

Class Notes

Introduction to fluid mechanics

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1 The Equation of Motion

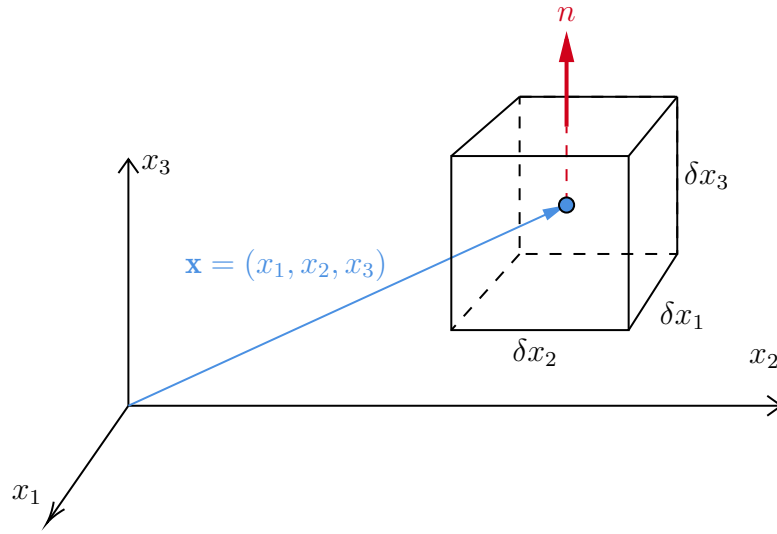
1.1 Introduction

1.1.1 Euler's equation:

Consider a fluid in a domain D in \mathbb{R}^n ($n = 2, n = 3$).

Let $x \in D$, and $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ be the fluid density, velocity vector field and the pressure at the point x and time t . Consider an infinitesimal element of the fluid of volume ∂V located at point x at time t with mass $\delta m = \rho(\mathbf{x}, t) \delta V$, which is moving $\mathbf{u}(\mathbf{x}, t)$ and momentum $\delta m \cdot \mathbf{u}(\mathbf{x}, t)$

The normal force directed into the infinitesimal volume across a face of area δa is $\mathbf{n} \cdot p(\mathbf{x}, t) \cdot \delta a$



Note that the pressure is the magnitude of the force per unit area or normal stress, imposed on the fluid from neighboring fluid elements.

1.1.2 Convective derivative

convective derivative 對流導數 / material derivative 物質導數 / advective derivative 隨流導數 / convective derivative 對流導數 / derivative following the motion 隨體導數 / hydrodynamic derivative 水動力導數 / Lagrange derivative 拉格朗日導數 / substantial derivative 隨質導數 Consider a fluid particle moving in fluid, whose position \mathbf{x} at time t is $\mathbf{x}(t)$. Then

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t) \quad (1)$$

Hence, if $f(\mathbf{x}, t)$ is a function on $D \times (0, T)$, then $f(\mathbf{x}(t), t)$ is the value of f at the moving fluid particle at $\mathbf{x}(t)$ at time t . We define the convective derivative of f :

$$\begin{aligned} \frac{Df(\mathbf{x}, t)}{Dt} &= \frac{\partial f(\mathbf{x}, t)}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f(\mathbf{x}, t) \\ &= f_t + \mathbf{u} \cdot \nabla f \end{aligned} \quad (2)$$

where $\nabla f = (f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

Hence, if $f(\mathbf{x}, t)$ is a function on $D \times (0, T)$, then $f(\mathbf{x}(t), t)$ is the value of f at the moving fluid particle at $\mathbf{x}(t)$ at time t .

We define the convective derivative of f as:

$$\begin{aligned} \frac{Df(x, t)}{Dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{x}}(t) \cdot \nabla f, \\ &= f_t + \mathbf{u} \cdot \nabla f \end{aligned} \quad (3)$$

where $\nabla f = (f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.1 Def.

For any vector field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ on D , we let

$$\int_D \mathbf{F} dV = \left(\int_D F_1 dV, \int_D F_2 dV, \dots, \int_D F_n dV \right). \quad (4)$$

1.1.2.2 Def.

We will assume that D is a smooth domain, i.e. for any $x_0 \in \partial D$, $\mathbb{R}^n = (x', x_n)$, $n = 2, 3$ $\exists \delta_0 > 0$ and a smooth function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, s.t.

$$\partial D \cap B(x_0, \delta_0) = \{(x', \varphi(x')) : \|x'\| < \delta_0, x' \in \mathbb{R}^{n-1}\} \cap B(x_0, \delta_0) \quad (5)$$

and

$$D \cap B(x_0, \delta_0) = \{(x', x_n) : x_n > \varphi(x'), x' \in \mathbb{R}^{n-1}, \|x'\| < \delta_0\} \cap B(x_0, \delta_0) \quad (6)$$

1.1.2.3 Claim

Consider the volume δV of an element of mass δm , which moves in the fluid by the fluid motion

$$\frac{d(\delta V)}{dt} = (\nabla \cdot \mathbf{u})(\mathbf{x}, t) \cdot \delta V \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0, \quad (7)$$

where $\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}$, $\mathbf{u} = (u_1, u_2, u_3)$.

1.1.2.4 proof

$$\begin{aligned} \frac{d(\delta V)}{dt} &= \frac{d}{dt}(\delta x_1, \delta x_2, \delta x_3) \\ &= \frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 + \frac{d(\delta x_2)}{dt} \delta x_1 \delta x_3 + \frac{d(\delta x_3)}{dt} \delta x_1 \delta x_2 \end{aligned} \quad (8)$$

For the first term

$$\begin{aligned} \frac{d(\delta x_1)}{dt} &\approx u_1 \left(x_1 + \frac{\delta x_1}{2}, x_2, x_3 \right) - u_1 \left(x_1 - \frac{\delta x_1}{2}, x_2, x_3 \right) \\ &= \frac{\partial u_1}{\partial x_1}(\xi_1, x_2, x_3) \delta x_1, \quad \text{for some } \xi_1 \in \left(x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2} \right) \end{aligned} \quad (9)$$

then

$$\frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \quad (10)$$

Similarly

$$\frac{d(\delta x_2)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_2}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \quad (11)$$

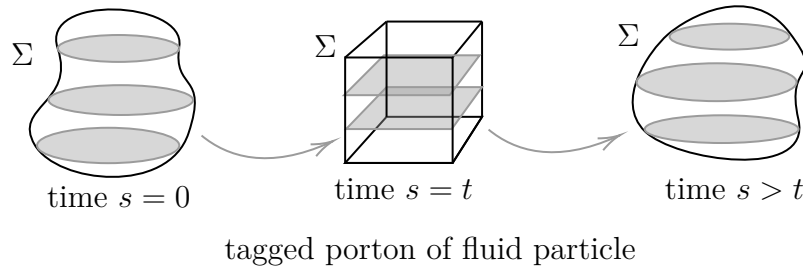
and

$$\frac{d(\delta x_3)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_3}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \quad (12)$$

