MATH 136

LINEAR ALGEBRA

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Future Modifications

1 Vectors in Euclidean Space

1.1 Vector Addition and Scalar Multiplication

Definition 1.1. $\mathbb{R}^n$ consists of n-tuples of real numbers, where $n \in \mathbb{N}$.

Definition 1.2. **Points/vectors** are elements of $\mathbb{R}^n$.

Notation

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

$$x_1 + x_2 = 3$$

$$2x_1 + 5x_2 = 4$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Two vectors in $\mathbb{R}^n$ are equal if all coordinates are equal.

Vector Operations

Let $\vec{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}$

Addition

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

Scalar Multiplication

$$\alpha\vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 1.3. $\vec{0}$ is the **additive identity**.

Definition 1.4. Given a vector $\vec{x} \in \mathbb{R}^n$, $-\vec{x}$ is the **additive inverse**.

Definition 1.5. A sum of scalar multiples of a combination of vectors is a **linear combination**

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : \, c_1 \ldots c_k \in \mathbb{R}$$

Theorem 1.1.1. If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$, then

- $\vec{x} + \vec{y} \in \mathbb{R}^n$
- $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + vw)$
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\exists\, \vec{0} \in \mathbb{R}^n$ such that $\vec{x} + \vec{0} = \vec{x} \qquad \forall \vec{x} \in \mathbb{R}^n$
- $\forall \vec{x} \in \mathbb{R}^n$, there exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$

- $c\vec{x} \in \mathbb{R}^n$
- $c(d\vec{x}) = (cd)\vec{x}$
- $(c+d)\vec{x} = c\vec{x} + d\vec{y}$
- $c(\vec{x} + vy) = c\vec{x} + c\vec{y}$
- $1\vec{x} = \vec{x}$

Definition 1.6. The set S of all possible linear combinations of a set of vectors $B = (\vec{v}_1, \ldots, \vec{v}_k)$ in $\mathbb{R}^n$ is called the **span** of the set B and we write

$$S = \text{Span B} = \{t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k\}$$

S is **spanned** by B and that B is a spanning set for S.

Note. A set in the form

$$\{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + \vec{b} | t_1, \ldots, t_k \in \mathbb{R}\}$$

can be written as

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + \vec{b}, t_1, \ldots, t_k \in \mathbb{R}$$

In $\mathbb{R}^n$, two linearly independent vectors $\vec{x}_1$ and $\vec{x}_2$ generate a plane.

Theorem 1.1.2. If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_{k-1}$, then

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$$

Definition 1.7. A set of vectors in $\mathbb{R}^n$ is said to be **linearly dependent** if there exists coefficients $c_1, \ldots, c_k$, not all 0, such that

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

Either 0 vector or two or more vectors are colinear (scalar multiple).

Definition 1.8. A set of vectors is **linearly independent** if the only solution is $c_1 = c_2 = \cdots = c_k = 0$ (**trivial solution**)

Theorem 1.1.3. If a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ contains the zero evctor, then it is linearly dependent.

Definition 1.9. If a subset of $\mathbb{R}^n$ can be written as a span of vectors $\vec{v}_1, \ldots, \vec{v}_k$ where $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a **basis** for S. The basis of the set $\{\vec{0}\}$ is the empty set.

Theorem 1.1.4. If $\beta = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for a subset S of $\mathbb{R}^n$, then every vector $\vec{x} \in S$ can be written as unique linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.

Definition 1.10. The **standard basis** in $\mathbb{R}^n$ is a set of vectors where each vector's ith component is 1, and all other components are 0.

Definition 1.11. Let $\vec{x}, vy \in \mathbb{R}^n$. The set with vector equation $\vec{w} = c_1\vec{x} + \vec{y}$ with $c_1 \in \mathbb{R}$ is a **line** in $\mathbb{R}^n$ that passes through $\vec{y}$.

Definition 1.12. Let $\vec{v}_1, \vec{v}_2, \vec{y} \in \mathbb{R}^n$ with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. The set with the vector equation $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{y}$ with $c_1, c_2 \in \mathbb{R}$ is a **plane** in $\mathbb{R}^n$ which passes through $\vec{y}$.

