

Rotation of The Fluid Flow

A brief introduction of rotation of fluid

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Presentation Overview

① Curl and Rotation

What is angular velocity?
Stream line and Path line
Example

② Circulation and Rotation

Definition of Circulation
Approximation of circulation

③ Circulation and Rotation

Vector calculus
Example for Cartesian
Example for Cylindrical

④ Conclusion

What is angular velocity?

What is the rotation of a particle?

First we look at a moving particle in a space \mathbb{R}^2 .

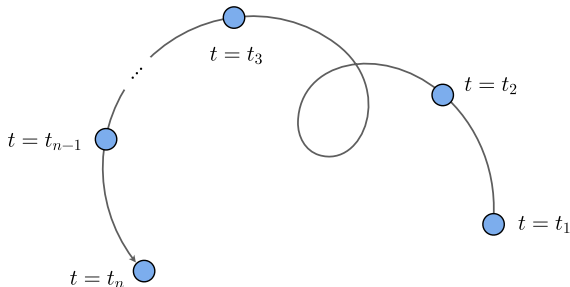
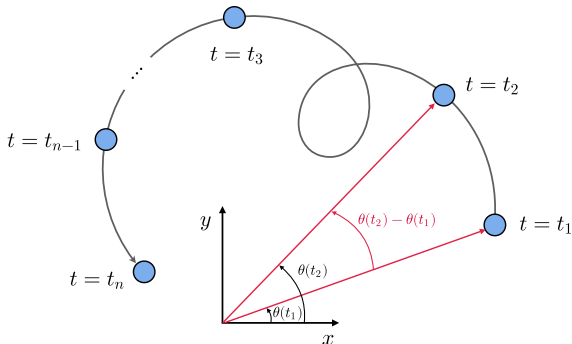


Figure: A moving particle in \mathbb{R}^2

What is angular velocity?

However, if we add a coordinate



And define the magnitude angular velocity to be

$$\omega = \lim_{t_1 \rightarrow t_2} \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1} = \frac{d\theta}{dt} \quad (1)$$

Notice that, the velocity can be calculated in polar coordinate (\mathbb{R}^2) by position vector $\vec{r} = r \cos \theta \hat{e}_x + r \sin \theta \hat{e}_y$, that is

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt} (r (\cos \theta \hat{e}_x + \sin \theta \hat{e}_y)) \\ &= \frac{dr}{dt} (\cos \theta \hat{e}_x + \sin \theta \hat{e}_y) + r (-\sin \theta \hat{e}_x + \cos \theta \hat{e}_y) \frac{d\theta}{dt} \\ &= \frac{dr}{dt} \hat{e}_r + r \frac{d\theta}{dt} \hat{e}_\theta\end{aligned}\quad (2)$$

Extending the space to z component, we define the angular velocity for a particle on the xy -plane in \mathbb{R}^3

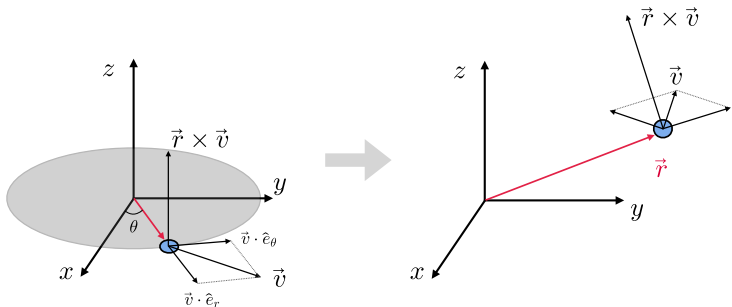
$$\vec{\omega} = \frac{\hat{e}_r \times \vec{v}}{r} = \frac{r(\hat{e}_r \times \hat{e}_\theta)}{r} \frac{d\theta}{dt} = \frac{d\theta}{dt} \hat{e}_z, \quad (3)$$

where \hat{e}_r , \hat{e}_θ and \hat{e}_z are the orthonormal basis of cylinder coordinate.

What is angular velocity?