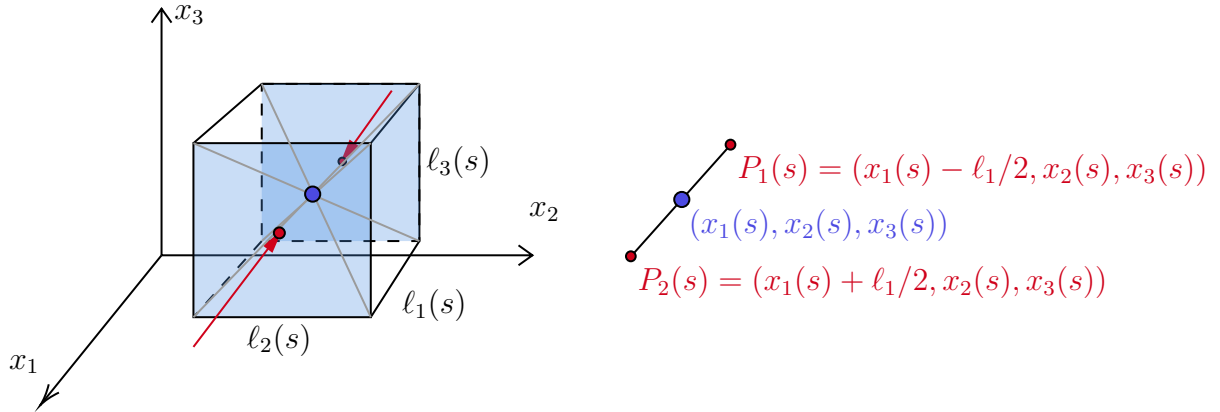
so that

$$\frac{d(\delta V)}{dt} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \delta x_1 \delta x_2 \delta x_3 = (\nabla \cdot \mathbf{u}) \delta V \quad (13)$$

1.1.2.5 Note



Consider a a tagged (marked) portion Σ of fluid with center of mass at $(x_1(s), x_2(s), x_3(s))$ at time s . Let $m(x)$ and $V(s)$ be the mass and volumn of this portion Σ of fluid at time s . The portion of fluid particle moves along with fluid. see as (2/21 fig1)



For a time $t > 0$, suppose at time t , the tagged portion Σ of fluid particles is a cube centered at (x_1, x_2, x_3) with side length ℓ_1, ℓ_2, ℓ_3 , see as (2/21 fig2 - textbook p.4), where

$$\begin{aligned} P_1(s) &= \left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \\ P_2(s) &= \left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \end{aligned} \quad (14)$$

We assume that Σ remain a cube for $s \approx t$ with side length, $\ell_1(s), \ell_2(s), \ell_3(s)$, then $V(s) = \ell_1(s) \cdot \ell_2(s) \cdot \ell_3(s)$

$$\left. \frac{dV(s)}{ds} \right|_{s=t} = \left. \frac{d\ell_1(s)}{ds} \right|_{s=t} \ell_2(s) \ell_3(s) + \left. \frac{d\ell_2(s)}{ds} \right|_{s=t} \ell_1(s) \ell_3(s) + \left. \frac{d\ell_3(s)}{ds} \right|_{s=t} \ell_1(s) \ell_2(s) \quad (15)$$

where

$$\begin{aligned}
\frac{d\ell_1(s)}{ds} &= u_1(P_2(t), t) - u_1(P_1(t), t) \\
&= u_1\left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) - u_1\left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) \quad (16) \\
&\approx \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3, t) \cdot \ell_1
\end{aligned}$$

Similarly

$$\left. \frac{d\ell_i}{ds} \right|_{s=t} = \frac{\partial u_i}{\partial x_i}(x_1, x_2, x_3) \cdot \ell_i, \quad \forall i = 1, 2, 3. \quad (17)$$

Now we write $\left. \frac{d}{ds} \right|_{s=t} = \frac{d}{dt}$, combined with equation

1.1.3 Continuity equation

Let $\rho(\mathbf{x}, t)$ be the density of fluid at time s . Since $M(s) = \text{const.}$, $\forall s > 0$ and $\frac{dM(s)}{ds} = 0$, $\forall s > 0$. Therefore, since it is similar to the cube, the density is

$$\rho(\mathbf{x}, s) \approx \frac{M(s)}{V(s)} \quad (18)$$

and the derivative is

$$\begin{aligned}
\left. \frac{d}{ds} \rho(\mathbf{x}, s) \right|_{s=t} &\approx \left. \frac{d}{ds} \frac{M(s)}{V(s)} \right|_{s=t} \\
&= \left. \frac{M'(s)V(s) - M(s)V'(s)}{V^2(s)} \right|_{s=t} \\
&= \left. \frac{0 - M(s) \frac{d}{ds} V(s)}{V^2(s)} \right|_{s=t} \\
&= - \left. \frac{M(s)(\text{div } \mathbf{u})V(s)}{V^2(s)} \right|_{s=t} \\
&= - \left. \frac{M(s)}{V(s)} (\text{div } \mathbf{u}(s)) \right|_{s=t} \\
&= - \left. \rho(\mathbf{x}(s), s) (\text{div } \mathbf{u}(s)) \right|_{s=t}
\end{aligned} \quad (19)$$

we get

$$-\frac{d}{dt} \rho(\mathbf{x}(t), t) = \rho \cdot (\nabla \cdot \mathbf{u}(t)) \quad (20)$$

On the other hand, by chain rule

$$\frac{d}{dt} \rho(\mathbf{x}(t), t) = \rho_t + (\nabla \rho) \cdot \mathbf{u}(t) \quad (21)$$

combining together we have

$$\begin{aligned}
 &\Rightarrow \rho_t + (\nabla \rho) \cdot \mathbf{u} = \rho \cdot (\nabla \cdot \mathbf{u}) \\
 &\Rightarrow \rho_t + (\nabla \rho) \cdot \mathbf{u} - \rho \cdot (\nabla \cdot \mathbf{u}) = 0 \\
 &\Rightarrow \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0
 \end{aligned} \tag{22}$$

and the equation $\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$ is called the *continuity equation*.

1.1.4 Heuristic proof of the Euler equation

In the ansense of an externally applied forces, the net force \mathbf{F} , acting on δV , is due to the pressure field.