Definition 1.13. Let $\vec{v}_1, \dots, \vec{v}_k, \vec{y} \in \mathbb{R}^n$ with the set being linearly independent. The set with the vector equation $\vec{x} = c_1v_1 + \cdots + c_k\vec{v}_k + \vec{y}$ with $c_1, \dots, c_k$ is a **k-plane** in R^n with passes through $\vec{y}$.

Definition 1.14. A **hyperplane** is a subspace of one dimension less than its ambient space.

1.2 Subspaces

Theorem 1.2.1. Subspace Test: Let $\mathbb{S}$ be a non-empty subset of $\mathbb{R}^n$. If $\vec{x} + \vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$

Quote. If $\vec{0}$ is not in the set, definitely not subset. If it is, further investigation needed.

Definition 1.15. $S \in \mathbb{R}^n$ is closed under scalar multiplication if for all $\vec{x} \in S$ and $\alpha \in \mathbb{R}$, $\alpha\vec{x} \in S$.

Theorem 1.2.2. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$.

1.3 Dot Product

Theorem 1.3.1. If $\vec{x}, \vec{y} \in \mathbb{R}^2$, and θ is the angle between them, then

$$\vec{x} \cdot \vec{y} = ||\vec{x}||\,||\vec{y}||\,\cos\theta$$

Definition 1.16. Given two vectors $\vec{x}, vy$, their dot product is defined by

$$\vec{x}\dot{\vec{y}} = x_1y_1 + x_2 + y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i$$

Theorem 1.3.2. Let $\vec{x}, \vec{y}\,\vec{z} \in \mathbb{R}^n$ and let $s, t \in \mathbb{R}$. Then

- $\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
- $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

Theorem 1.3.3. If $\vec{x} \cdot \vec{y} = 0$, then $\vec{x}$ and $\vec{y}$ are **orthogonal**.

Quote. The zero vector $\vec{0} \in \mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$.

Theorem 1.3.4. The **cross product** of $\vec{x}, \vec{y} \in \mathbb{R}^3$ is given by

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

Quote. Cross product is not associative. $\vec{v} \times (\vec{w} \times \vec{x}) \neq (\vec{v} \times \vec{w}) \times \vec{x}$.

Theorem 1.3.5. Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$ with $\{\vec{v}, \vec{w}\}$ being a linear independent set, and define $\vec{n} = \vec{v} \times \vec{w}$. If P is a plane with the vector equation

$$\vec{x} = c\vec{v} + d\vec{w} + \vec{b}, \qquad c, d \in \mathbb{R}$$

then an alternate equation for the plane is

$$(\vec{x} - \vec{b}) \cdot \vec{n} = 0$$

n is a normal vector to the plane P. Rearranging: $n_1x_1 + n_2x_2 + n_3x_3 = n_1a_1 + n_2a_2 + n_3a_3$.

1.4 Projections

Definition 1.17. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The **projection of $\vec{u}$ onto $\vec{v}$** is

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

Definition 1.18. The **perpendicular of $\vec{u}$ onto $\vec{v}$** is

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u})$$

Quote. To project a vector onto a plane, take the perpendicular of the vector projected onto the normal of the plane.

2 Systems of Linear Equations

2.1 Systems of Linear Equations

Definition 2.1. A system of linear equations in n variables

$$cx_1 + cx_2 + \cdots + cx_n = b_1 \tag{1}$$

$$cx_1 + cx_2 + \cdots + cx_n = b_2 \tag{2}$$

$$cx_1 + cx_2 + \cdots + cx_n = b_3 \tag{3}$$

$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix} \in \mathbb{R}^n$ is a solution to the system if all the equations are satisfied when x_i is set to s_i.

If a system has a solution, it is **consistent**. If not, it is **inconsistent**.

Theorem 2.1.1. Assume the system of linear equations with $a_1, \ldots, a_n, b \in \mathbb{R}$ has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$. Then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

Definition 2.2. A **solution set** is the set of all solutions of a system of linear equations. Two systems of equations are equivalent if they have the same solution set.

