Contents

0.1 The Equation of Motion

0.1.1 Euler's equation:

Consider a fluid in a domain D in $\mathbb{R}^n$ ($n = 2$, $n = 3$).

Let $x \in D$, and $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ be the fluid density, velocity vector field and the pressure at the point x and time t.Consider an infinitesimal element of the fluid of volumn ∂V located at point x at time t with mass $\delta m = \rho(\mathbf{x}, t)$, which is moving $\mathbf{u}(\mathbf{x}, t)$ and momentum $\delta m \cdot \mathbf{u}(\mathbf{x}, t)$

The normal force directed into the indeinetesmal volumn across a face of area δa is $\mathbf{n} \cdot p(\mathbf{x}, t) \cdot \delta a$

Fig1 - box

Note that the pressure is the magnitade of the torce per unit area or normal stress, imposed on the fluid from neighboring fluid elements.

0.1.2 Convective derivative 對流

Couvder a fluid particle moving in flaid, whose position $\mathbf{x}$ at time t is $\mathbf{x}(t)$. Then

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t) \tag{1}$$

Hence, if $f(\mathbf{x}, t)$ is a function on $D \times (0, T)$, then $f(\mathbf{x}(t), t)$ is the value if f at the moving fluid particle at $\mathbf{x}(t)$ at time t. We define the convective derivative of f:

$$\begin{aligned}\frac{Df(\mathbf{x}, t)}{Dt} &= \frac{\partial f(\mathbf{x}, t)}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f(\mathbf{x}, t) \\ &= f_t + \mathbf{u} \cdot \nabla f\end{aligned} \tag{2}$$

where $\nabla f = f(f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

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We define the convective derivative of f as:

$$\begin{aligned} \frac{Df(x,t)}{Dt} &= \frac{\partial f}{\partial t} + \dot{\mathbf{x}}(t) \cdot \nabla f \\ &= f_t + \mathbf{u} \cdot \nabla f \end{aligned}, \tag{3}$$

where $\nabla f = (f_x, f_y, f_z)$ and $\mathbf{u} = (u_1, u_2, u_3)$.

0.1.3 Def.

For any vector filed $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ on D, we let

$$\int_D \mathbf{F} dV = \left(\int_D F_1 dV, \int_D F_2 dV, \ldots, \int_D F_n dV \right). \tag{4}$$

0.1.4 Def.

We will assume that D is a smooth domain, i.e. for any $x_0 \in \partial D$, $\mathbb{R}^n = (x', x_n), n = 2, 3$

$\exists \delta_0 > 0$ and a smooth function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, s.t. $\partial D \cap B(x_0, \delta_0) = \{(x', \varphi(x')) : \|x'\| < \delta_0, x' \in \mathbb{R}^{n-1}\}$ $B(x_0, \delta_0)$ and $D \cap B(x_0, \delta_0) = \{(x', x_n) : x_n > \varphi(x'), x' \in \mathbb{R}^{n-1}, \|x'\| < \delta_0\} \cap B(x_0, \delta_0)$

0.1.5 Claim

Conside the volume δV of an element of mass δm, which moves in the fluid by the fluid motion

$$\frac{d(\delta V)}{dt} = (\nabla \cdot \mathbf{u})(\mathbf{x}, t) \cdot \delta V \quad \text{as} \quad \delta x_1, \delta x_2, \delta x_3 \to 0, \tag{5}$$

where $\nabla \cdot \mathbf{u} = \operatorname{div} \mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$, $\mathbf{u} = (u_1, u_2, u_3)$.

0.1.6 proof

$$\begin{aligned} \frac{d(\delta V)}{dt} &= \frac{d}{dt}(\delta x_1, \delta x_2, \delta x_3) \\ &= \frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 + \frac{d(\delta x_2)}{dt} \delta x_1 \delta x_3 + \frac{d(\delta x_3)}{dt} \delta x_1 \delta x_2 \end{aligned} \tag{6}$$

For the first term

$$\begin{aligned} \frac{d(\delta x_1)}{dt} &\approx u_1 \left(x_1 + \frac{\delta x_1}{2}, x_2, x_3 \right) - u_1 \left(x_1 - \frac{\delta x_1}{2}, x_2, x_3 \right) \\ &= \frac{\partial u_1}{\partial x_1}(\xi_1, x_2, x_3) \delta x_1, \quad \text{fot some } \xi_1 \in \left(x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2} \right) \end{aligned} \tag{7}$$

then

$$\frac{d(\delta x_1)}{dt} \delta x_2 \delta x_3 \to \frac{\partial u_1}{\partial x_1}(x_1, x_2, x_3) \delta x_1 \delta x_2 \delta x_3, \quad \text{as } \delta x_1, \delta x_2, \delta x_3 \to 0 \tag{8}$$

