\frac{\partial \rho}{\partial x_i}\right) = \sum_{i=1}^{3}\left(\frac{p'(\rho)}{\rho}\frac{\partial \rho}{\partial x_i}\right) = \frac{\nabla p}{\rho} \tag{125}$$

and hence, the fluid is isentropic.

In this case, the eternal energy $\epsilon = w - \dfrac{p}{\rho}$ satisfies

$$\begin{aligned} d\epsilon &= dw - \frac{dp}{\rho} + \frac{p}{\rho^2}d\rho \\ &= \frac{p'(\rho)}{\rho}d\rho - \frac{dp}{\rho} + \frac{p}{\rho^2}d\rho \\ &= \frac{p}{\rho^2}d\rho, \end{aligned} \tag{126}$$

Then

$$\frac{\partial \epsilon}{\partial \rho} = \frac{p}{\rho^2} \quad \Rightarrow \quad \epsilon = \int_{\rho_0}^{\rho} \frac{p}{\bar{\rho}^2}d\bar{\rho}. \tag{127}$$

Note that, by the transport thm. if $E = \int_{W_t} \left(\frac{1}{2}\|\mathbf{u}\|^2 + \rho\epsilon \right) dV$, then

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \frac{d}{dt}\int_{W_t} \left(\frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho\epsilon \right) dV \\
&= \int_{W_t} \left(\frac{\rho}{2}\frac{D}{Dt}\|\mathbf{u}\|^2 + \rho\frac{D\epsilon}{Dt} \right) dV \\
&= \int_{W_t} \rho\mathbf{u}\cdot(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u})\, dV + \int_{W_t} \rho\frac{D\epsilon}{Dt} dV \\
&= \int_{W_t} \rho\mathbf{u}\cdot\left(-\frac{\nabla p}{\rho} + \mathbf{b} \right) dV + \int_{W_t} \rho\frac{D\epsilon}{Dt} dV \\
&= \int_{W_t} \big(-\operatorname{div}(p\mathbf{u}) + p\mathbf{u}\cdot\mathbf{b} \big) dV + \int_{W_t} (\operatorname{div}\mathbf{u}) p dV + \int_{W_t} \rho\frac{D\epsilon}{Dt} dV \\
&= -\int_{W_t} p\mathbf{u}\cdot\mathbf{n} d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b} dV + \int_{W_t} (\operatorname{div}\mathbf{u}) p dV + \int_{W_t} \rho\frac{D\epsilon}{Dt} dV
\end{aligned} \tag{128}$$

Rmk: For isentropic flow with p, a function of ρ, where

$$\epsilon = \int^{\rho} \frac{p(\bar{\rho})}{\bar{\rho}} d\rho, \tag{129}$$

and then

$$\epsilon_t = \frac{p(\rho)}{\rho^2}\rho_t, \quad \text{and} \nabla\epsilon = \frac{p(\rho)}{\rho^2}\nabla\rho \tag{130}$$

Rmk:

$$\begin{aligned}
\frac{D\epsilon}{Dt} &= \epsilon_t + \mathbf{u}\cdot\nabla\epsilon \\
&= \frac{p(\rho)}{\rho^2}(\rho_t + \mathbf{u}\cdot\nabla\rho) \\
&= \frac{p(\rho)}{\rho^2}\,(\rho_t + \operatorname{div}(\rho\mathbf{u}) - (\operatorname{div}\mathbf{u})p) \\
&= -\frac{p(\rho)}{\rho^2}\,(\operatorname{div}\mathbf{u})
\end{aligned} \tag{131}$$

