

# Class Notes

## Introduction to fluid mechanics

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# 1 The Equation of Motion

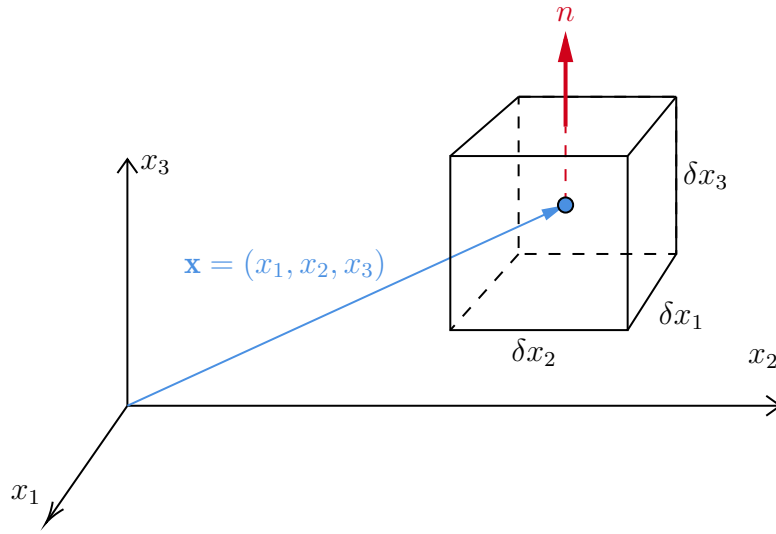
## 1.1 Introduction

### 1.1.1 Euler's equation:

Consider a fluid in a domain  $D$  in  $\mathbb{R}^n$  ( $n = 2, n = 3$ ).

Let  $x \in D$ , and  $\rho(\mathbf{x}, t)$ ,  $\mathbf{u}(\mathbf{x}, t)$ ,  $p(\mathbf{x}, t)$  be the fluid density, velocity vector field and the pressure at the point  $x$  and time  $t$ . Consider an infinitesimal element of the fluid of volume  $\partial V$  located at point  $x$  at time  $t$  with mass  $\delta m = \rho(\mathbf{x}, t) \delta V$ , which is moving  $\mathbf{u}(\mathbf{x}, t)$  and momentum  $\delta m \cdot \mathbf{u}(\mathbf{x}, t)$

The normal force directed into the infinitesimal volume across a face of area  $\delta a$  is  $\mathbf{n} \cdot p(\mathbf{x}, t) \cdot \delta a$



Note that the pressure is the magnitude of the force per unit area or normal stress, imposed on the fluid from neighboring fluid elements.

### 1.1.2 Convective derivative

convective derivative 對流導數 / material derivative 物質導數 / advective derivative 隨流導數 / convective derivative 對流導數 / derivative following the motion 隨體導數 / hydrodynamic derivative 水動力導數 / Lagrange derivative 拉格朗日導數 / substantial derivative 隨質導數 Consider a fluid particle moving in fluid, whose position  $\mathbf{x}$  at time  $t$  is  $\mathbf{x}(t)$ . Then

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t) \quad (1)$$

Hence, if  $f(\mathbf{x}, t)$  is a function on  $D \times (0, T)$ , then  $f(\mathbf{x}(t), t)$  is the value of  $f$  at the moving fluid particle at  $\mathbf{x}(t)$  at time  $t$ . We define the convective derivative of  $f$ :

$$\begin{aligned} \frac{Df(\mathbf{x}, t)}{Dt} &= \frac{\partial f(\mathbf{x}, t)}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f(\mathbf{x}, t) \\ &= f_t + \mathbf{u} \cdot \nabla f \end{aligned} \quad (2)$$

where  $\nabla f = (f_x, f_y, f_z)$  and  $\mathbf{u} = (u_1, u_2, u_3)$ .

Hence, if  $f(\mathbf{x}, t)$  is a function on  $D \times (0, T)$ , then  $f(\mathbf{x}(t), t)$  is the value of  $f$  at the moving fluid particle at  $\mathbf{x}(t)$  at time  $t$ .

We define the convective derivative of  $f$  as:

$$\begin{aligned}\frac{Df(x, t)}{Dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{x}}(t) \cdot \nabla f, \\ &= f_t + \mathbf{u} \cdot \nabla f\end{aligned}\tag{3}$$

where  $\nabla f = (f_x, f_y, f_z)$  and  $\mathbf{u} = (u_1, u_2, u_3)$ .

#### 1.1.2.1 Def.

For any vector field  $\mathbf{F} = (F_1, F_2, \dots, F_n)$  on  $D$ , we let

$$\int_D \mathbf{F} dV = \left( \int_D F_1 dV, \int_D F_2 dV, \dots, \int_D F_n dV \right).\tag{4}$$

#### 1.1.2.2 Def.

We will assume that  $D$  is a smooth domain, i.e. for any  $x_0 \in \partial D$ ,  $\mathbb{R}^n = (x', x_n)$ ,  $n = 2, 3$   $\exists \delta_0 > 0$  and a smooth function  $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , s.t.

$$\partial D \cap B(x_0, \delta_0) = \{(x', \varphi(x')) : \|x'\| < \delta_0, x' \in \mathbb{R}^{n-1}\} \cap B(x_0, \delta_0)\tag{5}$$

and

$$D \cap B(x_0, \delta_0) = \{(x', x_n) : x_n > \varphi(x'), x' \in \mathbb{R}^{n-1}, \|x'\| < \delta_0\} \cap B(x_0, \delta_0)\tag{6}$$

#### 1.1.2.3 Claim

Consider the volume  $\delta V$  of an element of mass  $\delta m$ , which moves in the fluid by the fluid motion

$$\frac{d(\delta V)}{dt} = (\nabla \cdot \mathbf{u})(\mathbf{x}, t) \cdot \delta V \quad \text{as} \quad \delta x_1, \delta x_2, \delta x_3 \rightarrow 0,\tag{7}$$

where  $\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}$ ,  $\mathbf{u} = (u_1, u_2, u_3)$ .