Now, consider the particle in \mathbb{R}^3 and we still using position vector \vec{r} rather than the r -component vector, since $\hat{e}_r = \vec{r}/r$, we also can write

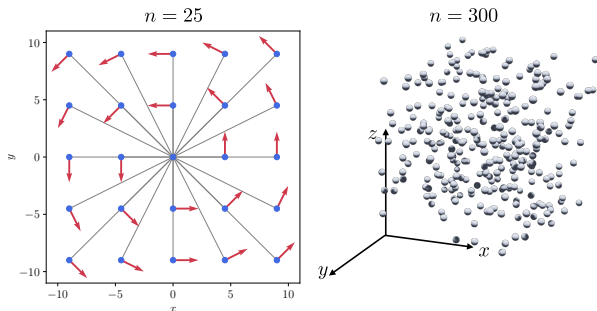
$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}. \quad (4)$$



What is angular velocity?

Then, how about a group of particles? Average angular velocity?

$$\tilde{\Omega} = \frac{1}{n} \sum_{i=1}^n \vec{\omega}_i = \sum_{i=1}^n \frac{\vec{r}_i \times \vec{v}}{\|\vec{r}_i\|^2} \quad (5)$$



It seems not to be a good estimating method.

Figure 1 shows a phase portrait in the (x, y) plane. The horizontal axis is labeled x and the vertical axis is labeled y , both ranging from -10 to 10. A vector field of blue arrows is plotted, representing the direction of the flow. A red trajectory is shown, starting at a red 'x' at approximately $(4, 3)$ and moving towards the origin. The trajectory is a smooth curve that follows the direction of the vector field.

Stream line $\frac{dy}{dx} = \frac{u_y(x, y)}{u_x(x, y)}$
at specific time $t = t_n$

Path line

$$\frac{d\vec{r}(x(t), y(t))}{dt} = \vec{u}(x(t), y(t))$$

So, a more intuitive way to describe how a group of particles (fluid) rotate, is to place a marker in these particles.

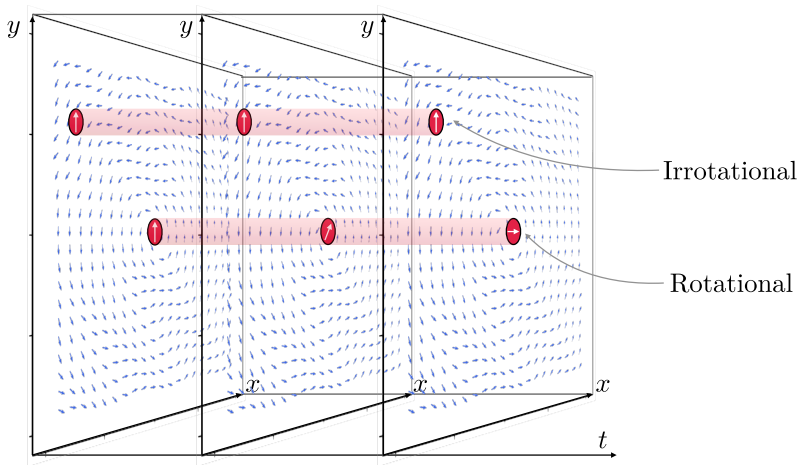
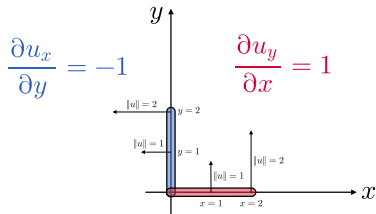
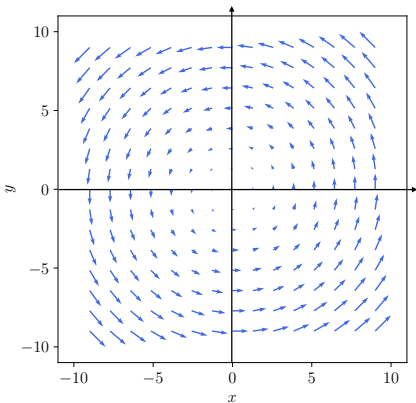


Figure: The *world line* of marker and vector field

Example

Now, consider a fluid flow given by

$$\vec{u}(x, y) = u_x \hat{e}_x + u_y \hat{e}_y = -y \hat{e}_x + x \hat{e}_y \quad (6)$$

