Note: Example for Circulation

Chang-Mao Yang 楊長茂

May 14, 2024

1 Line Vortex Flow

Consider a line vortex flow in cylindrical polar coordinate (r, θ, z) with the fluid velocity vector field

$$\mathbf{u} = \mathbf{u}(\mathbf{x}) = \frac{k}{r} \cdot \mathbf{e}_\theta, \tag{1}$$

where the basis are denotes as $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$. Also, in Cartesian coordinate (x, y, z), the fluid velocity vector field is given by

$$\mathbf{u} = \mathbf{u}(\mathbf{x}) = \frac{-ky}{x^2 + y^2} \cdot \mathbf{e}_x + \frac{-ky}{x^2 + y^2} \cdot \mathbf{e}_y, \tag{2}$$

where the basis are $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

Notice that the coordinates transformation between cylindrical polar coordinate and Cartesian coordinate are given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}, \tag{3}$$

and the basis transformation is

$$\begin{cases} \mathbf{e}_r = \cos\theta \cdot \mathbf{e}_x + \sin\theta \cdot \mathbf{e}_y \\ \mathbf{e}_\theta = -\sin\theta \cdot \mathbf{e}_x + \cos\theta \cdot \mathbf{e}_y \\ \mathbf{e}_z = \mathbf{e}_z \end{cases}, \quad \begin{cases} \mathbf{e}_x = \cos\theta \cdot \mathbf{e}_r - \sin\theta \cdot \mathbf{e}_\theta \\ \mathbf{e}_y = \sin\theta \cdot \mathbf{e}_r + \cos\theta \cdot \mathbf{e}_\theta \\ \mathbf{e}_z = \mathbf{e}_z \end{cases} \tag{4}$$

The velocity vector field in two different coordinate can be calculated by

$$\begin{aligned} \mathbf{u} = \mathbf{u}(\mathbf{x}) &= \frac{k}{r} \cdot \mathbf{e}_\theta = \frac{k}{\sqrt{x^2 + y^2}} \cdot (-\sin\theta \cdot \mathbf{e}_x + \cos\theta \cdot \mathbf{e}_y) \\ &= \frac{k}{\sqrt{x^2 + y^2}} \cdot \left(-\frac{-y}{\sqrt{x^2 + y^2}} \cdot \mathbf{e}_x + \frac{x}{\sqrt{x^2 + y^2}} \cdot \mathbf{e}_y \right) \\ &= \frac{-ky}{\sqrt{x^2 + y^2}} \cdot \mathbf{e}_x + \frac{kx}{\sqrt{x^2 + y^2}} \cdot \mathbf{e}_y \end{aligned} \tag{5}$$

2 Graph

If we graph the velocity vector field for $k = 1$, that is

$$\mathbf{u}\left(x, y, z\right) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right)$$

we get Figure 1.

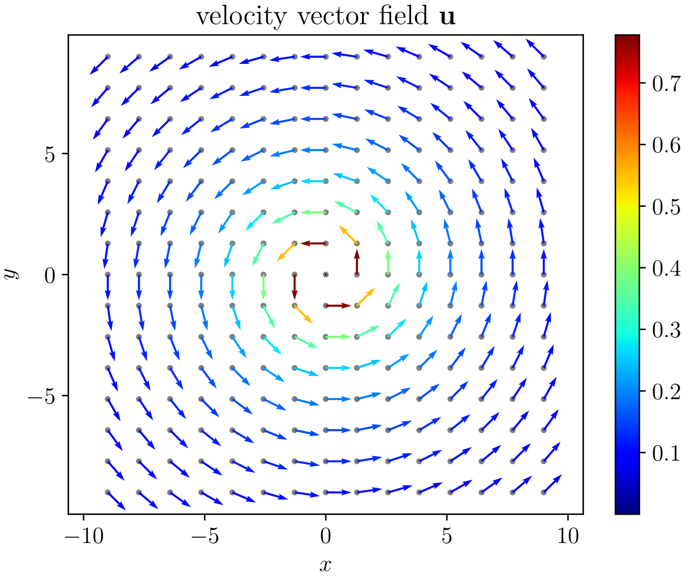

Figure 1: The length of vectors have been normalize, and the color represented the magnitude of velocity vector field $\|\mathbf{u}\|$.

