Class Notes
Introduction to fluid mechanics

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April 1, 2024

Contents

1 The Equation of Motion

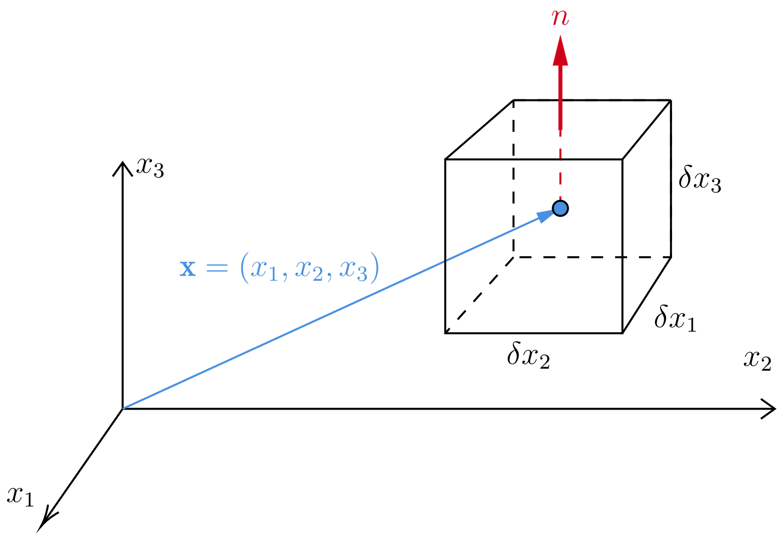
1.1 Introduction

1.1.1 Euler's equation:

Consider a fluid in a domain D in $\mathbb{R}^n$ ($n = 2$, $n = 3$).

Let $x \in D$, and $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ be the fluid density, velocity vector field and the pressure at the point x and time t.Consider an infinitesimal element of the fluid of volumn ∂V located at point x at time t with mass $\delta m = \rho(\mathbf{x}, t)$, which is moving $\mathbf{u}(\mathbf{x}, t)$ and momentum $\delta m \cdot \mathbf{u}(\mathbf{x}, t)$

The normal force directed into the indeinetesmal volumn across a face of area δa is $\mathbf{n} \cdot p(\mathbf{x}, t) \cdot \delta a$



Note that the pressure is the magnitade of the torce per unit area or normal stress, imposed on the fluid from neighboring fluid elements.

1.1.2 Convective derivative

convective derivative 對流導數 / material derivative 物質導數 / advective derivative 隨流導數 / convective derivative 對流導數 / derivative following the motion 隨體導數 / hydrodynamic derivative 水動力導數 / Lagrange derivative 拉格朗日導數 / substantial derivative 隨質導數 Couvder a fluid particle moving in flaid, whose position $\mathbf{x}$ at time t is $\mathbf{x}(t)$. Then

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t) \tag{1}$$

Hence, if $f(\mathbf{x},t)$ is a function on $D\times(0,T)$, then $f(\mathbf{x}(t),t)$ is the value if f at the moving fluid particle at $\mathbf{x}(t)$ at time t. We define the convective derivative of f:

$$\begin{aligned}\frac{Df(\mathbf{x},t)}{Dt} &= \frac{\partial f(\mathbf{x},t)}{\partial t} + \dot{\mathbf{x}}\cdot\nabla f(\mathbf{x},t)\\ &= f_t + \mathbf{u}\cdot\nabla f\end{aligned} \tag{2}$$

where $\nabla f = f(f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

Hence, if $f(\mathbf{x},t)$ is a function on $D\times(0,T)$, then $f(\mathbf{x}(t),t)$ is the vlaue of f at the moving fluid particle at $\mathbf{x}(t)$ at time t.

We define the convective derivative of f as:

$$\begin{aligned}\frac{Df(x,t)}{Dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{x}}(t)\cdot\nabla f\\ &= f_t + \mathbf{u}\cdot\nabla f\end{aligned}, \tag{3}$$

where $\nabla f = (f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.1 Def.