Write $\mathbf{F} = (F_1, F_2, F_3)$, we get

$$\begin{aligned}
 \mathbf{F}(x_1, x_2, x_3, t) &\approx \left(P \left(x_1 - \frac{\delta x_1}{2}, x_2, x_3, t \right) - P \left(x_1 + \frac{\delta x_1}{2}, x_2, x_3, t \right) \right) \delta x_2 \delta x_3 \\
 &= -\frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t) \delta x_1 \delta x_2 \delta x_3, \quad \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \\
 &= \frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t) \delta V
 \end{aligned} \tag{23}$$

for some $\zeta_1 \in (x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2})$.

By Newton's second law, the equation of motion for the elemnet of fund mass δm , at point $\mathbf{x}(t)$ is

$$\frac{d}{dt} (\delta m \cdot \mathbf{u}(\mathbf{x}, t)) = \mathbf{F} = -(\nabla P) \delta V \tag{24}$$

also

$$\frac{d}{dt} (\delta m \cdot \mathbf{u}(\mathbf{x}, t)) = \delta m \frac{d}{dt} \mathbf{u}(\mathbf{x}, t) = \delta m (\mathbf{u}_t + (\nabla \cdot \mathbf{u})) \mathbf{u} \tag{25}$$

then

$$\begin{aligned}
 \delta m (\mathbf{u}_t + (\nabla \cdot \mathbf{u})) \mathbf{u} &= -(\nabla P) \delta V \\
 \mathbf{u}_t + (\nabla \cdot \mathbf{u}) \mathbf{u} &= -(\nabla P) \frac{\delta V}{\delta m} = -(\nabla P) \frac{1}{\delta m / \delta V}
 \end{aligned} \tag{26}$$

we get a equation

$$\mathbf{u}_t + (\nabla \cdot \mathbf{u}) \mathbf{u} = -\frac{\nabla P}{\rho} \tag{27}$$

called *Euler's equation*.

Notice that

$$\begin{aligned}
 (\nabla \cdot \mathbf{u}) \mathbf{u} &= \left(\sum_{i=1}^3 u_i \frac{\partial}{\partial x_i} \right) \mathbf{u} \\
 &= \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\
 &= \begin{pmatrix} \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_1 \\ \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_2 \\ \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_3 \end{pmatrix}
 \end{aligned} \tag{28}$$

1.1.5 Lemma

Let D be a bounded domain and $F : \bar{D} \times [0, a_0] \rightarrow \mathbb{R}$ be a smooth function (or C^∞), then

$$\frac{d}{dt} \int_D F(x, t) dx = \int_D \frac{dF(x, t)}{dt} dx \quad (29)$$

1.1.5.1 proof

we have

$$\begin{aligned} \frac{d}{dt} \int_D F(x, t) dx &= \lim_{\Delta t \rightarrow 0} \left[\frac{1}{\Delta t} \int_D F(x, t + \Delta t) dx - \frac{d}{dt} \int_D F(x, t) dx \right] \\ &= \lim_{\Delta t \rightarrow 0} \frac{d}{dt} \int_D \frac{F(x, t + \Delta t) - F(x, t)}{\Delta t} dx \\ &= \text{By M.V.T.} \\ &= \lim_{\Delta t \rightarrow 0} \int_D \frac{\frac{\partial}{\partial t} F(x, \xi) \Delta t}{\Delta t} dx, \quad \text{for some } \xi, \text{ where } t < \xi < t + \Delta \\ &= \lim_{\Delta t \rightarrow 0} \int_D \frac{\partial}{\partial t} F(x, \xi) dx \end{aligned} \quad (30)$$

Denote, $\frac{\partial}{\partial t} F(x, t) = F_t(x, t)$ and $\frac{\partial^2}{\partial t^2} F(x, t) = F_{tt}(x, t)$, so

$$\begin{aligned} &\left| \frac{1}{\Delta t} \int_D [F(x, t + \Delta t) - F(x, t)] - \int_D \frac{\partial}{\partial t} F(x, t) dx \right| \\ &= \left| \int_D F_t(x, \xi) dx - \int_D F_t(x, t) dx \right| \\ &= \text{By MVT} \\ &= \left| \int_D [F_t(x, \xi) - F_t(x, t)] dx \right| \\ &= \text{By MVT} \\ &= \left| \int_D F_{tt}(x, z)(t - \xi) dz \right|, \quad z \text{ between } t \text{ and } \xi \\ &\leq M |t - \xi| |D| \rightarrow 0, \quad \text{where } |D| \text{ is volume of } D \end{aligned} \quad (31)$$

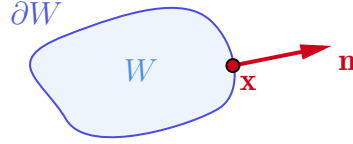
where $M = \sup_{(x, t) \in D \times (0, a)} F_{tt}(x, t)$.

1.1.6 The Continuity Equation

Recall that D is a region in \mathbb{R}^2 or \mathbb{R}^3 filled with fluid and $x = x(x_1, x_2, x_3) \in D$ is a particle of fluid moving through x at time t with velocity $\mathbf{u}(x, t)$. If W is a ant subregion of D , then the mass of fluid in W at time t is

$$m(W, t) = \int_W \rho(x, t) dV, \quad (32)$$

where $\rho(x, t)$ is the density of fluid at (x, t) . Then $\frac{d}{dt} m(W, t) = \int_W \rho_t(x, t) dV$.



Let ∂W be the boundary of W . Suppose ∂W is smooth, let $\mathbf{n}(x)$ be the normal vector to ∂W at $x \in \partial W$. Let dA denote the

Then the volume of fluid flow rate across ∂W per unit time, Since $\mathbf{u} \cdot \mathbf{n} \Delta A \Rightarrow$ the mass of fluid flow per unit time is $\rho \mathbf{u} \cdot \mathbf{n} \Delta A$

Since, by the conservation of mass, the rate of increase of mass in W is equal to the rate that mass is incoming ∂W

$$\int_W \rho_t dV = \frac{d}{dt} \int_W \rho dV = - \int_{\partial W} \rho (\mathbf{n} \cdot \mathbf{u}) dV = \text{by divergence theorem} = - \int_W \text{div}(\rho \mathbf{u}) dV \quad (33)$$

By divergence theorem, we have

$$\int_W (\rho_t + \text{div}(\rho \mathbf{u})) dV = 0, \quad \forall W \subset D \quad (34)$$

We now choose $W = B(x, r)$, and let $H(y, t) = \rho_t + \text{div}(\rho \mathbf{u}) = H$, by above equation, we have

$$\int_{B(x, r)} H(y, t) dV_x = 0, \quad \forall x \in D, B(x, r) \subset D \quad (35)$$