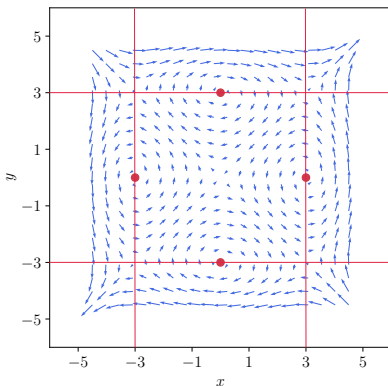


$$\text{curl } \vec{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 2$$

Example

Using another example fluid flow given by

$$\vec{u}(x, y) = (y^3 - 9y) \hat{e}_x + (x^3 - 9x) \hat{e}_y \quad (7)$$



$$\vec{u}(x(t), y(t)) = \frac{d\vec{r}}{dt} \quad \begin{cases} \frac{dx}{dt} = y^3 - 9y \\ \frac{dy}{dt} = x^3 - 9x \end{cases}$$

$$\vec{r}_3 = x(t_3)\hat{e}_x + y(t_3)\hat{e}_y$$

$$\vec{u} = \vec{u}(\vec{r}_3)$$

$t = t_3$

Path line

$$\vec{r}_2 = x(t_2)\hat{e}_x + y(t_2)\hat{e}_y$$

$$\vec{u} = \vec{u}(\vec{r}_2)$$

$t = t_2$

$$\vec{u} = \vec{u}(\vec{r}_1)$$

$t = t_1$

$$\vec{r}_1 = x(t_1)\hat{e}_x + y(t_1)\hat{e}_y$$

Example

After we solving the system of non-linear differential equations, we have the path line and moving particles (click [here](#) to see video).

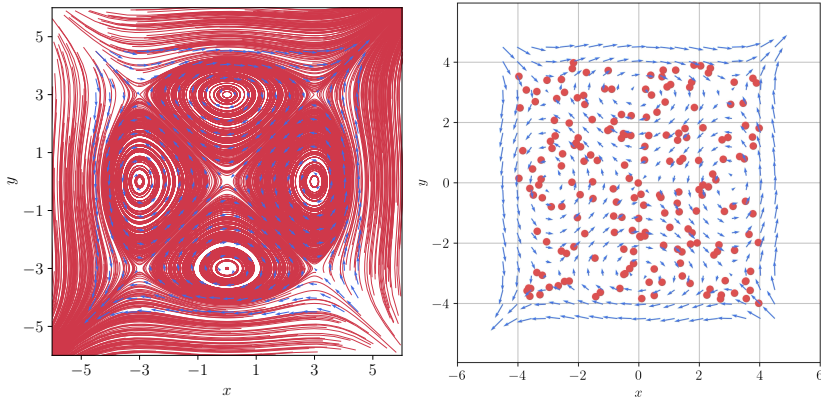
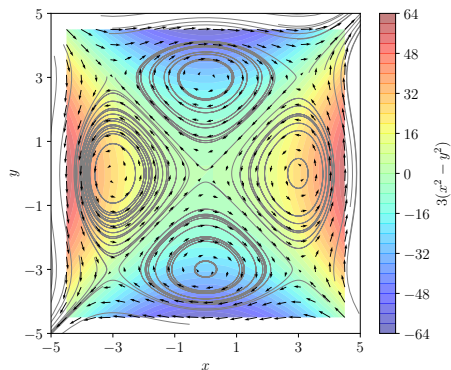


Figure: Left figure is the path line of vector field, right figure is a snapshot of the simulation video

Also calculate the curl of vector field

$$\begin{aligned}\operatorname{curl} \vec{u} &= \partial_x(x^3 - 9x) - \partial_y(y^3 - 9y) \\ &= 3(x^3 - y^2),\end{aligned}\tag{8}$$

and make a contour:



Here state a conclusion,

The curl is to measure the rotation at specific point.

