MATH 136
Linear Algebra

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Future Modifications

1 Vectors in Euclidean Space

1.1 Vector Addition and Scalar Multiplication

Theorem 1.1.1. If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$, then

- $\vec{x} + \vec{y} \in \mathbb{R}^n$
- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + vw)$
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\exists\, \vec{0} \in \mathbb{R}^n$ such that $\vec{x} + \vec{0} = \vec{x} \qquad \forall \vec{x} \in \mathbb{R}^n$
- $\forall \vec{x} \in \mathbb{R}^n$, there exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- $c\vec{x} \in \mathbb{R}^n$
- $c(d\vec{x}) = (cd)\vec{x}$
- $(c + d)\vec{x} = c\vec{x} + d\vec{y}$
- $c(\vec{x} + vy) = c\vec{x} + c\vec{y}$
- $1\vec{x} = \vec{x}$

Theorem 1.1.2. If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_{k-1}$, then

$$\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$$

Theorem 1.1.3. If a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ contains the zero evctor, then it is linearly dependent.

Theorem 1.1.4. If $\beta = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for a subset S of $\mathbb{R}^n$, then every vector $\vec{x} \in S$ can be written as unique linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.

1.2 Subspaces

Theorem 1.2.1. Subspace Test: Let $\mathbb{S}$ be a non-empty subset of $\mathbb{R}^n$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$

Theorem 1.2.2. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

1.3 Dot Product

Theorem 1.3.1. If $\vec{x}, \vec{y} \in \mathbb{R}^2$, and θ is the angle between them, then

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \, ||\vec{y}|| \, \cos\theta$$

Theorem 1.3.2. Let $\vec{x}, \vec{y}\,\vec{z} \in \mathbb{R}^n$ and let $s, t \in \mathbb{R}$. Then

- $\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
- $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

Theorem 1.3.3. If $\vec{x} \cdot \vec{y} = 0$, then $\vec{x}$ and $\vec{y}$ are **orthogonal**.

Theorem 1.3.4. The **cross product** of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

Theorem 1.3.5. Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$ with $\{\vec{v}, \vec{w}\}$ being a linear independent set, and define $\vec{n} = \vec{v} \times \vec{w}$. If P is a plane with the vector equation

$$\vec{x} = c\vec{v} + d\vec{w} + \vec{b}, \qquad c, d \in \mathbb{R}$$

then an alternate equation for the plane is

$$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$$

n is a normal vector to the plane P. Rearranging: $n_1x_1 + n_2x_2 + n_3x_3 = n_1a_1 + n_2a_2 + n_3a_3$.

1.4 Projections

2 Systems of Linear Equations

2.1 Systems of Linear Equations

Theorem 2.1.1. Assume the system of linear equations with $a_1, \ldots, a_n, b \in \mathbb{R}$ has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$. Then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

2.2 Solving Systems of Linear Equation

Theorem 2.2.1. If two augmented matrices are row equivalent, then the system of linear equations associated with each matrix are equivalent.

Theorem 2.2.2. The RREF of a matrix is unique.

Theorem 2.2.3. Let A be the $m \times n$ coefficient matrix of a system of linear equations.

1. IF the rank of A is less than the rank for the augmented matrix, then the system is inconsistent.
2. If the system is inconsistent, then the system contains $n -$ rank A free variables. A consistent system has a unique solution if and only if rank $A = n$.
3. rank $A = m$ if and only if the system is consistent for every $\vec{b} \in \mathbb{R}^m$

Theorem 2.2.4. The solution set of a homogeneneous systems of M linear equations in n variables is a subspace of R^n.