Similarly

$$\frac{d(\delta x_2)}{dt}\delta x_2\delta x_3 \to \frac{\partial u_2}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3, \quad \text{as } \delta x_1,\delta x_2,\delta x_3 \to 0 \tag{9}$$

and

$$\frac{d(\delta x_3)}{dt}\delta x_2\delta x_3 \to \frac{\partial u_3}{\partial x_1}(x_1,x_2,x_3)\delta x_1\delta x_2\delta x_3, \quad \text{as } \delta x_1,\delta x_2,\delta x_3 \to 0 \tag{10}$$

so that

$$\frac{d(\delta V)}{dt} = \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\delta x_1\delta x_2\delta x_3 = (\nabla\cdot\mathbf{u})\delta V \tag{11}$$

Consider a a tagged (marked) portion Σ of fluid with center of mass at $(x_1(s),x_2(s),x_3(s))$ at time s. Let $m(x)$ and $V(s)$ be the mass and volumn of this portion Σ of fluid at time s.The portion of fluid particle moves aling with fluid. see as (2/21 fig1)

For a time $t>0$, suppose at time t, the tagged portion Σ of fluid particles is a cube centered at (x_1,x_2,x_3) with side lengh ℓ_1,ℓ_2,ℓ_3, see as (2/21 fig2 - textbook p.4),where

$$\begin{aligned} P_1(s) &= \left(x_1(s)-\frac{\ell_1(s)}{2},x_2(s),x_3(s)\right)\\ P_2(s) &= \left(x_1(s)+\frac{\ell_1(s)}{2},x_2(s),x_3(s)\right)\end{aligned} \tag{12}$$

We assume that Σ remain a cube for $s\approx t$ with side length, $\ell_1(s),\ell_2(s),\ell_3(s)$, then $V(s)=\ell_1(s)\cdot\ell_2(s)\cdot\ell_3(s)$

$$\left.\frac{dV(s)}{ds}\right|_{s=t} = \left.\frac{d\ell_1(s)}{ds}\right|_{s=t}\ell_2(s)\ell_3(s)+\left.\frac{d\ell_2(s)}{ds}\right|_{s=t}\ell_1(s)\ell_3(s)+\left.\frac{d\ell_3(s)}{ds}\right|_{s=t}\ell_1(s)\ell_2(s) \tag{13}$$

where

$$\begin{aligned}\frac{d\ell_1(s)}{ds} &= u_1(P_2(t),t)-u_1(P_1(t),t)\\ &= u_1\left(x_1(s)+\frac{\ell_1(s)}{2},x_2(s),x_3(s),t\right)-u_1\left(x_1(s)-\frac{\ell_1(s)}{2},x_2(s),x_3(s),t\right)\\ &\approx \frac{\partial u_1}{\partial x_1}(x_1,x_2,x_3,t)\cdot\ell_1\end{aligned} \tag{14}$$

Similarly

$$\left.\frac{d\ell_i}{ds}\right|_{s=t} = \frac{\partial u_i}{\partial x_i}(x_1,x_2,x_3)\cdot\ell_i, \quad \forall i=1,2,3. \tag{15}$$

Now we write $\left.\frac{d}{ds}\right|_{s=t} = \frac{d}{dt}$.

Fig2 = tagged portion and face

0.1.7 Continuity equation

Let $\rho(\mathbf{x},t)$ be the density of fluid at time s.Since $M(s)=\text{const.}$, $\forall s>0$ and $\frac{dM(s)}{ds}=0$, $\forall s>0$.Therefore, since it is similar to the cube, the density is

$$\rho(\mathbf{x},s)\approx\frac{M(s)}{V(s)} \tag{16}$$

and the derevative is

$$
\begin{aligned}
\frac{d}{ds}\rho(\mathbf{x},s)\bigg|_{s=t} &\approx \frac{d}{ds}\frac{M(s)}{V(s)}\bigg|_{s=t} \\
&= \frac{M'(s)V(s)-M(s)V'(s)}{V^2(s)}\bigg|_{s=t} \\
&= \frac{0-M(s)\frac{d}{ds}V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)(\operatorname{div}\mathbf{u})V(s)}{V^2(s)}\bigg|_{s=t} \\
&= -\frac{M(s)}{V(s)}(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t} \\
&= -\rho(\mathbf{x}(s),s)(\operatorname{div}\mathbf{u}(s))\bigg|_{s=t}
\end{aligned} \tag{17}
$$

we get

$$
-\frac{d}{dt}\rho(\mathbf{x}(t),t) = \rho\cdot(\nabla\cdot\mathbf{u}(t)) \tag{18}
$$

On the other hand, by chain rule

$$
\frac{d}{dt}\rho(\mathbf{x}(t),t) = \rho_t + (\nabla\rho)\cdot\mathbf{u}(t) \tag{19}
$$

combining together we have

$$
\begin{aligned}
&\Rightarrow \rho_t + (\nabla\rho)\cdot\mathbf{u} = \rho\cdot(\nabla\cdot\mathbf{u}) \\
&\Rightarrow \rho_t + (\nabla\rho)\cdot\mathbf{u} - \rho\cdot(\nabla\cdot\mathbf{u}) = 0 \\
&\Rightarrow \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0
\end{aligned} \tag{20}
$$

and the equation $\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0$ is called the *contunuity equation.*

0.1.8 Heuristic proof of the Euler equation

In the ansense of an externally applied forces, the net force $\mathbf{F}$, acting on δV, is due to the pressure field.