Notice $H(y, t) = \frac{1}{|B(x, r)|} \int_{B(x, r)} H(y, t) dV = H(y, t) \frac{\int_{B(x, r)} dV}{|B(x, r)|}$, where $|B(x, r)|$ is the volume of $B(x, r)$. Now

$$\begin{aligned} \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} H(x, t) dV - H(x, t) \right| &= \frac{1}{|B(x, r)|} \left| \int_{B(x, r)} [H(y, t) - H(x, t)] dV \right| \\ &\leq \frac{1}{|B(x, r)|} \int_{B(x, r)} |H(y, t) - H(x, t)| dV \\ &\leq \frac{1}{|B(x, r)|} \max_{y \in B(x, r)} |H(y, t) - H(x, t)| \cdot |B(x, r)| \\ &\rightarrow 0, \quad \text{as } r \rightarrow 0 \end{aligned} \quad (36)$$

so that

$$\lim_{r \rightarrow 0} \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} H(x, t) dV - H(x, t) \right| = 0 \quad (37)$$

By equation (35) and (37), we have

$$H(x, t) = 0 \quad \Rightarrow \quad \rho_t + \text{div}(\rho \mathbf{u}) = 0 \quad (38)$$

which is called the continuity equation in $D \times (0, T)$.

1.2 Proof of Euler's Equation

1.2.1 Balance of Momentum 1 (BM1)

The force per unit area on a point $x \in \partial W$ is $-p \cdot \mathbf{n}$, since the total force on W due to the pressure on ∂W is

$$\begin{aligned}
 \mathbf{f} &= - \int_{\partial W} p \cdot \mathbf{n} dA \\
 &= \left(- \int_{\partial W} p n_1 dA, - \int_{\partial W} p n_2 dA, - \int_{\partial W} p n_3 dA \right) \\
 &= \left(- \int_{\partial W} (p, 0, 0) \cdot \mathbf{n} dA, - \int_{\partial W} (0, p, 0) \cdot \mathbf{n} dA, - \int_{\partial W} (0, 0, p) \cdot \mathbf{n} dA \right) \\
 &= \text{by divergence theorem} \\
 &= \left(- \int_W \operatorname{div}(p, 0, 0) dV, - \int_W \operatorname{div}(0, p, 0) dV, - \int_W \operatorname{div}(0, 0, p) dV \right) \\
 &= - \int_W (\nabla p) dV
 \end{aligned} \tag{39}$$

Now, the total force on the fluid due to the pressure

$$\partial W = - \int_W \nabla p dV \tag{40}$$

If $\mathbf{b}(\mathbf{x}, t)$ denotes the given body force per unit mass (ex: gravity), then the total body force is

$$F_B = \int_W \rho \cdot \mathbf{b} \cdot dV. \tag{41}$$

By equations (40) and (41), the force per unit volume is $-\nabla p + \rho \mathbf{b}$

By the Newton's second law,

$$\frac{D}{Dt}(\delta m \mathbf{u}) = \frac{D}{Dt}(\delta m \mathbf{u}(\mathbf{x}, t)) = (-p + \rho \mathbf{b}) \delta V \tag{42}$$

we then have

$$\Rightarrow \rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b} \quad \text{BM1 (Euler equation)} \tag{43}$$

1.2.2 Balance of Momentum 2 (BM2)

WLOG. we write \mathbf{u} for $\mathbf{u} = (u_1, u_2, u_3)$ and b for \mathbf{b} , Integral from of balance of momentum

By (BM1)

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \rho \mathbf{b} \tag{44}$$

then by 44 and continuity equation

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = \rho_t \mathbf{u} + \rho \mathbf{u}_t = -\operatorname{div}(\rho \mathbf{u}) \mathbf{u} - \rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \rho \mathbf{b} \tag{45}$$

Let \mathbf{e} be a fixed vector in space, then

$$\begin{aligned}
 \mathbf{e} \cdot \frac{\partial}{\partial t}(\rho \mathbf{u}) &= -\operatorname{div}(\rho \mathbf{u}) \mathbf{u} \cdot \mathbf{e} - \rho(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{e} - (\nabla p) \cdot \mathbf{e} + \rho \mathbf{b} \cdot \mathbf{e} \\
 &= -\operatorname{div}(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e})) - \operatorname{div}(p \mathbf{e}) + \rho \mathbf{b} \cdot \mathbf{e} \\
 &= -\operatorname{div}(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e}) + p \mathbf{e}) + \rho \mathbf{b} \cdot \mathbf{e}
 \end{aligned} \tag{46}$$

Since, we have

1. the divergence of $p\mathbf{e}$

$$\begin{aligned}\operatorname{div}(p\mathbf{e}) &= \operatorname{div}((pe_1, pe_2, pe_3)) \\ &= \frac{\partial p}{\partial x_1}e_1 + \frac{\partial p}{\partial x_2}e_2 + \frac{\partial p}{\partial x_3}e_3 \\ &= \sum_{i=1}^3 \frac{\partial p}{\partial x_i}e_i = \nabla p \cdot \mathbf{e}\end{aligned}\tag{47}$$

1. the divergence of $\rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})$

$$\begin{aligned}\operatorname{div}(\rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) &= \sum_{i=1}^3 \frac{\partial}{\partial x_i} [\rho u_i (\mathbf{u} \cdot \mathbf{e})] \\ &= \sum_{i=1}^3 \left(\frac{\partial}{\partial x_i} (\rho u_i) \right) (\mathbf{u} \cdot \mathbf{e}) + \sum_{i=1}^3 (\rho u_i) \left(\frac{\partial}{\partial x_i} (\mathbf{u} \cdot \mathbf{e}) \right) \\ &= \operatorname{div}(\rho\mathbf{u})(\mathbf{u} \cdot \mathbf{e}) + \rho(\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \mathbf{e})\end{aligned}\tag{48}$$

Hence, if W is a fixed region in space in the fluid

$$\begin{aligned}\mathbf{e} \cdot \frac{d}{dt} \int_W \rho \mathbf{u} dV &= \int_W \mathbf{e} \cdot \frac{d}{dt} (\rho \mathbf{u}) dV \\ &= - \int_W \operatorname{div}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) dV + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV \\ &= \text{By divergence theorem.} \\ &= - \int_{\partial W} (p\mathbf{e} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) \cdot \mathbf{n} dA + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV \\ &= - \int_{\partial W} p\mathbf{e} \cdot \mathbf{n} dA - \int_{\partial W} \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e}) \cdot \mathbf{n} dA + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV, \quad \forall \mathbf{e} \in \mathbb{R}^n, n = 2 \text{ or } 3\end{aligned}\tag{49}$$

then

$$\frac{d}{dt} \int_W \mathbf{u} dV = - \int_{\partial W} p \mathbf{n} dA - \int_{\partial W} \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dA + \int_{\partial W} \rho \mathbf{b} \cdot \mathbf{e} dV\tag{50}$$

or

$$\frac{d}{dt} \int_W \mathbf{u} dV = - \int_{\partial W} (p\mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u}) dA + \int_{\partial W} \rho \mathbf{b} \cdot \mathbf{e} dV, \quad \text{BM2}\tag{51}$$

and BM2 is also the Integral form of balance of momentum.