#### 1.1.2.4 proof

$$\begin{aligned}\frac{d(\delta V)}{dt} &= \frac{d}{dt}(\delta x_1, \delta x_2, \delta x_3) \\ &= \frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 + \frac{d(\delta x_2)}{dt} \delta x_1 \delta x_3 + \frac{d(\delta x_3)}{dt} \delta x_1 \delta x_2\end{aligned}\tag{8}$$

For the first term

$$\begin{aligned}\frac{d(\delta x_1)}{dt} &\approx u_1 \left( x_1 + \frac{\delta x_1}{2}, x_2, x_3 \right) - u_1 \left( x_1 - \frac{\delta x_1}{2}, x_2, x_3 \right) \\ &= \frac{\partial u_1}{\partial x_1}(\xi_1, x_2, x_3) \delta x_1, \quad \text{for some } \xi_1 \in \left( x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2} \right)\end{aligned}\tag{9}$$

then

$$\frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0\tag{10}$$

Similarly

$$\frac{d(\delta x_2)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_2}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \quad (11)$$

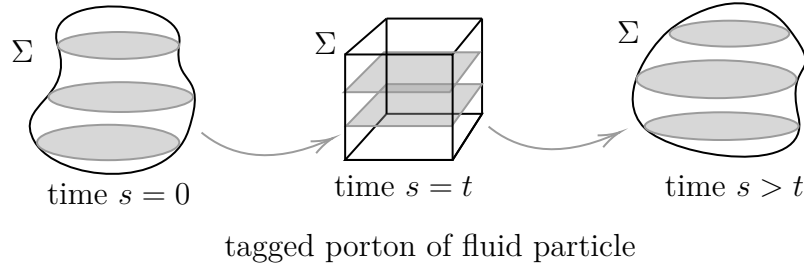
and

$$\frac{d(\delta x_3)}{dt} \delta x_2 \delta x_3 \rightarrow \frac{\partial u_3}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \quad (12)$$

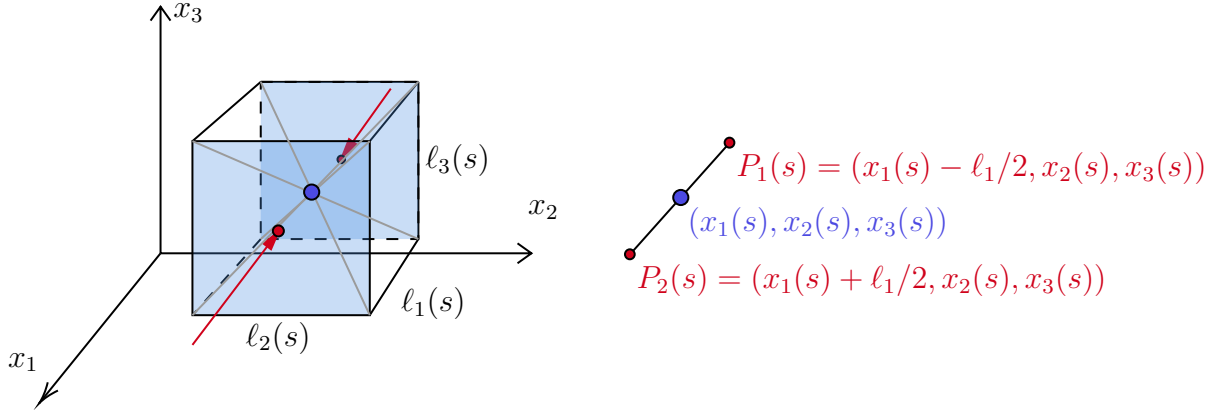
so that

$$\frac{d(\delta V)}{dt} = \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \delta x_1 \delta x_2 \delta x_3 = (\nabla \cdot \mathbf{u}) \delta V \quad (13)$$

#### 1.1.2.5 Note



Consider a a tagged (marked) portion  $\Sigma$  of fluid with center of mass at  $(x_1(s), x_2(s), x_3(s))$  at time  $s$ . Let  $m(x)$  and  $V(s)$  be the mass and volumn of this portion  $\Sigma$  of fluid at time  $s$ . The portion of fluid particle moves along with fluid. see as (2/21 fig1)



For a time  $t > 0$ , suppose at time  $t$ , the tagged portion  $\Sigma$  of fluid particles is a cube centered at  $(x_1, x_2, x_3)$  with side length  $\ell_1, \ell_2, \ell_3$ , see as (2/21 fig2 - textbook p.4), where

$$\begin{aligned} P_1(s) &= \left( x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \\ P_2(s) &= \left( x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s) \right) \end{aligned} \quad (14)$$