For any vector filed $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ on D, we let

$$\int_D \mathbf{F}dV = \left(\int_D F_1 dV, \int_D F_2 dV, \ldots, \int_D F_n dV\right). \tag{4}$$

1.1.2.2 Def.

We will assume that D is a smooth domain, i.e. for any $x_0 \in \partial D$, $\mathbb{R}^n = (x', x_n), n = 2, 3$ $\exists\delta_0 > 0$ and a smooth function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, s.t.

$$\partial D \cap B(x_0,\delta_0) = \left\{(x', \varphi(x')) : \|x'\| < \delta_0, x' \in \mathbb{R}^{n-1}\right\} \cap B(x_0,\delta_0) \tag{5}$$

and

$$D \cap B(x_0,\delta_0) = \left\{(x', x_n) : x_n > \varphi(x'), x' \in \mathbb{R}^{n-1}, \|x'\| < \delta_0\right\} \cap B(x_0,\delta_0) \tag{6}$$

1.1.2.3 Claim

Conside the volume δV of an element of mass δm, which moves in the fluid by the fluid motion

$$\frac{d(\delta V)}{dt} = (\nabla\cdot\mathbf{u})(\mathbf{x},t)\cdot\delta V \quad \text{as} \quad \delta x_1, \delta x_2, \delta x_3 \to 0, \tag{7}$$

where $\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{u} = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$, $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.4 proof

$$\begin{aligned}\frac{d(\delta V)}{dt} &= \frac{d}{dt}(\delta x_1, \delta x_2, \delta x_3)\\ &= \frac{d(\delta x_1)}{dt}\delta x_2\delta x_3 + \frac{d(\delta x_2)}{dt}\delta x_1\delta x_3 + \frac{d(\delta x_3)}{dt}\delta x_1\delta x_2\end{aligned} \tag{8}$$

For the first term

$$\begin{aligned}\frac{d(\delta x_1)}{dt} &\approx u_1\left(x_1+\frac{\delta x_1}{2},x_2,x_3\right)-u_1\left(x_1-\frac{\delta x_1}{2},x_2,x_3\right)\\ &=\frac{\partial u_1}{\partial x_1}(\xi_1,x_2,x_3)\delta x_1,\quad \text{fot some } \xi_1\in\left(x_1-\frac{\delta x_1}{2},x_1+\frac{\delta x_1}{2}\right)\end{aligned}\tag{9}$$

then

$$\frac{d(\delta x_1)}{dt}\delta x_2\delta x_3\to\frac{\partial u_1}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3,\quad \text{as } \delta x_1,\delta x_2,\delta x_3\to 0 \tag{10}$$

Similarly

$$\frac{d(\delta x_2)}{dt}\delta x_2\delta x_3\to\frac{\partial u_2}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3,\quad \text{as } \delta x_1,\delta x_2,\delta x_3\to 0 \tag{11}$$

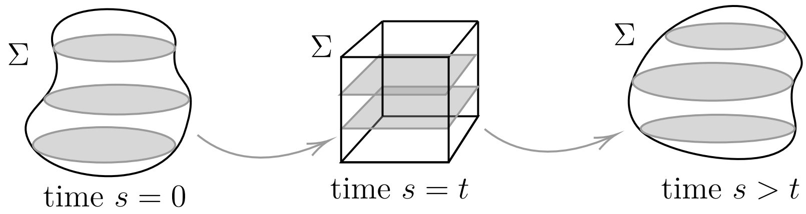
and

$$\frac{d(\delta x_3)}{dt}\delta x_2\delta x_3\to\frac{\partial u_3}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3,\quad \text{as } \delta x_1,\delta x_2,\delta x_3\to 0 \tag{12}$$

so that

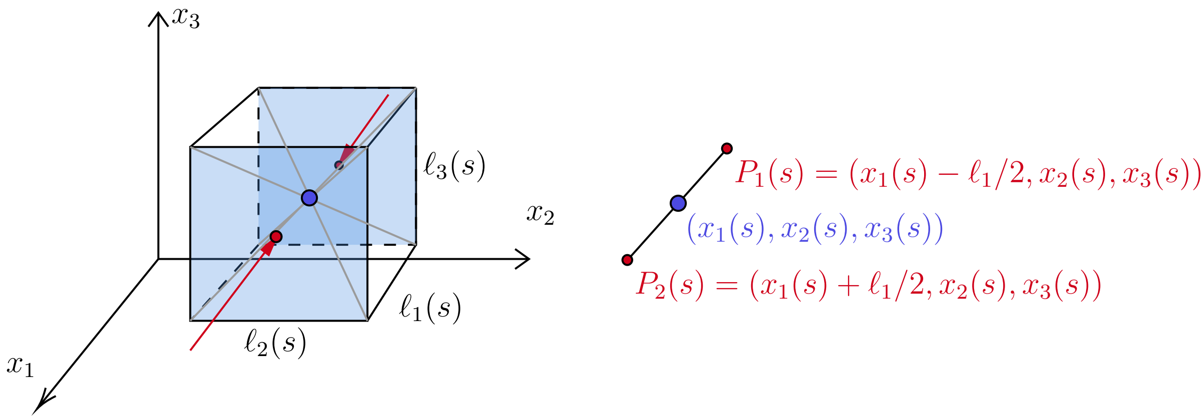
$$\frac{d(\delta V)}{dt}=\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\delta x_1\delta x_2\delta x_3=(\nabla\cdot\mathbf{u})\delta V \tag{13}$$