Note: The quantity $p\mathbf{n} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{n})$ is the momentum per unit area crossing ∂W when \mathbf{n} is unit vector outer normal to ∂W .

1.2.3 Balance of Momentum 3 (BM3)

Let D be a region that the fluid is moving and $x \in D$

Let $\varphi(\mathbf{x}, t)$ be the trajectory of the particle that is at point x , i.e. φ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} \varphi(\mathbf{x}, t) &= \mathbf{u}(\varphi(\mathbf{x}, t), t) \quad \forall t > 0 \text{ at time } t \\ \varphi(\mathbf{x}, 0) &= x \quad \varphi(\mathbf{x}, t) = \varphi_t(\mathbf{x}) \end{aligned} \quad (52)$$

We will assume that φ is smooth and for fixed t , $\varphi_t : t \rightarrow \varphi(\mathbf{x}, t)$ is invertible.

φ_t doesn't mean $\partial/\partial t$ here!

We called φ is the fluid flow map.

If W is the a region in D , then $W_t := \varphi_t(W)$ is the region of the fluid at time t whose initial position is in W at time t .

Then by the balance of momentum

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = \mathbf{F}_{\partial W_t} + \int_{W_t} \rho \mathbf{b} dV, \quad (53)$$

where $\mathbf{F}_{\partial W_t}$ is the force on ∂W_t due to pressure, i.e.

$$\mathbf{F}_{\partial W_t} = - \int_{\partial W_t} p \mathbf{n} dA = - \int_{W_t} \nabla p dV \quad (54)$$

so that

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV, \quad \text{BM3} \quad (55)$$

Recall

$$\frac{d}{dt}(\delta V) = (\text{div } \mathbf{u}) \delta V \quad (56)$$

for an infinitesimal volume δV of fluid moving in the fluid with the fluid velocity, We will now give a rigorous proof of the result.

Now that

$$\begin{aligned} \text{volume}(W_t) &= \int_{W_t} 1 dV \\ &= \int_{W_t} 1 dy, \quad \text{put } y \text{ to be } \varphi_t(\mathbf{x}) \\ &= \int_W J(\mathbf{x}, t) d\mathbf{x}, \end{aligned} \quad (57)$$

where $J(\mathbf{x}, t)$ is the Jacobian determinant of the map φ_t , so that

$$\frac{d}{dt} \text{volume}(W_t) = \int_W \frac{\partial}{\partial t} J(\mathbf{x}, t) d\mathbf{x} \quad (58)$$

1.3 Equivalence between BM1, BM2 and BM3

Quick Summary of *Balance of Momentum*

1. BM1: $\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b}$
2. BM2: $\frac{d}{dt} \int_W \rho \mathbf{u} dV = \int_{\partial W} (p \mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u}) dA + \int_W \rho \mathbf{b} dV$
3. BM3: $\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV$

1.3.1 Lemma

$$\frac{\partial J(\mathbf{x}, t)}{\partial t} = \operatorname{div}(\mathbf{u}(y, t)) \cdot J(\mathbf{x}, t) \quad (59)$$

1.3.1.1 proof:

$\mathbf{y} = \varphi(\mathbf{x}, t) = (y_1, y_2, y_3)$, and $\mathbf{x} = (x_1, x_2, x_3)$. Observe that

$$\frac{\partial \varphi}{\partial t} = \mathbf{u}(\varphi(\mathbf{x}, t), t), \quad \text{or} \quad \frac{\partial y_i}{\partial t} = u_i(y, t), \quad \forall i = 1, 2, 3 \quad (60)$$

then

$$\begin{aligned} J(\mathbf{x}, t) &= \operatorname{div} \left(\frac{\partial y_i}{\partial x_j} \right)_{1 \leq i \leq 3} \\ &= \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \end{aligned} \quad (61)$$

here S is the family of the permutations $\{1, 2, 3\}$, and

$$\operatorname{sign} \sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation;} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases} \quad (62)$$

then

$$\begin{aligned} \frac{\partial J(\mathbf{x}, t)}{\partial t} &= \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial^2 y_2}{\partial t \partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial^2 y_3}{\partial t \partial x_{\sigma(3)}} \\ &= I_1 + I_2 + I_3 \end{aligned} \quad (63)$$

then calculate I_1 first

$$\begin{aligned}
 I_1 &= \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \sum_{\sigma \in S} (\text{sign } \sigma) \left(\frac{\partial}{\partial x_{\sigma(2)}} \left(\frac{\partial y_1}{\partial t} \right) \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \sum_{\sigma \in S} (\text{sign } \sigma) \left(\frac{\partial u_1}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \sum_{\sigma \in S} (\text{sign } \sigma) \left(\sum_{k=1}^3 \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \sum_{\sigma \in S} \sum_{k=1}^3 (\text{sign } \sigma) \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \sum_{k=1}^3 \frac{\partial u_1}{\partial y_k} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \frac{\partial u_1}{\partial y_1} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &\quad + \frac{\partial u_1}{\partial y_2} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &\quad + \frac{\partial u_1}{\partial y_3} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_3}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
 &= \frac{\partial u_1}{\partial y_1} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} + 0 + 0 \\
 &= \frac{\partial u_1}{\partial y_1} J(\mathbf{x}, t)
 \end{aligned} \tag{64}$$

using the same we can get

$$\begin{aligned}
 \frac{\partial J(\mathbf{x}, t)}{\partial t} &= I_1 + I_2 + I_3 \\
 &= \frac{\partial u_1}{\partial y_1} J(\mathbf{x}, t) + \frac{\partial u_2}{\partial y_2} J(\mathbf{x}, t) + \frac{\partial u_3}{\partial y_3} J(\mathbf{x}, t) \\
 &= \left(\sum_{i=1}^3 \frac{\partial u_i}{\partial y_i} \right) J(\mathbf{x}, t) \\
 &= \text{div}_y(\mathbf{u}) J(\mathbf{x}, t)
 \end{aligned} \tag{65}$$