2.2 Solving Systems of Linear Equation

Definition 2.3. The **coefficient matrix** of a system is denoted by $A = \begin{bmatrix} a_{11} & a_{21} & \cdots \\ a_{21} & a_{22} & \cdots \end{bmatrix}$.

Definition 2.4. The **augment matrix** is

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Method 2.1. The three **elementary row operations** for solving a system of linear equations are:

1. Multiplying a row by a scalar
2. Adding a multiple of one row to another
3. Swapping two rows

Theorem 2.2.1. If two augmented matrices are row equivalent, then the system of linear equations associated with each matrix are equivalent.

Definition 2.5. A matrix is said to be in **reduced row echelon form** (RREF) if:

1. All rows containing a non-zero entry are above rows which only contain zeroes.
2. The first non-zero entry in each row is 1. (**leading one**).
3. Leading one on each zero row is to the right of the leading one on any row above it.

4. Leading one is the only non-zero entry in its column.

Theorem 2.2.2. The RREF of a matrix is unique.

Definition 2.6. Let R be the RREF of a coefficient matrix of a system of linear equations. If the jth column does not contain a leading one, x_j is a **free variable**.

Definition 2.7. The **rank** of a matrix is the number of leading ones in the RREF of the matrix.

Theorem 2.2.3. Let A be the $m \times n$ coefficient matrix of a system of linear equations.

1. IF the rank of A is less than the rank for the augmented matrix, then the system is inconsistent.
2. If the system is inconsistent, then the system contains $n -$ rank A free variables. A consistent system has a unique solution if and only if rank $A = n$.
3. rank $A = m$ if and only if the system is consistent for every $\vec{b} \in \mathbb{R}^m$

Definition 2.8. A system of linear equations is said to be **homogeneous system** if the right-hand side only contains zeroes. It has the form $\left[A \mid \vec{0}\right]$.

Theorem 2.2.4. The solution set of a homogeneneous systems of M linear equations in n variables is a subspace of R^n.

3 Matrices and Linear Mappings

3.1 Operations on Matrices

Definition 3.1. A $m \times n$ **matrix** is a rectangular array with m rows and n columns.

Definition 3.2. Addition and scalar multiplication of matrices: Let $A, B \in M_{m\times n}(\mathbb{R})$ and $c \in \mathbb{R}$. $A + B$ and cA are defined as

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$
$$(cA)_{ij} = c(A)_{ij}$$

Theorem 3.1.1. Let A, B, C be $m \times n$ matrices and let $c, d \in \mathbb{R}$

1. $A + B$ is an $m \times n$ matrix
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix such that $A + O_{m,n} = A$. This is called the **zero matrix**
5. There exists a matrix $(-A)$ such that $A + (-A) = O_{m,n}$
6. $cA \in M_{m\times n}$

7. $c(dA) = cd(A)$

8. $(c+d)A = cA + dA$

9. $c(A+B) = cA + cB$

10. $1A = A$

Definition 3.3. The **zero matrix**, denoted as $O_{m,n}$ is the matrix whose entries are all 0.

Definition 3.4. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A.

$$(A^T)_{ij} = (A)_{ji}$$

Theorem 3.1.2. For any $m \times n$ matrices A and B and scalar $c \in \mathbb{R}$,

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = c(A^T)$

Definition 3.5. Matrix-Vector multiplication: Let A be an $m \times n$ matrix whose rows are denoted $\vec{a}_i^T$ for $1 \leq i \leq m$. Then, for any $\vec{x} \in \mathbb{R}^n$, we define

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

An alternate form is

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Theorem 3.1.3. If $\vec{e}_i$ is the ith standard basis vector for $\mathbb{R}^i$ and $A = [\vec{a}_1, \ldots, \vec{a}_n]$ is an $m \times n$ matrix, then

$$A\vec{e}_i = \vec{a}_i$$

Method 3.1. Matrix Multiplication: Let A be an $m \times n$ matrix and let $B = [\vec{b}_1 \cdots \vec{b}_p]$ be an $n \times p$ matrix. Then, AB is the $m \times p$ matrix

$$AB = [A\vec{b}_1 \cdots A\vec{b}_p]$$

Note. The number of columns of A must equal to the number of rows of B for this to be defined. The resulting matrix will have the same rows as A and same columns as B.