Last, time we solving that

$$\frac{\partial E}{\partial t} = -\int_{W_t} p\mathbf{u}\cdot\mathbf{n} d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b} dV + \int_{W_t} (\operatorname{div}\mathbf{u}) p dV + \int_{W_t} \rho\frac{D\epsilon}{Dt} dV \tag{132}$$

where $\frac{D\epsilon}{Dt} = -\frac{p(\rho)}{\rho}\,(\operatorname{div}\mathbf{u})$, and pressure a the function of ρ, i.e. $p = p(\rho)$. Plugin we have

$$
\begin{aligned}
\frac{\partial E}{\partial t} &= -\int_{W_t} p\mathbf{u}\cdot\mathbf{n}d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}dV + \int_{W_t} (\operatorname{div}\mathbf{u})pdV + \int_{W_t} \rho\frac{D\epsilon}{Dt}dV \\
&= -\int_{W_t} p\mathbf{u}\cdot\mathbf{n}d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}dV + \int_{W_t} (\operatorname{div}\mathbf{u})pdV - \int_{W_t} \rho\frac{p}{\rho}(\operatorname{div}\mathbf{u})\,dV \\
&= -\int_{W_t} p\mathbf{u}\cdot\mathbf{n}d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}dV + \int_{W_t} (\operatorname{div}\mathbf{u})p - (\operatorname{div}\mathbf{u})\,pdV \\
&= -\int_{W_t} p\mathbf{u}\cdot\mathbf{n}d\sigma + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}dV, \quad \text{(BE)}
\end{aligned}
\tag{133}
$$

Thus the rate of change of energy $E = \displaystyle\int_{W_t}\left(\frac{1}{2}\|\mathbf{u}\|^2 + \rho\epsilon\right)dV$ is equal to the rate which work is done on it.

Euler equation for isentropic flow are

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla w + b \\
\dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0
\end{cases}, \quad \text{in } D \text{ and } \mathbf{u}\cdot\mathbf{n}\Big|_{\partial D} = 0
\tag{134}
$$

> **Rmk:** Gases can often be viewed a sisentropic with $p = c_0\rho^\gamma$, where c_0 and γ are constant, then
>
> $$w = \int_0^\rho \frac{c_0\gamma s^{\gamma-1}}{s}ds = \frac{c_0\gamma\rho^{\gamma-1}}{\gamma-1} \tag{135}$$
>
> and
>
> $$\epsilon = w - \frac{p}{\rho} = \frac{c_0\gamma\rho^{\gamma-1}}{\gamma-1} - c_0\rho^{\gamma-1} = c_0\rho^{\gamma-1}\left(\frac{\gamma}{\gamma-1} - 1\right) = c_0\rho^{\gamma-1}\frac{1}{\gamma-1}. \tag{136}$$

Definition

GIven a fluid flow with velocity field $\mathbf{u}(\mathbf{x}, t)$, a streamline, at fixed time t is an integral. curve of $\mathbf{u}$ that satistice the equation

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{u}(\mathbf{x}(s), t), \quad \forall s > 0. \tag{137}$$

1.7.1 Definition

We define a trajectory to a curve trace out by a particle as time progresses. More precisely, a trajectory is a solution of the equation.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(\mathbf{x}(t), t), \quad \forall t > 0. \tag{138}$$

> **Rmk:** If $\mathbf{u}$ is independent of time, i.e. $\partial_t\mathbf{u} = 0$, streenline and trajectory coincide. That is the solution of
>
> $$\frac{d\mathbf{x}(s)}{ds} = \mathbf{u}(\mathbf{x}(s), t) \quad \text{and} \quad \frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(\mathbf{x}(t), t) \tag{139}$$
>
> are the same.

1.7.2 Lemma

$$\frac{1}{2}\nabla\mathbf{u}^2 = (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{140}$$

where $\mathbf{u} = (u_1, u_2, u_3)$.