1.3.2 Transport Theorem

For any smooth function $f : D \times [0, T] \rightarrow \mathbb{R}$, we have

$$\frac{d}{dt} \int_{W_t} \rho f dV_y = \int_{W_t} \rho \frac{Df}{Dt} dV_y \tag{66}$$

1.3.2.1 proof

Change $W_t \rightarrow W$

$$\int_{W_t} \rho f dV_y = \int_W \rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) dx \quad (67)$$

we have

$$\begin{aligned} \frac{d}{dt} \int_{W_t} \rho f dV &= \frac{d}{dt} \int_W \rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) dx \\ &= \int_W \frac{d}{dt} \left(\rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} \left(\rho(y, t) f(y, t) J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} \left(\rho(y, t) f(y, t) \right) J(\mathbf{x}, t) dx + \int_W \rho(y, t) f(y, t) \frac{d}{dt} \left(J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} (\rho f) J(\mathbf{x}, t) dx + \int_W (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x}, t) dx \\ &= \int_W \left(\frac{d}{dt} (\rho f) + (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x}, t) \right) dx \\ &= \int_W \left(\frac{\partial}{\partial t} (\rho f) + \nabla_y(\rho f) \cdot \left(\frac{dy}{dt} \right) + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left(\frac{\partial}{\partial t} (\rho f) + \nabla_y(\rho f) \cdot \mathbf{u} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV_y \end{aligned} \quad (68)$$

and consider the first term in the integral

$$\begin{aligned}
 \frac{D(\rho f)}{Dt} &= \frac{\partial(\rho f)}{\partial t} + \mathbf{u} \cdot \nabla_y(\rho f) \\
 &= f \frac{\partial \rho}{\partial t} + \rho \frac{\partial f}{\partial t} + \mathbf{u} \cdot (f \nabla_y \rho + \rho \nabla_y f) \\
 &= f \left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla_y \rho \right) + \rho \left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla_y f \right) \\
 &= f \left(\rho_t + \mathbf{u} \cdot \nabla_y \rho \right) + \rho \left(f_t + \mathbf{u} \cdot \nabla_y f \right) \\
 &= f \left(\rho_t + \operatorname{div}(\rho \mathbf{u}) - \rho \operatorname{div}(\mathbf{u}) \right) + \rho \left(f_t + \mathbf{u} \cdot \nabla_y f \right) \\
 &= f \underbrace{\left(\rho_t + \operatorname{div}(\rho \mathbf{u}) \right)}_{\text{continuity equation}=0} - f \rho \operatorname{div}(\mathbf{u}) + \rho \left(f_t + \mathbf{u} \cdot \nabla_y f \right) \\
 &= -f \rho \operatorname{div}(\mathbf{u}) + \rho \left(f_t + \mathbf{u} \cdot \nabla_y f \right) \\
 &= \rho \left(f_t + \mathbf{u} \cdot \nabla_y f \right) - \rho f \operatorname{div}(\mathbf{u}) \\
 &= \rho \frac{Df}{Dt} - (\rho f) \operatorname{div}(\mathbf{u})
 \end{aligned} \tag{69}$$

then plugin to the integral

$$\begin{aligned}
 \frac{d}{dt} \int_{W_t} \rho f dV &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV \\
 &= \int_W \left(\rho \frac{Df}{Dt} - (\rho f) \operatorname{div}(\mathbf{u}) + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV \\
 &= \int_W \left(\rho \frac{Df}{Dt} \right) dV
 \end{aligned} \tag{70}$$

we get

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_W \left(\rho \frac{Df}{Dt} \right) dV, \quad \forall f \text{ is smooth function} \tag{71}$$

Notice that:

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_W \left(\rho \frac{Df}{Dt} \right) dV \quad (72)$$

so when we consider a vector function \mathbf{u} , we can write

$$\begin{aligned} \frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV &= \frac{d}{dt} \int_{W_t} \rho(u_1, u_2, u_3) dV \\ &= \left(\frac{d}{dt} \int_{W_t} \rho u_1 dV, \frac{d}{dt} \int_{W_t} \rho u_2 dV, \frac{d}{dt} \int_{W_t} \rho u_3 dV \right) \\ &= \left(\int_W \left(\rho \frac{Du_1}{Dt} \right) dV, \int_W \left(\rho \frac{Du_2}{Dt} \right) dV, \int_W \left(\rho \frac{Du_3}{Dt} \right) dV \right) \\ &= \int_W \left(\rho \frac{Du_1}{Dt}, \rho \frac{Du_2}{Dt}, \rho \frac{Du_3}{Dt} \right) dV \\ &= \int_W \rho \frac{D}{Dt} (u_1, u_2, u_3) dV \\ &= \int_W \rho \frac{D\mathbf{u}}{Dt} dV \end{aligned} \quad (73)$$

Now we rewrite BM3

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV \quad (74)$$

to be

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV = 0 \quad (75)$$

then using the result from above

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\ 0 &= \int_{W_t} \rho \frac{D\mathbf{u}}{Dt} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\ 0 &= \int_{W_t} \left(\rho \frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b} \right) dV, \quad \forall W_t \end{aligned} \quad (76)$$

imply

$$\rho \frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b} = 0 \quad (77)$$

which is just BM1, this means BM3 is equivalent to BM1.

Similarly,

$$\frac{d}{dt} \int_W \rho \mathbf{u} dV = \int_{\partial W_t} p \mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dA + \int_W \rho \mathbf{b} dV \quad (78)$$

so that

$$\text{BM1} \Leftrightarrow \text{BM2} \Leftrightarrow \text{BM3} \quad (79)$$

By an argument similar to the proof of the transport theorem, for any smooth function $f : D \times [0, T] \rightarrow \mathbb{R}$, we have

$$\frac{d}{dt} \left(\int_{W_t} f dV \right) = \int_{W_t} \left(\frac{Df}{Dt} + f(\text{div } \mathbf{u}) \right) dV \quad (80)$$

or in the other form as the textbook

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f \mathbf{u}) \right) dV, \quad (81)$$

since $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$, and these 2 equations (80) and (81) is called the *Transport theorem without mass density*.

Rmk: Here we consider the *Transport theorem* (with mass density):

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_{W_t} \rho \frac{Df}{Dt} dV, \quad (82)$$

if we take ρ to be a constant (ex: $\rho = 1$), we have

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \frac{Df}{Dt} dV, \quad (83)$$

however compare to the equation without no mass density (80),

$$\frac{d}{dt} \left(\int_{W_t} f dV \right) = \int_{W_t} \frac{Df}{Dt} dV + \int_{W_t} f(\operatorname{div} \mathbf{u}) dV \quad (84)$$

we have an extra term contain $f(\operatorname{div} \mathbf{u})$. **BUT**, if we consider more carefully, the mass density ρ here must satisfy the continuity equation, i.e.