Theorem 3.1.4. If A, B, C are matrices of the correct size so the required products are defined and $t \in \mathbb{R}$, then

- $A(B+C) = AB_AC$
- $t(AB) = (tA)B + A(tB)$
- $A(BC) = (AB)C$
- $(AB)^T = B^TA^T$

Quote. Matrix Multiplication is NOT commutative. $AB \neq BA$., if $AB = AC$, $B \neq C$.

Theorem 3.1.5. Suppose that A and B are $m \times n$ matrices such that $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$.

Definition 3.6. The $n \times n$ **identity matrix**, denoted as I, is the matrix containing a diagonal row of 1s and everything else set to 0. The columns of I_n are the standard basis vectors of $\mathbb{R}^n$.

For every $n \times n$ matrix, A, $AI = A = IA$.

Theorem 3.1.6. If I is the matrix $I = [\vec{e}_1, \ldots, \vec{e}_n]$ then for any $n \times n$ matrix where $IA = A = AI$

Theorem 3.1.7. The multiplicative identity for $M_{n\times n}(\mathbb{R})$ is unique.

Example 3.1. **Block matrix**: Let $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$. By reducing A into blocks, we can write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $A_{11} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 3 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 1 & 2 \end{bmatrix}$.
These are useful to distribute matrix multiplication over multiple computers to speed up the process.

3.2 Linear Mapping

Theorem 3.2.1. Let A be an $m \times n$ matrix, and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(\vec{x}) = A\vec{x}$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$ we have

$$f(b\vec{x} + c\vec{y}) = bf(\vec{x}) + cf(\vec{y})$$

Definition 3.7. A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a **linear mapping** if for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$, we have

$$L(b\vec{x} + c\vec{y}) = bL(\vec{x}) + cL(\vec{y})$$

Note. This definition can be used to prove linear mapping.

Example 3.2. Prove that the function $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x_1, x_2, x_3) = (3x_1 - x_2, 2x_1 + 2x_3)$ is a linear mapping.

Solution.

$$\begin{aligned}
L(b\vec{x} + c\vec{y}) &= L(b(x_1, x_2, x_3) + c(y_1, y_2, y_3)) \\
&= L(bx_1 + cy_1, bx_2 + cy_2, bx_3 + cy_3) \\
&= (3(bx_1 + cy_1) - (bx_2 + cy_2)), 2(bx_1 + cy_1) + 2(bx_3 + cy_3) \\
&= b(3x_1 - x_2, 2x_1 + 2x_3) + c(3y_1 - y_2, 2y_1 + 2y_3) \\
&= bL(\vec{x}) + cL(\vec{y})
\end{aligned}$$

Theorem 3.2.2. Every linear mapping can be represented as a matrix mapping whose columns are the images of the standard basis vector of $\mathbb{R}^n$ under L. $L(\vec{x}) = [L]\vec{x}$ where

$$[L] = [L(\vec{e}_1) \cdots L(\vec{e}_n)]$$

Example 3.3. Determine the standard matrix of $L(x_1, x_2, x_3) = (3x_1 - x_2, 2x_1 + 2x_3)$

Solution.

$$L(1, 0, 0) = (3, 2)$$

$$L(0, 1, 0) = (-1, 0)$$

$$L(0, 0, 1) = (0, 2)$$

$$[L] = [L(\vec{e}_1) \qquad L(\vec{e}_2) \qquad L(\vec{e}_3)] = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Definition 3.8. The **rotation** in $\mathbb{R}^2$ is

$$R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$$

The standard matrix of $\mathbb{R}_\theta$ is

$$[r_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Theorem 3.2.3. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation with rotation matrix $A = [\mathbb{R}_\theta]$. Then the columns of A are orthogonal unit vectors.