Write $\mathbf{F} = (F_1, F_2, F_3)$, we get

$$
\begin{aligned}
\mathbf{F}(x_1,x_2,x_3,t) &\approx \left(P\left(x_1-\frac{\delta x_1}{2},x_2,x_3,t\right) - P\left(x_1+\frac{\delta x_1}{2},x_2,x_3,t\right)\right)\delta x_2\delta x_3 \\
&= -\frac{\partial P}{\partial x_1}(\zeta_1,x_2,x_3,t)\delta x_1\delta x_2\delta x_3, \quad \delta x_1,\delta x_2,\delta x_3\to 0 \\
&= \frac{\partial P}{\partial x_1}(\zeta_1,x_2,x_3,t)\delta V
\end{aligned} \tag{21}
$$

for some $\zeta_1 \in \left(x_1 - \frac{\delta x_1}{2}, x_1 + \frac{\delta x_1}{2}\right)$.

By Newton's second law, the equation of motion for the elemnet of fund mass δm, at point $\mathbf{x}(t)$ is

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \mathbf{F} = -(\nabla P)\delta V \tag{22}$$

also

$$\frac{d}{dt}\left(\delta m \cdot \mathbf{u}(\mathbf{x},t)\right) = \delta m \frac{d}{dt}\mathbf{u}(\mathbf{x},t) = \delta m\left(\mathbf{u}_t + (\nabla\cdot\mathbf{u})\right)\mathbf{u} \tag{23}$$

then

$$\begin{aligned} \delta m\left(\mathbf{u}_t + (\nabla\cdot\mathbf{u})\right)\mathbf{u} &= -(\nabla P)\delta V \\ \mathbf{u}_t + (\nabla\cdot\mathbf{u})\,\mathbf{u} &= -(\nabla P)\frac{\delta V}{\delta m} = -(\nabla P)\frac{1}{\delta m/\delta V} \end{aligned} \tag{24}$$

we get a equation

$$\mathbf{u}_t + (\nabla\cdot\mathbf{u})\,\mathbf{u} = -\frac{\nabla P}{\rho} \tag{25}$$

called *Euler's equation.*

Notice that

$$(\nabla\cdot\mathbf{u})\,\mathbf{u} = \left(\sum_{i=0}^{3} u_i\frac{\partial}{\partial x_i}\right)\mathbf{u} = \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}\left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_1\\ \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_2\\ \left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)u_3\end{pmatrix} \tag{26}$$

0.1.9 Lemma

Let D be a bounded domain and $F : \bar{D}\times[0,a_0]\to\mathbb{R}$ be a smooth function (or C^∞), then

$$\frac{d}{dt}\int_D F(x,t)dx = \int_D \frac{dF(x,t)}{dt}dx \tag{27}$$

0.1.10 proof:

we have

$$\begin{aligned} \frac{d}{dt}\int_D F(x,t)dx &= \lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}\int_D F(x,t+\Delta t)dx - \frac{d}{dt}\int_D F(x,t)dx\right] \\ &= \lim_{\Delta t\to 0}\frac{d}{dt}\int_D \frac{F(x,t+\Delta t)-F(x,t)}{\Delta t}dx \\ &= \text{By M.V.T.} \\ &= \lim_{\Delta t\to 0}\int_D \frac{\frac{\partial}{\partial t}F(x,\xi)\Delta t}{\Delta t}dx, \quad \text{for some } \xi, \text{ where} t<\xi<t+\Delta \\ &= \lim_{\Delta t\to 0}\int_D \frac{\partial}{\partial t}F(x,\xi)dx \end{aligned} \tag{28}$$

Denote, $\dfrac{\partial}{\partial t}F(x,t) = F_t(x,t)$ and $\dfrac{\partial^2}{\partial t^2}F(x,t) = F_{tt}(x,t)$, so

$$
\begin{aligned}
&\left|\frac{1}{\Delta t}\int_D [F(x,t+\Delta t)-F(x,t)] - \int_D \frac{\partial}{\partial t}F(x,t)dx\right| \\
&= \left|\int_D F_t(x,\xi)dx - \int_D F_t(x,\xi)dx\right| \\
&= \text{By MVT} \\
&= \left|\int_D [F_t(x,\xi)-F_t(x,t)]\right|dx \\
&= \text{By MVT} \\
&= \left|\int_D F_{tt}(x,z)(t-\xi)dz\right|, \quad z \text{ between } t \text{ and } \xi \\
&\le M|t-\xi||D| \to 0, \quad \text{where } |D| \text{ is vlolumn of } D
\end{aligned}
\tag{29}
$$

where $M = \sup_{(x,t)\in D\times(0,a)} F_{tt}(x,t)$.

0.1.11 The Continuity Equation

Recall that D is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ filled with fluid and $x = x(x_1, x_2, x_3) \in D$ is a particle of fluid moving through x at time t with velocity $\mathbf{u}(x,t)$. If W is a ant subregion of D, then the mass of fluid in W at time t is

$$
m(W,t) = \int_W \rho(x,t)dV, \tag{30}
$$

where $\rho(x,t)$ is the density of fluid at (x,t). Then $\frac{d}{dt}m(W,t) = \int_W \rho_t(x,t)dV$.

