MATH 138

Calculus II

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Last Revision: December 14, 2014

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Future Modifications

1 Integration

1.1 Fundamental Theorem of Calculus

Theorem 1.1. Suppose f is continuous on $[a, b]$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Differentiation undoes integration. A generalization is

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Example 1.1. Find the derivative of

$$\int_5^x t^2 \ln(\pi t + 7)\,dt$$

Solution. Apply 1.1

$$\frac{d}{dx}\int_5^x t^2 \ln(\pi t + 7)\,dt = x^2 \ln(\pi x + 7)$$

Example 1.2. Evaluate

$$\frac{d}{dx}\int_2^{x^2} \frac{1}{t}\,dt$$

Solution. $f(t) = \frac{1}{t}$ and $g(x) = 2$ and $g'(x) = 0$ and $h(x) = x^2$ and $h'(x) = 2x$

$$\frac{d}{dx}\int_2^{x^2} \frac{1}{t}\,dt = \frac{1}{x^2}2x - (\frac{1}{2})(0) = \frac{2}{x}$$

Definition 1.1. An **antiderivative** of a function f(x) is a function F(x) satisfying F'(x) for all x. Also known as indefinite integral.

Example 1.3. Find an antiderivative of $\frac{1}{x} + cos(x) + x^2$ for $x > 0$

Solution.

$$\frac{1}{x} + \cos(x) + x^2 = \ln(x) + \sin(x) + \frac{x^3}{3}$$

If you have time, always check by differentiating.

Theorem 1.2 (Fundamental Theorem of Calculus II)**.** If f is continuous on the interval $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is **any** antiderivative of f.

Example 1.4. Evaluate $\int_1^2 (\frac{1}{x} + \cos(x) + x^2)dx$

Solution. Apply 1.2

$$F(x) = \ln(x) + \sin(x) + \frac{x^3}{3}$$

$$\begin{aligned} F(2) - F(1) &= \ln(2) + \sin(2) + \frac{2^3}{3} - (\ln(1) + \sin(1) + \frac{1}{3}) \\ &= \ln(2) + \sin(2) - \sin(1) + \frac{7}{3} \end{aligned}$$

Exercise 1.1. Evaluate

$$\frac{d}{dx}\int_2^{10} (t^2 e^t \ln(t))\, dt$$

1.2 Techniques of Integration

1.2.1 Integration by Substitution

While a function may not be easily integrable with respect to its current variable, there may exist another variable we can integrate with respect to for which the integration step is easier.

Example 1.5. Evaluate $\int (x\sqrt{2x-1})\, dx$

Solution. Define new variable u in terms of x
$u = 2x - 1$ then $\frac{du}{dx} = 2$ and $\frac{du}{2} = dx$
Since $\frac{u+1}{2} = x$ then

$$\begin{aligned} \int (x\sqrt{2x-1})\, dx &= \int (u^{\frac{1}{2}})(\frac{u+1}{2})\, \frac{du}{2} \\ &= \frac{1}{4}\int \sqrt{u}(u+1)\, du \\ &= \frac{1}{4}\int u^{\frac{3}{2}}\, du + \frac{1}{4}\int \sqrt{u}\, du \\ &= \frac{1}{10}u^{\frac{5}{2}} + \frac{1}{6}u^{\frac{3}{2}} + C \\ &= \frac{(2x-1)^{\frac{5}{2}}}{10} + \frac{(2x-1)^{\frac{3}{2}}}{6} + C \\ &= \frac{1}{2}[\frac{(2x-1)^{\frac{5}{2}}}{5} + \frac{(2x-1)^{\frac{3}{2}}}{3}] + C \end{aligned}$$

General Procedure for Integration by Substitution

1. Choose an appropriate new variable
2. Replace old variables with the new variables (including the differential (dx))
3. Integrate with respect to the new variable.

4. Replace the new variable with the original
5. Check by differentiating

Integration by substitution also can be applied to definite integral in the following two ways

- Keep the limits in the original variable. Integrate with new variable, replace back old variable, then solve.
- Find the new limits with respect to the new variable. Integrate and solve.

Example 1.6. Find the integral

$$\int_1^2 \frac{(\ln(x)^2)}{x}\, dx$$

Solution. Let u = ln(x) $\frac{du}{dx} = \frac{1}{x}$ then $du = \frac{1}{x} dx$
Try finding new limits
$x = 2- > u = \ln(2)$
$x = 1- > u = \ln(1) = 0$

$$\int_1^2 \frac{(\ln(x)^2)}{x}\, dx = \int_0^{\ln(2)} u^2\, du = u^3/3\Big|_0^{\ln(2)} = \frac{(\ln(2))^3}{3}$$

Try leaving the limits in terms of x

$$\int_1^2 \frac{(ln(x)^2)}{x}\, dx = \int_{x=1}^{x=2} u^2\, du = \frac{(ln(2))^3}{3}\Big|_{x=1}^{x=2})$$

Exercise 1.2. Solve

$$\int \frac{1}{5t^2+7}\, dt$$

Recall that

$$\int \frac{1}{u^2+1}\, du = \arctan(x) + C$$

Manipulate integral:

$$\frac{1}{7}\int \frac{1}{\frac{5t^2}{7}+1}\, dt$$

$$\frac{1}{7}\int \frac{1}{(\sqrt{\frac{5}{7}}t)^2+1}\, dt$$

Let $u = \sqrt{\frac{5}{7}}t$ then $\frac{du}{dt} = \sqrt{\frac{5}{7}}$

$$\frac{1}{7\sqrt{\frac{5}{7}}}\int \frac{1}{u^2+1}\, du$$

$$\frac{1}{35}\int \frac{1}{u^2+1}\, du$$

$$\frac{1}{35}\tan^{-1}(u) + C$$

$$\frac{\tan^{-1}(\sqrt{\frac{5}{7}})}{35} + C$$

1.2.2 Integration by Parts

Idea is similar to product rule but for integration.

Theorem 1.3 (Integration by Parts)**.** Let f(x) and g(x) be two functions
Applying the product rule to fg leads to

$$\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}gx + f(x)\frac{dg(x)}{dx}$$

$$\int \frac{d}{dx}(f(x)g(x)) = \int \frac{df(x)}{dx}gx + f(x)\int \frac{dg(x)}{dx}$$

$$f(g(x)) = \int f'g(x)\,dx + \int f(g'(x))\,dx$$

$$\int f'(g(x))\,dx = f(g(x)) - \int f(g'(x))\,dx$$

For definite integrals, Integration by Pats is

$$\int_a^b f'(g(x))\,dx = f(g(x)) - \int_a^b f(g'(x))\,dx$$

Note. Let $v = g(x)$ and let $u = f(x)$, Integration by Parts can be equivalently stated as

$$\int u\,dv = uv - \int v\,du$$

Example 1.7. Evaluate

$$\int_0^1 xe^x\,dx$$

Solution. Apply IBP: $Let f = x, f' = 1, Let g' = e^x, g = e^x$

$$xe^x\Big|_0^1 - \int_0^1 1 * e^x\,dx$$

$$= e^1 - e^x\Big|_0^1$$

$$= e^1 - (e^1 - e^0))$$

$$= 1$$

Exercise 1.3. Apply 1.3 to example 1.5

Some suggestions for using IBP

- g' is usually the more complicated component, if it can be integrated, then do it.
- f should have a simpler derivative.
- it may be necessary to apply IBP multiple times, in which case, let the factors play the same role in each step.
- Integration by Parts can be applied to a single term by setting f or g' to be 1.
- Sometimes the original integral re-appears after a few iterations in which case, solve for the original integral.
- May have to apply other integration techniques after first applying integration by parts.

Example 1.8. Determine the integral $\int \ln(x)\,dx$ Let $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$ and let $g'(x) = 1$

$$\begin{aligned}\int \ln(x)\,dx &= \int (\ln(x))(1)\,dx \\ &= x\ln(x) - x + C\end{aligned}$$

Example 1.9. Evaluate

$$\int e^{-2x}\cos(x)\,dx$$

$f(x) = e{-}2x$ then $f'(x) = -2e^{-2x}$ and $g'(x) = \cos(x)$

$$\int e^{-2x}\cos(x)\,dx = e^{-2x}\sin(x) + 2\int e^{-2x}\sin(x)\,dx$$

$f = e^{-2x}$ then $f'(x) = -2e^{-2x}$ and $g'(x) = \sin(x)$ then $g(x) = -\cos(x)$

$$e^{-2x}\sin(x) + 2[-e^{-2x}\cos(x) - 2\int e^{-2x}\cos(x)\,dx]$$

$$5\int e^{-2x}\cos(x)\,dx = e^{2x}\sin(x) - 2e^{-2x}\cos(x)$$

$$\int e^{-2x}\cos(x)\,dx = \frac{1}{5}e^{-2x}\sin(x) - \frac{2}{5}e^{-2x}\cos(x) + C$$

Example 1.10. Solve $\int x^3e^x\,dx$

Solution. $f(x) = x^3$ then $f'(x) = 3x^2$ and $g'(x) = e^x$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx$$

Apply 1.3: $f(x) = x^2$ then $f'(x) = 2x$ and $g'(x) = e^x$

$$x^3e^x = 3[x^2e^x - 2\int xe^x\,dx]$$

Apply 1.3: $f(x) = x$ then $f'(x) = 1$ and $g'(x) = e^x$

$$x^3e^x - 3x^2e^x + 6[xe^x - \int e^x\,dx]$$

$$x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

1.2.3 Trigonometric Integrals

Solving integrals of the form

$$\int \cos^n(x)\sin^m(x)\,dx$$

$$\int \tan^n(x)\sec^m(x)\,dx$$

$$\int \csc^n(x)\cot^m(x)\,dx$$

$m, n \in \mathbb{Z} \mid m, n > 0$

Helpful trigonometric axioms:

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1 + cos(2x)}{2}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Example 1.11. Evaluate $\int \cos^2(x)\sin^3(x)\,dx$

Solution. Let $u = \sin(x)$ then $\frac{du}{dx} = \cos(x)$

$$\int \cos^2(x)\sin^3(x)\,dx = \int u^3 cos(x)\,du$$

Note. Don't replace cos(x) with $\sqrt{1 - sin^2(x)}$

Very complicated to continue. Perhaps another substitution works better.

Let $u = \cos(x)$ then $\frac{du}{dx} = -\sin(x)$

$$\int u^2 sin^2(x)\,du$$

$$\int u^2(1-u^2)\,du$$

$$\int (-u^4 + u^2)\,du$$

$$-\frac{u^3}{3} + \frac{u^3}{5} + C$$

$$-\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C$$

Guidelines for solving $\int \cos^n(x)\sin^m(x)\,dx$

1. Where there is an odd and even power, factor out the odd power and make the appropriate substitution. See 1.11
2. When both powers are odd, then factor out either.
3. For both even powers, apply power reduction identity to lower the power.