1.7.2.1 proof

By use of the Levi-Civita symbols:

$$\varepsilon_{i,j,k} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is cyclic permutation,} \\ -1 & \text{if } (i,j,k) \text{ is anti-cyclic permutation,} \\ 0 & \text{others,} \end{cases} \tag{141}$$

and Kronecker delta:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases} \tag{142}$$

we first calculate

$$\begin{aligned} (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= \sum_{i=1}^{3} (\mathbf{u}\cdot\nabla)\, u_i \hat{e}_i \\ &= \sum_{i,j=1}^{3} u_j \partial_j u_i \hat{e}_i \end{aligned} \tag{143}$$

Then using the properties of Levi-Civita symbols $\varepsilon_{i,j,k}\varepsilon_{\ell,m,k} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$

$$\begin{aligned}
\mathbf{u} \times (\nabla \times \mathbf{u}) &= \sum_{i,j,k=1}^{3} \varepsilon_{i,j,k} u_i \left(\nabla \times \mathbf{u}\right)_j \hat{e}_k \\
&= \sum_{i,j,k,\ell,m=1}^{3} \varepsilon_{i,j,k} u_i \left(\varepsilon_{\ell,m,j} \partial_\ell u_m\right) \hat{e}_k \\
&= \sum_{i,j,k,\ell,m=1}^{3} \varepsilon_{i,j,k} \varepsilon_{\ell,m,j}\, u_i \partial_\ell u_m\, \hat{e}_k \\
&= - \sum_{i,j,k,\ell,m=1}^{3} \varepsilon_{i,k,j} \varepsilon_{\ell,m,j}\, u_i \partial_\ell u_m\, \hat{e}_k \\
&= - \sum_{i,k,\ell,m=1}^{3} \left(\delta_{i\ell}\delta_{km} - \delta_{im}\delta_{k\ell}\right)\, u_i \partial_\ell u_m\, \hat{e}_k \\
&= - \sum_{i,k,\ell,m=1}^{3} \left(\delta_{i\ell}\delta_{km} u_i \partial_\ell u_m\, \hat{e}_k - \delta_{im}\delta_{k\ell} u_i \partial_\ell u_m\, \hat{e}_k\right) \\
&= - \left(\sum_{i,k,\ell,m=1}^{3} \delta_{i\ell}\delta_{km}\, u_i \partial_\ell u_m\, \hat{e}_k - \sum_{i,k,\ell,m=1}^{3} \delta_{im}\delta_{k\ell}\, u_i \partial_\ell u_m\, \hat{e}_k \right) \\
&= - \left(\sum_{i,k=1}^{3} u_i \partial_i u_k\, \hat{e}_k - \sum_{i,k=1}^{3} u_i \partial_k u_i\, \hat{e}_k \right) \\
&= \sum_{i,k=1}^{3} u_i \partial_k u_i\, \hat{e}_k - \sum_{i,k=1}^{3} u_i \partial_i u_k\, \hat{e}_k \\
&= \sum_{i,j=1}^{3} u_i \partial_j u_i\, \hat{e}_j - \sum_{i,j=1}^{3} u_i \partial_i u_j\, \hat{e}_j
\end{aligned} \tag{144}$$

So that we have

$$\begin{aligned}
(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\mathbf{u}\times(\nabla\times\mathbf{u}) &= \sum_{i,j=1}^{3} u_j\partial_j u_i\hat{e}_i + \left(\sum_{i,j=1}^{3} u_i\partial_j u_i\,\hat{e}_j - \sum_{i,j=1}^{3} u_i\partial_i u_j\,\hat{e}_j\right)\\
&= \sum_{i,j=1}^{3} u_i\partial_i u_j\hat{e}_j + \left(-\sum_{i,j=1}^{3} u_i\partial_i u_j\,\hat{e}_j + \sum_{i,j=1}^{3} u_i\partial_j u_i\,\hat{e}_j\right)\\
&= \sum_{i,j=1}^{3} u_i\partial_j u_i\,\hat{e}_j\\
&= \frac{1}{2}\sum_{i,j=1}^{3} \partial_j u_i^2\,\hat{e}_j\\
&= \frac{1}{2}\sum_{j=1}^{3} \partial_j\hat{e}_j\left(\sum_{i=1}^{3} u_i^2\right)\\
&= \frac{1}{2}\sum_{j=1}^{3} \partial_j\hat{e}_j\|\mathbf{u}\|\\
&= \frac{1}{2}\nabla\|\mathbf{u}\|
\end{aligned} \tag{145}$$