3 Vorticity

Now, we may calculate the curl or vorticity $\xi = \nabla \times \mathbf{u}$ in different coordinate system, that is

$$\nabla \times \mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r & ru_\theta & u_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & k & 0 \end{vmatrix} = \frac{k}{r} \cdot 0 = 0 \tag{6}$$

and

$$\begin{aligned}
\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ \dfrac{-ky}{x^2+y^2} & \dfrac{kx}{x^2+y^2} & 0 \end{vmatrix} && (7) \\
&= \left(\frac{\partial}{\partial x}\left(\frac{kx}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-ky}{x^2+y^2}\right)\right)\mathbf{e}_z && (8) \\
&= \left(\left(\frac{k\left(x^2+y^2\right) - kx\left(2x\right)}{x^2+y^2}\right) + \left(\frac{k\left(x^2+y^2\right) - ky\left(2y\right)}{x^2+y^2}\right)\right)\mathbf{e}_z && (9) \\
&= \left(\frac{2k\left(x^2+y^2\right) - 2k\left(x^2+y^2\right)}{x^2+y^2}\right)\mathbf{e}_z = 0 && (10)
\end{aligned}$$

So that, the vorticity curl of line vortex fluid is zero $\xi = 0$.

4 Circulation

However, if we calculate the circulation of the fluid with a closed path C in Cartesian coordinate

$$\Gamma_C = \oint_C \mathbf{u} \cdot d\mathbf{r} \tag{11}$$

Notice that the z-component of $\mathbf{u}$ is zero, the circulation is

$$\Gamma_C = \oint_C \frac{-ky}{x^2+y^2} \cdot dx + \frac{kx}{x^2+y^2} \cdot dy = \oint_C \frac{k}{x^2+y^2} \cdot (xdy - ydx)\,. \tag{12}$$

Here, using the total differential for the $d(y/x)$, or consider a curve $(x(t), y(t))$ parametrized by t, we have

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{y^2} \quad \text{or} \quad \frac{d}{dt}\left(\frac{y(t)}{x(t)}\right) = \frac{x(t)\dfrac{dy(t)}{dt} - y(t)\dfrac{dx(t)}{dt}}{x^2(t)}. \tag{13}$$

Plugin to the circulation, we have

$$\Gamma_C = k \oint_C \frac{x^2}{x^2+y^2} \cdot \left(\frac{ydx - xdy}{x^2}\right) = k \oint_C \frac{1}{1+y^2/x^2} \cdot d\,(y/x)\,. \tag{14}$$

Then we easily can calculate the integral

$$\Gamma_C = k \oint_C d\tan^{-1}(y/x), \tag{15}$$

or in cylindrical polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$

$$\Gamma_C = k \oint_C d\tan^{-1}(\tan\theta) = k \oint_C d\theta, \tag{16}$$

Example.
If we consider a simple closed circle curve with radius R parametrized by t, in polar coordinate we have

$$C : \begin{cases} x(t) = R \cdot \cos\theta(t) \\ y(t) = R \cdot \sin\theta(t) \end{cases} , \quad R \in \mathbb{R}, \quad \theta = [0, 2\pi] \tag{17}$$

the circulation is easily $\Gamma_C = 2\pi \cdot k$.

Remark. Actually, we can calculate the *Winding Number* of the curve around point 0, that is

$$\Gamma_C = 2\pi \,\mathrm{wind}\,(C, 0)\,. \tag{18}$$

For more detail, see Do Carmo, M. (1976) *Differential geometry of curves and surfaces*, Section 5-7, p-392.

Remark.
In complex coordinate $z = x + iy$, also in polar coordinate, we write $z = re^{i\theta}$, then

$$dz = dx + idy = e^{i\theta}dr + ire^{i\theta}d\theta. \tag{19}$$

Consider

$$\frac{dz}{z} = \frac{dx + idy}{x + iy} = \frac{e^{i\theta}dr + ire^{i\theta}d\theta}{re^{i\theta}} = \frac{dr}{r} + id\theta = d\ln r + id\theta, \tag{20}$$

then integral along a closed path curve γ in the complex plane is given by

$$\oint_\gamma \frac{dz}{z} = \oint_\gamma d\ln r + i\oint_\gamma d\theta = i\oint_\gamma d\theta, \tag{21}$$

r-part vanish, since γ is a closed curve. Now, we can rewrite the circulation for the path γ in the complex plane,

$$\Gamma_\gamma = \frac{1}{i}\oint_\gamma \frac{dz}{z}. \tag{22}$$

Actually, this is a more common way to introduce *Winding Number* about $z = 0$, which is

$$\mathrm{wind}(\gamma, 0) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z - 0} \tag{23}$$

so the circulation is

$$\Gamma_\gamma = \frac{1}{i}\oint_\gamma \frac{dz}{z} = \frac{1}{i}2\pi i\,\mathrm{wind}(\gamma, 0) = 2\pi\,\mathrm{wind}(\gamma, 0). \tag{24}$$