Exercise 1.4. Evaluate $\int \sin^2(\theta)\cos^2(\theta)\,d\theta$

Solution. Apply $\sin^2(\theta) = \frac{1-cos(2\theta)}{2}$
Apply $\cos^2(\theta) = \frac{1+cos(2\theta)}{2}$

$$\frac{1}{4}\int (1 - \cos^2(2\theta))\,d\theta$$

$$\frac{1}{4}\int (1 - [\frac{1+cos(4\theta)}{2}]), d\theta$$

$$\frac{1}{8}\int (2 - 1 - cos(4\theta)), d\theta$$

$$\frac{1}{8}\int 1 - \frac{1}{8}\int \cos(4\theta)\,d\theta$$

$$\frac{1}{8}\int 1 - \frac{1}{8}\int \cos(4\theta)\,d\theta$$

$$\frac{1}{8}(\theta - \frac{\sin 4\theta}{4}) + C$$

Example 1.12. Evaluate $\int \sec^7(x)\tan^3(x)\,dx$

Don't let $u = \tan(x)$ because $sec^5(x)$ is odd power.

$$\int \sec^7(x)\tan^3(x)\,dx = \int sec^5(x)u^3\,du$$

So try $u = \sec(x)$ then $\frac{du}{dx} = \sec(x)\tan(x)\,dx$

$$\int \sec^7(x)\tan^3(x)\,dx = \int u^6\tan^2(x)\,du$$

$$\int u^6(u^2-1)\,du$$

$$\int u^8 - u^6\,du$$

$$\frac{u^9}{9} - \frac{u^7}{7} + C$$

$$\frac{(\sec^2(x))^9}{9} - \frac{(\sec^2(x))^7}{7} + C$$

Exercise 1.5. Compute

$$\int \sec^6(\theta)\tan^2(\theta)\,d\theta$$

Let $u = \tan(x)$ then $\frac{du}{dx} = \sec^2(x)$

$$\int \sec^4(\theta)u^2\,du$$

$$\int (1+u^2)^2u^2\,du$$

$$\int u^2 + 2u^4 + u^8\,du$$

$$\frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^9}{9} + C$$

$$\frac{\tan^3(x)}{3} + \frac{2\tan^5(x)}{5} + \frac{\tan^9(x)}{9} + C$$

Products of cot/csc

Apply similar approach as for tan/sec and use

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

1.2.4 Trigonometric Substitution

Recall:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$$
$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$$

Example 1.13. Evaluate

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution. Let $x = \sin\theta$ then $dx = \cos\theta\, d\theta$

$$\int \frac{\cos\theta\, d\theta}{\sqrt{cos^2\theta}}$$

$$\int \frac{\cos\theta\, d\theta}{|\cos\theta|}$$

Assume $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\int 1\, d\,\theta$$

$$\arcsin x + C$$

1.2.5 Table of Trigonometry Substitutions

$$\sqrt{a^2-x^2}$$

$$x = a\sin\theta$$

Expression	Substitution	Domain	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1-\sin^2\theta = \cos\theta$
	$x = a\cos\theta$	$0 < \theta < \frac{\pi}{2}$	
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 < \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan\theta$

Example 1.14. Solve $\int \frac{1}{x\sqrt{x^2+3}}\,dx$

Solution. Given $\sqrt{x^2\sqrt{3}^2}$, we choose $x = \sqrt{3}\tan\theta$

$$dx = \sqrt{3}\sec^2\theta\, d\theta$$

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2+3}}\,dx &= \int \frac{\sqrt{3}^2 \sec^2\theta\,d\theta}{3\tan\theta\sec\theta} \\ &= \frac{\sqrt{3}}{3}\int \csc\theta\,d\theta \\ &= \frac{\sqrt{3}}{3}\int \csc\theta\frac{\csc\theta+\cot\theta}{csc\theta+\cot\theta}\,d\theta\end{aligned}$$

Let $u = \csc\theta + \cot\theta$ then $du = (-\csc\theta\cot\theta - \csc^2\theta)\,d\theta$

$$\frac{\sqrt{3}}{3}\int \frac{du}{u} = \frac{\sqrt{3}}{3}\ln|u| + C$$

$$\frac{\sqrt{3}}{3}\int \frac{du}{u} = \frac{\sqrt{3}}{3}\ln|\csc\theta + \cot\theta| + C$$

Since $\tan\theta = \frac{x}{\sqrt{3}}$ then by drawing and labelling and right angle triangle, $\sin\theta = \frac{x}{\sqrt{x^2+3}}$ and $\csc\theta = \frac{\sqrt{x^2+3}}{x}$

$$\frac{\sqrt{3}}{3}\ln(\frac{\sqrt{x^2+3}}{x} + \frac{\sqrt{3}}{x}) + C$$

Example 1.15. Determine

$$\int \frac{x}{\sqrt{27+6x-x^2}}\,dx$$

Solution. Rewrite the radical ino a sum of squares by using completing the square.

$$\begin{aligned}&-x^2+6x+27 \\ =&-(x^2-6x)+27 \\ =&-(x^2-6x+9)+9+27 \\ =&-(x-3)^2+6^2\end{aligned}$$

$$\int \frac{x}{\sqrt{27+6x-x^2}}\,dx = \int \frac{x}{\sqrt{6^2-(x-3)^2}}\,dx$$

Let $x-3 = 6\sin\theta$ then $dx = 6\cos\theta\,d\theta$

$$\int \frac{6\sin\theta+3}{6\cos\theta}(6\cos\theta)\,d\theta$$

$$\int 6\sin\theta + 3\,d\theta$$

$$-6\cos\theta + 3\theta + C$$

Rewrite theta in terms of x. Draw a triangle if necessary.

$$-6\frac{\sqrt{6^2-(x-3)^2}}{6} + 3\theta + C$$

$$-\sqrt{36-(x-3)^2} + 3\arcsin(\frac{x-3}{6}) + C$$

Exercise 1.6. Use an appropriate integral to prove the area of a circle with radius r is πr^2.

Solution. The area of a circle is given by $\pm\sqrt{r^2 - x^2}$ We can just find the area of a quarter of the circle and multiply it by 4.

$$y = r\sqrt{\frac{1 - x^2}{r^2}}\,dx$$

$$A = 4\int_0^r r\sqrt{1 - \frac{x^2}{r^2}}\,dx$$

Let $\frac{x}{r} = \sin t$ then $dx = r\cos t\,dt$

$$A = 4\int_0^{\frac{\pi}{2}} r^2\sqrt{1 - \sin^2 t}\cos t\,dt$$

$$A = 4\int_0^{\frac{\pi}{2}} r^2\sqrt{\cos^2 t}\cos^2 t\,dt$$

Since in our example, $\cos t$ ranges from 0 to $\frac{\pi}{2}$ then $\sqrt{\cos^2 t} = \cos t$

$$A = 4\int_0^{\frac{\pi}{2}} r^2\cos^2 t\,dt$$

$$A = 4r^2\int_0^{\frac{\pi}{2}} \frac{\cos 2t + 1}{2}\,dt$$

$$4r^2\left(\frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}t\Big|_0^{\frac{\pi}{2}}\right)$$

$$A = 4r^2\left(0 + \frac{1}{2}\frac{\pi}{2}\right)$$

$$A = \pi r^2$$

1.2.6 Integration of Rational Functions by Partial Fractions (PFD)

Consider the following rational function:

$$\int \frac{x}{(x-1)(x+1)(x+3)}\,dx$$

If we can rewrite this as $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$ where A, B, C are constants, then we can easily integrate. Consider the general form $\int \frac{P(x)}{Q(x)}\,dx$ where P and Q are polynomials.

1. If degree of P is less than the degree of Q, skip this step. Else, if degree of P is greater than the degree of Q, rewrite $\frac{P(x)}{Q(x)}$ as $S(x) + \frac{R(x)}{Q(x)}$ where $R(x)$'s degree will be less than $Q(x)$'s degree.

2. Factor Q as far as possible into products of linear and irreducible quadratic factors. It turns out any polynomial can be factored uniquely into chains of terms of the form $(ax+b)^n$ or $(\beta x^2 + \alpha x + \lambda)^m$

3. Partial Fraction Decomposition. Express terms of the form

$$\frac{R_1(x)}{(ax+b)^n}$$

as

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{ax+b)^n}$$

and express terms of the form

$$\frac{R_2(x)}{(\beta x^2 + \alpha x + \lambda)^m}$$

as

$$\frac{B_1 x + C1}{\beta x^2 + \alpha x + \lambda} + \frac{B_2 x + C_2}{(\beta x^2 + \alpha x + \lambda)^2} + ... + \frac{B_m x + C_m}{(\beta x^2 + \alpha x + \lambda)^m}$$

eg: Apply PFD to

$$\frac{x^4 + 2x^2 + x + 1}{x^3(x-2)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{K1}{x} + \frac{K2}{x^2} + \frac{K3}{x^3} + \frac{K4}{x-2} + \frac{K5}{x^2+x+1} + \frac{K6}{x^2+1} + \frac{K7}{(x^2+1)^2} + \frac{K8}{(x^2+1)^3}$$

4. Determine the unknown coefficients in Step 3 by equating like powers of x

eg:

$$\begin{aligned} \frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\ &= \frac{A(x+1) + B(x)}{x(x+1)} \end{aligned}$$

$$1 = A(x+1) + Bx$$

$$x^0 = 1 - A$$

$$x^1 = 0 = A + B$$

Since $A = 1$ then $B = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

5. We know how to integrate all of the terms in $(*)$ and $(**)$. Terms in $(***)$ may require completing the square before being in an integrable form.

Example 1.16. Evaluate

$$\int \frac{x^4 + x^3 + x^2 - x}{x^3 - 1}\, dx$$

Solution. $x^3 - 1\overline{)x^4 + x^3 + x^2 - x}$ After solving, $S(x) = x + 1$ and $R(x) = x^2 + 1$

$$\int (x+1)\,dx + \int \frac{x^2+1}{x^3-1}\,dx$$

$$\frac{x^2}{2} + x + \int \frac{x^2+1}{x^3-1}\,dx$$

$$\frac{x^2}{2} + x + \int \frac{x^2+1}{(x-1)(x^2+x+1)}\,dx$$

Step 3:

$$\frac{x^2+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Step 4:

$$x^2 + 1 = A(x^2+x+1) + (Bx+C)(x-1)$$

Equating like powers:

$$x^0 : 1 = A - C$$

$$x^1 : 0 = A - B + C$$

$$x^2 : 1 = A + B$$

After solving, $A = \frac{2}{3}, B = \frac{1}{3}, C = -\frac{1}{3}$

$$\frac{x^2}{2} + x + \frac{2}{3}\int \frac{1}{x-1} + \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx$$

Step 5: Completing the square

$$x^2 + x + 1 = (x^2 + x + \frac{1}{4}) - \frac{1}{4} + 1$$

$$(x + \frac{1}{2})^2 + \frac{3}{4}$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{3}\int \frac{x-1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{4}{3}\int \frac{x-1}{(2x+1)^2 + 3}\,dx$$

Let $u = 2x + 1$ then $du = 2dx$ also $x - 1 = \frac{u-1}{2} - 1 = \frac{u-3}{2}$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{3}\int \frac{u}{u^2+3}\,du + \frac{1}{3}\int \frac{3}{u^2+3}\,du$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|u^2+3| + \frac{1}{3}\int \frac{3}{u^2+3}\,du$$

$u^2 + 3 > 0$ always so absolute not needed.

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln(u^2+3) + \frac{1}{3}\int \frac{1}{(\frac{1}{\sqrt{3}})^2+1}\,du$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln(u^2+3) + \frac{1}{3}\int \frac{1}{y^2+1}\,dy$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln(u^2+3) + -\frac{\sqrt{3}}{3}\arctan(\frac{1}{\sqrt{3}}(2x+1)) + C$$

$$\frac{x^2}{2} + x + \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln((2x+1)^2+3) + -\frac{\sqrt{3}}{3}\arctan(\frac{1}{\sqrt{3}}(2x+1)) + C$$

1.3 Volumes of Solids

1.3.1 Slicing Method

Consider a solid (eg a loaf of bread), what is its volume?

Solution. Place the solid along the x-axis and cut it into n pieces of equal thickness.
Let $\triangle x = \frac{b-a}{n}$ denote the thickness of one slice. Consider one slice at $[x_i - 1, x_1]$.
Assume $\triangle x$ is small $\implies$ area on the right side of the slice is approximately equal to the area on the left side of the slice. Let $A(x_i)$ be the area.

Volume of one slice $= A(x_i)\triangle x$
Volume of all slices $=$

$$\sum_{i=1}^{n} A(x_i)\triangle x$$

The approximation becomes exact as $n \to \infty$
Volume of solid $=$

$$\lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\triangle x$$

Example 1.17. Consider the region between $f(x) = x^2$ and the x-axis from $x = 0$ to $x = 1$. Find the volume of this region rotated about the x-axis.

Solution. The shape of the solid is a funnel-like shape. A slice of the solid is the circle.

$$A = \pi\int_0^1 (x^2)^2\,dx$$

$$A = \pi\int_0^1 x^4\,dx$$

$$A = \pi(\frac{x^5}{5})\Big|_0^1$$

$$A = \pi(\frac{x^5}{5})\Big|_0^1$$

$$A = \frac{\pi}{5}$$

In general, volumes by revolution are determined by

$$\int_a^b \pi(r(s))^2 \, ds$$

where $r(s)$ is the radius of the circle obtained by slicing the solid.

Example 1.18. Consider the region between $y = x^2 + 1$ and the y-axis from $y = 1$ to $y = 5$. Find the volume of this region rotated about the y axis.

Solution.

$$A = \pi \int_1^5 \sqrt{y-1}^2 \, dy$$
$$A = \pi \int_1^5 y - 1 \, dy$$
$$A = \pi(\frac{y^2}{2} - y)\Big|_1^5$$
$$A = 8\pi$$

Example 1.19. Consider the region between $f(x) = x$ and $g(s) = \sqrt{x}$ rotated about the x-axis. Find the volume of this solid.