Definition 3.9. Let $\text{refl}_{\vec{n}} : \mathbb{R}^n \to \mathbb{R}^n$ denote the linear mapping which maps a vector $\vec{x}$ to its mirror image in the hyperplane with normal vector $\vec{n}$. The reflection of $\vec{x}$ over the line witth the normal vector $\vec{n}$ is given by

$$\text{refl}_{\vec{n}} = \vec{x} - 2\text{proj}_{\vec{n}}\vec{x}$$

3.3 Special Subspaces

Definition 3.10. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The **kernel** is defined by

$$\ker(L) = \{\vec{x} \in \mathbb{R}^n | L(\vec{x}) = \vec{0}\}$$

The set of all vectors in $\mathbb{R}^n$ (domain) where when L is applied, becomes the zero vector.

Theorem 3.3.1. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $L(\vec{0}) = \vec{0}$.

Theorem 3.3.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $\ker(L)$ is a subspace of $\mathbb{R}^n$.

Definition 3.11. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The **range** is

$$R(L) = \{L(\vec{x}) \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$$

The set of all vectors in the codomain where $L(\vec{x})$ is defined.

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Theorem 3.3.3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then $R(L)$ is a subspace of $\mathbb{R}^m$.

3.3.1 Four Fundamental Subspaces of a Matrix

Theorem 3.3.4. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $A = [L]$ be the standard matrix of L. Then, $\vec{x} \in \ker(L)$ if and only if $A\vec{x} = \vec{0}$.

Definition 3.12. Let A be an $m \times n$ matrix. The set of all $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$ is called the **nullspace** of A. We write

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$$

Theorem 3.3.5. Let A be an $m \times n$ matrix. A consistent system of linear equations $A\vec{x} = \vec{b}$ has a unique solution if and only if $\text{Null}(A) = \{\vec{0}\}$.

Theorem 3.3.6. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mpping with standard matrix $[L] = A = [\vec{a}_1 \cdots \vec{a}_n]$. Then

$$R(L) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

Definition 3.13. Let $A = [\vec{a}_1 \cdots \vec{a}_n]$. The **columnspace** of A s the subspace of $\mathbb{R}^m$ defined by

$$\text{Col}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$$

It is the span of a set created from the columns of A.

Theorem 3.3.7. Let A be an $m \times n$ matrix. Then $\text{Col}(A) = \mathbb{R}^m$ if and only if $\text{rank}(A) = m$.

Definition 3.14. Let A be an $m \times n$ matrix. The **rowspace** of A is the subspace of $\mathbb{R}^n$ defined by

$$\text{Row}(A) = \{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}$$

It is the span of the rows of A.

Definition 3.15. Let A be an $m \times n$ matrix. The **left nullspace** of A is the subspace of $\mathbb{R}^m$ defined by

$$\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}$$

It is the nullspace of the transpose of A.

Theorem 3.3.8. Let A be an $m \times n$ matrix. If $\vec{a} \in \text{Row}(A)$ and $\vec{x} \in \text{Null}(A)$, then $\vec{a} \cdot \vec{x} = 0$.

Theorem 3.3.9. Let A be an $m \times n$ matrix. If $\vec{a} \in \text{Col}(A)$ and $\vec{x} \in \text{Null}(A^T)$, then $\vec{a} \cdot \vec{x} = 0$.

3.4 Operations on Linear Mapping

Definition 3.16. Addition & Scalar Multiplication:

$$(L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$$

$$(cL)(\vec{x}) = cL(\vec{x})$$

Note. Two linear mappings L and M are equal if and only if they have the same domain, same range, and $L(\vec{x}) = M(\vec{x})$ for all $\vec{x}$ in the domain.

Theorem 3.4.1. Let $L, M, N \in \mathbb{L}$ and let c_1, c_2 be real scalars. Then

- $L + M \in \mathbb{L}$
- $(L+M)+N = L+(M+N)$
- $L+M = M+L$
- There exists a linear mapping $O : \mathbb{R}^n \to \mathbb{R}^m$ such that $L + O = L$. This means $O(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.
- There exists $(-L)$ such that $L + (-L) = O$.
- $c_1 L \in \mathbb{L}$
- $c_1(c_2 L) = (c_1 c_2)L$
- $(c_1 + c_2)L = c_1 L + c_2 L$
- $c_1(L+M) = c_1 L + c_1 M$
- $1L = L$