Fig3 - boundary of W

Let ∂W be the boundary of W. Suppose ∂W is smooth, let $\mathbf{n}(x)$ be the normal vector to ∂W at $x \in \partial W$, Let dA denote the

Then the volumn of fluid flow rate across ∂W per unit time, Since $\mathbf{u}\cdot\mathbf{n}\,\Delta A \Rightarrow$ the mass of fluid flow per unit time is $\rho\mathbf{u}\cdot\mathbf{n}\,\Delta A$

Since, by the conservation of mass, the rate of increase of mass in W is equal to the rate that mass is incoming ∂W

$$
\int_W \rho_t dV = \frac{d}{dt}\int_W \rho dV = -\int_{\partial W} \rho\,(\mathbf{n}\cdot\mathbf{u})dV = \text{by divergence theorem} = -\int_W \operatorname{div}(\rho\mathbf{u})dV \tag{31}
$$

By divergence theorem, we have

$$
\int_W \left(\rho_t + \operatorname{div}(\rho\mathbf{u})\right)dV = 0, \quad \forall W \subset D \tag{32}
$$

We now choose $W = B(x,r)$, and let $H(y,t) = \rho_t + \operatorname{div}(\rho\mathbf{u}) = H$, by above equation, we have

$$
\int_{B(x,r)} H(y,t)dV_x = 0, \quad \forall x \in D, B(x,r) \subset D \tag{33}
$$

Notice $H(y,t) = \frac{1}{|B(x,r)|}\int_{B(x,r)} H(y,t)dV = H(y,t)\frac{\int_{B(x,r)} dV}{|B(x,r)|}$, where $|B(x,r)|$ is the volumn of $B(x,r)$. Now

$$\begin{aligned}
\left|\frac{1}{B(x,r)}\int_{B(x,r)} H(x,t)dV - H(x,t)\right| &= \frac{1}{|B(x,r)|}\left|\int_{B(x,r)}[H(y,t)-H(x,t)]dV\right| \\
&\leq \frac{1}{|B(x,r)|}\int_{B(x,r)}|H(y,t)-H(x,t)|dV \\
&\leq \frac{1}{|B(x,r)|}\max_{y\in B(x,r)}|H(y,t)-H(x,t)|\cdot|B(x,r)| \\
&\to 0, \quad \text{as } r\to 0
\end{aligned} \tag{34}$$

so that

$$\lim_{r\to 0}\left|\frac{1}{B(x,r)}\int_{B(x,r)} H(x,t)dV - H(x,t)\right| = 0 \tag{35}$$

By equation (33) and (35), we have

$$H(x,t) = 0 \quad \Rightarrow \quad \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \tag{36}$$

which is called the continuity equation in $D\times(0,T)$.

0.2 Proof of Euler's Equation

0.2.1 Balance of Momentum 1 (BM1)

The force per unit area on a point $x\in\partial W$ is $-p\cdot\mathbf{n}$, since the total force on W due force the pressure on ∂W is

$$\begin{aligned}
\mathbf{f} &= -\int_{\partial W} p\cdot\mathbf{n}dA \\
&= \left(-\int_{\partial W} pn_1 dA, -\int_{\partial W} pn_2 dA, -\int_{\partial W} pn_3 dA\right) \\
&= \left(-\int_{\partial W}(p,0,0)\cdot\mathbf{n}dA, -\int_{\partial W}(0,p,0)\cdot\mathbf{n}dA, -\int_{\partial W}(0,0,p)\cdot\mathbf{n}dA\right) \\
&= \text{by divergence theorem} \\
&= \left(-\int_W \operatorname{div}(p,0,0)dV, -\int_W \operatorname{div}(0,p,0)dV, -\int_W \operatorname{div}(0,0,p)dV\right) \\
&= -\int_W(\nabla p)dV
\end{aligned} \tag{37}$$

Now, the total force on the fluid due to the pressure

$$\partial W = -\int_W \nabla p dV \tag{38}$$

If $\mathbf{b}(\mathbf{x},t)$ denotes the given body force per unit mass (ex: gravity), then the toal body force is

$$F_B = \int_W \rho\cdot\mathbf{b}\cdot dV. \tag{39}$$

By equations (38) and (39), the force per unit volume is $-\nabla p + \rho\mathbf{b}$

By the Newton's second law,

$$\frac{D}{Dt}(\delta m\,\mathbf{u}) = \frac{D}{Dt}(\delta m\,\mathbf{u}(\mathbf{x},t)) = (-p+\rho\mathbf{b})\delta V \tag{40}$$

we then have

$$\Rightarrow \rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b} \quad \text{BM1 (Euler equation)} \tag{41}$$

0.2.2 Balance of Momentum 2 (BM2)

WLOG. we write $\mathbf{u}$ for $\mathbf{u} = (u_1, u_2, u_3)$ and b for $\mathbf{b}$, Integral from of balance of momentum