Solution. Point of intersection is $(1, 1)$.

$$A = \pi \left[\int_0^1 x^2 \, dx - \int_0^1 (\sqrt{x})^2 \, dx\right]$$
$$A = \pi \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1$$
$$A = \frac{\pi}{6}$$

In general, for washer shaped volumes, we have

$$V = \pi \int_a^b (r_1)^2 - (r_2)^2 \, ds$$

Exercise 1.7. Repeat Example 1.19 but rotate about y-axis.

Solution. $x = y$ and $x = y^2$. Point of intersection: $y = 1$

$$\pi \int_0^1 (y^2 - y^4) dy$$
$$= \pi(\frac{y^3}{3} - \frac{y^5}{5})\Big|_0^1$$
$$= \pi(\frac{5y^3 - 3y^5}{15})\Big|_0^1$$
$$= \frac{2\pi}{15}$$

Example 1.20. For $x \geq 0$, then region bounded by $y = x^3$ and $y = x$ is rotated about $y = 1$. Compute the resulting volume.

Solution. $y = 1 - x^3$ and $y = 1 - x$

$$V = \pi \int_0^1 (1 - x^3)^2 - (1 - x)^2 \, dx$$
$$V = \pi \int_0^1 (1 - 2x^3 + x^6) - (1 - 2x + x^2) \, dx$$
$$V = \pi \int_0^1 -2x^3 + x^6 + 2x - x^2 \, dx$$
$$V = \pi(-\frac{x^4}{2} + \frac{x^7}{7} + x^2 - \frac{x^3}{3})\Big|_0^1$$
$$V = \frac{13\pi}{42}$$

1.3.2 Cylindrical Shells Method

Let r_1 represent radius of outer circle.
Let r_2 represent radius of inner circle. If we assume $r_1 \approx r_2$ (denote by r), then the volume of the cylindrical shell.

$$\underbrace{2\pi r}_{Circumference} \times \underbrace{h}_{height} \times \underbrace{\triangle x}_{thickness}$$

For volumes of revolution problems, we can use the volumes of cylindrical shell to approximate the volume of many solids.

Discretize the x-domain into n intervals with thickness $\triangle x = \frac{b-a}{n}$
We have created nested cylindrical shells and the sum of their volumes approximates the volume of the solid. Volume of solid $\approx$

$$\sum_{i=1}^{n} 2\pi r(x_i) h(x_i) \triangle x$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r(x_i) h(x_i) \triangle x$$

$$\int_a^b 2\pi r(x) h(x) \, dx$$

where $r(x)$ is the radius and $h(x)$ is the height of the shell.

Example 1.21. Consider the region bounded by $y = \frac{1}{2}, y = 0, x = 1$ and $x = 2$ is rotated about the y-axis, thus creating a solid. Compute the volume

Solution.

$$V = \int \_1^2 2\pi x \frac{1}{x} \, dx$$

$$V = 2\pi \int \_1^2 \, dx$$

$$V = 2\pi$$

Note. One method may be easier than other one.

Solution.

$$V = \pi \int_0^{\frac{1}{2}} 2^2 - 1^2 \, dx$$

Example 1.22. Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 0, x = 1$ about the line $y = 1$.

Solution. Radius of shell $= 1 - y$
Height of shell $= 1 - y^{\frac{1}{3}}$

$$V = \int_0^1 2\pi(1-y)(1-y^{\frac{1}{3}}) \, dy$$

$$V = 2\pi \int_0^1 (1 - y^{\frac{1}{3}} - y + y^{\frac{4}{3}}) \, dy$$

$$V = 2\pi(y - \frac{3y^{\frac{4}{3}}}{4} - \frac{y^2}{2} + \frac{3y^{\frac{7}{3}}}{7}\Big|_0^1$$

$$V = \frac{5\pi}{14}$$

Exercise 1.8. Consider the disc governed by

$$(x-R)^2 + y^2 = a^2$$

rotated about the y-axis where $R > a$. Ths solid created is called a torus. Determine the volume using both the slicing method and cylindrical shell method.

Ans should be $2\pi^2 Ra^2$

Finish exercise

Exercise 1.9. Find the volume of a pyramid with a square base where the base length is $a > 0$ and height $h > 0$. Answer $\frac{a^2h}{3}$

Try exercise

1.4 Improper Integrals

There are two types

- Integrals on an infinite length (type 1)
- Integrals with discontinuity. (type 2)

Type 1 integrals are of the form:

$$\int_a^\infty f(x)\,dx \int_{-\infty}^b f(x)\,dx \int_{-\infty}^\infty f(x)\,dx$$

These are defined as

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists as a finite number. Similarly,

$$\int_{-\infty}^b f(x)\,dx = \lim_{s\to-\infty}\int_s^b f(x)\,dx$$

These improper integrals are called **convergent** if the corresponding limit exists, and **divergent** otherwise.

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x) + \int_a^\infty f(x)$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{s\to-\infty}\int_s^a f(x) + \lim_{t\to\infty}\int_a^t f(x)$$

Note.

$$\int_{-\infty}^\infty f(x)\,dx \neq \lim_{t\to\infty}\int_{-t}^t f(x)\,dx$$

$\int_{-\infty}^\infty x\,dx$ diverges. If the first term diverges, no need to check second term and we can immediately conclude that the integral diverges. $-\infty+\infty \neq 0$. Both limits must exist for the integral to be convergent.

Theorem 1.4. Claim: If $\int_{-\infty}^\infty f(x)\,dx$ converges, then $\int_{-\infty}^\infty f(x) = \lim_{t\to\infty}\int_{-t}^t f(x)\,dx$ is true.

Proof. By definition

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x) + \lim_{t\to\infty}\int_a^t f(x)$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty}\int_{-t}^a f(x) + \lim_{t\to\infty}\int_a^t f(x)$$

Since both integrals converge, then each limit exists and we can take the sum over one limit.

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty}[\int_{-t}^a f(x) + \int_a^t f(x)]$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty}\int_{-t}^t f(x)\,dx$$

□

Example 1.23. For what values of $p\in\mathbb{R}$ is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Solution.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p} &= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx\\ &= \lim_{t\to\infty}(fx^{-p+1} - p+1)\Big|_1^t\\ &= \lim_{t\to\infty}[\frac{1}{1-p}[\frac{1}{t^{p-1}}-1]]\end{aligned}$$

If $p>1$ then $\lim_{t\to\infty}\frac{1}{t^{p-1}}=0$ and $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ which converges.
If $p<1$ then $\lim_{t\to\infty}\frac{1}{t^{p-1}}=\infty$ and $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.
If $p=1$ then $\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}(\ln(x))\Big|_1^t = \lim_{t\to\infty}\ln(t)=\infty$

Summary:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if } p>1 \text{ and diverges if } p\le 1$$

Type 2

Type 2 improper integrals occur when f is discontinuous along $[a,b]$.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

eg: $\int_{-1}^0 \frac{1}{x}\,dx = \lim_{t\to 0^-}\int_a^t \frac{1}{x}\,dx$

.

Discontinuity at a:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

eg: $\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx$

Discontinuity at c where $a<c<b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{s\to c^-}\int_a^s f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

eg:

$$\int_{-1}^1 \frac{1}{x}\,dx = \lim_{s\to 0^-}\int_{-1}^s \frac{1}{x}\,dx + \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx$$

These types of improper integrals are called convergent if the limit exists, and divergent if the limit does not exist.

Example 1.24. Evaluate

$$\int_1^5 \frac{1}{x-2}\,dx$$

Solution. At $x = 2$ function has a vertical asymptote.
Suppose didn't realize asymptote.

$$\begin{aligned} &= \ln|x-2|\Big|_1^5 \\ &= \ln 3 \end{aligned}$$

Be cautious and determine whether a given integral is improper.

$$\int_1^5 \frac{1}{x-2}\,dx = \int_1^2 \frac{1}{x-2}\,dx + \int_2^5 \frac{1}{x-2}\,dx$$

Look at first part.

$$\lim_{t\to 2^-} \int_1^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \ln|t-2| = \lim_{t\to 2^-} \ln(2-t) = -\infty$$

$\therefore \int_1^5 \frac{1}{x-2}\,dx$ diverges since $\int_1^2 \frac{1}{x-2}\,dx$ diverges.

Theorem 1.5 (Comparison Theorem)**.** Suppose f and g are continuous functions with $f(x) > g(x)$ for $x \geq 0$.
If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.
If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example 1.25. Determine whether

$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$$

converges or diverges.

Solution. $\sin^2(x) \leq 1 \implies \frac{\sin^2(x)}{x^2} \leq \frac{1}{x^2}$ because x^2 is positive.

$\int_1^\infty \frac{1}{x}\,dx$ converges (in exmp 1.23). By the comparison theorem, $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges.

2 Differential Equations

Definition 2.1. A **differential equation** (DE) is an equation that contains an unknown function, say $y(x)$, and 1 or more of its derivatives.

$$\frac{d^2y}{dx^2} + 2\frac{dy(x)}{dx} + 2y(x) = \cos(x) + 11.2$$

$$y"(x) + y'(x) + 2y(x) = \cos(x) + 11.2$$

Definition 2.2. The **order** of a DE is the order of the highest derivative that appears in the DE. In the above example, order 2 or second order.

$$x^3\frac{dy}{dx} + y^2(x) + y(x) = e^x + \cos(x) + x$$

has a degree of 1, order 1 or first order.

A first order DE is linear if it can be written in the form

$$\frac{dy(x)}{dx} + p(x)y(x) = q(x)$$

where $p(x)$ and $q(x)$ are given functions. Otherwise, it is non-linear.
$y'(x) + x^2y(x) = e^x\cos(x)$ is a linear DE.
$(1 + x^2)y'(x) + 2y(x) = e^x\cos(x) + 10$ can be written into linear DE form.
$y'(x) + \frac{2y(x)}{(1+x^2)} = \frac{e^x\cos(x)+10}{(1+x^2)}$ is also a linear DE.
$y'(x) + x^2y^2(x) = e^x\sin(x)$ is a non-linear DE.

A function f is called a solution to a DE if the equation satisfied when f and its derivatives are substituted into the equation.

Example 2.1. Show $y(x) = \cos(2x)$ is a solution to $y"(x) + 4y(x) = 0$.

Solution. $y' = -2\sin(2x)$, $y" = -4\cos(2x)$

$$-4\cos(2x) + 4\cos(2x) = 0$$

Exercise 2.1. Show that $y = \sin(2x)$ is also a solution to the DE. Show that any function of the form $c_1\cos(2x) + c_2\sin(2x)$ is a solution to the DE where c_1, c_2 are arbitrary constants.

Note. A DE describes a family of functions. These are known as general solutions. Problems of finding a solution to a DE that satisfies a given initial condition are called initial value problems. (IVP)

Example 2.2. Find the solution to the IVP $y"(x) + 4y(x) = 0$. $y(0) = 5, y'(0) = 0$

Solution. From 2.1, the general solution is $c_1\cos(2x) + c_2\sin(2x)$.

$$y(0) = \underbrace{c_1\cos(2x)}_{1} + \underbrace{c_2\sin(2x)}_{0}, c_1 = 5$$

$$y'(0) = \underbrace{-2c_1\cos(2x)}_{0} + \underbrace{2c_2\sin(2x)}_{0}, c_2 = 0$$

$\therefore$ The solution to the IVP is $y(x) = 5\cos(2x)$

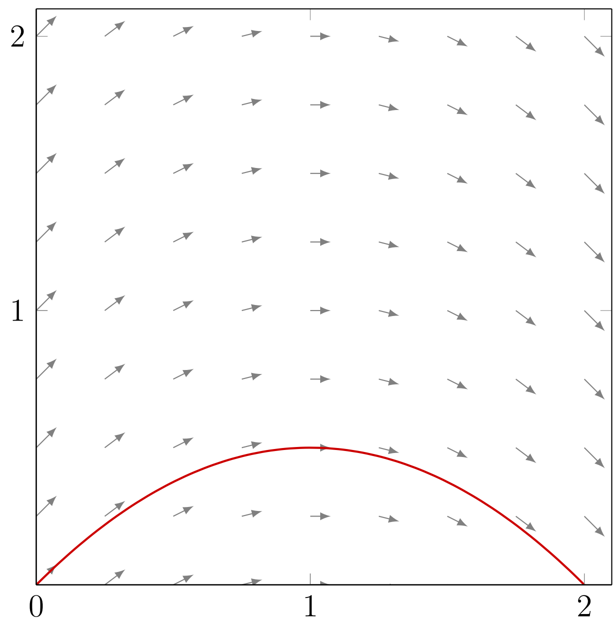
2.1 Direction Fields

Consider a first order $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ is a function that depends on x, y. The DE tells us that the slope at some arbitrary point, (x_0, y_0) on the solution curve $y(x)$, is $f(x_0, y_0)$

Definition 2.3. A **direction field** creates a field of line segments indicating the direction of a function $y(x)$.