3 Matrices and Linear Mappings

3.1 Operations on Matrices

Theorem 3.1.1. Let A, B, C be $m \times n$ matrices and let $c, d \in \mathbb{R}$

1. $A + B$ is an $m \times n$ matrix
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix such that $A + O_{m,n} = A$. This is called the **zero matrix**
5. There exists a matrix $(-A)$ such that $A + (-A) = O_{m,n}$
6. $cA \in M_{m \times n}$
7. $c(dA) = cd(A)$
8. $(c + d)A = cA + dA$
9. $c(A + B) = cA + cB$
10. $1A = A$

Theorem 3.1.2. For any $m \times n$ matrices A and B and scalar $c \in \mathbb{R}$,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$

- $(cA)^T = c(A^T)$

Theorem 3.1.3. If $\vec{e}_i$ is the ith standard basis vector for $\mathbb{R}^i$ and $A = [\vec{a}_1, \ldots, \vec{a}_n]$ is an $m \times n$ matrix, then

$$A\vec{e}_i = \vec{a}_i$$

Theorem 3.1.4. If A, B, C are matrices of the correct size so the required products are defined and $t \in \mathbb{R}$, then

- $A(B + C) = AB_AC$
- $t(AB) = (tA)B + A(tB)$
- $A(BC) = (AB)C$
- $(AB)^T = B^TA^T$

Theorem 3.1.5. Suppose that A and B are $m \times n$ matrices such that $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$.

Theorem 3.1.6. If I is the matrix $I = [\vec{e}_1, \ldots, \vec{e}_n]$ then for any $n \times n$ matrix where $IA = A = AI$

Theorem 3.1.7. The multiplicative identity for $M_{n\times n}(\mathbb{R})$ is unique.

3.2 Linear Mapping

Theorem 3.2.1. Let A be an $m \times n$ matrix, and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(\vec{x}) = A\vec{x}$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$ we have

$$f(b\vec{x} + c\vec{y}) = bf(\vec{x}) + cf(\vec{y})$$

Theorem 3.2.2. Every linear mapping can be represented as a matrix mapping whose columns are the images of the standard basis vector of $\mathbb{R}^n$ under L. $L(\vec{x}) = [L]\vec{x}$ where

$$[L] = [L(\vec{e}_1) \cdots L(\vec{e}_n)]$$

Theorem 3.2.3. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation with rotation matrix $A = [\mathbb{R}_\theta]$. Then the columns of A are orthogonal unit vectors.

3.3 Special Subspaces

Theorem 3.3.1. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $L(\vec{0}) = \vec{0}$.

Theorem 3.3.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $\ker(L)$ is a subspace of $\mathbb{R}^n$.

Theorem 3.3.3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $R(L)$ is a subspace of $\mathbb{R}^m$.

Theorem 3.3.4. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $A = [L]$ be the standard matrix of L. Then, $\vec{x} \in \ker(L)$ if and only if $A\vec{x} = \vec{0}$.

Theorem 3.3.5. Let A be an $m \times n$ matrix. A consistent system of linear equations $A\vec{x} = \vec{b}$ has a unique solution if and only if Null$(A) = \{\vec{0}\}$.

Theorem 3.3.6. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mpping with standard matrix $[L] = A = [\vec{a}_1 \cdots \vec{a}_n]$. Then

$$R(L) = \operatorname{span}\{\vec{a}_1, \dots, \vec{a}_n\}$$

Theorem 3.3.7. Let A be an $m \times n$ matrix. Then Col$(A) = \mathbb{R}^m$ if and only if rank$(A) = m$.

Theorem 3.3.8. Let A be an $m \times n$ matrix. If $\vec{a} \in$ Row(A) and $\vec{x} \in$ Null(A), then $\vec{a} \cdot \vec{x} = 0$.

Theorem 3.3.9. Let A be an $m \times n$ matrix. If $\vec{a} \in$ Col(A) and $\vec{x} \in$ Null(A^T), then $\vec{a} \cdot \vec{x} = 0$.

3.4 Operations on Linear Mapping

Theorem 3.4.1. Let $L, M, N \in \mathbb{L}$ and let c_1, c_2 be real scalars. Then

- $L + M \in \mathbb{L}$
- $(L + M) + N = L + (M + N)$
- $L + M = M + L$
- There exists a linear mapping $O : \mathbb{R}^n \to \mathbb{R}^m$ such that $L + O = L$. This means $O(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.
- There exists $(-L)$ such that $L + (-L) = O$.
- $c_1 L \in \mathbb{L}$
- $c_1(c_2 L) = (c_1 c_2)L$
- $(c_1 + c_2)L = c_1 L + c_2 L$
- $c_1(L + M) = c_1 L + c_1 M$
- $1L = L$