1.1.2.5 Note



tagged porton of fluid particle

Consider a a tagged (marked) portion Σ of fluid with center of mass at $(x_1(s),x_2(s),x_3(s))$ at time s. Let $m(x)$ and $V(s)$ be the mass and volumn of this portion Σ of fluid at time s.The portion of fluid particle moves aling with fluid. see as (2/21 fig1)



For a time $t > 0$, suppose at time t, the tagged portion Σ of fluid particles is a cube centered at (x_1, x_2, x_3) with side lengh ℓ_1, ℓ_2, ℓ_3, see as (2/21 fig2 - textbook p.4),where

$$P_1(s) = \left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s)\right)$$
$$P_2(s) = \left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s)\right) \tag{14}$$

We assume that Σ remain a cube for $s \approx t$ with side length, $\ell_1(s), \ell_2(s), \ell_3(s)$, then $V(s) = \ell_1(s) \cdot \ell_2(s) \cdot \ell_3(s)$

$$\left.\frac{dV(s)}{ds}\right|_{s=t} = \left.\frac{d\ell_1(s)}{ds}\right|_{s=t} \ell_2(s)\ell_3(s) + \left.\frac{d\ell_2(s)}{ds}\right|_{s=t} \ell_1(s)\ell_3(s) + \left.\frac{d\ell_3(s)}{ds}\right|_{s=t} \ell_1(s)\ell_2(s) \tag{15}$$

where

$$\begin{aligned}\frac{d\ell_1(s)}{ds} &= u_1(P_2(t), t) - u_1(P_1(t), t) \\ &= u_1\left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) - u_1\left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) \\ &\approx \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3, t) \cdot \ell_1\end{aligned} \tag{16}$$

Similarly

$$\left.\frac{d\ell_i}{ds}\right|_{s=t} = \frac{\partial u_i}{\partial x_i}(x_1, x_2, x_3) \cdot \ell_i, \quad \forall i = 1, 2, 3. \tag{17}$$

Now we write $\left.\frac{d}{ds}\right|_{s=t} = \frac{d}{dt}$, combinded with equation

1.1.3 Continuity equation

Let $\rho(\mathbf{x}, t)$ be the density of fluid at time s.Since $M(s) = \text{const.}$, $\forall s > 0$ and $\frac{dM(s)}{ds} = 0$, $\forall s > 0$.Therefore, since it is similar to the cube, the density is

$$\rho(\mathbf{x}, s) \approx \frac{M(s)}{V(s)} \tag{18}$$

and the derevative is

$$
\begin{aligned}
\frac{d}{ds}\rho(\mathbf{x},s)\bigg|_{s=t} &\approx \frac{d}{ds}\frac{M(s)}{V(s)}\bigg|_{s=t} \\
&= \frac{M'(s)V(s)-M(s)V'(s)}{V^2(s)}\bigg|_{s=t} \\
&= \frac{0-M(s)\frac{d}{ds}V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)(\operatorname{div}\mathbf{u})V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)}{V(s)}(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t} \\
&= -\rho(\mathbf{x}(s),s)(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t}
\end{aligned} \tag{19}
$$

we get

$$-\frac{d}{dt}\rho(\mathbf{x}(t),t) = \rho\cdot(\nabla\cdot\mathbf{u}(t)) \tag{20}$$

On the other hand, by chain rule

$$\frac{d}{dt}\rho(\mathbf{x}(t),t) = \rho_t + (\nabla\rho)\cdot\mathbf{u}(t) \tag{21}$$

combining together we have

$$
\begin{aligned}
&\Rightarrow \rho_t + (\nabla\rho)\cdot\mathbf{u} = \rho\cdot(\nabla\cdot\mathbf{u}) \\
&\Rightarrow \rho_t + (\nabla\rho)\cdot\mathbf{u} - \rho\cdot(\nabla\cdot\mathbf{u}) = 0 \\
&\Rightarrow \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0
\end{aligned} \tag{22}
$$

and the equation $\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0$ is called the *contunuity equation.*

1.1.4 Heuristic proof of the Euler equation

In the ansense of an externally applied forces, the net force $\mathbf{F}$, acting on δV, is due to the pressure field.