Definition of Circulation

Now, consider a simple closed oriented curve C , the *circulation* Γ_C of vector field \vec{u} on C is given by

$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell}, \quad (9)$$

where $d\vec{\ell}$ is the infinitesimal displacement vector on C . More precisely,

Definition of circulation in \mathbb{R}^2

For a vector field $\vec{u} : S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given a piecewise smooth oriented curve C parametrized by $\vec{\ell}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$, where $a \leq t \leq b$ and $\vec{\ell}(a) = \vec{\ell}(b)$, the circulation Γ of \vec{u} on C is given by

$$\Gamma_C = \oint_C \vec{u} \cdot \frac{d\vec{\ell}(t)}{dt} dt = \int_a^b \vec{u}(x(t), y(t)) \cdot \frac{d\vec{\ell}(t)}{dt} dt. \quad (10)$$

But why are we using circulation ? By Stokes' theorem we have

$$\oint_{\partial\Sigma} \vec{u} \cdot d\vec{\ell} = \int_{\Sigma} (\text{curl } \vec{u}) \cdot d\vec{\Sigma}, \quad (11)$$

where $\ell = \partial\Sigma$ is the boundary of Σ , and Σ is smooth oriented surface. If we only consider \mathbb{R}^2 , we have the circulation

$$\Gamma_{\partial\Sigma} = \oint_{\partial\Sigma} \vec{u} \cdot d\vec{\ell} = \iint_{\Sigma} (\text{curl } \vec{u}) \, dx \, dy. \quad (12)$$

Definition of Circulation

Let the area of Σ to be $A = \iint_{\Sigma} dx dy$, we have

$$\Gamma_{\partial\Sigma} = A \cdot \frac{\iint_{\Sigma} (\text{curl } \vec{u}) \, dx \, dy}{\iint_{\Sigma} dx dy} \approx \text{Area} \times \text{mean of } \text{curl } \vec{u} \text{ in } \Sigma. \quad (13)$$

Here state a conclusion,

The circulation is to measure the rotation in a region.

So, now the method to estimate the rotation of the fluid flow with fluid velocity \vec{u} are

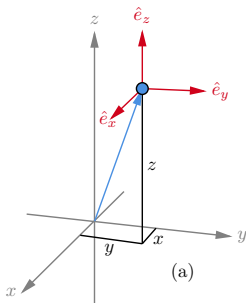
- ① Curl $\nabla \times \vec{u}(\vec{r}) = \text{curl } \vec{u}(\vec{r})$ is to measure the rotation at point \vec{r} .
- ② Circulation $\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell}$ is to measure the rotation in a region C .

So, now let us consider some special case for $\nabla \times \vec{u} = 0$.

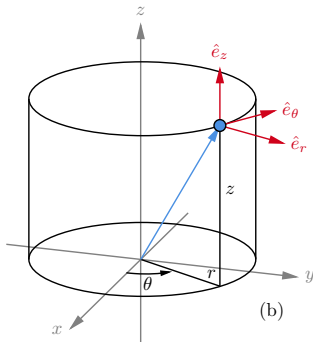
Vector calculus

First, let us specify our coordinates, that is

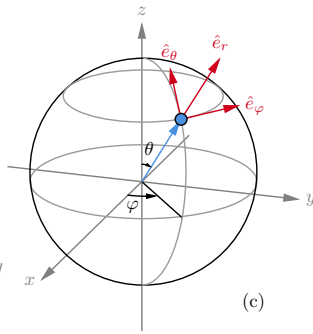
Cartesian
cooridante



Cylindrical
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Spherical
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The curl in three different coordinate are

- ① For a vector field $\vec{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \quad (14)$$

- ② For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix} \quad (15)$$

- ③ For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\varphi \hat{e}_\varphi$

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r \sin \theta \hat{e}_\varphi \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & r \sin \theta A_\varphi \end{vmatrix} \quad (16)$$

Or, expanding ...

- ① For a vector field $\vec{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{e}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{e}_z \quad (17)$$

- ② For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z$

$$\nabla \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{e}_z \quad (18)$$

- ③ For a vector field $\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\varphi \hat{e}_\varphi$

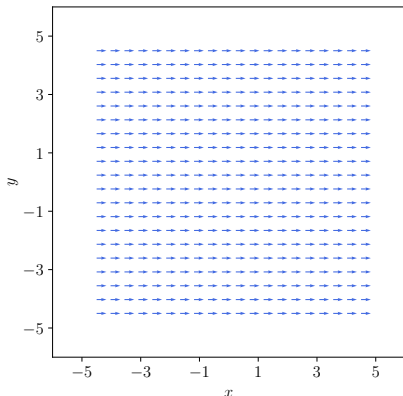
$$\begin{aligned} \nabla \times \vec{A} = & \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (A_z \sin \theta) - \frac{\partial A_\theta}{\partial \varphi} \right) \hat{e}_r \\ & + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{e}_\varphi \end{aligned} \quad (19)$$