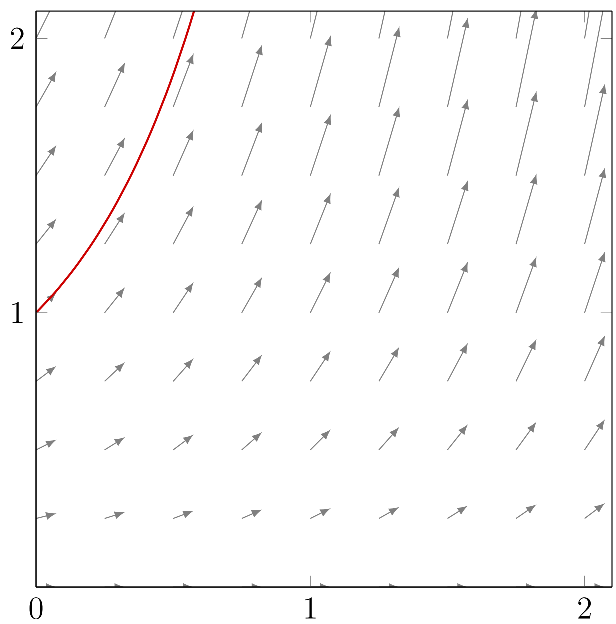
Example 2.3. Sketch the direction field of $y' = 1 - x$ and then draw the solution curve that passes through $(0, 0)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x$$



Example 2.4. Sketch the direction field for $y' = y + xy$, and then draw the solution curve that passes through the point $(0, 1)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + xy$$



2.2 Separable Equations

Separable DEs are first order DEs that c an be written in the form:

$$\frac{dy}{dx} = g(x) \times f(y) \text{ for some function of } f \text{ and } g$$

Separable DEs can be solved as follows:

$$\frac{dy}{dx} = g(x) \times f(y) \implies \frac{1}{f(y)}\frac{dy}{dx} = g(x)$$

$$\implies \frac{dy}{f(y)} = g(x)\,dx$$

Example 2.5. Solve $y' = y(1+x)$

Solution.

$$y' = y(1+x) \implies \frac{dy}{dx} = (1+x)y$$

$$\int \frac{dy}{y} = \int (1+x)\,dx$$

Assume $y \neq 0$.

$$\implies \ln|y| = x + \frac{x^2}{2} + C$$

$$y = \pm e^{x+\frac{x^2}{2}+C}$$
$$y = \pm k e^{x+\frac{x^2}{2}}$$

How about $Y(x) = 0$. $y = 0 \implies y' = 0$. DE $\implies 0 = 0$ true.

Exercise 2.2. Find the solution to the IVP $2xy + (x^2+1)$, $\frac{dy}{dx} = 0$. $y(0) = 2$. Answer $y = \frac{2}{x^2+1}$.

fill

2.3 Integrating Factor

Motivating example: Consider the following first order DE.

$$y'(x) + y(x) = x \text{ which is not separable}$$

How do we solve this DE?

$$e^x(y' + y) = xe^x$$
$$\underbrace{e^x y' + e^x y}_{\frac{d}{dx}[e^x y]} = xe^x$$

This gives $\frac{d}{dx}[e^x y] = xe^x$
Integrating with respect to x, $\int \frac{d}{dx}[e^x y]\,dx = \int xe^x\,dx$.
FTC I $\implies e^x y = \int xe^x\,dx = xe^x - \int e^x\,dx = xe^x + e^x + C$
The solution is $y(x) = x - 1 + Ce^{-x}$

Note. Constants of integration is crucial to the form of solution.

The key to solving DE was multiplying by e^x (called an integrating factor). Method of finding integrating factor: Consider first order linear DE in standard form:

$$y'(x) = p(x)y(x) - q(x)$$

where p and q are constants, nonzero function of x.

Suppose $\mu(x)$ is an integrating factor and multiply DE by $\mu(x)$.

$$\mu(x)y'(x) + \mu(x)y(x) = \mu(x) + q(x)$$

We want LHS $= \frac{d}{dx}[\mu(x)y(x)]$ like in previous examples.
If this is true, then DE becomes $\frac{d}{dy}[\mu(x)y(x)] = \mu(x)q(x)$

$$FTCI \implies \mu(x)y(x) = \int \mu(x)q(x)\,dx$$
$$\implies y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)\,dx$$

This could be the solution if we can find such $\mu(x)$. To find such $\mu(x)$, we want $\mu(x)y'(x) + \mu(x)p(x)y(x) = \frac{d}{dx}[\mu(x)y(x)]$

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu`(x)y(x) + \mu(x)y'(x)$$

$$\implies \mu(x)p(x)y(x) = \mu(x)y(x)$$

$$y'(x) = \mu(x)p(x) \text{ if } y(x) \neq 0$$

If we know the integrating factor, $\mu(x)$, then the DE can be solved. We saw that $\mu(x)$ satisfies $\mu'(x) = \mu(x)p(x)$

$$\frac{d\mu}{dx} = \mu(x)p(x)$$

$$\implies \frac{d\mu}{\mu} = p(x)\,dx$$

$$\implies \int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\implies \ln|\mu| = \int p(x)\,dx + C$$

$$|u| = e^c e^{\int p(x)\,dx}$$

$$u = \pm k e^{\int p(x)\,dx}$$

Therefore the integrating factor for $y'(x) + p(x)y(x) = q(x)$ is $\mu(x) = ke^{\int p(x)\,dx}$. Pick one choice for μ. Pick $\mu(x) = e^{\int p(x)\,dx}$

Example 2.6. Find the solution to

$$xy' = x^2 + 4 - y, x \neq 0$$

Solution. This DE is not separable.
The DE **MUST** be in the form $y'(x) + p(x)y(x) = q(x)$ where the coefficient for $y'(x)$ is just 1, before determining the integrating factor. This is known as the standard form.

$$\implies y' + \frac{y}{x} = x + \frac{4}{x}$$

$$\implies \mu = e^{\int \frac{1}{x}\,dx}$$

$$\implies \mu = e^{\ln|x|+C}$$

$$\implies \mu = e^c|x|$$

Just pick one c and one x. Easiest is to pick $\mu(x) = x$.
Multiply the standard form DE by μ.

$$x(y' + \frac{y}{x} = x + \frac{4}{x})$$

$$\underbrace{xy' + y}_{=(xy)' \text{ Check}} = x^2 + 4$$

$$(xy)' = x^2 + 4$$

$$\int (xy)' = \int x^2 + 4$$

$$xy = \frac{x^3}{3} + 4x + C$$

The solution to the DE is

$$y = \frac{x^2}{3} + 4 + \frac{C}{x}$$

2.4 Applications of DEs

Population Growth.
Let $P(t)$ be the population of a particular species at time $t > 0$. Let $P(0$ be the initial turtle population. Physically $P(t) > 0$ for $t \geq 0$. Assume the population grows at a rate proportional to the size of the population. As population increases, the rate of its growth increases.

$$\implies \frac{dp}{dt} = rp(t)$$

where r is a constant representing the growth rate and $r > 0$. This DE models population growth.

Example 2.7. Solve $\frac{dp}{dt} = rp(t)$.

Solution. The DE is separable.

$$\implies \frac{dp}{p} = r\,dt$$

$$\implies \int \frac{dp}{p} = \int r\,dt$$

$$\implies \ln|p| = rt + C$$

$$\implies p(t) = ke^{rt}$$

Apply initial condition: $P(0) = P_0$.

$$\implies P(0) = k$$

$$\implies p(t) = P_0 e^{rt}$$

This means that the population of turtles is growing at an exponential rate.

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} P_0 e^{rt} = \infty$$

The DE needs to be modified because infinite turtles is unrealistic.

How much the environment can provide for the population is known as the carrying capacity, denoted as $k > 0$.These population dynamics are governed by

$$\frac{dP}{dt} = rP(1 - \frac{p}{k})$$

where $P(t) > 0$ for $t > 0$ and $r > 0$, representing the rate of growth, and $k > 0$ representing the carrying capacity. This is known as the logistic model.
If $p < k \implies \frac{p}{k} < 0 \implies$ ** becomes *
If $p < k \implies \frac{p}{k} > 1 \implies 1 - \frac{p}{k} < 0$
If $p = k \implies \frac{p}{k} = 1 \implies \frac{dP}{dt} = 0 \implies p(t) =$ constant for all time.

Example 2.8. Solve the logistic model.

Solution. The equation is separable.

$$\frac{dP}{P(1 - \frac{p}{k})} = r\,dt$$

$$\int \frac{dP}{P(1 - \frac{p}{k})} = \int r\,dt$$

Apply PFD

$$\int \frac{dP}{P(1 - \frac{p}{k})} = \frac{A}{p} + \frac{B}{1 - \frac{p}{k}}$$

$$1 = a(1 - \frac{p}{k}) + BP$$

$$P^0 : 1 = A$$

$$P^1 0 = \frac{-A}{k} + B$$

$$B = \frac{1}{k}$$

Rewrite equation as

$$\int \frac{1}{p}\,dP + \int \frac{\frac{1}{k}}{1 - \frac{p}{k}}\,dP = \int r\,dt$$

$$\ln P - \ln|1 - \frac{p}{k}| = rt + C$$

$$\ln(\frac{P}{|1 - \frac{p}{k}|}) = rt + C$$

$$\frac{P}{|1 - \frac{p}{k}|} = e^c e^{rt}$$

$$\frac{P}{1 - \frac{p}{k}} = \pm c_1 e^{rt}$$

Rearrange and

$$c_1 = \frac{P_0}{1 - \frac{P}{k}}$$

And then

$$P = (1 - \frac{P}{k})c_1 e^r t$$

$$P + \frac{P}{k}(c_1 e^r t) = c_1 e^r t$$

$$P = \frac{c_1 e^r t}{1 + \frac{1}{k} c_1 e^r t}$$

$$P(t) = \frac{kP_0 e^{rt}}{k - P_0 + P_0 e^{rt}}$$

When $P(t) = k$, $\frac{dP}{dt} = 0$ so $P(t) \equiv$ constant. $P(0) = P_0$ and $P_0 = k$ Sub this into the above equation and simplify.

$$P(t) = \frac{k^2 e^{rt}}{k - k + e^r t} = k$$

Aside:

$$\lim_{t \to \infty} P(t) = k$$

$$\lim_{t \to \infty} \frac{e^{rt}}{e^{rt}} \frac{kP_0}{(k - P_0)e^{-rt} + P_0} = k$$

2.5 Newton's Law of Cooling

The rate of cooling of an object is proportional to the temperature difference between the object and its surrounding.

Let $T(t)$ be the temperature of the object at time $t > 0$ and T_s be the constant that represents the surrounding temperature.

$$\frac{dT}{dt} = K(T(t) - T_s)$$

and $T(0) = T_0$

Example 2.9. Suppose the initial temperature of a cup of coffee is 80 degrees Celsius and after 5 mins, the temperature is 50 degrees. Assume the room is a constant temperature of 20 degrees C.
How long until the coffee reaches a temperature of 25 degrees?

Solution. Let $T(t)$ be the temperature of the coffee.

$$\frac{dT}{dt} = k(T(t) - 20)$$

$T(0) = 80$. Want $t_1 =?$ such that $T(t_1) = 25$ degrees.
Solve the DE. It is separable.

$$\int \frac{dT}{T - 20} = \int k\, dt$$

$$\ln |T - 20| = kt + C$$

$$T - 20 = \underbrace{\pm e^c}_{c_1} e^{kt}$$

$$T - 20 = c_1 e^{kt}$$

$$T(t) = c_1 e^{kt} + 20$$

Solve for $T(0)$ to get c_1:

$$80 = c_1 e^{k(0)} + 20$$

$$c_1 = 60$$

$$T(t) = 60e^{kt} + 20$$

Sub in $T(5)$ to get k:

$$50 = 60e^{-5k} + 20$$

$$\frac{1}{2} = e^5 k$$

$$\ln(\frac{1}{2}) = 5k$$

$$-\frac{1}{5}\ln(2) = k$$

$$T(t) = 60e^{-\frac{1}{5}\ln(2)t} + 20$$

Now that we have the equation, need $t \ni T(t) = 25$.

$$25 = 60e^{-\frac{1}{5}\ln(2)t} + 20$$

$$\frac{1}{12} = e^{-\frac{1}{5}\ln(2)t}$$

$$\ln(\frac{1}{12}) = -\frac{1}{5}\ln(2)t$$

$$\ln(12) = \frac{1}{5}\ln(2)t$$

$$t = \frac{5\ln(12)}{\ln(2)} \approx 17.9 \text{ minutes}$$

$\therefore$ It will take approximately 17.9 minutes for the coffee to reach 25 degrees.