1.7.3 Bernoullis' Theorem

For stationary. isentropic flows and in the absence of external forces,

$$\mathbf{u}\cdot\nabla\left(w+\frac{1}{2}\|\mathbf{u}\|^2\right)=0 \Leftrightarrow \frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)(\mathbf{x}(t))=0 \tag{146}$$

which means $\frac{1}{2}\|\mathbf{u}\|^2+w$ is constant along stream, where $x(t)$ satisfies $\frac{d}{dt}\mathbf{x}(t)=\mathbf{u}(\mathbf{x}(t))$ (Streamline of flow).

The same result holds if the force $\mathbf{b}$ is conservative, i.e. $\mathbf{b}=-\nabla\varphi$ for some function φ, w replaced by $w+\varphi$ in above statement (146).

Rmk: Note, if flow is stationary, homogeneous (i.e. $\rho=\rho_0$ is constant in D) and incompressible, the flow is isentropic with $w=p/\rho_0$ hence the statement (146) will holds.

1.7.3.1 Proof

By Lemma 1,

$$\frac{1}{2}\nabla\|\mathbf{u}\|^2=(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\mathbf{u}\times(\nabla\times\mathbf{u})\,. \tag{147}$$

Since the flow is steady, we have

$$\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\,\mathbf{u}=-\nabla w \tag{148}$$

so that

$$\nabla w+(\mathbf{u}\cdot\nabla)\,\mathbf{u}=-\mathbf{u}_t=0 \quad\Rightarrow\quad (\mathbf{u}\cdot\nabla)\,\mathbf{u}=-\nabla w. \tag{149}$$

Consider,

$$\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)=(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\mathbf{u}\times(\nabla\times\mathbf{u})+\nabla w \tag{150}$$

using the result of (149), we have

$$\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)=\mathbf{u}\times(\nabla\times\mathbf{u}) \tag{151}$$

so that

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)=\mathbf{u}\cdot(\mathbf{u}\times(\nabla\times\mathbf{u}))=0 \tag{152}$$

Taking the divergence,

$$\nabla w=-\operatorname{div}\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right),\quad \text{in } D. \tag{153}$$

By (149), $\left.\dfrac{\partial W}{\partial \mathbf{n}}\right|_{\partial D}=\nabla w\cdot\mathbf{n}=-(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{n}$, since w satisfies the elliptic PDE with boundary condition, it shows that w is a finction independent of the time t:

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)(\mathbf{x}(t))=\mathbf{u}\cdot\nabla\left(w+\frac{1}{2}\|\mathbf{u}\|^2\right)(\mathbf{x}(t))=0 \tag{154}$$

or

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)(\mathbf{x}(2))-\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)(\mathbf{x}(1))=\int_{t_1}^{t_2}\left(\frac{1}{2}\|\mathbf{u}\|^2+w\right)dt=0 \tag{155}$$

That is $\frac{1}{2}\|\mathbf{u}\|^2+w$ is a constant along stream line.

1.7.3.2 Example

Consider a fluid flow in a channel

FIg

Suppose the pressure is a function of $\mathbf{x}$ only and the pressure p_1 at $\mathbf{x}=0$ is greater than the pressure p_2. Then the fluid will flow from left to right with the velocity $\mathbf{u}(x,y,t)=(u_1(x,y),0,0)$ and the pressure $\mathbf{p}(x,y,t)=p(\mathbf{x})$. Suppose the density of fluid $\rho=\rho_0$ is a constant and the fluid is incompressible.

