MATH 136

Linear Algebra

Professor M. Rubinstein • Winter 2014 • University of Waterloo

Last Revision: February 18, 2015

Table of Contents

Future Modifications

1 Vectors

Linear Algebra

- Systems of linear equations
- Related geometry
- Matrices
- Vector spaces, $\mathbb{R}^n$

$\mathbb{R}^n$ consists of n-tuples of real numbers, where $n \in \mathbb{N}$.

1.1 Vector Properties

Definition 1.1. **Points/vectors** are elements of $\mathbb{R}^n$.

Notation

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

$$x_1 + x_2 = 3$$

$$2x_1 + 5x_2 = 4$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Two vectors in $\mathbb{R}^n$ are equal if all coordinates are equal.

Vector Operations

Let $\vec{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}$

Addition

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

Scalar Multiplication

$$\alpha\vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 1.2. $\vec{0}$ is the **additive identity**

Definition 1.3. Given a vector $\vec{x} \in \mathbb{R}^n$, $-\vec{x}$ is the **additive inverse**.

Definition 1.4. A sum of scalar multiples of a combination of vectors is a **linear combination**

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \,:\, c_1 \ldots c_k \in \mathbb{R}$$

Theorem 1.1. If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$, then

- $\vec{x} + \vec{y} \in \mathbb{R}^n$
- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + vw)$
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\exists \vec{0} \in \mathbb{R}^n$ such that $\vec{x} + \vec{0} = \vec{x} \qquad \forall \vec{x} \in \mathbb{R}^n$
- $\forall \vec{x} \in \mathbb{R}^n$, there exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- $c\vec{x} \in \mathbb{R}^n$
- $c(d\vec{x}) = (cd)\vec{x}$
- $(c + d)\vec{x} = c\vec{x} + d\vec{y}$
- $c(\vec{x} + vy) = c\vec{x} + c\vec{y}$
- $1\vec{x} = \vec{x}$

Definition 1.5. The set S of all possible linear combinations of a set of vectors $B = (\vec{v}_1, \ldots, \vec{v}_k)$ in $\mathbb{R}^n$ is called the **span** of the set B and we write

$$S = \text{Span B} = \{t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k\}$$

S is **spanned** by B and that B is a spanning set for S.

For a set

$$\{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + \vec{b} | t_1, \ldots, t_k \in \mathbb{R}\}$$

can be written as

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + \vec{b}, t_1, \ldots, t_k \in \mathbb{R}$$

In $\mathbb{R}^n$, two linearly independent vectors $\vec{x}_1$ and $\vec{x}_2$ generate a plane.

Definition 1.6. A set of vectors in $\mathbb{R}^n$ is said to be **linearly dependent** if there exists coefficients $c_1, \ldots, c_k$, not all 0, such that

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

Either 0 vector or two or more vectors are colinear (scalar multiple).

Definition 1.7. A set of vectors is **linearly independent** if the only solution is $c_1 = c_2 = \cdots = c_k = 0$ (**trivial solution**)

Definition 1.8. If a subset of $\mathbb{R}^n$ can be written as a span of vectors $\vec{v}_1, \ldots, \vec{v}_k$ where $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a **basis** for S. The basis of the set $\{\vec{0}\}$ is the empty set.

Theorem 1.2. If $\beta = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for a subset S of $\mathbb{R}^n$, then every vector $\vec{x} \in S$ can be written as unique linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.

Definition 1.9. The **standard basis** in $\mathbb{R}^n$ is a set of vectors where each vector's ith component is 1, and all other components are 0.

Definition 1.10. Let $\vec{x}, vy \in \mathbb{R}^n$. The set with vector equation $\vec{w} = c_1\vec{x} + \vec{y}$ with $c_1 \in \mathbb{R}$ is a **line** in $\mathbb{R}^n$ that passes through $\vec{y}$.

Definition 1.11. Let $\vec{v}_1, \vec{v}_2, \vec{y} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. The set with the vector equation $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{y}$ with $c_1, c_2 \in \mathbb{R}$ is a **plane** in $\mathbb{R}^n$ which passes through $\vec{y}$.

Definition 1.12. Let $\vec{v}_1, \ldots, \vec{v}_k, \vec{y} \in \mathbb{R}^n$ with the set being linearly independent. The set with the vector equation $\vec{x} = c_1 v_1 + \cdots + c_k\vec{v}_k + \vec{y}$ with $c_1, \ldots, c_k$ is a **k-plane** in R^n with passes through $\vec{y}$.

Definition 1.13. A **hyperplane** is a subspace of one dimension less than its ambient space.

1.2 Subspaces

Theorem 1.3. Subspace Test: Let $\mathbb{S}$ be a non-empty subset of $\mathbb{R}^n$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$

Quote. If $\vec{0}$ is not in the set, definitely not subset. If it is, further investigation needed.

Definition 1.14. $S \in \mathbb{R}^n$ is closed under scalar multiplication if for all $\vec{x} \in S$ and $\alpha \in \mathbb{R}$, $\alpha\vec{x} \in S$.

Theorem 1.4. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

Theorem 1.5. If $\vec{x}, \vec{y} \in \mathbb{R}^2$, and θ is the angle between them, then

$$\vec{x} \cdot \vec{y} = ||\vec{x}||\,||\vec{y}||\,\cos\theta$$

Theorem 1.6. Given two vectors $\vec{x}, vy$, their dot product is defined by

$$\vec{x}\dot{\vec{y}} = x_1y_1 + x_2 + y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i$$

Theorem 1.7. If $\vec{x} \cdot \vec{y} = 0$, then $\vec{x}$ and $\vec{y}$ are **orthogonal**.

