

MATH 136

LINEAR ALGEBRA

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Future Modifications

1 Introduction

Linear Algebra

- Systems of linear equations
- Related geometry
- Matrices
- Vector spaces, \mathbb{R}^n

\mathbb{R}^n consists of n-tuples of real numbers, where $n \in \mathbb{N}$.

Definition 1.1. Points/vectors are elements of \mathbb{R}^n .

Notation

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$$

$$x_1 + x_2 = 3$$

$$2x_1 + 5x_2 = 4$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Two vectors in \mathbb{R}^n are equal if all coordinates are equal.

Vector Operations

Let $\vec{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}$

Addition

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

Scalar Multiplication

$$\alpha \vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 1.2. $\vec{0}$ is the **additive identity**

Definition 1.3. Given a vector $\vec{x} \in \mathbb{R}^n$, $-\vec{x}$ is the **additive inverse**.

Definition 1.4. A sum of scalar multiples of a combination of vectors is a **linear combination**

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k : c_1 \dots c_k \in \mathbb{R}$$

Theorem 1.1. If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$, then

- $\vec{x} + \vec{y} \in \mathbb{R}^n$

- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\exists \vec{0} \in \mathbb{R}^n$ such that $\vec{x} + \vec{0} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$
- $\forall \vec{x} \in \mathbb{R}^n$, there exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- $c\vec{x} \in \mathbb{R}^n$
- $c(d\vec{x}) = (cd)\vec{x}$
- $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
- $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
- $1\vec{x} = \vec{x}$

Definition 1.5. The set S of all possible linear combinations of a set of vectors $B = (\vec{v}_1, \dots, \vec{v}_k)$ in \mathbb{R}^n is called the **span** of the set B and we write

$$S = \text{Span } B = \{t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k\}$$

S is **spanned** by B and that B is a spanning set for S .

For a set

$$\{t_1\vec{v}_1 + \dots + t_k\vec{v}_k + \vec{b} \mid t_1, \dots, t_k \in \mathbb{R}\}$$

can be written as

$$\vec{x} = t_1\vec{v}_1 + \dots + t_k\vec{v}_k + \vec{b}, t_1, \dots, t_k \in \mathbb{R}$$

In \mathbb{R}^n , two linearly independent vectors \vec{x}_1 and \vec{x}_2 generate a plane.

Definition 1.6. A set of vectors in \mathbb{R}^n is said to be **linearly dependent** if there exists coefficients c_1, \dots, c_k , not all 0, such that

$$\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$$

Either 0 vector or two or more vectors are colinear (scalar multiple).

Definition 1.7. A set of vectors is **linearly independent** if the only solution is $c_1 = c_2 = \dots = c_k = 0$ (**trivial solution**)

Definition 1.8. If a subset of \mathbb{R}^n can be written as a span of vectors $\vec{v}_1, \dots, \vec{v}_k$ where $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a **basis** for S . The basis of the set $\{\vec{0}\}$ is the empty set.

Theorem 1.2. If $\beta = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subset S of \mathbb{R}^n , then every vector $\vec{x} \in S$ can be written as unique linear combination of $\vec{v}_1, \dots, \vec{v}_k$.

Definition 1.9. The **standard basis** in \mathbb{R}^n is a set of vectors where each vector's i th component is 1, and all other components are 0.

Definition 1.10. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. The set with vector equation $\vec{w} = c_1\vec{x} + \vec{y}$ with $c_1 \in \mathbb{R}$ is a **line** in \mathbb{R}^n that passes through \vec{y} .

Definition 1.11. Let $\vec{v}_1, \vec{v}_2, \vec{y} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. The set with the vector equation $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{y}$ with $c_1, c_2 \in \mathbb{R}$ is a **plane** in \mathbb{R}^n which passes through \vec{y} .

Definition 1.12. Let $\vec{v}_1, \dots, \vec{v}_k, \vec{y} \in \mathbb{R}^n$ with the set being linearly independent. The set with the vector equation $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k + \vec{y}$ with c_1, \dots, c_k is a **k-plane** in \mathbb{R}^n which passes through \vec{y} .

Definition 1.13. A **hyperplane** is a subspace of one dimension less than its ambient space.

Theorem 1.3. Subspace Test: Let S be a non-empty subset of \mathbb{R}^n . If $\vec{x} + \vec{y} \in S$ and $c\vec{x} \in S$ for all $\vec{x}, \vec{y} \in S$ and $c \in \mathbb{R}$, then S is a subspace of \mathbb{R}^n .

Quote. If $\vec{0}$ is not in the set, definitely not subset. If it is, further investigation needed.

Definition 1.14. $S \subseteq \mathbb{R}^n$ is closed under scalar multiplication if for all $\vec{x} \in S$ and $\alpha \in \mathbb{R}$, $\alpha\vec{x} \in S$.

Theorem 1.4. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in \mathbb{R}^n , then $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of \mathbb{R}^n .

Theorem 1.5. If $\vec{x}, \vec{y} \in \mathbb{R}^2$, and θ is the angle between them, then

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \theta$$

Theorem 1.6. Given two vectors \vec{x}, \vec{y} , their dot product is defined by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{i=1}^n x_iy_i$$

Theorem 1.7. If $\vec{x} \cdot \vec{y} = 0$, then \vec{x} and \vec{y} are **orthogonal**.

Quote. The zero vector $\vec{0} \in \mathbb{R}^n$ is orthogonal to every vector in \mathbb{R}^n .

Theorem 1.8. The **cross product** of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}$$