We assume that  $\Sigma$  remain a cube for  $s \approx t$  with side length,  $\ell_1(s), \ell_2(s), \ell_3(s)$ , then  $V(s) = \ell_1(s) \cdot \ell_2(s) \cdot \ell_3(s)$

$$\left. \frac{dV(s)}{ds} \right|_{s=t} = \left. \frac{d\ell_1(s)}{ds} \right|_{s=t} \ell_2(s) \ell_3(s) + \left. \frac{d\ell_2(s)}{ds} \right|_{s=t} \ell_1(s) \ell_3(s) + \left. \frac{d\ell_3(s)}{ds} \right|_{s=t} \ell_1(s) \ell_2(s) \quad (15)$$

where

$$\begin{aligned}
\frac{d\ell_1(s)}{ds} &= u_1(P_2(t), t) - u_1(P_1(t), t) \\
&= u_1\left(x_1(s) + \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) - u_1\left(x_1(s) - \frac{\ell_1(s)}{2}, x_2(s), x_3(s), t\right) \quad (16) \\
&\approx \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3, t) \cdot \ell_1
\end{aligned}$$

Similarly

$$\left. \frac{d\ell_i}{ds} \right|_{s=t} = \frac{\partial u_i}{\partial x_i}(x_1, x_2, x_3) \cdot \ell_i, \quad \forall i = 1, 2, 3. \quad (17)$$

Now we write  $\left. \frac{d}{ds} \right|_{s=t} = \frac{d}{dt}$ , combined with equation

### 1.1.3 Continuity equation

Let  $\rho(\mathbf{x}, t)$  be the density of fluid at time  $s$ . Since  $M(s) = \text{const.}$ ,  $\forall s > 0$  and  $\frac{dM(s)}{ds} = 0$ ,  $\forall s > 0$ . Therefore, since it is similar to the cube, the density is

$$\rho(\mathbf{x}, s) \approx \frac{M(s)}{V(s)} \quad (18)$$

and the derivative is

$$\begin{aligned}
\left. \frac{d}{ds} \rho(\mathbf{x}, s) \right|_{s=t} &\approx \left. \frac{d}{ds} \frac{M(s)}{V(s)} \right|_{s=t} \\
&= \left. \frac{M'(s)V(s) - M(s)V'(s)}{V^2(s)} \right|_{s=t} \\
&= \left. \frac{0 - M(s) \frac{d}{ds} V(s)}{V^2(s)} \right|_{s=t} \\
&= - \left. \frac{M(s)(\text{div } \mathbf{u})V(s)}{V^2(s)} \right|_{s=t} \\
&= - \left. \frac{M(s)}{V(s)} (\text{div } \mathbf{u}(s)) \right|_{s=t} \\
&= - \left. \rho(\mathbf{x}(s), s) (\text{div } \mathbf{u}(s)) \right|_{s=t}
\end{aligned} \quad (19)$$

we get

$$-\frac{d}{dt} \rho(\mathbf{x}(t), t) = \rho \cdot (\nabla \cdot \mathbf{u}(t)) \quad (20)$$

On the other hand, by chain rule

$$\frac{d}{dt} \rho(\mathbf{x}(t), t) = \rho_t + (\nabla \rho) \cdot \mathbf{u}(t) \quad (21)$$

combining together we have

$$\begin{aligned}
&\Rightarrow \rho_t + (\nabla \rho) \cdot \mathbf{u} = \rho \cdot (\nabla \cdot \mathbf{u}) \\
&\Rightarrow \rho_t + (\nabla \rho) \cdot \mathbf{u} - \rho \cdot (\nabla \cdot \mathbf{u}) = 0 \\
&\Rightarrow \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0
\end{aligned} \tag{22}$$

and the equation  $\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$  is called the *continuity equation*.

#### 1.1.4 Heuristic proof of the Euler equation

In the ansense of an externally applied forces, the net force  $\mathbf{F}$ , acting on  $\delta V$ , is due to the pressure field.

Write  $\mathbf{F} = (F_1, F_2, F_3)$ , we get

$$\begin{aligned}
\mathbf{F}(x_1, x_2, x_3, t) &\approx \left( P \left( x_1 - \frac{\delta x_1}{2}, x_2, x_3, t \right) - P \left( x_1 + \frac{\delta x_1}{2}, x_2, x_3, t \right) \right) \delta x_2 \delta x_3 \\
&= -\frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t) \delta x_1 \delta x_2 \delta x_3, \quad \delta x_1, \delta x_2, \delta x_3 \rightarrow 0 \\
&= \frac{\partial P}{\partial x_1}(\zeta_1, x_2, x_3, t) \delta V
\end{aligned} \tag{23}$$

for some  $\zeta_1 \in (x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2})$ .

By Newton's second law, the equation of motion for the elemnet of fund mass  $\delta m$ , at point  $\mathbf{x}(t)$  is

$$\frac{d}{dt}(\delta m \cdot \mathbf{u}(\mathbf{x}, t)) = \mathbf{F} = -(\nabla P) \delta V \tag{24}$$

also

$$\frac{d}{dt}(\delta m \cdot \mathbf{u}(\mathbf{x}, t)) = \delta m \frac{d}{dt} \mathbf{u}(\mathbf{x}, t) = \delta m (\mathbf{u}_t + (\nabla \cdot \mathbf{u})) \mathbf{u} \tag{25}$$

then

$$\begin{aligned}
\delta m (\mathbf{u}_t + (\nabla \cdot \mathbf{u})) \mathbf{u} &= -(\nabla P) \delta V \\
\mathbf{u}_t + (\nabla \cdot \mathbf{u}) \mathbf{u} &= -(\nabla P) \frac{\delta V}{\delta m} = -(\nabla P) \frac{1}{\delta m / \delta V}
\end{aligned} \tag{26}$$

we get a equation

$$\mathbf{u}_t + (\nabla \cdot \mathbf{u}) \mathbf{u} = -\frac{\nabla P}{\rho} \tag{27}$$

called *Euler's equation*.