Quote. The zero vector $\vec{0} \in \mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$.

Theorem 1.8. The **cross product** of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

Quote. Cross product is not associative. $\vec{v} \times (\vec{w} \times \vec{x}) \neq (\vec{v} \times \vec{w}) \times \vec{x}$.

Theorem 1.9. Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$ with $\{\vec{v}, \vec{w}\}$ being a linear independent set, and define $\vec{n} = \vec{v} \times \vec{w}$. If P is a plane with the vector equation

$$\vec{x} = c\vec{v} + d\vec{w} + \vec{b}, \qquad c, d \in \mathbb{R}$$

then an alternate equation for the plane is

$$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$$

n is a normal vector to the plane P. Rearranging: $n_1x_1 + n_2x_2 + n_3x_3 = n_1a_1 + n_2a_2 + n_3a_3$.

1.3 Projections

Definition 1.15. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The **projection of $\vec{u}$ onto $\vec{v}$** is

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

The **perpendicular of $\vec{u}$ onto $\vec{v}$** is

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u})$$

Quote. To project a vector onto a plane, take the perpendicular of the vector projected onto the normal of the plane.

2 Systems of Linear Equations

Definition 2.1. A system of linear equations in n variables

$$cx_1 + cx_2 + \cdots + cx_n = b_1 \tag{1}$$

$$cx_1 + cx_2 + \cdots + cx_n = b_2 \tag{2}$$

$$cx_1 + cx_2 + \cdots + cx_n = b_3 \tag{3}$$

$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix} \in \mathbb{R}^n$ is a solution to the system if all the equations are satisfied when x_i is set to s_i.

If a system has a solution, it is **consistent**. If not, it is **inconsistent**.

Definition 2.2. A **solution set** is the set of all solutions of a system of linear equations. Two systems of equations are equivalent if they have the same solution set.

Definition 2.3. The **coefficient matrix** of a system is denoted by $A = \begin{bmatrix} a_{11} & a_{21} & \cdots \\ a_{21} & a_{22} & \cdots \end{bmatrix}$.

Definition 2.4. The **augment matrix** is

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Theorem 2.1. If two augmented matrices are row equivalent, then the system of linear equations associated with each matrix are equivalent.

Method 2.1. The three **elementary row operations** for solving a system of linear equations are:

1. Multiplying a row by a scalar
2. Adding a multiple of one row to another
3. Swapping two rows

Definition 2.5. A matrix is said to be in **reduced row echelon form** if:

1. All rows containing a non-zero entry are above rows which only contain zeroes.
2. The first non-zero entry in each row is 1. (**leading one**).
3. Leading one on each zero row is to the right of the leading one on any row above it.
4. Leading one is the only non-zero entry in its column.

Definition 2.6. Let R be the RREF of a coefficient matrix of a system of linear equations. If the jth column does not contain a leading one, x_j is a **free variable**.

Definition 2.7. The **rank** of a matrix is the number of leading ones in the RREF of the matrix.

Theorem 2.2. Let A be the $m \times n$ coefficient matrix of a system of linear equations.

1. IF the rank of A is less than the rank for the augmented matrix, then the system is inconsistent.
2. If the system is inconsistent, then the system contains $n -$ rank A free variables. A consistent system has a unique solution if and only if rank $A = n$.
3. rank $A = m$ if and only if the system is consistent for every $\vec{b} \in \mathbb{R}^m$

Definition 2.8. A system of linear equations is said to be **homogeneous system** if the right-hand side only contains zeroes. It has the form $\left[A \,|\, \vec{0}\right]$.

Theorem 2.3. The solution set of a homogeneneous systesm of M linear equations in n variables is a subspace of R^n.

3 Matrices

Definition 3.1. A $m \times n$ **matrix** is a rectangular array with m rows and n columns.

Definition 3.2. The **zero matrix**, denoted as $O_{m,n}$ is the matrix whose entries are all 0.

Definition 3.3. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A.

$$(A^T)_{ij} = (A)_{ji}$$

Theorem 3.1. For any $m \times n$ matrices A and B and scalar $c \in \mathbb{R}$,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$

Definition 3.4. Matrix-Vector multiplication: Let A be an $m \times n$ matrix whose rows are denoted $\vec{a}_i^T$ for $1 \leq i \leq m$. Then, for any $\vec{x} \in \mathbb{R}^n$, we define

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

An alternate form is

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Method 3.1. Matrix Multiplication: Let A be an $m \times n$ matrix and let $B = [\vec{b}_1 \cdots \vec{b}_p]$ be an $n \times p$ matrix. Then, AB is the $m \times p$ matrix

$$AB = [A\vec{b}_1 \cdots A\vec{b}_p]$$

Note. The number of columns of A must equal to the number of rows of B for this to be defined. The resulting matrix will have the same rows as A and same columns as B.

Quote. Matrix Multiplication is NOT commutative. $AB \neq BA$., if $AB = AC$, $B \neq C$.

Definition 3.5. The $n \times n$ **identity matrix**, denoted as I, is the matrix containing a diagonal row of 1s and everything else set to 0. The columns of I_n are the standard basis vectors of $\mathbb{R}^n$.

For every $n \times n$ matrix, A, $AI = A$.

Definition 3.6. Block matrix:

Example 3.1. Let $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$. By reducing A into blocks, we can write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 3 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 1 & 2 \end{bmatrix}$.
These are useful to distribute matrix multiplication over multiple computers to speed up the process.

3.1 Linear Mapping

Definition 3.7. A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a **linear mapping** if for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$, we have

$$L(b\vec{x} + c\vec{y}) = bL(\vec{x}) + cL(\vec{y})$$