By (BM1)

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \rho\mathbf{b} \tag{42}$$

then by 42 and continuity equation

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) = \rho_t\mathbf{u} + \rho\mathbf{u}_t = -\operatorname{div}(\rho\mathbf{u})\mathbf{u} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \rho\mathbf{b} \tag{43}$$

Let $\mathbf{e}$ be a fixed vector in space, then

$$\begin{aligned}\mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u}) &= -\operatorname{div}(\rho\mathbf{u})\mathbf{u}\cdot\mathbf{e} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{e} - (\nabla p)\cdot\mathbf{e} + \rho\mathbf{b}\cdot\mathbf{e}\\ &= -\operatorname{div}\left(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})\right) - \operatorname{div}\left(p\mathbf{e}\right) + \rho\mathbf{b}\cdot\mathbf{e}\\ &= -\operatorname{div}\left(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e}) + p\mathbf{e}\right) + \rho\mathbf{b}\cdot\mathbf{e}\end{aligned} \tag{44}$$

Since, we have

1. the divergence of $p\mathbf{e}$

$$\begin{aligned}\operatorname{div}(p\mathbf{e}) &= \operatorname{div}\left((pe_1, pe_2, pe_3)\right)\\ &= \frac{\partial p}{\partial x_1}e_1 + \frac{\partial p}{\partial x_2}e_2 + \frac{\partial p}{\partial x_3}e_3\\ &= \sum_{i=1}^{3}\frac{\partial p}{\partial x_i}e_i = \nabla p\cdot\mathbf{e}\end{aligned} \tag{45}$$

1. the divergence of $\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})$

$$\begin{aligned}\operatorname{div}(\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})) &= \sum_{i=0}^{3}\frac{\partial}{\partial x_i}\left[\rho u_i(\mathbf{u}\cdot\mathbf{e})\right]\\ &= \sum_{i=0}^{3}\left(\frac{\partial}{\partial x_i}(\rho u_i)\right)(\mathbf{u}\cdot\mathbf{e}) + \sum_{i=0}^{3}(\rho u_i)\left(\frac{\partial}{\partial x_i}(\mathbf{u}\cdot\mathbf{e})\right)\\ &= \operatorname{div}(\rho\mathbf{u})(\mathbf{u}\cdot\mathbf{e}) + \rho(\mathbf{u}\cdot\nabla)(\mathbf{u}\cdot\mathbf{e})\end{aligned} \tag{46}$$

Hence, if W is a fixed region in space in the fluid

$$\begin{aligned}
\mathbf{e} \cdot \frac{d}{dt}\int_W \rho \mathbf{u} dV &= \int_W e \cdot \frac{d}{dt}(\rho \mathbf{u}) dV \\
&= -\int_W \operatorname{div}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e}))dV + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV \\
&= \text{By divergence theorem.} \\
&= -\int_{\partial W} (p\mathbf{e} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e}))\cdot\mathbf{n}dA + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV \\
&= -\int_{\partial W} p\mathbf{e}\cdot\mathbf{n}dA - \int_{\partial W}\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})\cdot\mathbf{n}dA + \int_W \rho\mathbf{b}\cdot\mathbf{e}dV, \quad \forall \mathbf{e}\in\mathbb{R}^n, n = 2 \text{ or } 3
\end{aligned} \tag{47}$$

then

$$\frac{d}{dt}\int_W \mathbf{u}dV = -\int_{\partial W} p\mathbf{n}dA - \int_{\partial W}\rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}dA + \int_{\partial W}\rho\mathbf{b}\cdot\mathbf{e}dV \tag{48}$$

or

$$\frac{d}{dt}\int_W \mathbf{u}dV = -\int_{\partial W}\left(p\mathbf{n} + \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\right)dA + \int_{\partial W}\rho\mathbf{b}\cdot\mathbf{e}dV, \quad \text{BM2} \tag{49}$$

and BM2 is also the Integral form of balance of momentum.

Note: The quantity $p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ is the momentum per unit area crossing ∂W when $\mathbf{n}$ is unit vector outer normal to ∂W.

0.2.3 Balance of Momentum 3 (BM3)

Let D be a region that the fluid is moving and $x \in D$

Let $\varphi(\mathbf{x}, t)$ be the trajectory of the partiacle that is at point x, i.e. φ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi(\mathbf{x},t) = \mathbf{u}\left(\varphi(\mathbf{x},t),t\right) && \forall t > 0 \text{ at time } t \\
&\varphi(\mathbf{x},0) = x && \varphi(\mathbf{x},t) = \varphi_t(\mathbf{x})
\end{aligned} \tag{50}$$

We will assume that φ is smooth and for fixed t, $\varphi_t : t \to \varphi(\mathbf{x}, t)$ is invertible.

φ_t doesn't mean $\partial/\partial t$ here!

We called φ is the fluid flow map.

If W is the a region in D, then $W_t := \varphi_t(W)$ is the region of the fluid at time t whose initial position is in W at time t.