Notice that

$$\begin{aligned}
(\nabla \cdot \mathbf{u}) \mathbf{u} &= \left( \sum_{i=1}^3 u_i \frac{\partial}{\partial x_i} \right) \mathbf{u} \\
&= \left( u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} \left( u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_1 \\ \left( u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_2 \\ \left( u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} \right) u_3 \end{pmatrix}
\end{aligned} \tag{28}$$

### 1.1.5 Lemma

Let  $D$  be a bounded domain and  $F : \bar{D} \times [0, a_0] \rightarrow \mathbb{R}$  be a smooth function (or  $C^\infty$ ), then

$$\frac{d}{dt} \int_D F(x, t) dx = \int_D \frac{dF(x, t)}{dt} dx \quad (29)$$

#### 1.1.5.1 proof

we have

$$\begin{aligned} \frac{d}{dt} \int_D F(x, t) dx &= \lim_{\Delta t \rightarrow 0} \left[ \frac{1}{\Delta t} \int_D F(x, t + \Delta t) dx - \frac{d}{dt} \int_D F(x, t) dx \right] \\ &= \lim_{\Delta t \rightarrow 0} \frac{d}{dt} \int_D \frac{F(x, t + \Delta t) - F(x, t)}{\Delta t} dx \\ &= \text{By M.V.T.} \\ &= \lim_{\Delta t \rightarrow 0} \int_D \frac{\frac{\partial}{\partial t} F(x, \xi) \Delta t}{\Delta t} dx, \quad \text{for some } \xi, \text{ where } t < \xi < t + \Delta \\ &= \lim_{\Delta t \rightarrow 0} \int_D \frac{\partial}{\partial t} F(x, \xi) dx \end{aligned} \quad (30)$$

Denote,  $\frac{\partial}{\partial t} F(x, t) = F_t(x, t)$  and  $\frac{\partial^2}{\partial t^2} F(x, t) = F_{tt}(x, t)$ , so

$$\begin{aligned} &\left| \frac{1}{\Delta t} \int_D [F(x, t + \Delta t) - F(x, t)] - \int_D \frac{\partial}{\partial t} F(x, t) dx \right| \\ &= \left| \int_D F_t(x, \xi) dx - \int_D F_t(x, t) dx \right| \\ &= \text{By MVT} \\ &= \left| \int_D [F_t(x, \xi) - F_t(x, t)] dx \right| \\ &= \text{By MVT} \\ &= \left| \int_D F_{tt}(x, z)(t - \xi) dz \right|, \quad z \text{ between } t \text{ and } \xi \\ &\leq M |t - \xi| |D| \rightarrow 0, \quad \text{where } |D| \text{ is volume of } D \end{aligned} \quad (31)$$

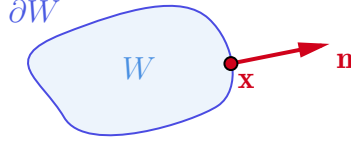
where  $M = \sup_{(x, t) \in D \times (0, a)} F_{tt}(x, t)$ .

### 1.1.6 The Continuity Equation

Recall that  $D$  is a region in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  filled with fluid and  $x = x(x_1, x_2, x_3) \in D$  is a particle of fluid moving through  $x$  at time  $t$  with velocity  $\mathbf{u}(x, t)$ . If  $W$  is a ant subregion of  $D$ , then the mass of fluid in  $W$  at time  $t$  is

$$m(W, t) = \int_W \rho(x, t) dV, \quad (32)$$

where  $\rho(x, t)$  is the density of fluid at  $(x, t)$ . Then  $\frac{d}{dt} m(W, t) = \int_W \rho_t(x, t) dV$ .



Let  $\partial W$  be the boundary of  $W$ . Suppose  $\partial W$  is smooth, let  $\mathbf{n}(x)$  be the normal vector to  $\partial W$  at  $x \in \partial W$ , Let  $dA$  denote the

Then the volumn of fluid flow rate across  $\partial W$  per unit time, Since  $\mathbf{u} \cdot \mathbf{n} \Delta A \Rightarrow$  the mass of fluid flow per unit time is  $\rho \mathbf{u} \cdot \mathbf{n} \Delta A$

Since, by the conservation of mass, the rate of increase of mass in  $W$  is equal to the rate that mass is incoming  $\partial W$

$$\int_W \rho_t dV = \frac{d}{dt} \int_W \rho dV = - \int_{\partial W} \rho (\mathbf{n} \cdot \mathbf{u}) dV = \text{by divergence theorem} = - \int_W \text{div}(\rho \mathbf{u}) dV \quad (33)$$

By divergence theorem, we have

$$\int_W (\rho_t + \text{div}(\rho \mathbf{u})) dV = 0, \quad \forall W \subset D \quad (34)$$

We now choose  $W = B(x, r)$ , and let  $H(y, t) = \rho_t + \text{div}(\rho \mathbf{u}) = H$ , by above equation, we have

$$\int_{B(x, r)} H(y, t) dV_x = 0, \quad \forall x \in D, B(x, r) \subset D \quad (35)$$

Notice  $H(y, t) = \frac{1}{|B(x, r)|} \int_{B(x, r)} H(y, t) dV = H(y, t) \frac{\int_{B(x, r)} dV}{|B(x, r)|}$ , where  $|B(x, r)|$  is the volumn of  $B(x, r)$ . Now

$$\begin{aligned} \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} H(x, t) dV - H(x, t) \right| &= \frac{1}{|B(x, r)|} \left| \int_{B(x, r)} [H(y, t) - H(x, t)] dV \right| \\ &\leq \frac{1}{|B(x, r)|} \int_{B(x, r)} |H(y, t) - H(x, t)| dV \\ &\leq \frac{1}{|B(x, r)|} \max_{y \in B(x, r)} |H(y, t) - H(x, t)| \cdot |B(x, r)| \\ &\rightarrow 0, \quad \text{as } r \rightarrow 0 \end{aligned} \quad (36)$$

so that

$$\lim_{r \rightarrow 0} \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} H(x, t) dV - H(x, t) \right| = 0 \quad (37)$$

By equation (35) and (37), we have

$$H(x, t) = 0 \quad \Rightarrow \quad \rho_t + \text{div}(\rho \mathbf{u}) = 0 \quad (38)$$

which is called the continuity equation in  $D \times (0, T)$ .