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (85)$$

so that if mass density is a constant (ex: $\rho = 1$), we have

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 + \operatorname{div}(\mathbf{u}) = 0 \quad (86)$$

so that $\operatorname{div} \mathbf{u} = 0$, which means the extra term vanishing, and the *transport theorem with mass density* and *transport theorem without mass density* are equivalent.

1.3.3 Def:

And ideal fluid is one in which there are no shear stresses. Hence Euler's equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{b} \quad (87)$$

holds for ideal fluid.

1.4 Incompressible

We call a flow incompressible if for any subregion $W \subset D$

$$\operatorname{volumn}(W_t) = \int_{W_t} dV = \text{constant}. \quad (88)$$

int time t , $W_t = \varphi_t(W)$, where φ_t is flow map.

Then a flow is incompressible if and only if

$$\begin{aligned}
 0 &= \frac{d}{dt} \int_{W_t} dV_y \\
 &= \frac{d}{dt} \int_W J(\mathbf{x}, t) dV_x \\
 &= \int_W \frac{\partial}{\partial t} J(\mathbf{x}, t) dV_x \\
 &= \int_W (\operatorname{div} \mathbf{u}) J(\mathbf{x}, t) dV_x \\
 &= \int_W (\operatorname{div} \mathbf{u}) dV_y
 \end{aligned} \tag{89}$$

imply

$$\operatorname{div} \mathbf{u} = 0 \Leftrightarrow \frac{\partial J}{\partial t} = (\operatorname{div} \mathbf{u})J = 0 \tag{90}$$

so that $J(\mathbf{x}, t)$ is a constant, notice that $J(\mathbf{x}, 0) = 1$, we have

$$J(\mathbf{x}, t) = 1, \quad \forall x \in D, t > 0 \tag{91}$$

Rmk: Since

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \Rightarrow \frac{D\rho}{Dt} = \rho_t + \mathbf{u} \cdot \nabla \rho = -\rho \operatorname{div} \mathbf{u} = 0 \tag{92}$$

so that $D\rho/Dt = 0$. Hence, the mass density is constant following the fluid for incompressible fluid.

1.5 Homogeneous

1.5.0.1 Def:

A fluid is said to be homogeneous, if $\rho(\mathbf{x}, t) = \rho(t), \forall x \in D$

Rmk: For incompressible homogeneous fluid,

$$\rho(\mathbf{x}, t) = \rho(t) \quad \text{and} \quad \frac{D\rho}{Dt} = 0 \tag{93}$$

so that

$$\begin{aligned}
 0 &= \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \\
 &= \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \operatorname{div} \mathbf{u} \\
 &= \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \mathbf{0} + \rho \cdot 0 = \rho_t
 \end{aligned} \tag{94}$$

that is $\rho_t = 0$. Then $\rho(t) = \text{constant} = \rho(0), \forall t > 0$

Rmk: For any subregion $W \subset D$, Let $W_t = \varphi_t(W)$, where φ_t is the flow map. If the fluid is incompressible,

$$\frac{d}{dt} \int_{W_t} \rho dV = \int_{W_t} \frac{D\rho}{Dt} dV = 0, \quad (95)$$

then

$$\int_{W_t} \rho(\mathbf{x}, t) dV_y = \int_W \rho(\varphi, t) J(\mathbf{x}, t) dV_x = \int_W \rho(\mathbf{x}, 0) dV_x \quad (96)$$

Now, we consider

$$0 = \frac{1}{\text{Volumn}(W)} \int_W \left(\rho(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0) \right) dV_x, \quad \forall W \subset D \quad (97)$$

Let $W = B(\mathbf{x}, r)$ and letting $r \rightarrow 0$, we have

$$\lim_{r \rightarrow 0} \frac{1}{\text{Volumn}(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \left(\rho(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0) \right) dV_x. \quad (98)$$

Yield that

$$\rho(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) - \rho(\mathbf{x}, 0) = 0. \quad (99)$$

Hence

$$\rho(\varphi(\mathbf{x}, t), t) = \rho(\mathbf{x}, 0), \quad \forall \mathbf{x} \in D, t > 0 \quad (100)$$

if the fluid is incompressible.

Rmk:

- Incompressible: $\rho(\varphi(\mathbf{x}, t), t) = \rho(\mathbf{x}, 0), \quad \forall \mathbf{x} \in D, t > 0$
- Homogeneous: $\rho(\mathbf{x}, t) = \rho(t), \quad \forall x \in D$

e.g. For $\varphi(\mathbf{x}, t) = \varphi((x_1, x_2, x_3), t) = ((1+t)x_1, x_2, x_3)$, so the Jacobian

$$J(\mathbf{x}, t) = \begin{vmatrix} 1+t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1+t \quad (101)$$

We can choose $\rho(\mathbf{x}, t) = \frac{1}{1+t}$, then the fluid is compressible but homogeneous.

1.5.0.2 Example

Consider an incompressible homogeneous fluid in a region then the density ρ is a constant.

Then $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p / \rho = 0$ (Euler's equation) in D , since it is incompressible, $\text{div } \mathbf{u} = 0$ in D . That is

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\nabla p}{\rho} = 0 \\ \text{div } \mathbf{u} = 0 \end{cases}, \quad \text{in } D. \quad (102)$$

Taking the derivative of Euler's equation

$$\begin{aligned}
& \operatorname{div} \left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\nabla p}{\rho} \right) = 0 \\
& \operatorname{div}(\mathbf{u}_t) + \operatorname{div}((\mathbf{u} \cdot \nabla) \mathbf{u}) + \operatorname{div} \left(\frac{\nabla p}{\rho} \right) = 0 \\
& \frac{\partial}{\partial t} \underbrace{\operatorname{div}(\mathbf{u})}_{=0} + \operatorname{div}((\mathbf{u} \cdot \nabla) \mathbf{u}) + \operatorname{div} \left(\frac{\nabla p}{\rho} \right) = 0 \\
& \operatorname{div}((\mathbf{u} \cdot \nabla) \mathbf{u}) + \operatorname{div} \left(\frac{\nabla p}{\rho} \right) = 0
\end{aligned} \tag{103}$$