Write $\mathbf{F} = (F_1, F_2, F_3)$, we get

$$
\begin{aligned}
\mathbf{F}(x_1,x_2,x_3,t) &\approx \left(P\left(x_1 - \frac{\delta x_1}{2}, x_2, x_3, t\right) - P\left(x_1 + \frac{\delta x_1}{2}, x_2, x_3, t\right)\right)\delta x_2\delta x_3 \\
&= -\frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t)\delta x_1\delta x_2\delta x_3, \quad \delta x_1, \delta x_2, \delta x_3 \to 0 \\
&= \frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t)\delta V
\end{aligned} \tag{23}
$$

for some $\zeta_1 \in \left(x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2}\right)$.

By Newton's second law, the equation of motion for the elemnet of fund mass δm, at point $\mathbf{x}(t)$ is

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \mathbf{F} = -(\nabla P)\delta V \tag{24}$$

also

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \delta m \frac{d}{dt}\mathbf{u}(\mathbf{x},t) = \delta m\left(\mathbf{u}_t + (\nabla \cdot \mathbf{u})\right)\mathbf{u} \tag{25}$$

then

$$\begin{aligned} \delta m\left(\mathbf{u}_t + (\nabla \cdot \mathbf{u})\right)\mathbf{u} &= -(\nabla P)\delta V \\ \mathbf{u}_t + (\nabla \cdot \mathbf{u})\,\mathbf{u} &= -(\nabla P)\frac{\delta V}{\delta m} = -(\nabla P)\frac{1}{\delta m/\delta V} \end{aligned} \tag{26}$$

we get a equation

$$\mathbf{u}_t + (\nabla \cdot \mathbf{u})\,\mathbf{u} = -\frac{\nabla P}{\rho} \tag{27}$$

called *Euler's equation*.

Notice that

$$\begin{aligned} (\nabla \cdot \mathbf{u})\,\mathbf{u} &= \left(\sum_{i=0}^{3} u_i \frac{\partial}{\partial x_i}\right)\mathbf{u} \\ &= \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &= \begin{pmatrix} \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_1 \\ \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_2 \\ \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_3 \end{pmatrix} \end{aligned} \tag{28}$$

1.1.5 Lemma

Let D be a bounded domain and $F : \bar{D} \times [0, a_0] \to \mathbb{R}$ be a smooth function (or C^∞), then

$$\frac{d}{dt}\int_D F(x,t)dx = \int_D \frac{dF(x,t)}{dt}dx \tag{29}$$

1.1.5.1 proof

we have

$$\begin{aligned} \frac{d}{dt}\int_D F(x,t)dx &= \lim_{\Delta t \to 0}\left[\frac{1}{\Delta t}\int_D F(x,t+\Delta t)dx - \frac{d}{dt}\int_D F(x,t)dx\right] \\ &= \lim_{\Delta t \to 0}\frac{d}{dt}\int_D \frac{F(x,t+\Delta t) - F(x,t)}{\Delta t}dx \\ &= \text{By M.V.T.} \\ &= \lim_{\Delta t \to 0}\int_D \frac{\frac{\partial}{\partial t}F(x,\xi)\Delta t}{\Delta t}dx, \quad \text{for some } \xi, \text{ where} t < \xi < t + \Delta \\ &= \lim_{\Delta t \to 0}\int_D \frac{\partial}{\partial t}F(x,\xi)dx \end{aligned} \tag{30}$$

Denote, $\dfrac{\partial}{\partial t}F(x,t) = F_t(x,t)$ and $\dfrac{\partial^2}{\partial t^2}F(x,t) = F_{tt}(x,t)$, so

$$\begin{aligned}
&\left|\frac{1}{\Delta t}\int_D [F(x,t+\Delta t) - F(x,t)] - \int_D \frac{\partial}{\partial t}F(x,t)dx\right| \\
&= \left|\int_D F_t(x,\xi)dx - \int_D F_t(x,\xi)dx\right| \\
&= \text{By MVT} \\
&= \left|\int_D [F_t(x,\xi) - F_t(x,t)]\right| dx \\
&= \text{By MVT} \\
&= \left|\int_D F_{tt}(x,z)(t-\xi)dz\right|, \quad z \text{ between } t \text{ and } \xi \\
&\le M|t-\xi||D| \to 0, \quad \text{where } |D| \text{ is vlolumn of } D
\end{aligned} \tag{31}$$

where $M = \sup\limits_{(x,t)\in D\times(0,a)} F_{tt}(x,t)$.