## 1.2 Proof of Euler's Equation

### 1.2.1 Balance of Momentum 1 (BM1)

The force per unit area on a point  $x \in \partial W$  is  $-p \cdot \mathbf{n}$ , since the total force on  $W$  due force the pressure on  $\partial W$  is

$$\begin{aligned}
 \mathbf{f} &= - \int_{\partial W} p \cdot \mathbf{n} dA \\
 &= \left( - \int_{\partial W} p n_1 dA, - \int_{\partial W} p n_2 dA, - \int_{\partial W} p n_3 dA \right) \\
 &= \left( - \int_{\partial W} (p, 0, 0) \cdot \mathbf{n} dA, - \int_{\partial W} (0, p, 0) \cdot \mathbf{n} dA, - \int_{\partial W} (0, 0, p) \cdot \mathbf{n} dA \right) \\
 &= \text{by divergence theorem} \\
 &= \left( - \int_W \operatorname{div}(p, 0, 0) dV, - \int_W \operatorname{div}(0, p, 0) dV, - \int_W \operatorname{div}(0, 0, p) dV \right) \\
 &= - \int_W (\nabla p) dV
 \end{aligned} \tag{39}$$

Now, the total force on the fluid due to the pressure

$$\partial W = - \int_W \nabla p dV \tag{40}$$

If  $\mathbf{b}(\mathbf{x}, t)$  denotes the given body force per unit mass (ex: gravity), then the total body force is

$$F_B = \int_W \rho \cdot \mathbf{b} \cdot dV. \tag{41}$$

By equations (40) and (41), the force per unit volume is  $-\nabla p + \rho \mathbf{b}$

By the Newton's second law,

$$\frac{D}{Dt}(\delta m \mathbf{u}) = \frac{D}{Dt}(\delta m \mathbf{u}(\mathbf{x}, t)) = (-p + \rho \mathbf{b}) \delta V \tag{42}$$

we then have

$$\Rightarrow \rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b} \quad \text{BM1 (Euler equation)} \tag{43}$$

### 1.2.2 Balance of Momentum 2 (BM2)

WLOG. we write  $\mathbf{u}$  for  $\mathbf{u} = (u_1, u_2, u_3)$  and  $b$  for  $\mathbf{b}$ , Integral from of balance of momentum

By (BM1)

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \rho \mathbf{b} \tag{44}$$

then by 44 and continuity equation

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = \rho_t \mathbf{u} + \rho \mathbf{u}_t = -\operatorname{div}(\rho \mathbf{u}) \mathbf{u} - \rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \rho \mathbf{b} \tag{45}$$

Let  $\mathbf{e}$  be a fixed vector in space, then

$$\begin{aligned}
 \mathbf{e} \cdot \frac{\partial}{\partial t}(\rho \mathbf{u}) &= -\operatorname{div}(\rho \mathbf{u}) \mathbf{u} \cdot \mathbf{e} - \rho(\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{e} - (\nabla p) \cdot \mathbf{e} + \rho \mathbf{b} \cdot \mathbf{e} \\
 &= -\operatorname{div}(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e})) - \operatorname{div}(p \mathbf{e}) + \rho \mathbf{b} \cdot \mathbf{e} \\
 &= -\operatorname{div}(\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{e}) + p \mathbf{e}) + \rho \mathbf{b} \cdot \mathbf{e}
 \end{aligned} \tag{46}$$

Since, we have

1. the divergence of  $p\mathbf{e}$

$$\begin{aligned}
 \operatorname{div}(p\mathbf{e}) &= \operatorname{div}((pe_1, pe_2, pe_3)) \\
 &= \frac{\partial p}{\partial x_1}e_1 + \frac{\partial p}{\partial x_2}e_2 + \frac{\partial p}{\partial x_3}e_3 \\
 &= \sum_{i=1}^3 \frac{\partial p}{\partial x_i}e_i = \nabla p \cdot \mathbf{e}
 \end{aligned} \tag{47}$$

1. the divergence of  $\rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})$

$$\begin{aligned}
 \operatorname{div}(\rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) &= \sum_{i=1}^3 \frac{\partial}{\partial x_i} [\rho u_i (\mathbf{u} \cdot \mathbf{e})] \\
 &= \sum_{i=1}^3 \left( \frac{\partial}{\partial x_i} (\rho u_i) \right) (\mathbf{u} \cdot \mathbf{e}) + \sum_{i=1}^3 (\rho u_i) \left( \frac{\partial}{\partial x_i} (\mathbf{u} \cdot \mathbf{e}) \right) \\
 &= \operatorname{div}(\rho\mathbf{u})(\mathbf{u} \cdot \mathbf{e}) + \rho(\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \mathbf{e})
 \end{aligned} \tag{48}$$