Theorem 3.4.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mapping and let $c \in \mathbb{R}$. Then

$$[L + M] = [L] + [M]$$

$$[cL] = c[L]$$

Theorem 3.4.3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. then $M \circ L$ is a linear mapping and

$$[M \circ L] = [M][L]$$

4 Vector Spaces

4.1 Vector Spaces

Theorem 4.1.1. Let $\mathbb{V}$ be a vector space with addition defined by $\vec{x} + \vec{y}$ and scalar multiplication defined by $c\vec{x}$ for all $\vec{x}, \vec{y} \in \mathbb{V}$, and $c \in \mathbb{R}$, Then

- $0\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{V}$
- $-\vec{x} = (-1)\vec{x}$ for all $\vec{x} \in \mathbb{V}$

Theorem 4.1.2. Let $\mathbb{S}$ be a non-empty subset of $\mathbb{V}$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$, and $c \in \mathbb{R}$ under the operations of $\mathbb{V}$, then $\mathbb{S}$ is a subspace of $\mathbb{V}$

Theorem 4.1.3. If If $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a span of vectors in a vector space $\mathbb{V}$, then $\operatorname{span} B$ is a subspace of $\mathbb{V}$.

Theorem 4.1.4. Let $\mathbb{V}$ be a vector space and $\vec{v}_1, \ldots, vv_k \in \mathbb{V}$. Then $v_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$.

Theorem 4.1.5. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ is linearly dependent if and only if there exists $1 \leq i \leq k$ such that

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$$

Theorem 4.1.6. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ which contains the zero vector is linearly dependent.

4.2 Bases and Dimension

Theorem 4.2.1. Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a vector space $\mathbb{V}$ and let $C = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be a set in $\mathbb{V}$. If $k > n$ (rank) then C is linearly dependent.

Proof. Consider $0 = c_1\vec{w}_1 + \cdots + c_k\vec{w}_k$.

Since B is a basis for $\mathbb{V}$, we can write every vector $\vec{w}$ as a linear combination of the vectors in B.

$$w_i = a_{i1}\vec{v}_1 + \cdots + a_{in}\vec{v}_n, \text{ for } 1 \leq i \leq k$$

Substituting,

$$0 = (c_1a_{11} + \cdots + c_ka_{k1})\vec{v}_1 + \cdots + (c_1a_{1n} + \cdots + c_ka_{kn})\vec{v}_n$$

Since B is a basis, it is linearly independent, and the only solution is when

$$c_1a_{11} + \cdots + c_ka_{k1} = 0$$

$$\vdots$$

$$c_1a_{1n} + \cdots + c_ka_{kn} = 0$$

Since $k > n$, the system has infinitely many solutions by the system rank theorem, so the equation has infinitely many solutions, and hence C is linearly independent. $\square$

Theorem 4.2.2. If $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $C = \{\vec{w}_1, \ldots, \vec{w}_k\}$ are bases for the vector space $\mathbb{V}$, then $k = n$.

Proof. Since B is a basis and C is linearly independent, $k \leq n$ by the previous theorem. Similary, since C is a basis and B is linearly independent, $n \neq k$. Hence $n = k$. □

Theorem 4.2.3. Let $\mathbb{V}$ be an n-dimensional vector space. Then

1. A set of more than n vectors in $\mathbb{V}$ must be inearly dependent.
2. A set of fewer than n vectors in $\mathbb{V}$ cannot span $\mathbb{V}$.
3. A set of n vectors in $\mathbb{V}$ is linearly independent if and only if it spans $\mathbb{V}$.

Theorem 4.2.4. If $\mathbb{V}$ is an n-dimensional vector space and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{V}$ with $k < n$, then there exists vectors $\vec{w}_{k+1}, \ldots, \vec{w}_n$ in $\mathbb{V}$ such that $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n\}$ is a basis for $\mathbb{V}$.