Example for Cartesian

Here, for simplicity, we only consider the flow on the xy -plane and using the Cartesian coordinate to describe. We consider a flow velocity

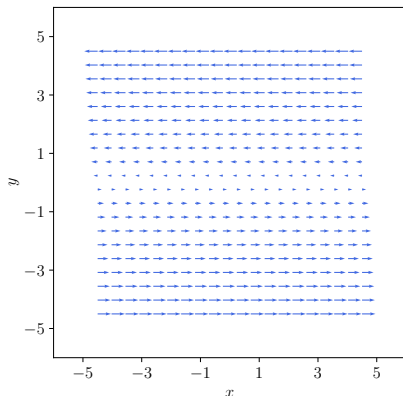
$$\vec{u}(x, y) = 1 \cdot \hat{e}_x. \quad (20)$$

Obviously, the curl is zero, $\nabla \times \vec{u} = 1$, since it is a uniform flow.



so that the velocity grows when y becomes larger. Now, the curl is

Although, the curl is not zero, we still can not see any rotation in the arrow graph of vector field.



Example for Cartesian

Even the stream and path line, we still cannot see any rotation.

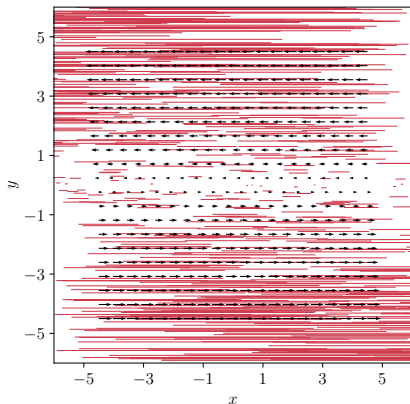


Figure: Given random initial position condition to solve *path line*.

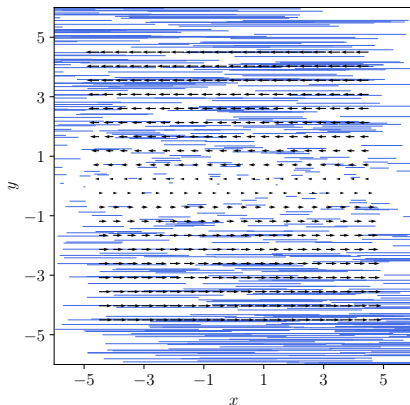
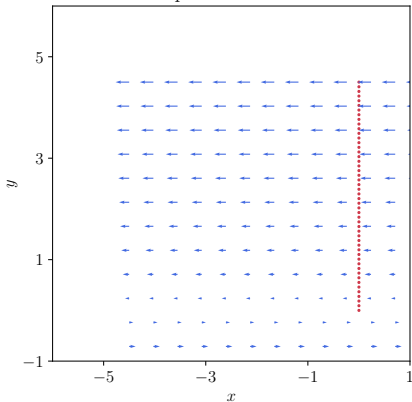
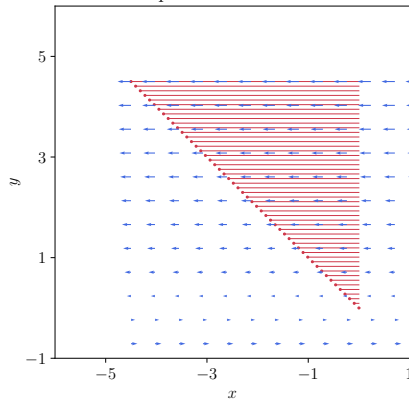


Figure: Given random initial position condition to solve *stream line*.