Hence, if  $W$  is a fixed region in space in the fluid

$$\begin{aligned}
 \mathbf{e} \cdot \frac{d}{dt} \int_W \rho \mathbf{u} dV &= \int_W \mathbf{e} \cdot \frac{d}{dt} (\rho \mathbf{u}) dV \\
 &= - \int_W \operatorname{div}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) dV + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV \\
 &= \text{By divergence theorem.} \\
 &= - \int_{\partial W} (p\mathbf{e} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e})) \cdot \mathbf{n} dA + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV \\
 &= - \int_{\partial W} p\mathbf{e} \cdot \mathbf{n} dA - \int_{\partial W} \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{e}) \cdot \mathbf{n} dA + \int_W \rho \mathbf{b} \cdot \mathbf{e} dV, \quad \forall \mathbf{e} \in \mathbb{R}^n, n = 2 \text{ or } 3
 \end{aligned} \tag{49}$$

then

$$\frac{d}{dt} \int_W \mathbf{u} dV = - \int_{\partial W} p\mathbf{n} dA - \int_{\partial W} \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u} dA + \int_{\partial W} \rho \mathbf{b} \cdot \mathbf{e} dV \tag{50}$$

or

$$\frac{d}{dt} \int_W \mathbf{u} dV = - \int_{\partial W} (p\mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u}) dA + \int_{\partial W} \rho \mathbf{b} \cdot \mathbf{e} dV, \quad \text{BM2} \tag{51}$$

and BM2 is also the Integral form of balance of momentum.

Note: The quantity  $p\mathbf{n} + \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{n})$  is the momentum per unit area crossing  $\partial W$  when  $\mathbf{n}$  is unit vector outer normal to  $\partial W$ .

### 1.2.3 Balance of Momentum 3 (BM3)

Let  $D$  be a region that the fluid is moving and  $x \in D$

Let  $\varphi(\mathbf{x}, t)$  be the trajectory of the particle that is at point  $x$ , i.e.  $\varphi$  satisfies

$$\begin{aligned} \frac{\partial}{\partial t} \varphi(\mathbf{x}, t) &= \mathbf{u}(\varphi(\mathbf{x}, t), t) \quad \forall t > 0 \text{ at time } t \\ \varphi(\mathbf{x}, 0) &= x \quad \varphi(\mathbf{x}, t) = \varphi_t(\mathbf{x}) \end{aligned} \quad (52)$$

We will assume that  $\varphi$  is smooth and for fixed  $t$ ,  $\varphi_t : t \rightarrow \varphi(\mathbf{x}, t)$  is invertible.

$\varphi_t$  doesn't mean  $\partial/\partial t$  here!

We called  $\varphi$  is the fluid flow map.

If  $W$  is the a region in  $D$ , then  $W_t := \varphi_t(W)$  is the region of the fluid at time  $t$  whose initial position is in  $W$  at time  $t$ .

Then by the balance of momentum

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = \mathbf{F}_{\partial W_t} + \int_{W_t} \rho \mathbf{b} dV, \quad (53)$$

where  $\mathbf{F}_{\partial W_t}$  is the force on  $\partial W_t$  due to perssure, i.e.

$$\mathbf{F}_{\partial W_t} = - \int_{\partial W_t} p \mathbf{n} dA = - \int_{W_t} \nabla p dV \quad (54)$$

so that

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV, \quad \text{BM3} \quad (55)$$

Recall

$$\frac{d}{dt}(\delta V) = (\text{div } \mathbf{u}) \delta V \quad (56)$$

for an infitesmal volume  $\delta V$  of fluid moving in te fluid with the fluid velocity,

We will now given a rigorous proof of the result.

Now that

$$\begin{aligned} \text{volume}(W_t) &= \int_{W_t} 1 dV \\ &= \int_{W_t} 1 dy, \quad \text{put } y \text{ to be } \varphi_t(\mathbf{x}) \\ &= \int_W J(\mathbf{x}, t) d\mathbf{x}, \end{aligned} \quad (57)$$

where  $J(\mathbf{x}, t)$  is the Jacobian determinant of the map  $\varphi_t$ , so that

$$\frac{d}{dt} \text{volume}(W_t) = \int_W \frac{\partial}{\partial t} J(\mathbf{x}, t) d\mathbf{x} \quad (58)$$

### 1.3 Equivalence between BM1, BM2 and BM3

Quick Summary of *Balance of Momentum*

1. BM1:  $\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{b}$
2. BM2:  $\frac{d}{dt} \int_W \rho \mathbf{u} dV = \int_{\partial W} (p \mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u}) dA + \int_W \rho \mathbf{b} dV$
3. BM3:  $\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV$

### 1.3.1 Lemma

$$\frac{\partial J(\mathbf{x}, t)}{\partial t} = \operatorname{div}(\mathbf{u}(y, t)) \cdot J(\mathbf{x}, t) \quad (59)$$