Proof. By Theorem 4.2.3, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ does not span $\mathbb{V}$.Let $\vec{w}_{k+1}$ be a vector in $\mathbb{V}$ such that $\vec{w}_{k+1} \notin \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. If $k + 1 = n$, then by Theorem 4.2.3, $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}\}$ is a basis. Else, repeat the procedure until it is true, and the set will be

$$\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n\}$$

□

4.3 Coordinates

Theorem 4.3.1. If $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for a vector space $\mathbb{V}$, then every vector $\vec{v} \in \mathbb{V}$ can be represented as a **unique** linear combination of $\vec{v}_1, \ldots, \vec{v}_n$.

Proof. Since B is a basis, it a spanning set. Ths for every vector $\vec{v} \in \mathbb{V}$ there exists constants such that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{v}$$

Assume that there also exists constants such that $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = \vec{v}$. Then

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = \vec{v}$$

$$(c_1 - d_1)\vec{v}_1 + \cdots + (c_n - d_n)\vec{v}_n = \vec{0}$$

But this implies $c_i = d_i$ for all $1 \leq i \leq n$ since B is linearly independent. Thus there exists only one linear combination of the vectors in B that equals $\vec{v}$. □

Theorem 4.3.2. If $\mathbb{V}$ is a vector space with $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$, then for any $\vec{v}, \vec{w} \in \mathbb{V}$, and $s, t \in \mathbb{R}$, we have

$$[s\vec{v} + t\vec{w}]_B = s[\vec{v}]_b + t[\vec{w}]_B$$

Proof. Let $\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$ and $w = c_1\vec{v}_1 + \cdots c_n\vec{v}_n$. Then we have

$$s\vec{v} + t\vec{w} = (sb_1 + tc_1)\vec{v}_1 + \cdots + (sb_n + tc_n)\vec{v}_n$$

Therefore,

$$[s\vec{v} + t\vec{w}]_B = \begin{bmatrix} sb_1 + tc_1 \\ \vdots \\ sb_n + tc_n \end{bmatrix} = s\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + t\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = s[\vec{v}]_B + t[\vec{w}]_B$$

□

Theorem 4.3.3. If B and C are bases for an n-dimensional vector space $\mathbb{V}$, then the change of coordinate matrices ${}_CP_B$ and ${}_BP_C$ satisfy

$${}_CP_B\, {}_BP_C = I = {}_BP_C\, {}_CP_B$$

5 Inverses and Determinants

5.1 Matrix Inverses

Theorem 5.1.1. If A is an $m \times n$ matrix with $n > m$, then A cannot have a right inverse.

Proof.

$$I = AB = [A\vec{b}_1 \cdots A\vec{b}_m] = [\vec{e}_1 \cdots \vec{e}_m]$$

Need to find $\vec{b}_i$ such that $A\vec{b}_i = \vec{e}_i$. But this is just solving m systems of linear equations with the same coefficient matrix. If we row reduce $[A|I_m]$ to RREF, we can find a solution to each equation. For this to be reduced to the identity matrix, we require rank$(A) = M$. Therefore we require that $n \geq m$. □

Theorem 5.1.2. If A is an $m \times n$ matrix with $n > m$, then A cannot have a left inverse.

Proof. If A has a left inverse C, then C is an $n \times n$ matrix with $n > m$ with a right inverse, contradicting the previous theorem. □

Theorem 5.1.3. The inverse of a matrix is unique.

Proof. Assume B and C are both inverses of A, then

$$B = BI = B(AC) = (BA)C = C$$

Therefore $B = C$, and the inverse is unique. □

Theorem 5.1.4. If A and B are $n \times n$ matrices such that $AB = I$, then A and B are invertible and rank$(A) = $ rank$(B) = n$.

Proof. Assume that $AB = I$ and consider the homogeneous system

$$B(\vec{x}) = \vec{0}$$

$$A(B\vec{x}) = A\vec{0}$$
$$(AB)\vec{x} = \vec{0}$$
$$I\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0}$$

So the system has a unique solution and by the system rank theorem, the coefficient matrix B has rank(n). This implies that $B\vec{x} = \vec{y}$ is consistent for all $y \in \mathbb{R}^n$.

$$BA\vec{y} = BA(B\vec{x}) = B(AB)\vec{x} = BI(vx) = B\vec{x} = \vec{y} = I\vec{y}$$

By the matrices equal theorem, $BA = I$, and we can repeat the previous procedure to obtain rank$(A) = n$, ☐

Theorem 5.1.5. IF A and B are invertible matrices, and $c \in \mathbb{R}$ with $c \neq 0$, then

- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 5.1.6. If A is an $n \times n$ matrix such that rank$(A) = n$, then A is invertible.