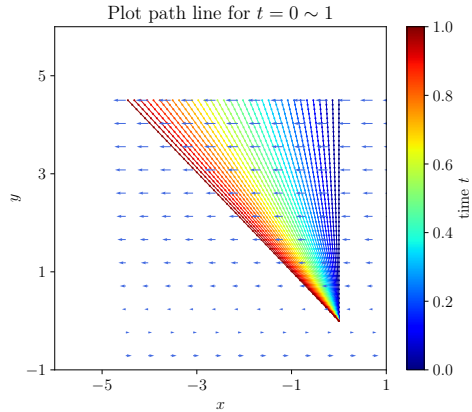
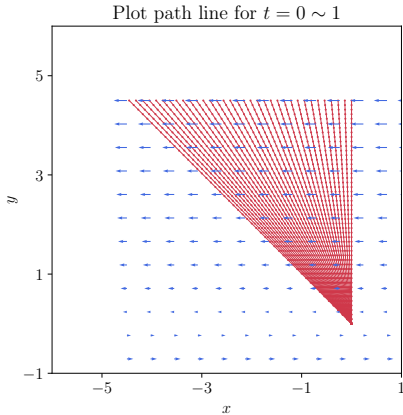
Example for Cartesian

But if we look at some specific particle in space:

 Plot path line for $t = 0$

 Plot path line for $t = 0 \sim 1$


Example for Cartesian

And link the particles in each time, and color them in different colors:



Here, for simplicity, we only consider the flow on the xy -plane and using the cylindrical coordinate to describe. So, the fluid velocity now becomes

$$\vec{u}(r, \theta) = u_r(r, \theta)\hat{e}_r + u_\theta(r, \theta)\hat{e}_\theta, \quad (23)$$

and then the curl

$$\nabla \times \vec{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{e}_z \quad (24)$$

becomes

$$\nabla \times \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{e}_z, \quad (25)$$

also notice that the differential displacement in this case is

$$d\vec{\ell} = dr \hat{e}_r + r d\theta \hat{e}_\theta. \quad (26)$$

Example for Cylindrical

Now, let $\vec{u}(r, \theta) = u_\theta(r, \theta)\hat{e}_\theta$, and solve for $0 = \nabla \times \vec{u}$, we have

$$0 = \frac{1}{r} \left(\frac{\partial}{\partial r} (ru_\theta(r, \theta)) - \frac{\partial 0}{\partial \theta} \right) \hat{e}_z \quad (27)$$

$$\Rightarrow 0 = u_\theta(r, \theta) + r \frac{\partial u_\theta(r, \theta)}{\partial r}, \quad \text{for } r \neq 0 \quad (28)$$

$$\Rightarrow \frac{\partial u_\theta(r, \theta)}{\partial r} = -\frac{u_\theta(r, \theta)}{r}. \quad (29)$$

By experience, the general solution is $u_\theta(r, \theta) = r^{-1}f(\theta)$, where f is a function of θ . We can check by plug into the equation

$$\frac{\partial u_\theta(r, \theta)}{\partial r} = -r^{-2}f(\theta) = -\frac{r^{-1}f(\theta)}{r} = -\frac{u_\theta(r, \theta)}{r}.$$

Then we can calculate the circulation for $\vec{u}(r, \theta) = u_\theta(r, \theta)\hat{e}_\theta$ such that $\nabla \times \vec{u} = 0$, the general solution is $u_\theta(r, \theta) = r^{-1}f(\theta)$, then the circulation in cylindrical coordinate (polar coordinate) is

$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell} = \oint_C u_\theta(r, \theta) r d\theta = \oint_C f(\theta) d\theta. \quad (30)$$

Now, we can observing that, although the curl is zero, if $f(\theta) > 0$ on C , then $\Gamma_C > 0$, on the other hand, if $f(\theta) < 0$ on C , then $\Gamma_C < 0$.

Now, consider a more general case, let $\vec{u}(r, \theta) = u_r(r, \theta)\hat{e}_r + u_\theta(r, \theta)\hat{e}_\theta$, and solve for $\nabla \times \vec{u} = 0$, we have

$$0 = \frac{\partial}{\partial r} (ru_\theta(r, \theta)) - \frac{\partial u_r(r, \theta)}{\partial \theta}, \quad (31)$$

Also, by experience, let $u_r(r, \theta) = r^{-1}f(r, \theta)$, we have

$$\frac{\partial f(r, \theta)}{\partial r} = \frac{\partial u_r(r, \theta)}{\partial \theta}, \quad (32)$$

which is the first equation in *Cauchy-Riemann equations*.