Then by the balance of momentum

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = \mathbf{F}_{\partial W_t} + \int_{W_t}\rho\mathbf{b}dV, \tag{51}$$

where $\mathbf{F}_{\partial W_t}$ is the force on ∂W_t due to perssure, i.e.

$$\mathbf{F}_{\partial W_t} = -\int_{\partial W_t} p\mathbf{n}dA = -\int_{W_t}\nabla p dV \tag{52}$$

so that

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = -\int_{W_t}\nabla p dV + \int_{W_t}\rho\mathbf{b}dV, \quad \text{BM3} \tag{53}$$

Recall

$$\frac{d}{dt}(\delta V) = (\operatorname{div}\mathbf{u})\delta V \tag{54}$$

for an infitesmal volume δV of fluid moving in te fluid with the fluid velocity, We will now given a rigorous proof of the result.

Now that

$$\begin{aligned}
\text{volume}(W_t) &= \int_{W_t} 1dV \\
&= \int_{W_t} 1dy, \quad \text{put } y \text{ to be } \varphi_t(\mathbf{x}) \\
&= \int_{W} J(\mathbf{x}, t)d\mathbf{x},
\end{aligned} \tag{55}$$

where $J(\mathbf{x}, t)$ is the Jocobian determinant of the map φ_t , so that

$$\frac{d}{dt}\text{volume}(W_t) = \int_W \frac{\partial}{\partial t}J(\mathbf{x}, t)d\mathbf{x} \tag{56}$$

0.3 Equivalence between BM1, BM2 and BM3

Quick Summary of *Balance of Momentum*

1. BM1: $\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{b}$
2. BM2: $\frac{d}{dt}\int_W \rho\mathbf{u}dV = \int_{\partial W}(p\mathbf{n} + \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u})\, dA + \int_W \rho\mathbf{b}dV$
3. BM3: $\frac{d}{dt}\int_{W_t} \rho\mathbf{u}dV = -\int_{W_t}\nabla p dV + \int_{W_t}\rho\mathbf{b}dV$

0.3.1 Lemma

$$\frac{\partial J(\mathbf{x}, t)}{\partial t} = \operatorname{div}(\mathbf{u}(y, t))\cdot J(\mathbf{x}, t) \tag{57}$$

0.3.2 proof:

$\mathbf{y} = \varphi(\mathbf{x}, t) = (y_1, y_2, y_3)$, and $\mathbf{x} = (x_1, x_2, x_3)$. Observe that

$$\frac{\partial\varphi}{\partial t} = \mathbf{u}(\varphi(\mathbf{x}, t), t), \quad \text{or} \quad \frac{\partial y_i}{\partial t} = u_i(y, t)\,, \forall i = 1, 2, 3 \tag{58}$$

then

$$\begin{aligned}
J(\mathbf{x}, t) &= \operatorname{div}\left(\frac{\partial y_i}{\partial x_j}\right)_{1\le i\le 3} \\
&= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\frac{\partial y_1}{\partial x_{\sigma(1)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}}
\end{aligned} \tag{59}$$

here S is the family of the permutations $\{1, 2, 3\}$, and

$$\operatorname{sign}\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation;} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases} \tag{60}$$

then

$$\begin{aligned}\frac{\partial J(\mathbf{x},t)}{\partial t} &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial^2 y_1}{\partial t\partial x_{\sigma(1)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad+\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_1}{\partial x_{\sigma(1)}}\frac{\partial^2 y_2}{\partial t\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad+\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_1}{\partial x_{\sigma(1)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial^2 y_3}{\partial t\partial x_{\sigma(3)}} \\ &= I_1+I_2+I_3\end{aligned} \tag{61}$$

then calculate I_1 first

$$\begin{aligned}I_1 &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial^2 y_1}{\partial t\partial x_{\sigma(1)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\left(\frac{\partial}{\partial x_{\sigma(2)}}\left(\frac{\partial y_1}{\partial t}\right)\right)\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\left(\frac{\partial u_1}{\partial x_{\sigma(2)}}\right)\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S}(\operatorname{sign}\sigma)\left(\sum_{k=1}^{3}\frac{\partial u_1}{\partial y_k}\frac{\partial y_k}{\partial x_{\sigma(2)}}\right)\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{\sigma\in S}\sum_{k=1}^{3}(\operatorname{sign}\sigma)\,\frac{\partial u_1}{\partial y_k}\frac{\partial y_k}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \sum_{k=1}^{3}\frac{\partial u_1}{\partial y_k}\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_k}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \frac{\partial u_1}{\partial y_1}\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_1}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad+\frac{\partial u_1}{\partial y_2}\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &\quad+\frac{\partial u_1}{\partial y_3}\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_3}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}} \\ &= \frac{\partial u_1}{\partial y_1}\sum_{\sigma\in S}(\operatorname{sign}\sigma)\,\frac{\partial y_1}{\partial x_{\sigma(2)}}\frac{\partial y_2}{\partial x_{\sigma(2)}}\frac{\partial y_3}{\partial x_{\sigma(3)}}+0+0 \\ &= \frac{\partial u_1}{\partial y_1}J(\mathbf{x},t)\end{aligned} \tag{62}$$

using the same we can get

$$\begin{aligned}\frac{\partial J(\mathbf{x},t)}{\partial t} &= I_1+I_2+I_3 \\ &= \frac{\partial u_1}{\partial y_1}J(\mathbf{x},t)+\frac{\partial u_2}{\partial y_2}J(\mathbf{x},t)+\frac{\partial u_3}{\partial y_3}J(\mathbf{x},t) \\ &= \left(\sum_{i=1}^{3}\frac{\partial u_i}{\partial y_i}\right)J(\mathbf{x},t) \\ &= \operatorname{div}_y(\mathbf{u})\,J(\mathbf{x},t)\end{aligned} \tag{63}$$