Mixing Problem

Consider a tank of fixed capacity filled with a thoroughly mixed solution of some substance. Let $m(t)$ denote amount of salt in the tank at $t > 0$. There is an inflow of salt solution of specified concentration, and an outflow of salt solution whose concentration depends on $m(t)$.

Example 2.10. A tank m_0 kg of salt dissolved in 100 L of water. A salt solution containing 0.25 kg of salt of L of water enters the tank at a rate of $3L$/min. The well stirred mixture leaves the tank at the same rate. Find the amount of salt at t.

Solution. $r_{in} = 3 \times 0.25 = 0.75$ kg/min.
$r_{out} = 3 \times \frac{1}{100} = 0.03$ kg/min.

$$\frac{dm}{dt} = 0.75 - 0.03m(t)$$
$$m(0) = m_0$$
$$\int \frac{dm}{0.75 - 0.03m} = \int dt$$
$$-\frac{\ln|0.75 - 0.03m|}{0.03} = t + C$$
$$|0.75 - 0.03m| = e^{-0.03C}e^{-0.03t}$$
$$|0.75 - 0.03m| = e^{c_2}e^{-0.03t}$$
$$-0.03m = c_3e^{-0.03t} - 0.75$$
$$m(t) = c_4e^{-0.03t} + 25$$

Solve c_4 for m_0:

$$m(0) = c_4e^0 + 25$$
$$c_4 = m_0 - 25$$
$$m(t) = (m_0 - 25)(e^{-0.03t}) + 25$$

3 Sequences

3.1 Formal Definition

Recall functions, $f(x)$, where $x \in \mathbb{R}$ Let n be a positive integer.

Definition 3.1. A **sequence** $a_1, a_2, ..., a_n$ is denoted as $\{a_n\}$. A sequence can be defined as a function whose domain is $n \in \mathbb{N}$, that is,

$$a_n = f(n)$$

The sequence has limit L if for every $\epsilon > 0$, there exists a number $N > 0$ such that $n > N \implies |a_n - L| < \epsilon$

Definition 3.2. **Explicit** sequences occur where the n^{th} term is given as a function n.

Definition 3.3. **Recursive** sequences require that the n^{th} term depends on the term or terms before it.

$$a_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n$$

The sequence $\{a_n\}$ has $L < \infty$ if the a_n are as close to L as we wish by taking n sufficiently large. This is denoted as $\lim_{x\to\infty} a_n = L$. If L is finite, then the sequence converges, else it diverges.

Example 3.1.

$$\{c_n\}_{n=1}^{\infty} = \{\frac{6n}{3n-2}\}_{n=1}^{\infty}$$

Solution. We need to find $N > 0$ such that $n > N$ implies $|c_n - L| < \epsilon$ for any $\epsilon > 0$.

$$|c_n - L| = |\frac{6n}{3n-2} - 2|$$
$$= \frac{4}{3n-2}$$

We can choose N so rewrite $\frac{4}{3n-2}$ in terms of N.

$$n > N \implies 3n - 2 > 3N - 2$$

$$n > N \implies 3n - 2 > 3N - 2$$
$$\implies \frac{1}{3n-2} < \frac{1}{3N-2}$$
$$\implies \frac{4}{3n-2} < \frac{4}{3N-2}$$
$$c_n - L < \frac{4}{3N-2}$$

Choose $N = \frac{4}{3\epsilon} + \frac{2}{3}$

$$\frac{4}{3(\frac{4}{3\epsilon} + \frac{2}{3}) - 2}$$
$$= \frac{4}{\frac{4}{\epsilon}} = \epsilon$$

Therefore, for any $\epsilon > 0$ with $N = \frac{4}{3\epsilon} + \frac{2}{3}$, we have $n > N$ implies $|c_n - 2| < \epsilon$

Note. Pick N such that $\epsilon = \frac{4}{3N-2}$. Solve this expression for N to get the desired value for N.

Example 3.2. Prove $\lim_{n\to\infty}(-1)^n$ does not exist.

Solution. Apply a proof by contradiction. Assume that the limit exists. By definition, for any $\epsilon > 0$ there exists $N > 0$ such that if $n > N \implies |(-1)^n - L| < \epsilon$. Since this supposed to be true for all $\epsilon > 0$, let $\epsilon = \frac{1}{2}$. Pick some fixed $N > 0$ and when $n > N \implies |(-1)^n - L| < \frac{1}{2}$. If n is even, then $|1 - L| < \frac{1}{2}$

$$\frac{-1}{2} < 1 - L < \frac{1}{2}$$
$$\frac{-3}{2} < -L < \frac{-1}{2}$$
$$\frac{3}{2} > L > \frac{1}{2}$$

If n is odd then $|-1-L| < \frac{1}{2}$

$$\frac{-1}{2} < -1 - L < \frac{1}{2}$$
$$\frac{1}{2} < -L < \frac{3}{2}$$
$$\frac{-1}{2} > L > \frac{-3}{2}$$

Therefore $L \in (\frac{-3}{2}, \frac{-1}{2}) \cap (\frac{1}{2}, \frac{3}{2})$ which is $\emptyset$. Therefore L doesn't exist, assumption was false, and $\lim_{n\to\infty}(-1)^n$ does not exist.

3.2 Properties of Limits for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and c is a constant, then

$$\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} c \times a_n = c \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} a_n \times b_n = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad \lim_{n\to\infty} b_n \neq 0$$
$$\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p$$

Theorem 3.1. If $\lim_{x\to\infty} f(x) = L, x \in \mathbb{R}$ and $f(n) = a_n, n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = L$.

Example 3.3. Determine whether the following sequences converge or diverge. If it converges, find the limit.

$$\left\{\frac{n\ln(n)}{e^n}\right\}_{n=1}^{\infty} \text{ and } \{\cos(n\pi)\}_{n=1}^{\infty}$$

Solution. L'Hopital's Rule cannot be used for a discrete domain, but it can be indirectly used with Theorem 3.1.
Let $f(x) = \frac{x\ln(x)}{e^x}$.

$$\lim_{x\to\infty} \frac{x\ln(x)}{e^x} \qquad \text{Apply L'Hopital's Rule}$$
$$\lim_{x\to\infty} \frac{\ln(x)+1}{e^x} \qquad \text{Apply L'Hopital's Rule}$$
$$\lim_{x\to\infty} \frac{1}{xe^x} = 0$$

By Theorem 3.1, we can conclude that the first sequence converges to 0.

$$\lim_{n\to\infty} \cos(n\pi) = \lim_{n\to\infty} (-1)^n \implies \text{ diverges (proved previously)}$$

Theorem 3.2. If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example 3.4. Find $\lim_{n\to\infty} \cos(\frac{\pi^2}{n})^n$.

Solution. Let $a_n = \frac{\pi^2}{n}$ and f will be cosine.

$$\lim_{n\to\infty} \frac{\pi^2}{n} = 0$$

Since cosine is continuous at $L = 0$, then applying Theorem 3.2,

$$\lim_{n\to\infty} \cos\left(\frac{\pi^2}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi^2}{n}\right) = \cos(0) = 1$$

Theorem 3.3. Squeeze Theorem for Sequences:
If $a_n \le b_n \le c_n$ for all $n \ge n_0$ where n_0 is some number, and $\lim_{n\to\infty} a_n = lim_{n\to\infty} c_n = L$ then

$$\lim_{n\to\infty} b_n = L$$

Example 3.5. Use the squeeze theorem for sequences to prove:
If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ then use it to prove $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution. *Proof.*

$$-|a_n| \le a_n \le |a_n|$$

$$\lim_{n\to\infty} -|a_n|$$

$$= -\lim_{n\to\infty} |a_n|$$

$$= 0$$

$$\lim_{n\to\infty} |a_n| = 0$$

By the squeeze theorem, then

$$\lim_{n\to\infty} a_n = 0$$

Furthermore,

$$\lim_{n\to\infty} |\frac{(-1)^n}{n}| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by the proved theorem, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$. □

Example 3.6. For what values of $r \in \mathbb{R}$ is $\{r^n\}$ convergent? $n \in \mathbb{N}$.

Solution. Consider various cases for r.
for $r > 1$ and $r < -1$,$\{r^n\}$ diverges.
for $r = 1$, then $\{r^n\}$ converges to 1.
for $r = -1$, then $\{r^n\}$ diverges.
for $-1 < r < 1$, then $\{r^n\}$ converges to 0.
Therefore, $\{r^n\}$ converges if $-1 < r \le 1$, and it diverges for all other values of r.

3.3 Monotonic Sequence Theorem

Definition 3.4. $\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_n < a_{n+1}$. A sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if $a_n > a_{n+1}$

Definition 3.5. The sequence a_n is **monotonic** if it is either always decreasing or increasing but not both.

Definition 3.6. $\{a_n\}_{n=1}^{\infty}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all n. Similarly it is **bounded below** if there is a number m such that $a_n \geq m$ for all n.
$\{a_n\}_{n=1}^{\infty}$ is a **bounded sequence** if it is bounded above and bounded below.

Theorem 3.4 (MST)**. Monotonic sequence theorem** states that if a sequence is bounded and monotonic, it is convergent.

Example 3.7. Define $a_1 = 0, a_{n+1} = 1 + \sqrt{6 + a_n}$ for $n \geq 1$. Does $\{a_n\}_{n=1}^{\infty}$ converge? If yes, what is the limit?

Solution. Consider the first few terms

$$a_1 = 0, a_2 = 1 + \sqrt{6}, a_3 = 1 + \sqrt{1 + \sqrt{6}}$$

Guess: The sequence is increasing and has a lower bound 0 and upper bound 5. The proof for this can be divided into three parts:

1. $a_n \geq 0, \forall n \in \mathbb{N}$
2. $a_n \leq 5, \forall n \in \mathbb{N}$
3. $a_{n+1} > a_n, \forall n \in \mathbb{N}$

Prove $a_n \geq 0, \forall n \in \mathbb{N}$

Proof. BC: $a_1 = 0 \geq 0$
IH: Assume $a_k \geq 0$ for some $k \geq 1$
IC: $a_{k+1} = 1 + \sqrt{6 + a_k} \geq \sqrt{1 + \sqrt{6}} \geq 0$
$\therefore \{a_n\}$ is bounded below. □

Prove $a_n \leq 5, \forall n \in \mathbb{N}$

Proof. BC: $a_1 = 0 \leq 5$
IH: Assume $a_k \leq 5$ for some $k \geq 1$
IC: $a_k \leq 5$

$$\begin{aligned} 6 + a_k &\leq 11 \\ \sqrt{6 + a_k} &\leq \sqrt{11} \\ 1 + \sqrt{6 + a_k} &\leq 1 + \sqrt{11} \\ a_{k+1} &\leq 1 + \sqrt{11} \\ a_{k+1} &\leq 1 + \sqrt{16} \\ a_{k+1} &\leq 5 \end{aligned}$$

$\therefore$ the sequence $\{a_n\}$ is bounded above, and since it is bounded below as well, it is a bounded sequence. □

Show $a_{n+1} > a_n, \forall n \in \mathbb{N}$

Proof. BC: $a_1 = 0, a_2 = 1 + \sqrt{6}$. Therefore $a_2 > a_1$
IH: Assume $a_{k+1} > a_k$ for some $k \in N, k \geq 1$.
IC: $a_{k+1} > a_k$

$$6 + a_{k+1} > 6 + a_k$$
$$\sqrt{6 + a_{k+1}} > \sqrt{6 + a_k}$$
$$\underbrace{1 + \sqrt{6 + a_{k+1}}}_{a_{k+2}} > \underbrace{1 + \sqrt{6 + a_k}}_{a_{k+1}}$$

$\therefore \{a_n\}$ is increasing $\forall n \in \mathbb{N}$. □

Since $\{a_n\}$ is monotonic and bounded, by MST, $\{a_n\}$ converges.