Proof. If rank$(A) = n$, then the system of equations $A\vec{b}_i = \vec{e}_i$, $1 \leq i \leq n$ are all consistent by Theorem 2.2.3. Let $B = [\vec{b}_1 \ \cdots \ \vec{b}_n]$, then we get

$$AB = A[\vec{b}_1 \ \cdots \ \vec{b}_n] = [A\vec{b}_1 \ \cdots \ A\vec{b}_n] = I$$

This by Theorem 5.1.4, A is invertible. ☐

Theorem 5.1.7. Invertible Matrix Theorem: For an $n \times n$ matrix A, the following are equivalent

1. A is invertible
2. THE RREF of A is I
3. rank$(A) = n$
4. The system of equations $A\vec{x} = \vec{b}$ is consistent with a unique solution for all $\vec{b} \in \mathbb{R}^n$
5. The nullspace of A is $\vec{0}$
6. The columns of A form a basis for $\mathbb{R}^n$
7. The rows of A for ma basis for $\mathbb{R}^n$
8. A^T is invertible

To solve a system $A\vec{x} = \vec{b}$, we can simply rearrange to get $\vec{x} = A^{-1}\vec{b}$.

5.2 Elementary Matrices

Theorem 5.2.1. If E is an elementary matrix, then E is invertible. Moreover E^{-1} is the elementary matrix corresponding to the reverse elementary row operations of E.

Theorem 5.2.2. If A is an $m \times m$ matrix, and E is an $m \times m$ matrix corresponding to the row operation of $R_i + cR_j$, for $i \neq j$, then EA is the matrix obtained from A by performing the row operation $R_i + cR_j$ on A.

Theorem 5.2.3. Let A be an $m \times n$ matrix and let E be the $m \times m$ elementary matrix corresponding to the row operation of cR_i. Then EA is the matrix obtained from A by performing the row operation cR_i on A.

Theorem 5.2.4. Let A be an $m \times n$ matrix and let E be the $m \times m$ elementary matrix corresponding to the row operation $R_i \leftrightarrow R_j$, for $i \neq j$. Then EA is the matrix obtained from A by performing the row operation $R_i \leftrightarrow R_j$ on A.

Theorem 5.2.5. Let A be an $m \times n$ matrix, and let E be an $m \times m$ elementary matrix. Then

$$\text{rank}(EA) = \text{rank}(A)$$

Theorem 5.2.6. If A is an $m \times n$ matrix in its RREF form, then there exists a sequence $E_k, \ldots, E_2, E_1$, of $m \times m$ matrices such that $E_k \cdots E_2 E_1 A = R$. In particular

$$A = E_1^{-1} \cdots E_k^{-1} R$$

Proof. We know that A can be reduced to its RREF, R with a sequence of elementary row operations. Let E_1 be the first operation, E_2 be the second row operation, and so on. By the previous few theorems,

$$E_k \cdots E_2 E_1 A = R$$

□

Theorem 5.2.7. If A is an invertible matrix, then A and A^{-1} can be written as a product of elementary matrices.

5.3 Determinants

Theorem 5.3.1. Let A be an $n \times n$ matrix.

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik}$$

$$\det A = \sum_{k=1}^{n} a_{ij} C_{kj}$$

The cofactor expansion can be called upon any row or column to obtain the determinant.

