MATH 136

LINEAR ALGEBRA

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Linear Algebra

- Systems of linear equations
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- Vector spaces, $\mathbb{R}^n$

$\mathbb{R}^n$ consists of n-tuples of real numbers, where $n \in \mathbb{N}$.

Definition 1.1. **Points/vectors** are elements of $\mathbb{R}^n$.

Notation

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \,|\, x_1, x_2, \dots, x_n \in \mathbb{R}\}$$

$$x_1 + x_2 = 3$$

$$2x_1 + 5x_2 = 4$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Two vectors in $\mathbb{R}^n$ are equal if all coordinates are equal.

Vector Operations

Let $\vec{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}$

Addition

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

Scalar Multiplication

$$\alpha\vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 1.2. $\vec{0}$ is the **additive identity**

Definition 1.3. Given a vector $\vec{x} \in \mathbb{R}^n$, $-\vec{x}$ is the **additive inverse**.

Definition 1.4. A sum of scalar multiples of a combination of vectors is a **linear combination**

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : \, c_1 \dots c_k \in \mathbb{R}$$

Theorem 1.1. If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$, then

- $\vec{x} + \vec{y} \in \mathbb{R}^n$

- $(\vec{x}+\vec{y})+\vec{w}=\vec{x}+(\vec{y}+vw)$
- $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
- $\exists\,\vec{0}\in\mathbb{R}^n$ such that $\vec{x}+\vec{0}=\vec{x} \qquad \forall\vec{x}\in\mathbb{R}^n$
- $\forall\vec{x}\in\mathbb{R}^n$, there exists a vector $(-\vec{x})\in\mathbb{R}^n$ such that $\vec{x}+(-\vec{x})=\vec{0}$
- $c\vec{x}\in\mathbb{R}^n$
- $c(d\vec{x})=(cd)\vec{x}$
- $(c+d)\vec{x}=c\vec{x}+d\vec{y}$
- $c(\vec{x}+vy)=c\vec{x}+c\vec{y}$
- $1\vec{x}=\vec{x}$

Definition 1.5. The set S of all possible linear combinations of a set of vectors $B=(\vec{v}_1,\dots,\vec{v}_k)$ in $\mathbb{R}^n$ is called the **span** of the set B and we write

$$S=\text{Span B}=\{t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k\}$$

S is **spanned** by B and that B is a spanning set for S.

For a set

$$\{t_1\vec{v}_1+\cdots+t_k\vec{v}_k+\vec{b}|t_1,\dots,t_k\in\mathbb{R}\}$$

can be written as

$$\vec{x}=t_1\vec{v}_1+\cdots+t_k\vec{v}_k+\vec{b},t_1,\dots,t_k\in\mathbb{R}$$

In $\mathbb{R}^n$, two linearly independent vectors $\vec{x}_1$ and $\vec{x}_2$ generate a plane.

Definition 1.6. A set of vectors in $\mathbb{R}^n$ is said to be **linearly dependent** if there exists coefficients $c_1,\dots,c_k$, not all 0, such that

$$\vec{0}=c_1\vec{v}_1+\cdots+c_k\vec{v}_k$$

Either 0 vector or two or more vectors are colinear (scalar multiple).

Definition 1.7. A set of vectors is **linearly independent** if the only solution is $c_1=c_2=\cdots=c_k=0$ (**trivial solution**)

Definition 1.8. If a subset of $\mathbb{R}^n$ can be written as a span of vectors $\vec{v}_1,\dots,\vec{v}_k$ where $\{\vec{v}_1,\dots,\vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1,\dots,\vec{v}_k\}$ is a **basis** for S. The basis of the set $\{\vec{0}\}$ is the empty set.

Theorem 1.2. If $\beta=\{\vec{v}_1,\dots,\vec{v}_k\}$ is a basis for a subset S of $\mathbb{R}^n$, then every vector $\vec{x}\in S$ can be written as unique linear combination of $\vec{v}_1,\dots,\vec{v}_k$.

Definition 1.9. The **standard basis** in $\mathbb{R}^n$ is a set of vectors where each vector's ith component is 1, and all other components are 0.

Definition 1.10. Let $\vec{x}, vy \in \mathbb{R}^n$. The set with vector equation $\vec{w} = c_1\vec{x} + \vec{y}$ with $c_1 \in \mathbb{R}$ is a **line** in $\mathbb{R}^n$ that passes through $\vec{y}$.

Definition 1.11. Let $\vec{v}_1, \vec{v}_2, \vec{y} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. The set with the vector equation $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{y}$ with $c_1, c_2 \in \mathbb{R}$ is a **plane** in $\mathbb{R}^n$ which passes through $\vec{y}$.

Definition 1.12. Let $\vec{v}_1, \ldots, \vec{v}_k, \vec{y} \in \mathbb{R}^n$ with the set being linearly independent. The set with the vector equation $\vec{x} = c_1 v_1 + \cdots + c_k\vec{v}_k + \vec{y}$ with $c_1, \ldots, c_k$ is a **k-plane** in R^n with passes through $\vec{y}$.

Definition 1.13. A **hyperplane** is a subspace of one dimension less than its ambient space.

Theorem 1.3. Subspace Test: Let $\mathbb{S}$ be a non-empty subset of $\mathbb{R}^n$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$

Quote. If $\vec{0}$ is not in the set, definitely not subset. If it is, further investigation needed.

Definition 1.14. $S \in \mathbb{R}^n$ is closed under scalar multiplication if for all $\vec{x} \in S$ and $\alpha \in \mathbb{R}$, $\alpha\vec{x} \in S$.

Theorem 1.4. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

Theorem 1.5. If $\vec{x}, \vec{y} \in \mathbb{R}^2$, and θ is the angle between them, then

$$\vec{x}\dot{\vec{y}} = ||\vec{x}||\, ||\vec{y}||\, \cos\theta$$

Theorem 1.6. Given two vectors $\vec{x}, vy$, their dot product is defined by

$$\vec{x}\dot{\vec{y}} = x_1y_1 + x_2 + y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i$$

Theorem 1.7. If $\vec{x}\dot{\vec{y}} = 0$, then $\vec{x}$ and $\vec{y}$ are **orthogonal**.

Quote. The zero vector $\vec{0} \in \mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$.

Theorem 1.8. The **cross product** of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}$$