Example for Cylindrical

In order to solve $\partial f(r, \theta)/\partial r = \partial u_r(r, \theta)/\partial \theta$, we can suppose there exists a potential function $\phi(r, \theta)$ such that

$$f(r, \theta) = \frac{\partial \phi(r, \theta)}{\partial \theta} \quad \text{and} \quad u_r(r, \theta) = \frac{\partial \phi(r, \theta)}{\partial r}. \quad (33)$$

With this assumption, the equation is automatically satisfied:

$$\frac{\partial f(r, \theta)}{\partial r} = \frac{\partial^2 \phi(r, \theta)}{\partial r \partial \theta} = \frac{\partial^2 \phi(r, \theta)}{\partial \theta \partial r} = \frac{\partial u_r(r, \theta)}{\partial \theta}.$$

Now, the solution is

$$\vec{u} = \frac{\partial \phi(r, \theta)}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi(r, \theta)}{\partial \theta} \hat{e}_\theta = \nabla \phi \quad (34)$$

where ϕ is any function of r and θ .

Example for Cylindrical

Then we can calculate the circulation for $\vec{u} = u_r(r, \theta)\hat{e}_r + u_\theta(r, \theta)\hat{e}_\theta$ such that $\nabla \times \vec{u} = 0$. The general solution is $u_\theta(r, \theta) = \partial_r \phi \hat{e}_r + r^{-1} \partial_\theta \phi \hat{e}_r$, where ϕ is any function of r and θ . The circulation in cylindrical coordinate (polar coordinate) is

$$\Gamma_C = \oint_C \vec{u} \cdot d\vec{\ell} = \oint_C \left(u_r(r, \theta) dr + u_\theta(r, \theta) r d\theta \right) \quad (35)$$

$$= \oint_C \frac{\partial \phi(r, \theta)}{\partial r} dr + \frac{1}{r} \frac{\partial \phi(r, \theta)}{\partial \theta} r d\theta \quad (36)$$

$$= \oint_C \frac{\partial \phi(r, \theta)}{\partial r} dr + \frac{\partial \phi(r, \theta)}{\partial \theta} d\theta \quad (37)$$

$$= \oint_C \nabla \phi \cdot d\vec{\ell} = \oint_C d\phi, \quad (38)$$

where $d\phi$ is the total differential.

Usually, ϕ is a continuous function; however, not always. If a piecewise smooth oriented path C can be parameterized by

$$\vec{\ell}(t) = x(r(t), \theta(t))\hat{e}_x + y(r(t), \theta(t))\hat{e}_y, \quad (39)$$

where $t \in [a, b]$ and $\vec{\ell}(a) = \vec{\ell}(b)$, then the circulation becomes

$$\Gamma_C = \phi(\vec{r}(b)) - \phi(\vec{r}(a)). \quad (40)$$

However, if $\phi(r, \theta) = \theta$, the vector field is $\vec{u} = \nabla\phi = 1/r\hat{e}_\theta$. If the path C surrounds the origin, which is the pole $r = 0$, the circulation becomes $\Gamma_C = 2\pi - 0 = 2\pi > 0$, which means the circulation still can be measurement of rotation.

Here, I list some potential function ϕ in cylindrical coordinate, so that $\nabla \times \vec{u} = 0$:

$\phi(r, \theta)$	$\vec{u} = \nabla \phi$	$\nabla \times \vec{u}$	Γ_C
θ	$(1/r) \hat{e}_\theta$	0	$\neq 0$
$c_0 + c_1 \cdot \theta$	$(c_1/r) \hat{e}_\theta$	0	$\neq 0$
$\sum_{s=0}^n c_s \cdot \theta^s$	$(\sum_{s=1}^n s c_s \cdot \theta^{s-1}/r) \hat{e}_\theta$	0	$\neq 0$
$f(\theta)$	$(f'(\theta)/r) \hat{e}_\theta$	0	?
$g(r) \cdot f(\theta)$	$(g'(r)f(\theta)) \hat{e}_r + (g(r)f'(\theta)/r) \hat{e}_\theta$	0	?
\vdots	\vdots	0	\vdots

where c_0, c_1, \dots, c_n are constants, and f is any function of θ , g is any function of r .

In conclude,

- ① The ***curl*** is to measure the rotation at specific point.
- ② The ***circulation*** is to measure the rotation in a region.
- ③ If the potential function is discontinuous on the closed path, even if curl is zero, the circulation can still be nonzero.

Thanks!

Questions? Comments?