**1.3.1.1 proof:**

$\mathbf{y} = \varphi(\mathbf{x}, t) = (y_1, y_2, y_3)$ , and  $\mathbf{x} = (x_1, x_2, x_3)$ . Observe that

$$\frac{\partial \varphi}{\partial t} = \mathbf{u}(\varphi(\mathbf{x}, t), t), \quad \text{or} \quad \frac{\partial y_i}{\partial t} = u_i(y, t), \forall i = 1, 2, 3 \quad (60)$$

then

$$\begin{aligned} J(\mathbf{x}, t) &= \operatorname{div} \left( \frac{\partial y_i}{\partial x_j} \right)_{1 \leq i \leq 3} \\ &= \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \end{aligned} \quad (61)$$

here  $S$  is the family of the permutations  $\{1, 2, 3\}$ , and

$$\operatorname{sign} \sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation;} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases} \quad (62)$$

then

$$\begin{aligned} \frac{\partial J(\mathbf{x}, t)}{\partial t} &= \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial^2 y_2}{\partial t \partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad + \sum_{\sigma \in S} (\operatorname{sign} \sigma) \frac{\partial y_1}{\partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial^2 y_3}{\partial t \partial x_{\sigma(3)}} \\ &= I_1 + I_2 + I_3 \end{aligned} \quad (63)$$

then calculate  $I_1$  first

$$\begin{aligned}
I_1 &= \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial^2 y_1}{\partial t \partial x_{\sigma(1)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \sum_{\sigma \in S} (\text{sign } \sigma) \left( \frac{\partial}{\partial x_{\sigma(2)}} \left( \frac{\partial y_1}{\partial t} \right) \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \sum_{\sigma \in S} (\text{sign } \sigma) \left( \frac{\partial u_1}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \sum_{\sigma \in S} (\text{sign } \sigma) \left( \sum_{k=1}^3 \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \right) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \sum_{\sigma \in S} \sum_{k=1}^3 (\text{sign } \sigma) \frac{\partial u_1}{\partial y_k} \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \sum_{k=1}^3 \frac{\partial u_1}{\partial y_k} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_k}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \frac{\partial u_1}{\partial y_1} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&\quad + \frac{\partial u_1}{\partial y_2} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&\quad + \frac{\partial u_1}{\partial y_3} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_3}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} \\
&= \frac{\partial u_1}{\partial y_1} \sum_{\sigma \in S} (\text{sign } \sigma) \frac{\partial y_1}{\partial x_{\sigma(2)}} \frac{\partial y_2}{\partial x_{\sigma(2)}} \frac{\partial y_3}{\partial x_{\sigma(3)}} + 0 + 0 \\
&= \frac{\partial u_1}{\partial y_1} J(\mathbf{x}, t)
\end{aligned} \tag{64}$$

using the same we can get

$$\begin{aligned}
\frac{\partial J(\mathbf{x}, t)}{\partial t} &= I_1 + I_2 + I_3 \\
&= \frac{\partial u_1}{\partial y_1} J(\mathbf{x}, t) + \frac{\partial u_2}{\partial y_2} J(\mathbf{x}, t) + \frac{\partial u_3}{\partial y_3} J(\mathbf{x}, t) \\
&= \left( \sum_{i=1}^3 \frac{\partial u_i}{\partial y_i} \right) J(\mathbf{x}, t) \\
&= \text{div}_y(\mathbf{u}) J(\mathbf{x}, t)
\end{aligned} \tag{65}$$

### 1.3.2 Transport Theorem

For any smooth function  $f : D \times [0, T] \rightarrow \mathbb{R}$ , we have

$$\frac{d}{dt} \int_{W_t} \rho f \, dV_y = \int_{W_t} \rho \frac{Df}{dt} \, dV_y \tag{66}$$

### 1.3.2.1 proof

Change  $W_t \rightarrow W$

$$\int_{W_t} \rho f dV_y = \int_W \rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) dx \quad (67)$$

we have

$$\begin{aligned} \frac{d}{dt} \int_{W_t} \rho f dV &= \frac{d}{dt} \int_W \rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) dx \\ &= \int_W \frac{d}{dt} \left( \rho(\varphi(\mathbf{x}, t), t) f(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} \left( \rho(y, t) f(y, t) J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} \left( \rho(y, t) f(y, t) \right) J(\mathbf{x}, t) dx + \int_W \rho(y, t) f(y, t) \frac{d}{dt} \left( J(\mathbf{x}, t) \right) dx \\ &= \int_W \frac{d}{dt} (\rho f) J(\mathbf{x}, t) dx + \int_W (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x}, t) dx \\ &= \int_W \left( \frac{d}{dt} (\rho f) + (\rho f) \operatorname{div}_y(\mathbf{u}) J(\mathbf{x}, t) \right) dx \\ &= \int_W \left( \frac{\partial}{\partial t} (\rho f) + \nabla_y(\rho f) \cdot \left( \frac{dy}{dt} \right) + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left( \frac{\partial}{\partial t} (\rho f) + \nabla_y(\rho f) \cdot \mathbf{u} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left( \frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) J(\mathbf{x}, t) dx \\ &= \int_W \left( \frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV_y \end{aligned} \quad (68)$$

and consider the first term in the integral

$$\begin{aligned}
\frac{D(\rho f)}{Dt} &= \frac{\partial(\rho f)}{\partial t} + \mathbf{u} \cdot \nabla_y(\rho f) \\
&= f \frac{\partial \rho}{\partial t} + \rho \frac{\partial f}{\partial t} + \mathbf{u} \cdot (f \nabla_y \rho + \rho \nabla_y f) \\
&= f \left( \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla_y \rho \right) + \rho \left( \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla_y f \right) \\
&= f \left( \rho_t + \mathbf{u} \cdot \nabla_y \rho \right) + \rho \left( f_t + \mathbf{u} \cdot \nabla_y f \right) \\
&= f \left( \rho_t + \operatorname{div}(\rho \mathbf{u}) - \rho \operatorname{div}(\mathbf{u}) \right) + \rho \left( f_t + \mathbf{u} \cdot \nabla_y f \right) \\
&= f \underbrace{\left( \rho_t + \operatorname{div}(\rho \mathbf{u}) \right)}_{\text{continuity equation}=0} - f \rho \operatorname{div}(\mathbf{u}) + \rho \left( f_t + \mathbf{u} \cdot \nabla_y f \right) \\
&= -f \rho \operatorname{div}(\mathbf{u}) + \rho \left( f_t + \mathbf{u} \cdot \nabla_y f \right) \\
&= \rho \left( f_t + \mathbf{u} \cdot \nabla_y f \right) - \rho f \operatorname{div}(\mathbf{u}) \\
&= \rho \frac{Df}{Dt} - (\rho f) \operatorname{div}(\mathbf{u})
\end{aligned} \tag{69}$$