We now know that there exists a L such that $\lim_{n\to\infty} a_n = L$.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \sqrt{6 + a_n}$$
$$= 1 + \sqrt{6 + \lim_{n\to\infty} a_n}$$
$$L = 1 + \sqrt{6 + L}$$
$$(L-1)^2 = (\sqrt{6 + L})^2$$
$$L^2 - 2L + 1 = 6 + L$$
$$L^2 - 3L - 5 = 0$$
$$L = \frac{3 \pm \sqrt{3^2 + 20}}{2}$$
$$L = \frac{3 \pm \sqrt{29}}{2}$$

Since the sequence is bounded between 0 and 5, the negative solution is extraneous.

$$L = \frac{3 + \sqrt{29}}{2}$$

The sequence converges to $\frac{3+\sqrt{29}}{2}$.

Note. In the previous examples, we showed boundedness and then used it to show monotonicity. It may be better to prove one so the proof can be used in the proof for another.

4 Infinite Series

Consider the sequence

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots, a_n, \dots$$

Let $\{S_n\}_{n=1}^{\infty}$ represent the sum of the a_n sequence where S_n represents all the terms in a up to n.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

These numbers, $S_1, S_2, \dots, S_n$, etc are called partial sums We define

$$\lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} a_i \qquad \text{(known as sum of the series)}$$

and this is called an **infinite series**. If the limit exists, then $\sum_{i=1}^{\infty} a_i$ converges, else it diverges.

4.1 Examples of Infinite Series

Definition 4.1. Geometric Series:

$$1 + r + r^2 + r^3 + \cdots + r^{i-1} + \cdots = \sum_{i=1}^{\infty} r^{i-1}$$

where $r \in \mathbb{R}$. For what values of r does the geometric series converge?

Solution. Consider the partial sums of the geometric series.

$$S_n = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$$
$$rS_n = r + r^2 + r^3 + r^4 + \cdots + r^{n-1} + r^n$$
$$S_n - rSn = 1 - r_n$$
$$S_n = \frac{1-r^n}{1-r}, r \neq 1$$
$$\sum_{i=1}^{\infty} r^{i-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} \lim_{n\to\infty} 1 - r^n$$
$$\text{If } -1 < r < 1, \sum_{i=1}^{\infty} r^{i-1} = \frac{1}{1-r}$$

Otherwise for all other values of r (except 1), the geometric series diverges.

For $r = 1$, $S_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$

$$S_n = n, \sum_{i=1}^{\infty} 1^{i-1} = \lim_{n\to\infty} n = \infty$$

Method 4.1. Conclusion:
The geometric series $\sum_{i=1}^{\infty} r^{i-1}$ converges to $\frac{1}{1-r}$ if $-1 < r < 1$.
If $|r| \geq 1$, then $\sum_{i=1}^{\infty} r^{i-1}$ diverges.

Example 4.1. Set $r = \frac{1}{2}$. Does it converge?

Solution.

$$\sum_{i=1}^{\infty} (\frac{1}{2})^{i-1} = \frac{1}{1-\frac{1}{2}} = 2$$

Example 4.2. Find the sum of

$$\sum_{i=1}^{\infty} e^{-i}$$

Solution.

$$\begin{aligned}\sum_{i=1}^{\infty} e^{-i} &= \sum_{i=1}^{\infty} (\frac{1}{e})^{i} \\ &= \frac{1}{e}\sum_{i=1}^{\infty} (\frac{1}{e})^{i-1} \\ &= \frac{1}{e} \times \frac{1}{1-\frac{1}{e}} \\ &= \frac{1}{e-1}\end{aligned}$$

Exercise 4.1. Does this infinite series converge/

$$\sum_{i=3}^{\infty} e^{-i}$$

Solution. It converges because there is a finite difference between this one and example 4.2.

$$\begin{aligned}\sum_{i=3}^{\infty} e^{-i} &= \sum_{i=1}^{\infty} e^{-i} - e^{-1} - e^{-2} \\ &= \frac{1}{e-1} - \frac{1}{e} - \frac{1}{e^2}\end{aligned}$$

Example 4.3. Does this series converge?

$$\sum_{i=572}^{\infty} \frac{3^2 i}{7^i}$$

Solution. Consider

$$\sum_{i=1}^{\infty} \frac{3^2 i}{7^i} = \frac{9}{7} \sum_{i=1}^{\infty} (\frac{9}{7})^{i-1}$$

This series diverges because $\frac{9}{7}$ diverges. Therefore the original series also diverges because there is a finite difference between the two series.

Example 4.4. The **harmonic series** is

$$\sum_{i=1}^{\infty} \frac{1}{i}$$

Show this series diverges.

Solution. Compare the harmonic series with another divergent series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

The sum of this series is infinite.

$$1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

$$S_{2^n} \geq 1 + \frac{n}{2}$$

Exercise 4.2. Prove last line of previous example using induction

Have fun!

4.2 Tests for Convergence of an Infinite Series

Properties of Convergent Series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Note.

$$\sum_{n=1}^{\infty} a_n \times b_n \neq \sum_{n=1}^{\infty} a_n \times \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} \neq \frac{\sum_{n=1}^{\infty} a_n}{\sum_{n=1}^{\infty} b_n}$$

Exercise 4.3. Prove the above note.

Good luck!

Example 4.5. Does this series converge? If so, what does it converge to?

$$\sum_{k=0}^{\infty} \frac{2^k + 3^k}{5^k}$$

Solution.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{2^k + 3^k}{5^k} &= \sum_{k=1}^{\infty} \frac{2^{k-1}}{5^{k-1}} + \sum_{k=1}^{\infty} \frac{3^{k-1}}{5^{k-1}} \\ &= \frac{1}{1-\frac{2}{5}} + \frac{1}{1-\frac{3}{5}} \\ &= \frac{5}{3} + \frac{5}{2} \\ &= \frac{25}{6} \end{aligned}$$

Note. For the above example, **both** series must converge in order to separate the terms.

Theorem 4.1.

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0$$

Proof.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Since S_n converges, then let s be such that $\lim_{n\to\infty} S_n = s$.
We also know $S_n = \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{S_{n-1}} + a_n$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= s - s \\ &= 0 \end{aligned}$$

□

4.3 Divergence Test

Definition 4.2 (DT)**. Divergence Test** states:

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Example 4.6. Show this series diverges.

$$\sum_{n=1}^{\infty} \frac{17n^5}{n^5 + 17n + 5}$$

Solution. Let $a_n = \frac{17n^5}{n^5+17n+5}$

$$\lim_{n\to\infty} \frac{17n^5}{n^5 + 17n + 5} = 17 \neq 0$$

By the divergence test, the series diverges.

Example 4.7. Does this series diverge?

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

Solution. Let $a_n = \frac{\ln(n)}{n^2}$. Apply L'Hopital's rule with $f(x) = \frac{\ln(x)}{x^2}$ for $x \in \mathbb{R}$.

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

By Theorem 3.1, $\lim_{n\to\infty} a_n = 0$, but cannot apply DT, and we do not know whether the series converges.

4.3.1 Integral Test

Suppose f is a continuous positive decreasing function on $[N, \infty)$ where N is some number. Let $a_n = f(n)$, then

- If $\int_n^{\infty} f(x)\, dx$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.
- If $\int_n^{\infty} f(x)\, dx$ diverges, then $\sum_{n=N}^{\infty} a_n$ diverges.

Example 4.8. Use the integral test to determine whether this series converges

$$\sum_{n=1}^{\infty} \frac{1}{(n-\pi)^2}$$

Solution. Let $f(x) = \frac{1}{(x-\pi)^2}$ on $[1, \infty)$. f is not continuous on the domain. However, f is continuous, positive, decreasing on $[4, \infty)$.

$$\int_4^{\infty} f(x)\, dx = \int_4^{\infty} \frac{1}{(x-\pi)^2}\, dx$$

$$\lim_{t\to\infty} \int_4^t \frac{1}{(x-\pi)^2}\, dx$$

$$\lim_{t\to\infty} \frac{-1}{x-\pi}\bigg|_4^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{t-\pi}+\frac{1}{4-\pi}\right]$$
$$= \frac{1}{4-\pi}$$

By the integral test, $\displaystyle\sum_{n=4}^{\infty}\frac{1}{(n-\pi)^2}$ converges

Since a finite number of terms does not affect convergence, then

$$\sum_{n=1}^{\infty}\frac{1}{(n-\pi)^2} \text{ also converges.}$$

Note.

$$\sum_{n=4}^{\infty}\frac{1}{(n-\pi)^2} \neq \frac{1}{4-\pi}$$

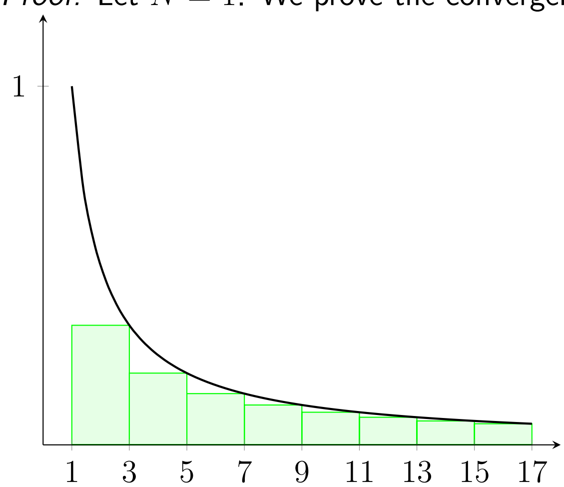
The integral test does not provide what the series converges to.

Theorem 4.2. Suppose $\sum_{n=1}^{\infty} a_n$ converges to s. Any partial sum, S_n is an approximation to s because $\lim_{n\to\infty} S_n = s$. How good is the approximation? If $S_n - s$ is small, then S_n approximates s well.

Suppose $f(k) = a_k, k \in \mathbb{N}$ where f is a continuous positive decreasing function for $x \geq n$, and $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq s - S_n \leq \int_{n}^{\infty} f(x)\,dx \tag{1}$$

Proof. Let $N = 1$. We prove the convergence part of the Integral Test. Consider right Riemann sums.



From the figure

$$\underbrace{a_2 + a_3 + \cdots + a_n}_{\sum_{i=2}^{n} a_i} \leq \int_1^n f(x)\,dx \leq \int_1^{\infty} f(x)\,dx$$

$$a_1 + \sum_{i=2}^{n} a_i \leq a_1 + \int_1^{\infty} f(x)\,dx$$

If the second part converges, then there exists a constant $M < \infty$ such that $S_n \leq M$ for all n. This means we have $\{S_n\}_{n=1}^{\infty}$ is bounded above. Furthermore $\underbrace{S_n + a_n}_{S_n+1} > a_n$ since $a_{n+1} > 0$. We have $\{S_n\}_{n=1}^{\infty}$ is increasing (monotonic). Since $S_n \geq S_1 = a_1$ for n, then $\{S_n\}_{n=1}^{\infty}$ is bounded below. Since $\{S_n\}_{n=1}^{\infty}$ is bounded and monotonic, then by MST, $\{S_n\}_{n=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$ converges.

To prove equation 1:

$$s - S_n = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{n} a_i = a_{n+1}a_{n+2}a_{n+3}\cdots \leq \int_n^{\infty} f(x)\,dx$$

□

Remark 4.1. If we rearrange Equation 1,

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq S_n + \int_n^{\infty} f(x)\,dx$$

Add in little bit, Fri Oct 31

Example 4.9. Consider

$$\sum_{n=1}^{\infty} \frac{1}{(3n+2)^2}$$

a) Show the series converges
b) Find an upper bound o the error in using the partial sum, S_{100}.
c) Find n such that the error is less than 0.01

Solution. Use the integral test

$$\int_n^{\infty} \frac{1}{(3x^2)^2}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{(3x+2)^2}\,dx = \lim_{t\to\infty} -\frac{1}{3(3x+2)}\Big|_n^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{3(3t+2)} + \frac{1}{3(3n+2)}\right] = \frac{1}{9n+6}$$

a. Apply Integral Test with $f(x) = \frac{1}{(3x+2)^2}$.

- f is continuous on $[1, \infty)$
- f is positive since the denominator is squared
- f is decreasing on $[1, \infty)$ since $f'(x) = -\frac{6}{(3x+2)^3} \leq 0$ for $x \geq 1$

$$\int_1^{\infty} \frac{1}{(3x+2)^2}\,dx = \frac{1}{25}$$

so by the integral test, $\sum_{n=1}^{\infty} \frac{1}{(3x+2)^2}$ converges.

b. Error $= s - S_n$

$$s - S_{100} \leq \int_{100}^{\infty} \frac{1}{(3x+2)^2}\, dx = \frac{1}{906} \approx 0.0011$$

c. $s - S_n \leq 0.01$

$$\frac{1}{9n+6} \leq 0.01$$
$$9n + 6 \geq 100$$
$$n \geq \frac{94}{9} \approx 10.84$$

$n = 11$ or larger will work.