0.3.3 Transport Theorem

For any smooth function $f : D \times [0,T] \to \mathbb{R}$, we have

$$\frac{d}{dt}\int_{W_t} \rho f\, dV_y = \int_{W_t} \rho \frac{Df}{dt}\, dV_y \tag{64}$$

0.3.4 proof

Change $W_t \to W$

$$\int_{W_t} \rho f dV_y = \int_W \rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\, dx \tag{65}$$

we have

$$\begin{aligned}
\frac{d}{dt}\int_{W_t} \rho f dV &= \frac{d}{dt}\int_W \rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\, dx \\
&= \int_W \frac{d}{dt}\Big(\rho(\varphi(\mathbf{x},t),t) f(\varphi(\mathbf{x},t),t) J(\mathbf{x},t)\Big)\, dx \\
&= \int_W \frac{d}{dt}\Big(\rho(y,t) f(y,t) J(\mathbf{x},t)\Big)\, dx \\
&= \int_W \frac{d}{dt}\Big(\rho(y,t) f(y,t)\Big) J(\mathbf{x},t)\, dx + \int_W \rho(y,t) f(y,t) \frac{d}{dt}\Big(J(\mathbf{x},t)\Big)\, dx \\
&= \int_W \frac{d}{dt}\,(\rho f)\, J(\mathbf{x},t)\, dx + \int_W (\rho f)\, \mathrm{div}_y(\mathbf{u}) J(\mathbf{x},t)\, dx \\
&= \int_W \left(\frac{d}{dt}\,(\rho f) + (\rho f)\, \mathrm{div}_y(\mathbf{u}) J(\mathbf{x},t)\right)\, dx \\
&= \int_W \left(\frac{\partial}{\partial t}(\rho f) + \nabla_y(\rho f) \cdot \left(\frac{dy}{dt}\right) + (\rho f)\, \mathrm{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\
&= \int_W \left(\frac{\partial}{\partial t}(\rho f) + \nabla_y(\rho f) \cdot \mathbf{u} + (\rho f)\, \mathrm{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\
&= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f)\, \mathrm{div}_y(\mathbf{u})\right) J(\mathbf{x},t)\, dx \\
&= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f)\, \mathrm{div}_y(\mathbf{u})\right) dV_y
\end{aligned} \tag{66}$$

and consider the first term in the integral

$$
\begin{aligned}
\frac{D(\rho f)}{Dt} &= \frac{\partial(\rho f)}{\partial t} + \mathbf{u} \cdot \nabla_y(\rho f) \\
&= f\frac{\partial \rho}{\partial t} + \rho\frac{\partial f}{\partial t} + \mathbf{u} \cdot (f\,\nabla_y \rho + \rho\,\nabla_y f) \\
&= f\left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla_y \rho\right) + \rho\left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla_y f\right) \\
&= f\Bigg(\rho_t + \mathbf{u} \cdot \nabla_y \rho\Bigg) + \rho\Bigg(f_t + \mathbf{u} \cdot \nabla_y f\Bigg) \\
&= f\Bigg(\rho_t + \operatorname{div}(\rho\mathbf{u}) - \rho\operatorname{div}(\mathbf{u})\Bigg) + \rho\Bigg(f_t + \mathbf{u} \cdot \nabla_y f\Bigg) \\
&= f\,\underbrace{\Bigg(\rho_t + \operatorname{div}(\rho\mathbf{u})\Bigg)}_{\text{continuity equation=0}} - f\rho\operatorname{div}(\mathbf{u}) + \rho\Bigg(f_t + \mathbf{u} \cdot \nabla_y f\Bigg) \\
&= -f\rho\operatorname{div}(\mathbf{u}) + \rho\Bigg(f_t + \mathbf{u} \cdot \nabla_y f\Bigg) \\
&= \rho\Bigg(f_t + \mathbf{u} \cdot \nabla_y f\Bigg) - \rho f\operatorname{div}(\mathbf{u}) \\
&= \rho\frac{Df}{Dt} - (\rho f)\operatorname{div}(\mathbf{u})
\end{aligned} \tag{67}
$$

then plugin to the integral

$$
\begin{aligned}
\frac{d}{dt}\int_{W_t} \rho f dV &= \int_W \left(\frac{D(\rho f)}{Dt} + (\rho f)\operatorname{div}_y(\mathbf{u})\right) dV \\
&= \int_W \left(\rho\frac{Df}{Dt} - (\rho f)\operatorname{div}(\mathbf{u}) + (\rho f)\operatorname{div}_y(\mathbf{u})\right) dV \\
&= \int_W \left(\rho\frac{Df}{Dt}\right) dV
\end{aligned} \tag{68}
$$

we get

$$
\frac{d}{dt}\int_{W_t} \rho f dV = \int_W \left(\rho\frac{Df}{Dt}\right) dV, \quad \forall f \text{ is smooth function} \tag{69}
$$