Now, we first calculate

$$\begin{aligned}
\operatorname{div}((\mathbf{u} \cdot \nabla) \mathbf{u}) &= \sum_i \frac{\partial}{\partial x_i} (\mathbf{u} \cdot \nabla) u_i \\
&= \sum_i \frac{\partial}{\partial x_i} \left(\sum_j u_j \frac{\partial}{\partial x_j} \right) u_i \\
&= \sum_i \sum_j \frac{\partial}{\partial x_i} \left(\left(u_j \frac{\partial}{\partial x_j} \right) u_i \right) \\
&= \sum_i \sum_j \left(\frac{\partial}{\partial x_i} \left(u_j \frac{\partial}{\partial x_j} \right) \right) u_i + \left(u_j \frac{\partial}{\partial x_j} \right) \left(\frac{\partial}{\partial x_i} u_i \right) \\
&= \sum_i \sum_j \left(\left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial}{\partial x_j} \right) \right) u_i + \left(u_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) u_i + \left(u_j \frac{\partial}{\partial x_j} \right) \left(\frac{\partial}{\partial x_i} u_i \right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right) + 2 \left(u_j \frac{\partial}{\partial x_j} \right) \left(\frac{\partial u_i}{\partial x_i} \right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right) + 2 \sum_i \sum_j \left(u_j \frac{\partial}{\partial x_j} \right) \left(\frac{\partial u_i}{\partial x_i} \right) \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right) + 2 \sum_j \left(u_j \frac{\partial}{\partial x_j} \right) \operatorname{div} \mathbf{u} \\
&= \sum_i \sum_j \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right) + 2 (\mathbf{u} \cdot \nabla) \operatorname{div} \mathbf{u}
\end{aligned} \tag{104}$$

notice $\operatorname{div} \mathbf{u} = 0$, we then have

$$\operatorname{div}((\mathbf{u} \cdot \nabla) \mathbf{u}) + \frac{\nabla p}{\rho} = \sum_{i,j} \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right) + \frac{\nabla p}{\rho} = 0 \tag{105}$$

So the Euler's equation becomes

$$\nabla p = -\rho \sum_{i,j} \left(\frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} \right). \tag{106}$$

Since the fluid is confined in fixed region of space by D , therefore the fluid cannot move into $\mathbb{R}^3 \setminus D$ (out side), so the normal component of the fluid satisfied

$$\mathbf{u} \cdot \mathbf{n} \Big|_{\partial D} = 0 \quad \Rightarrow \quad \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{n} \Big|_{\partial D} = 0, \quad (107)$$

then by Euler's equation, we have

$$\frac{\partial p}{\partial \mathbf{n}} \Big|_{\partial D} = \nabla p \cdot \mathbf{n} = -\rho(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{n} \Big|_{\partial D}. \quad (108)$$

For the flat boundary surfaces, e.g. the wall of fixed rectangular box, $\mathbf{n} = (n_1, n_2, n_3)$ and

$$\begin{aligned} \mathbf{n} \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) &= \sum_{i,j} n_i \left(u_i \frac{\partial}{\partial x_j} u_i \right) \\ &= \sum_{i,j} u_j \frac{\partial}{\partial x_j} (u_i n_i) \\ &= \sum_j u_j \frac{\partial}{\partial x_j} (\mathbf{u} \cdot \mathbf{n}) = 0, \quad \text{Since } \mathbf{u} \cdot \mathbf{n} \Big|_{\partial D} = 0 \end{aligned} \quad (109)$$

on each flat surface. Since

$$\frac{\partial p}{\partial n} \Big|_{\partial D} = 0 \quad (110)$$

on each flat surface. Then one can solve

$$\begin{cases} \nabla p = -\rho \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}, \\ \frac{\partial p}{\partial \mathbf{n}} \Big|_{\partial D} = 0, \quad (\text{If the surface is flat.}) \end{cases} \quad (111)$$

using standard PDE method.

Rmk: If the surface are not flat, the equations become

$$\begin{cases} \nabla p = -\rho \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}, \\ \frac{\partial p}{\partial \mathbf{n}} \Big|_{\partial D} = -(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{n} \Big|_{\partial D} \end{cases} \quad (112)$$

1.6 Consevation of Energy

For fluid moving in a domain D with velocity \mathbf{u} , the kinetic energy of the fluid in W is

$$E_{\text{kinetic}} = \frac{1}{2} \int_W \rho \cdot \|\mathbf{u}\|^2 dV, \quad (113)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $\|\mathbf{u}\|^2 = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

Rmk: When $W = D$ and the fluid is incompressible with no external force acting on fluid, we have

$$\begin{aligned}
 \frac{d}{dt} E_{\text{kinetic}} &= \frac{1}{2} \frac{d}{dt} \int_W \rho \cdot \|\mathbf{u}\|^2 dV \\
 &= \frac{1}{2} \int_D \frac{\partial}{\partial t} (\rho \|\mathbf{u}\|^2) dV \\
 &= \frac{1}{2} \int_D \rho_t \|\mathbf{u}\|^2 + \rho \frac{\partial}{\partial t} \|\mathbf{u}\|^2 dV \\
 &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u}) \|\mathbf{u}\|^2 + 2\rho \mathbf{u} \cdot \mathbf{u}_t \right) dV \\
 &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) + (\rho \mathbf{u}) \nabla \|\mathbf{u}\|^2 + 2\rho \mathbf{u} \cdot \mathbf{u}_t \right) dV
 \end{aligned} \tag{114}$$

also notice that the euler equation $\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p / \rho = 0$, pluin

$$\begin{aligned}
 \frac{d}{dt} E_{\text{kinetic}} &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) + (\rho \mathbf{u}) \cdot \nabla \|\mathbf{u}\|^2 + 2\rho \mathbf{u} \cdot \mathbf{u}_t \right) dV \\
 &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) + (\rho \mathbf{u}) \cdot 2((\mathbf{u} \cdot \nabla) \mathbf{u}) + 2\rho \mathbf{u} \cdot \left(-(\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{\nabla p}{\rho} \right) \right) dV \\
 &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) + 2\rho \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} - 2\rho \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} - 2\rho \mathbf{u} \cdot \frac{\nabla p}{\rho} \right) dV \\
 &= \frac{1}{2} \int_D \left(-\operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) - 2\rho \mathbf{u} \cdot \frac{\nabla p}{\rho} \right) dV \\
 &= -\frac{1}{2} \int_D \operatorname{div}(\rho \mathbf{u} \|\mathbf{u}\|^2) dV - \int_D (\mathbf{u} \cdot \nabla p) dV \\
 &= -\frac{1}{2} \int_{\partial D} \rho \|\mathbf{u}\|^2 \mathbf{u} \cdot \mathbf{n} d\sigma - \int_D (\mathbf{u} \cdot \nabla p) dV
 \end{aligned} \tag{115}$$

Since, $\mathbf{u} \cdot \mathbf{n} = 0$,

Rmk: When E_{kinetic}