Fig

For the flow

$$\mathbf{u}(x,y,t)=(u_1(x,t),0,0) \tag{156}$$

and the equation

$$\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{\nabla p}{\rho} \tag{157}$$

becomes

$$u_{1,t}+u_1\frac{\partial u_1}{\partial x}=-\frac{p_x}{\rho_0}\quad\Rightarrow\quad u_{1,t}+u_1u_{1,x}=-\frac{p_x}{\rho_0} \tag{158}$$

with the incompressibility $\operatorname{div}\mathbf{u}=u_{1,x}=0$, we have

$$u_{1,t}=-\frac{p_x}{\rho_0}\quad\Rightarrow\quad p_x=-\rho_0u_{1,t} \tag{159}$$

Notice that

$$p_{xx} = -\rho_0 u_{1,t,x} = -\rho_0 \frac{\partial}{\partial t} u_{1,x} = 0 \tag{160}$$

so that the pressure p is given by

$$p_{x,x} = 0 \quad \Rightarrow \quad p_x = C_1 \quad \Rightarrow \quad p(x) = C_1 x + C_2, \tag{161}$$

where C_1, C_2 is constant.

Plugin the boundary condition

$$\begin{cases} p(x=0) = p(0) = p_1 = C_2 \\ p(x=L) = p(L) = p_2 = C_1 L + p_1 \end{cases}, \quad \Rightarrow \quad C_2 = \frac{p_2 - p_1}{L} \tag{162}$$

solving that the pressure

$$p(x) = p_1 + (p_2 - p_1)\frac{x}{L} \tag{163}$$

or

$$p(x) = \frac{(1-x)p_1 + xp_2}{L}. \tag{164}$$

Notice that $p_x = (p_2 - p_1)/L$, we can solve that

$$u_{1,t} = \frac{p_2 - p_1}{L\rho} \quad \Rightarrow \quad u_1(t) = \frac{p_2 - p_1}{L\rho} t + C_3, \tag{165}$$

for some constant C_3.

Rmk: We can observe that, when $t \to \infty$

$$\lim_{t\to\infty} u_1(t) = \infty \tag{166}$$

which is impossible in real flow. Thus the Euler equation is not a good model for this flow.

This is because we have ignored frictional force in the modelling. This situation will be remedied by the "*Navier-Stokes equation*", which take account for friction force later.

1.8 Rotation and Vorticity

Definition If the velocity filed of a fluid is $\mathbf{u} = (u_1, u_2, u_3)$. We define the vorticity of the fluid

$$\zeta = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \tag{167}$$

1.8.1 Proposition

Suppose the fluid is in the region D and $x \in D$. Let $\mathbf{y} = \mathbf{x} + \mathbf{h}$ be a nearby point, then

$$\mathbf{u}(\mathbf{y}, t) = \mathbf{u}(\mathbf{x}, t) + \mathbf{D}(\mathbf{x}, t) \cdot \mathbf{h} + \frac{1}{2}\xi(\mathbf{x}, t) \cdot \mathbf{h} + \mathcal{O}\left(\|\mathbf{h}\|^2\right), \tag{168}$$

where $\mathbf{D}(\mathbf{x}, t)$ is a symmetric 3×3 matrix, $\mathbf{h} = (h_1, h_2, h_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

1.8.1.1 proof

Note that

$$\nabla \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)_{1\le i,j\le 3} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{169}$$