Example 4.10. For what values of $p \in \mathbb{R}$ is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution. Case 1: $p \leq 0$

$$\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty$$

By the divergence test, for $p \leq 0$, the infinite series diverges.

Case 2: $p > 0$. Divergence test is inconclusive. Try integral test Let $f(x) = \frac{1}{x^p}$

- f is positive on $[1, \infty)$
- f is continuous on $[1, \infty)$
- f is always decreasing on $[1, \infty)$ since $f'(x) = -\frac{p}{x^{p-1}} < 0$ for $x \geq 1$.
  By the integral test, $\int_1^{\infty} \frac{1}{x^p}\, dx$ converges if $p > 1$ and diverges if $p \leq 1$

4.3.2 Comparison Test for Infinite Series

1. Suppose $0 \leq a_n \leq b_n \,\forall\, n \geq N$ where N is some finite number and if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.
2. Suppose $0 \leq b_n \leq a_n \,\forall\, n \geq N$. If the infinite series associated with b_n diverges, then the $\sum_{n=1}^{\infty} a_n$ also diverges.

Example 4.11. Determine whether the following series converges.

$$\sum_{n=1}^{\infty} e^{-n^2} \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} \tag{3}$$

Solution. Let $a_n = e^{-n^2}$. Use comparison theorem with $b_n = e^{-n}$. $\sum_{n=1}^{\infty} b_n$ converges.

$$\forall n \geq 1, n^2 \geq n$$

$$-n^2 \leq -n$$

$$e^{-n^2} \leq e^{-n} \text{ since } e^n \text{ is always increasing}$$

$$\forall n \geq 1, a_n \leq b_n$$

By the Comparison Test, $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

Let $a_n = \frac{1}{n!}$

$$n! = \underbrace{\underbrace{n}_{\geq 2}\underbrace{(n-1)}_{\geq 2}\underbrace{(n-2)}_{\geq 2}\cdots\underbrace{2}_{\geq 2}}_{\geq 2^{n-1}}$$

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

Let $b_n = 2^{n-1}$. $\sum_{n=1}^{\infty}(\frac{1}{2})^{n-1}$ converges. By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

4.3.3 Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series with all positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ where C is a positive finite number, then either both infinite series converge or both diverge.

Proof. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, for n sufficiently large, $a_n \approx Cb_n$. The constant C does not affect the convergence, so if a_n converges or diverges, b_n will do the same thing. □

Remark 4.2. If $C = 0$, then a_n is significantly smaller than b_n for n sufficiently large. Try comparison test in this case.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then a_n is significantly larger than b_n for n sufficiently large. Try comparison test too.

Example 4.12. Determine whether the infinite series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3+1}{3n^5-1}$$

Solution.

$$\text{Let } a_n = \frac{n^3+1}{3n^5-1} \approx \frac{n^3}{3n^5} = \frac{1}{3n^2}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ which is a p-series with $p = 2$ and hence converges. Let $b_n = \frac{1}{3n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Since b_n converges, $\sum_{n=1}^{\infty} \frac{n^3+1}{3n^5-1}$ converges as well by the Limit Comparison Test.

Example 4.13. Use LCT to determine the convergence of the following series

$$\sum_{n=1}^{\infty}(1+\frac{1}{n})^2 e^n$$

Solution. Let $a_n = (1+\frac{1}{n})^2 e^n$. Compare with $b_n = e^n$
$\sum_{n=1}^{\infty} b_n$ diverges.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 1$$

Therefore by LCT, $\sum_{n=1}^{\infty}(1+\frac{1}{n})^2 e^n$ diverges.

Definition 4.3. An **alternating series** is a series whose terms are alternately positive and negative.

4.3.4 Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

Must satisfy:

- $b_{n+1} \leq b_n \, \forall n \in N$
- $\lim_{n\to\infty} b_{n+1} = 0$ Then the series converges.

The distance between S_n and S_{n+1} is bigger than the distance between S_n and S.

$$|S - S_n| \leq |S_{n+1} - S_n|$$

$$|S - S_n| \leq b_{n+1}$$

Alternating Series Estimation Theorem

If $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

- $b_{n+1} \leq b_n \, \forall n \in N$
- $\lim_{n\to\infty} b_{n+1} = 0$

Then $|S - S_n| \leq b_{n+1}$ where S_n is the partial sum of the alternating series.

Remark 4.3. Rearrange the above equation:

$$-b_{n+1} \leq S - S_n \leq b_{n+1}$$

$$S_n - b_{n+1} \leq S \leq S_n + b_{n+1}$$

We can find a lower bound and upper bound based on S_n

Example 4.14. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n - 1}$$

1. Show the series converges
2. Determine the smallest positive integer n such that the partial sum $S_n = \sum_{i=1}^{n} \frac{(-1)^n}{e^n-1}$ is guaranteed to approximate the exact sum, s with $|s - S_n| \le 0.01$.

Solution. Apply AST. Let $b_n = \frac{1}{e^n-1}$ for all $n \in \mathbb{N}$.
Must show $b_{n+1} \le b_n$.

$$n+1 \ge n$$
$$e^{n+1} \ge e^n$$
$$e^{n+1} - 1 \ge e^n - 1$$
$$\frac{1}{e^{n+1}-1} \le \frac{1}{e^n - 1}$$
$$b_{n+1} \le b_n$$
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{e^n-1} = 0$$

By AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n-1}$ converges.

ASET implies that $|S_n - s| \le b_{n+1}$

$$b_{n+1} \le 0.01$$
$$\frac{1}{e^n-1} \le 0.01$$
$$e^n - 1 \le 100$$
$$n+1 \le \ln(101)$$
$$n \ge \ln(101) - 1 \approx 3.6151$$

$\therefore n = 4$ will guarantee an accuracy within 0.01

Example 4.15. Alternating Harmonic Series: Does the series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Solution. Let $b_n = \frac{1}{n} \, \forall n \in \mathbb{N}$.
Apply AST.

$$n+1 \ge n$$
$$\frac{1}{n+1} \le \frac{1}{n}$$
$$b_{n+1} \le b_n$$
$$\lim_{n\to\infty} \frac{1}{n} = 0$$

By AST, the alternating harmonic series converges.

Definition 4.4. A series $\sum a_n$ is **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Example 4.16. Is the series absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

Solution.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2$ so the series is absolutely convergent.

Definition 4.5. A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent

Theorem 4.3. If a series is absolutely convergent, then it is convergent.

Example 4.17. Does this series converge?

$$\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^3}$$

Solution.

$$\sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p-series, so it is absolutely convergent by the Comparison Test. By the previous theorem, since it is absolutely convergent, it is convergent.

4.3.5 Ratio Test

Theorem 4.4 (Ratio Test)**.** If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, then if $L < 1, \sum a_n$ converges. If $L > 1, \sum a_n$ diverges. If $L = 1$, the Ratio Test is inconclusive.

Example 4.18. Determine whether the following converge.

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

$$\sum_{k=1}^{\infty} k \left(\frac{2}{3} \right)^k$$

Solution.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |(n+1)\frac{1}{100}$$
$$= \lim_{n\to\infty} \frac{n+1}{100} = \infty$$

By the ratio test, the first series diverges.

Let $a_k = k(\frac{2}{3})^k$

$$\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \lim_{k\to\infty} \frac{(k+1)(\frac{2}{3})^{k+1}}{k(\frac{2}{3})^k}$$
$$\frac{2}{3}\lim_{k\to\infty} \frac{k+1}{k} = \frac{2}{3}$$

Since $L = \frac{2}{3} < 1$, then the series converges.

Root Test

Theorem 4.5. Root Test: If

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent.
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or ∞, then the series $\sum a_n$ is divergent.
- If it equals 1, it's inconclusive.

Example 4.19. Determine the convergence of

$$\sum_{n=1}^{\infty}(\frac{n^2+1}{2n^2+2n+1})^{2n}$$

Solution. Let $a_n = (\frac{n^2+1}{2n^2+2n+1})^{2n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \left(\lim_{n\to\infty} \frac{n^2+1}{2n^2+2n+1}\right)^2$$
$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

By the root test, this infinite series converges.

5 Power Series

Example 5.1. Consider

$$\sum_{n=1}^{\infty} \frac{(x-7)^n}{2^n n}$$

. For what values of $x \in \mathbb{R}$ is this series convergent?

Solution. Apply Ratio Test.

$$\lim_{n\to\infty} \frac{\frac{(x-7)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-7)^n}{2^n n}} = \lim_{n\to\infty} \frac{2^n n}{2^{n+1}(n+1)}|x-7|$$

$$= |x-7| \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{|x-7|}{2} < 1$$

$$|x-7| < 2$$

$$-2 < x-7 < 2$$

$$5 < x < 9, x \neq 7$$

When $x = 7$, the series converges to 0.
When $x = 5$, the series becomes an alternating harmonic series which converges.
When $x = 9$, the series becomes a regular harmonic series, which diverges.

$\therefore$ The series converges for $x \in [5, 9)$ and diverges otherwise.

Definition 5.1. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

a is the centre of the power series.
$x \in \mathbb{R}$ is the variable
C_n are constants with respect to x, but they may depend on n.

The above power series can be reerred to as a power series centered at a.

Theorem 5.1. For a given series $\sum_{n=0}^{\infty} C_n (x-a)^n$ there are only three possibilities

- The series converges if $x = a$
- The series converges for all x.
- There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Example 5.2. Find the radius and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}, x \in \mathbb{R}$$

Solution. Let $a_n = \frac{(x-2)^n}{n!}$
Apply ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$\lim_{n\to\infty} \frac{\frac{(x-2)^{n+1}}{(n+1)!}}{\frac{(x-2)^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{n!}{n+1!}|x-2| = |x-2| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

By the Ratio Test, the power series converges for any value of x except $x = 2$. Consider $x = 2$, then the series converges to 1. Therefore the series converges for all values of x.

Remark 5.1. In general, when $x = a$ in a power series, the power series converges.

Exercise 5.1. Consider

$$\sum_{n=0}^{\infty} n!(x-3)^n$$

Show the radius of convergence is $R = 0$ and the interval of convergence is $\{3\}$

Power series can be used to represent functions. Recall the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

This can be viewed as a power series with $r = x$

$$\sum_{n=0}^{\infty} x^n = \underbrace{\frac{1}{1-x}}_{f(x)}, |x| < 1$$

$f(x) = \frac{1}{1-x}$ s a function whose domain is the set of all x for which $\sum_{n=0}^{\infty} x^n$ converges.

Example 5.3. Find a power series representation for the function $\frac{1}{3+2x}$, and state the radius of convergence and interval of convergence.

Solution.

$$\frac{1}{3(1+\frac{2}{3}x)} = \frac{1}{3(1-(-\frac{2}{3}x))}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n$$

$$|-\frac{2}{3}x| < 1 \implies |x| < \frac{3}{2}$$
$$R = \frac{3}{2}$$

The interval of convergence is $-\frac{3}{2}, \frac{3}{2}$.

Example 5.4. Determine the power series representation of

$$\frac{1}{(1-x)^2}$$

Solution.

$$\left(\frac{1}{1-x}\right)^2 = \left(\sum_{n=0}^{\infty} x^n\right)^2$$
$$= \sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n = (1 + x + x^2 + x^3 + x^4 + \dots)(1 + x + x^2 + x^3 + x^4 + \dots)$$
$$= 1 + x + x^2 + x^3 + x^4 + \dots + x + x^+ x^3 + x^4 + x^5 + \dots + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=0}^{\infty} (n+1)x^n$$

Furthermore

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

and the first equation is the derivative of the above equation.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n, |x| < 1$$

Power series may be differentiated, and this is called **term by term differentiation**.

Theorem 5.2. If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function is defined by

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n, |x-a| < R$$

is differentiable (and hence continuous) on $(a-R, a+R)$ and

$$\frac{d}{dx}f(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx}c_n(x-a)^n$$

$$= \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

Similarly,

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \int (x-a)^n$$
$$= \left[\sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}\right] + C$$

Note. For the above theorem, the radius of convergence are both R. but this does not mean that the interval of convergence remains the same.

Returning to the example,

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$= \frac{d}{dx}\sum_{n=0}^{\infty} x^n, |x| < 1$$
$$= \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$
$$= \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

This is equivalent to the previous solution.