then plugin to the integral

$$\begin{aligned}
\frac{d}{dt} \int_{W_t} \rho f dV &= \int_W \left( \frac{D(\rho f)}{Dt} + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV \\
&= \int_W \left( \rho \frac{Df}{Dt} - (\rho f) \operatorname{div}(\mathbf{u}) + (\rho f) \operatorname{div}_y(\mathbf{u}) \right) dV \\
&= \int_W \left( \rho \frac{Df}{Dt} \right) dV
\end{aligned} \tag{70}$$

we get

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_W \left( \rho \frac{Df}{Dt} \right) dV, \quad \forall f \text{ is smooth function} \tag{71}$$

Notice that:

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_W \left( \rho \frac{Df}{Dt} \right) dV \tag{72}$$

so when we consider a vector function  $\mathbf{u}$ , we can write

$$\begin{aligned}
\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV &= \frac{d}{dt} \int_{W_t} \rho(u_1, u_2, u_3) dV \\
&= \left( \frac{d}{dt} \int_{W_t} \rho u_1 dV, \frac{d}{dt} \int_{W_t} \rho u_2 dV, \frac{d}{dt} \int_{W_t} \rho u_3 dV \right) \\
&= \left( \int_W \left( \rho \frac{Du_1}{Dt} \right) dV, \int_W \left( \rho \frac{Du_2}{Dt} \right) dV, \int_W \left( \rho \frac{Du_3}{Dt} \right) dV \right) \\
&= \int_W \left( \rho \frac{Du_1}{Dt}, \rho \frac{Du_2}{Dt}, \rho \frac{Du_3}{Dt} \right) dV \\
&= \int_W \rho \frac{D}{Dt} (u_1, u_2, u_3) dV \\
&= \int_W \rho \frac{D\mathbf{u}}{Dt} dV
\end{aligned} \tag{73}$$

Now we rewrite BM3

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV = - \int_{W_t} \nabla p dV + \int_{W_t} \rho \mathbf{b} dV \tag{74}$$

to be

$$\frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV = 0 \tag{75}$$

then using the result from above

$$\begin{aligned}
0 &= \frac{d}{dt} \int_{W_t} \rho \mathbf{u} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\
0 &= \int_{W_t} \rho \frac{D\mathbf{u}}{Dt} dV + \int_{W_t} \nabla p dV - \int_{W_t} \rho \mathbf{b} dV \\
0 &= \int_{W_t} \left( \rho \frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b} \right) dV, \quad \forall W_t
\end{aligned} \tag{76}$$

imply

$$\rho \frac{D\mathbf{u}}{Dt} dV + \nabla p dV - \rho \mathbf{b} = 0 \tag{77}$$

which is just BM1, this means BM3 is equivalent to BM1.

Similarly,

$$\frac{d}{dt} \int_W \rho \mathbf{u} dV = \int_{\partial W_t} p \mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u} dA + \int_W \rho \mathbf{b} dV \tag{78}$$

so that

$$\text{BM1} \Leftrightarrow \text{BM2} \Leftrightarrow \text{BM3} \tag{79}$$

By an argument similar to the proof of the transport theorem, for any smooth function  $f : D \times [0, T] \rightarrow \mathbb{R}$ , we have

$$\frac{d}{dt} \left( \int_{W_t} f dV \right) = \int_{W_t} \left( \frac{Df}{Dt} + f(\text{div } \mathbf{u}) \right) dV \tag{80}$$



or in the other form as the textbook

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \left( \frac{\partial f}{\partial t} + \operatorname{div}(f \mathbf{u}) \right) dV, \quad (81)$$

since  $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$ , and these 2 equations (80) and (81) is called the *Transport theorem without mass density*.

Rmk: Here we consider the *Transport theorem* (with mass density):

$$\frac{d}{dt} \int_{W_t} \rho f dV = \int_{W_t} \rho \frac{Df}{Dt} dV, \quad (82)$$

if we take  $\rho$  to be a constant (ex:  $\rho = 1$ ), we have

$$\frac{d}{dt} \int_{W_t} f dV = \int_{W_t} \frac{Df}{Dt} dV, \quad (83)$$

however compare to the equation without no mass density (80),

$$\frac{d}{dt} \left( \int_{W_t} f dV \right) = \int_{W_t} \frac{Df}{Dt} dV + \int_{W_t} f (\operatorname{div} \mathbf{u}) dV \quad (84)$$

we have an extra term contain  $f(\operatorname{div} \mathbf{u})$ . **BUT**, if we consider more carefully, the mass density  $\rho$  here must satisfy the continuity equation, i.e.

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (85)$$

so that if mass density is a constant (ex:  $\rho = 1$ ), we have

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 + \operatorname{div}(\mathbf{u}) = 0 \quad (86)$$

so that  $\operatorname{div} \mathbf{u} = 0$ , which means the extra term vanishing, and the *transport theorem with mass density* and *transport theorem without mass density* are equivalent.