and

$$\begin{aligned}
u_i(\mathbf{y},t) - u_i(\mathbf{x},t) &= u_i(\mathbf{x}+\mathbf{h},t) - u_i(\mathbf{x},t) \\
&= \int_0^1 \frac{\partial}{\partial s} u_i(\mathbf{x}+s\mathbf{h},t)ds, \quad \text{let } \mathbf{z} = \mathbf{x}+s\mathbf{h} \\
&= \int_0^1 \sum_{j=1}^3 \frac{\partial u_i(\mathbf{z},t)}{\partial z_j}\frac{\partial z_j}{\partial s} ds \\
&= \sum_{j=1}^3 \int_0^1 \frac{\partial}{\partial z_j} u_i(\mathbf{x}+s\mathbf{h},t) h_j\, ds \\
&= \sum_{j=1}^3 \int_0^1 \left(\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s\mathbf{h},t) + \frac{\partial}{\partial x_j}u_i(\mathbf{x},t) - \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)\right) h_j\, ds \\
&= \sum_{j=1}^3 \int_0^1 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)ds + \sum_{j=1}^3 \int_0^1 \left(\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s\mathbf{h},t) - \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)\right) h_j\, ds \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j=1}^3 \int_0^1 h_j \left(\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s\mathbf{h},t) - \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)\right) ds \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j=1}^3 \int_0^1 h_j \left(\int_0^1 \frac{\partial}{\partial s'}\left(\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s's\mathbf{h},t)\right) ds'\right) ds, \\
&\quad \text{let } \mathbf{w} = \mathbf{x}+s's\mathbf{h} \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j=1}^3 \int_0^1 h_j \left(\int_0^1 \sum_{m=1}^3 \frac{\partial}{\partial w_m}\left(\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s's\mathbf{h},t)\right)\frac{\partial w_m}{\partial s'} ds'\right) ds \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j,m=1}^3 \int_0^1 h_j \left(\int_0^1 \frac{\partial}{\partial w_m}\frac{\partial}{\partial z_j} u_i(\mathbf{x}+s's\mathbf{h},t)(sh_m)ds'\right) ds \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j,m=1}^3 h_j h_m \int_0^1\int_0^1 s\frac{\partial^2}{\partial w_m \partial z_j} u_i(\mathbf{x}+s's\mathbf{h},t)ds'\, ds \\
&= \sum_{j=1}^3 \frac{\partial}{\partial x_j}u_i(\mathbf{x},t)h_j + \sum_{j,m=1}^3 h_j h_m \int_0^1\int_0^1 s\frac{\partial^2}{\partial x_m \partial x_j} u_i(\mathbf{x}+s's\mathbf{h},t)ds'\, ds
\end{aligned} \tag{170}$$

Then we let

$$\begin{cases} E = \displaystyle\sum_{j,m=1}^3 h_j h_m \int_0^1\int_0^1 s\frac{\partial^2}{\partial x_m \partial x_j} u_i(\mathbf{x}+s's\mathbf{h},t)ds'\, ds \\ C_{j,m} = \displaystyle\int_0^1\int_0^1 s\frac{\partial^2}{\partial x_m \partial x_j} u_i(\mathbf{x}+s's\mathbf{h},t)ds'\, ds \end{cases} \tag{171}$$

consider

$$|E| \leq \sum_{j,m=1}^{3} C_{j,m}|h_j||h_m| \leq C'\|\mathbf{h}\|^2 \tag{172}$$

we have

$$u_i(\mathbf{y},t) = u_i(\mathbf{x},t) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} u_i(\mathbf{x},t) h_j + \mathcal{O}(\|\mathbf{h}\|^2) \tag{173}$$

that is

$$\mathbf{u}(\mathbf{y},t) = \mathbf{u}(\mathbf{x},t) + \nabla\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h} + \mathcal{O}(\|\mathbf{h}\|^2) \tag{174}$$

where $\nabla\mathbf{u}(\mathbf{x},t)\cdot\mathbf{h}$ is a matrix multiplication with

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \tag{175}$$

Let

$$\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \quad \text{and} \quad \mathbf{S} = \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right) \tag{176}$$

then $\nabla\mathbf{u} = \mathbf{S} + \mathbf{D}$, and

$$\begin{aligned} \mathbf{D}^T &= \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)^T = \frac{1}{2}\left((\nabla\mathbf{u})^T + \nabla\mathbf{u}\right) &&= \mathbf{D} \\ \mathbf{S}^T &= \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right) = \frac{1}{2}\left((\nabla\mathbf{u})^T - \nabla\mathbf{u}\right) &&= -\mathbf{S} \end{aligned} \tag{177}$$

So that, here $\mathbf{D}$ is a symmetric matrix $\mathbf{D}^T = \mathbf{D}$ and S is a anti-symmetric matrix $\mathbf{S} = -\mathbf{S}^T$.