Theorem 5.3.2. If a matrix is upper or lower triangular, the determinant is just the product of the numbers along the diagonal

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

Proof. If A is a 2×2 upper triangular matrix, then det $A = a_{11}a_{22}$ as required. Assume that if B is an $(n-1) \times (n-1)$ upper triangular matrix, then det $B = b_{11}\cdots b_{(n-1)(n-1)}$. Let A be an $n \times n$ upper triangular matrix. Expanding the determinant along the first column gives

$$\det A = a_{11}(-1)^{1+1}C_{11} + 0 + \cdots + 0 = a_{11}\det A(1,1)$$

But $A(1,1)$ is the $(n-1) \times (n-1)$ upper triangular matrix formed by deeting the first row and first column of A. Thus, det $A(1,1) = a_{11}a_{22}\cdots a_{(n-1)(n-1)}$ by the inductive hypothesis. Thus det $A = a_{11}a_{22}\cdots a_{nn}$ as required. □

Theorem 5.3.3. If B is the matrix obtained by multiplying one row of A by $c \in \mathbb{R}$, then det $B = c$ det A.

Theorem 5.3.4. If B is the matrix obtained from A by swapping two rows of A. Then det $B = -$det A.

Theorem 5.3.5. If a matrix A has two identical rows, then det $A = 0$.

Theorem 5.3.6. If B is the matrix obtained from A by adding a multiple of one row of A to another, then det $B =$det A.

Theorem 5.3.7. Let A be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then det $EA =$det E det A.

Theorem 5.3.8. An $n \times n$ matrix is invertible if and only if its determinant is not 0.

Theorem 5.3.9. If A and B are $n \times n$ matrices, then det $(AB) =$det A det B.

Theorem 5.3.10. If A is an invertible matrix, then det $A^{-1} = \frac{1}{\det A}$.

Theorem 5.3.11. If A is an $n \times n$ matrix, then det $A =$ det A^T.

6 Diagonalization

6.1 Similar Matrices

Theorem 6.1.1. Let A and B be $n \times n$ matrices such that $P^{-1}AP = B$ for some invertible matrix P. Then

- rank $A =$ rank B
- det $A =$ det B
- tr $A =$ tr B

6.2 Eigenvalues & Eigenvectors

Theorem 6.2.1. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if $C(\lambda) = 0$

Theorem 6.2.2. If A is an $n \times n$ upper or lower trianglar matrix, then the eigenvalues of A are the diagonal entries of A.

Theorem 6.2.3. If A and B are similar matrices, then A and B have the same characteristic polynomial, and hence the same eigenvalues.

Theorem 6.2.4. If A and B are similar matrices, then A and B have the same characteristic polynomial, and hence the same eigenvalues.

Theorem 6.2.5. IF A is an $n \times n$ matrix, with eigenvalue λ_1, then

$$1 \leq g_{\lambda_1} \leq a_{\lambda_1}$$

6.3 Diagonalization

Theorem 6.3.1. A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there exists a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of A.

Theorem 6.3.2. If A is an $n \times n$ matrix with eigenpairs, $(\lambda_1, \vec{v}_1), \ldots, (\lambda_k, \vec{v}_k)$ where $\lambda_i \neq \lambda_j$, for $i \neq j$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Theorem 6.3.3. Of A is an $n \times n$ matrix with distinct eigenvaleus, and B is a basis for the eigenspace of λ_1 for $i \leq i \leq k$, then $B_1 \cup \cdots \cup B_k$ is a linearlly independent set.

Theorem 6.3.4. If A is an $n \times n$ matrix with distinct eigenvallues $\lambda_1. \ldots, \lambda_k$, then A is diagonalizable if and only if $g_{\lambda_i} = g_{\lambda_k}$ for $1 \leq i \leq k$.

Theorem 6.3.5. If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Theorem 6.3.6. If A is an $n \times n$ matrix, then

- the determinant of A equals the product of the eigenvalues
- λ is an eigenvalue of A if and only if $\det(A) = 0$. If it is, then the geometric multiplicity of 0 equals $1 - \text{rank}(A)$.
- The trace of A equals the sum of the eigenvalues.

6.4 Powers of Matrices

Theorem 6.4.1. If $D = \text{diag}\,(d_1, \ldots, d_n)$, then $D^m = \text{diag}\,(d_1^m, \ldots, d_n^m)$.

Theorem 6.4.2. If there exists an invertible matrix p such that $P^{-1}AP = D$ is diagonal, then

$$A^m = PD^mP^{-1}$$