Notice that:

$$
\frac{d}{dt}\int_{W_t} \rho f dV = \int_W \left(\rho\frac{Df}{Dt}\right) dV \tag{70}
$$

so when we consider a vector fucntion $\mathbf{u}$, we can write

$$\begin{aligned}
\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV &= \frac{d}{dt}\int_{W_t}\rho(u_1,u_2,u_3)dV \\
&= \left(\frac{d}{dt}\int_{W_t}\rho u_1 dV, \frac{d}{dt}\int_{W_t}\rho u_2 dV, \frac{d}{dt}\int_{W_t}\rho u_3 dV\right) \\
&= \left(\int_W\left(\rho\frac{Du_1}{Dt}\right)dV, \int_W\left(\rho\frac{Du_2}{Dt}\right)dV, \int_W\left(\rho\frac{Du_3}{Dt}\right)dV\right) \\
&= \int_W\left(\rho\frac{Du_1}{Dt}, \rho\frac{Du_2}{Dt}\rho\frac{Du_3}{Dt}\right)dV \\
&= \int_W\rho\frac{D}{Dt}\left(u_1,u_2,u_3\right)dV \\
&= \int_W\rho\frac{D\mathbf{u}}{Dt}dV
\end{aligned} \tag{71}$$

Now we rewrite BM3

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV = -\int_{W_t}\nabla p dV + \int_{W_t}\rho\mathbf{b}dV \tag{72}$$

to be

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV + \int_{W_t}\nabla p dV - \int_{W_t}\rho\mathbf{b}dV = 0 \tag{73}$$

then using the result from above

$$\begin{aligned}
0 &= \frac{d}{dt}\int_{W_t}\rho\mathbf{u}dV + \int_{W_t}\nabla p dV - \int_{W_t}\rho\mathbf{b}dV \\
0 &= \int_{W_t}\rho\frac{D\mathbf{u}}{Dt}dV + \int_{W_t}\nabla p dV - \int_{W_t}\rho\mathbf{b}dV \\
0 &= \int_{W_t}\left(\rho\frac{D\mathbf{u}}{Dt}dV + \nabla p dV - \rho\mathbf{b}\right)dV, \quad \forall W_t
\end{aligned} \tag{74}$$

imply

$$\rho\frac{D\mathbf{u}}{Dt}dV + \nabla p dV - \rho\mathbf{b} = 0 \tag{75}$$

which is just BM1, this means BM3 is equivalent to BM1.

Similarly,

$$\frac{d}{dt}\int_W\rho\mathbf{u}dV = \int_{\partial W_t}p\mathbf{n} + \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}dA + \int_W\rho\mathbf{b}dV \tag{76}$$

so that

$$\text{BM1} \Leftrightarrow \text{BM2} \Leftrightarrow \text{BM3} \tag{77}$$

By an arrgument similart ot the proof of the transport theorem, for any smooth function $f : D \times [0,T] \to \mathbb{R}$, we have

$$\frac{d}{dt}\left(\int_{W_t}f dV\right) = \int_{W_t}\left(\frac{Df}{Dt} + f(\operatorname{div}\mathbf{u})\right)dV \tag{78}$$

or in the other form as the textbook

$$\frac{d}{dt}\int_{W_t} f dV = \int_{W_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u})\right) dV, \tag{79}$$

since $\dfrac{Df}{Dt} = \dfrac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f$, and these 2 equations (78) and (79) is called the *Transport theorem without mass density.*

Rmk: Here we consider the *Transport theorem* (with mass density):

$$\frac{d}{dt}\int_{W_t} \rho f dV = \int_{W_t} \rho\frac{Df}{Dt} dV, \tag{80}$$

if we take ρ to be a constant (ex: $\rho = 1$), we have

$$\frac{d}{dt}\int_{W_t} f dV = \int_{W_t} \frac{Df}{Dt} dV, \tag{81}$$

however compare to the equation without no mass density (78),

$$\frac{d}{dt}\left(\int_{W_t} f dV\right) = \int_{W_t} \frac{Df}{Dt} dV + \int_{W_t} f(\mathrm{div}\,\mathbf{u}) dV \tag{82}$$

we have an extra term contain $f(\mathrm{div}\,\mathbf{u})$. **BUT**, if we consider more carefully, the mass density ρ here must satisfy the continuity equation, i.e.

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \tag{83}$$

so that if mass density is a constant (ex: $\rho = 1$), we have

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 + \mathrm{div}(\mathbf{u}) = 0 \tag{84}$$

so that $\mathrm{div}\,\mathbf{u} = 0$, which means the extra term vanishing, and the *transport theorem with mass density* and *transport theorem without mass density* are equivalent.