By direct computation

$$\mathbf{S} = \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right) = \frac{1}{2}\begin{pmatrix} 0 & \xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \quad \text{where } \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \nabla\times\mathbf{u} \tag{178}$$

Then $\mathbf{S}\cdot\mathbf{h} = \frac{1}{2}\xi\cdot\mathbf{h}$, i.e.

$$\ldots \tag{179}$$

then substituting into (174) we have

$$\mathbf{u}(\mathbf{y},t) = \mathbf{u}(\mathbf{x},t) + \mathbf{D}(\mathbf{x},t)\cdot\mathbf{h} + \frac{1}{2}\xi(\mathbf{x},t)\cdot\mathbf{h} + \mathcal{O}\left(\|\mathbf{h}\|^2\right), \tag{180}$$

Rmk 1: Let $\Psi((\mathbf{x},t),\mathbf{h}) = \frac{1}{2}\big(\mathbf{D}(\mathbf{x},t)\mathbf{h}\big)\cdot\mathbf{h}$,or

$$\Psi((\mathbf{x},t),\mathbf{h}) = \frac{1}{2}\mathbf{h}^T\mathbf{D}(\mathbf{x},t)\mathbf{h} = \sum_{i,j=1}^{3}\frac{1}{2}h_i d_{i,j} h_j \tag{181}$$

be the quadratic form associate with $\mathbf{D}$, where $(\mathbf{D})_{i,j} = d_{i,j}$, then

$$\begin{aligned}
\frac{\partial\Psi}{\partial h_k} &= \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial}{\partial h_k}\left(h_i d_{i,j} h_j\right)\\
&= \frac{1}{2}\sum_{i,j=1}^{3}\left(\delta_{i,k} d_{i,j} h_j + h_i d_{i,j}\delta_{j,k}\right)\\
&= \frac{1}{2}\sum_{i,j=1}^{3}\delta_{i,k} d_{i,j} h_j + \frac{1}{2}\sum_{i,j=1}^{3} h_i d_{i,j}\delta_{j,k}\\
&= \frac{1}{2}\sum_{j=1}^{3} d_{k,j} h_j + \frac{1}{2}\sum_{i=1}^{3} h_i d_{i,k}\\
&= \frac{1}{2}\sum_{j=1}^{3} d_{k,j} h_j + \frac{1}{2}\sum_{j=1}^{3} h_j d_{j,k}\\
&= \frac{1}{2}\sum_{j=1}^{3}(d_{k,j} + d_{j,k}) h_j
\end{aligned} \tag{182}$$

since $\mathbf{D}$ is symmetric $(d_{k,j} = d_{j,k})$, we have

$$\frac{\partial\Psi}{\partial h_k} = \frac{1}{2}\sum_{j=1}^{3}(d_{k,j} + d_{j,k}) h_j = \frac{1}{2}\sum_{j=1}^{3} 2(d_{k,j} + d_{k,j}) h_j = \sum_{j=1}^{3} d_{k,j} h_j \tag{183}$$

in vector representation, we have

$$\nabla_{\mathbf{h}}\Psi((\mathbf{x},t),\mathbf{h}) = \mathbf{D}(\mathbf{x},t)\cdot\mathbf{h}. \tag{184}$$

We call $\mathbf{D}$ the *deformation tensor*.

Rmk 2: Since $\mathbf{D}$ is symmetric, $\mathbf{D}$ is diagonoalizable, that is $\exists$ orthorgonal matrix $\mathbf{U}$, such that

$$\widetilde{\mathbf{U}} = \mathbf{U}\mathbf{D}\mathbf{U}^T = \begin{pmatrix} d_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & d_3 \end{pmatrix} \tag{185}$$

and $\mathbf{U}^T\mathbf{U} = \mathbf{I} = \mathbf{U}\mathbf{U}^T$