Example 5.5. Find a power series representation for $\ln(1-x)$

Solution.

$$\ln(1-x) = -\int \frac{1}{1-x}\,dx$$
$$= -\int \left(\sum_{n=0}^{\infty} x^n\right)\,dx$$
$$= -\sum_{n=0}^{\infty} \int x^n\,dx$$
$$= \left[\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right] + k$$

We need to determine k. Let $x = 0$,

$$\underbrace{\ln(1)}_{0} = \underbrace{-\sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1}}_{0} + k$$

$$k = 0$$

Note. Pick simple numbers to test. Typically choosing $x = a$ works well.

Example 5.6. What is the PS representation for $\ln(1+x)$.

Solution. Replace x with $-x$ in the previous example.

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}, |-x| < 1 \implies |x| < 1$$

6 Taylor and MacLaurin Series

Use Taylor and MacLaurin series to approximate functions.

$$y_3 = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3$$

Higher degree polynomials and appropriate c_i selections will lead to a better approximation. Each c_i will depend on f.

To get c_0, set $x = a, f(a) \approx c_0$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + nc_n(x-a)^{n-1}$$

Again, let $x = a, f'(a) \approx c$

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots + n(n-1)(x-a)^{n-2}$$

Let $x = a, f''(a) = 2c_2$ and $c_2 = \frac{f''(a)}{2}$.

By rewriting $f(a)$ as $f^{(0)}(a)$

$$c_0 = \frac{f^{(0)}(a)}{0!} \qquad c_1 = \frac{f^{(1)}(a)}{1!} \qquad c_1 = \frac{f^{(2)}(a)}{2!}$$

In general

$$c_i = \frac{f^{(i)}(a)}{i!}$$

$f(x)$ can then be approximated with the following polynomial

$$\begin{aligned}
f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n \\
&= \frac{f^{(0)}(a)}{0!} + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\
&= Tn(x)
\end{aligned}$$

If $a = 0$, Taylor polynomial is called MacLaurin polynomial

$$Tn(x) = \sum_{i=0}^{\infty} \frac{f^{(0)}(0)}{i!} x^i$$

Example 6.1. Find the second degree Taylor polynomial of the following polynomial centred at 4.

$$f(x) = \sqrt{x}$$

Solution.

$$T_2(x) = \sum_{i=0}^{2} \frac{f^{(i)}(4)}{i!}(x-4)^i$$

$$\begin{aligned}
f^{(0)}(4) &= \sqrt{4} = 2 \\
f^{(1)}(x) &= \frac{1}{2\sqrt{x}} \implies f'(4) = \frac{1}{4} \\
f^{(2)}(x) &= -\frac{1}{4\sqrt{x^3}} \implies f''(4) = -\frac{1}{4\sqrt{2^3}} = -\frac{1}{32}
\end{aligned}$$

$$\begin{aligned}
T_2(x) &= 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2\frac{1}{2!} \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2
\end{aligned}$$

Example 6.2. Find the nth degree Maclaurin polynomial for $f(x) = e^x$.

Solution. $f^{(0)}(x) = f^{(1)}(x) == f^{(2)}(x) = f^{(i)}(x) = e^x$ Since f is centered around $a = 0, f^{(i)}(0) = e^0 = 1$.

$$T_n(x) = \sum_{i=0}^{n} \frac{1}{i!}x^i$$

Aside. Does the approximation get better as $n \to \infty$?
Let $\underbrace{Rn(x)}_{\text{Taylor's remainder}} = \underbrace{f(x)}_{\text{exact}} - \underbrace{Tn(x)}_{\text{approximation}}$

Theorem 6.1. If $f(x)$ is $n+1$ times differentiable, and $R_n(x) = f(x) - T_n(x)$ then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Proof. Start with $n = 1$. Show $R_1(x) = f(x) - T_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$

$$\text{By Mean Value Theorem, } f''(c) = \frac{f'(x) - f'(a)}{x-a}$$

$$f'(a)f''(c)(x-a) = f'(x)$$

Replace x with t.

$$f'(a)f''(c)(t-a) = f'(t)$$

Integrate t from a to x

$$\int_a^x f'(t)\,dt = \int_a^x f'(a)\,dt + \int -a^x f''(c)\frac{(x-a)^2}{2}$$

$$f(x) - f(a) = f'(a)(x-a) + f''(c)\frac{(t-a)^2}{2}\Big|_a^x$$

$$f(x) - f(a) = f'(a)(x-a) + f''(c)\frac{(x-a)^2}{2}$$

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{T_1(x)} + f''(c)\frac{(x-a)^2}{2}$$

$$\underbrace{f(x) - T_1(x)}_{R_n(x)} = \frac{f''(c)(x-a)^2}{2}$$

$$R_n(x) = \frac{f''(c)(x-a)^2}{2}$$

□

Theorem 6.2. Taylor's Inequality: Let M represent the maximum of $|f^{(n+1)}(t)|$ for t between a and x, then

$$|R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 6.3. Use $T_2(x)$ from example 6.1 to approximate $\sqrt{4.1}$ and use Taylor's inequality to find an upper bound for the error.

Solution.

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\sqrt{4.1} \approx T_2(4.1) = 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2 = 2.02484375$$

From Taylor's inequality:

$$|R_2(x)| < \frac{M}{3!}|x-4|^3$$

M is the maximum of $f^{(3)}(t)$ for $4 < t < 4.1$.

$$f^{(3)}(x) = \frac{3}{8\sqrt{t^5}}$$

This function is always decreasing so the maximum is at $t = 4$.

$$f^{(3)}(4) = \frac{3}{8\sqrt{4^5}} = \frac{3}{256}$$

$$|R_2(x)| < \frac{\frac{3}{256}}{3!}|0.1|^3$$

$$|R_2(x)| < \frac{1}{512000} \approx 0.00000195$$

Example 6.4. Find the upper bound for the error of $R_2(x)$ in using $T_2(x)$ to approximate $f(x) = \sqrt{x}$ centred at 4 for $3 \le x \le 6$.

Solution.

$$|R_2(x)| < \frac{M}{3!}|x-4|^3$$

Since $f^{(3)}(x)$ is always decreasing, $t = 3$ gives the maximum.

$$M = \frac{3}{8\sqrt{3^5}}$$

$$|R_2(x)| < \frac{\frac{3}{8\sqrt{3^5}}}{3!} \underbrace{|x-4|^3}_{\text{Maximize, } x=6}$$

$$|R_2(x) < \frac{8}{16\sqrt{3^5}} \approx 0.032$$

Example 6.5. Find the MacLaurin series of $f(x) = \ln(x+1)$.

Solution. $f^{(0)}(x) = \ln(x+1) \implies f(0) = \ln 1 = 0$. This contributes nothing, so the series can begin at 1.

$$\sum_{i=1}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

$$f'(x) = (x+1)^{-1}$$

$$f''(x) = -(x+1)^{-2}$$

$$f'''(x) = (-1)(-2)(x+1)^{-3}$$

$$f^{(i)}(x) = (-1)^{i-1}(i-1)!(x+1)^{-i} \qquad \text{for } i \ge 1$$

The series is

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}(i-1)!}{i!} x^i = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i$$

Remark 6.1. The power series of $\ln(x+1)$ was

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i \text{ for } |x| < 1$$

usin the geometric series. In general, if f has a power series representation, centred at a for $|x-a| < R$, it will be the Taylor series centred at a.

$$f(x) = \sum_{i=0}^{\infty} C_i (x-a)^i \implies C_i = \frac{f^{(i)}(a)}{i!}$$

When does $f(x)$ equal its Taylor Series. Consider $Rn(x) = f - Tn(x)$.

$$\lim_{n\to\infty} Rn(x) = \lim_{n\to\infty}(f(x) - Tn(x))$$
$$= f(x) - \lim_{n\to\infty} Tn(x)$$

If $Rn(x) = 0$, then

$$f(x) = \lim_{n\to\infty} Tn(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Theorem 6.3. If $f(x) = Tn(x) + Rn(x)$ and $\lim_{n\to\infty} Rn(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor Series on $|x-a| < R$.

Example 6.6. Prove that e^x is equal to its MacLaurin series for all x.

Solution. Recall that the MacLaurin series for e^x is

$$\sum_{i=0}^{\infty} \frac{1}{i!}x^i$$

Taylor's Inequality $\implies$ $|Rn(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}$ where M is the maximum for $\{|f^{(n+1)}(t)|\}$ for t between 0 and x.

$$f^{(n+1)}(t) = e^t$$
$$M = \max\{1, e^x\}$$
$$0 \leq |Rn(x)| \leq \frac{M|x|^{n+1}}{n!}$$
$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} |Rn(x)| \leq \lim_{n\to\infty} \frac{M|x|^{n+1}}{n!}$$
$$0 \leq \lim_{n\to\infty} |Rn(x)| \leq M \lim_{n\to\infty} \frac{|x|^{n+1}}{n!}$$
$$0 \leq \lim_{n\to\infty} |Rn(x)| \leq 0$$

By the squeeze theorem, $\lim_{n\to\infty} |Rn(x)| = 0$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \qquad \forall\, x \in \mathbb{R}$$

Example 6.7. Prove that $\cos(x)$ is equal to its MacLaurin series for all x.

Solution. Prove $\cos(x) = \sum_{i=1}^{\infty} \frac{f^{(i)}(0)}{i!}x^i$.

$$f^{(0)}(x) = \cos(x) \implies \cos(0) = 1$$
$$f^{(1)}(x) = -\sin(x) \implies -\sin(0) = 0$$

$$f^{(2)}(x) = -\cos(x) \implies -\cos(0) = -1$$
$$f^{(3)}(x) = \sin(x) \implies \cos(0) = 0$$
$$\frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \dots$$
$$= \sum_{i=0}^{\infty}(-1)^i \frac{x^{2i}}{(2i)!}$$

Taylor's Inequality $\implies |Rn(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}$ where M is the maximum for $\{|f^{(n+1)}(t)|\}$ for t between 0 and x. Since the derivative $|f|$ is either $\|cos(x)|$ or $|\sin(x)|$ implies that M can be chosen to be 1.

$$0 \leq |Rn(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

By the squeeze theorem, $\lim_{n\to\infty}|Rn(x)| = 0 \implies \lim_{n\to\infty} Rn(x) = 0$ by Taylor's Theorem

$$\cos(x) = \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}}{(2i)!} \qquad \forall\, x \in \mathbb{R}$$

Other functions that equal the sum of its Taylor series.

- $\ln(1+x)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, |x| < 1$
- $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, |x| < 1$
- $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n, k \in \mathbb{R}, |x| < 1$

Exercise 6.1. Find the taylor series of $f(x) = |x|$ centered at 2 and prove that f is not equal to its taylor series for all $x \in \mathbb{R}$.

Example 6.8. Approximate $\int_0^1 x\cos(x^3)\,dx$ to within an accuracy of 0.0001.

Solution. Recall that $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

$$\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}$$

$$x\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(2n)!}$$

$$\int_0^1 x\cos(x^3)\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int_0^1 (x^{6n+1})\,dx$$

$$\int_0^1 x\cos(x^3)\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{x^{6n+2}}{6n+2}\Big|_0^1$$

$$\int_0^1 x\cos(x^3)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(6n+2)}$$

Apply ASET, Let $b_n = \frac{1}{(2n)!(6n+2)}$

$$(2n+1)!(6(n+1)+2) \geq (2n)!(6n+2)$$

$$\frac{1}{(2n+1)!(6(n+1)+2)} \leq \frac{1}{(2n)!(6n+2)}$$

$$b_{n+1} \leq b_n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n)!(6n+2)} = 0$$

$$|S - S_n| \leq b_{n+1}$$

$$0.00001 \leq b_{n+1}$$

$$0.00001 \leq \frac{1}{(2n+1)!(6(n+1)+2)}$$

$$(2n+1)!(6(n+1)+2) \geq 10000$$

Guess and check because cannot isolate for n.

$$(2\times(2+1))!(6(2+1)+2) = 720\times 20 = 14400 \geq 10000$$

$$\int_6^1 x\cos(x^3)\,dx = \sum_{i=0}^{2} \frac{(-1)^i}{(2i)!(6i+2)}$$

Find the first three terms to get the approximation.

$$= \frac{(-1)^0}{0!2} + \frac{(-1)^1}{2!(8)} + \frac{(-1)^2}{4!(14)} = \frac{1}{2} - \frac{1}{16} + \frac{1}{336} = \frac{168-21+1}{336} = \frac{148}{336} = \frac{37}{84}$$