Theorem 3.4.2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mapping and let $c \in \mathbb{R}$. Then

$$[L+M] = [L] + [M]$$

$$[cL] = c[L]$$

Definition 3.17. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. Then M composed of L is the function $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ defined by

$$(M \circ L)(\vec{x}) = M(L(\vec{x}))$$

Theorem 3.4.3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. then $M \circ L$ is a linear mapping and

$$[M \circ L] = [M][L]$$

4 Vector Spaces

4.1 Vector Spaces

Definition 4.1. Let $\mathbb{V}$ be a set. The elements of $\mathbb{V}$ are vectors denoted as $\vec{x}$. $\mathbb{V}$ is called a **vector space over** $\mathbb{R}$ if there is an operation of addition and scalar multiplication such that for any $\vec{x}, \vec{y}, \vec{v} \in \mathbb{V}$ and $a, b \in \mathbb{R}$,

1. $\vec{x} + \vec{y} \in \mathbb{V}$
2. $(\vec{x} + \vec{y}) + \vec{v} = \vec{x} + (\vec{y} + \vec{v})$
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
4. The zero vector exists in $\mathbb{V}$, $\vec{x} + \vec{0} = \vec{x}$
5. For each $\vec{x} \in \mathbb{V}$, there exists $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$, known as the **additive inverse**
6. $a\vec{x} \in \mathbb{V}$
7. $a(b\vec{x}) = (ab)\vec{x}$
8. $(a + b)\vec{x} = a\vec{x} + b\vec{x}$
9. $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$
10. $1\vec{x} = \vec{x}$

Example 4.1. Is the empty set a vector space?

Solution. No. It does not contain $\vec{0}$ even though the other statements are vacuously true.

Example 4.2. Let $\mathbb{V} = \{\vec{0}\}$ and define addition by $\vec{0} + \vec{0} = \vec{0}$ and scalar multiplication by $c\vec{0} = \vec{0}$. Show that $\mathbb{V}$ is a vector space.

Solution. Must show that it satisfies all ten axioms.

1. The only element in $\mathbb{V}$ is $\vec{0}$ and $\vec{0} + \vec{0} = \vec{0} \in \mathbb{V}$
2. $(\vec{0} + \vec{0}) + \vec{0} = \vec{0} + (\vec{0} + \vec{0})$
3. $\vec{0} + \vec{0} = \vec{0} = \vec{0} + \vec{0}$
4. $\vec{0} + \vec{0} = \vec{0}$ so the zero vector is in the set.
5. Additive inverse of $\vec{0}$ is $\vec{0}$.
6. $a\vec{0} = \vec{0} \in \mathbb{V}$
7. $a(b\vec{0}) = a\vec{0} = \vec{0} = (ab)\vec{0}$
8. $(a + b)\vec{0} = \vec{0} = \vec{0} + \vec{0} = a\vec{0} + b\vec{0}$

9. $a(\vec{0}+\vec{0}) = a\vec{0} = \vec{0} = \vec{0}+\vec{0} = a\vec{0}+a\vec{0}$
10. $1\,\vec{0} = \vec{0}$

Example 4.3. Let $\mathbb{S} = \{x \in \mathbb{R} | x > 0\}$. Define addition in $\mathbb{S}$ by $x \oplus y = xy$ and define sclar multiplication by $c \odot x = x^c$ for all $x, y \in \mathbb{S}$ and all $c \in \mathbb{R}$. Prove that $\mathbb{S}$ is a vector space under these operations.

Solution. Must should that $\mathbb{S}$ satisfies all ten vector space axioms. For any $x, y, z \in \mathbb{S}$ and $a, b \in \mathbb{R}$ we have

1. $x \oplus y = xy > 0$ since $x > 0$ and $y > 0$, hence $x \oplus y \in \mathbb{S}$
2. $(x \oplus y) \oplus z = (xy) \oplus z = (xy)zx(yz) = x \oplus (yz) = x \oplus (y \oplus z)$
3. $x \oplus y = xy = yx = y \oplus x$
4. The zero vector is 1 because $1 \in \mathbb{S}$ and $x \oplus 1 = x1 = x$
5. $\frac{1}{x}$ is the additive inverse of x since $\frac{1}{x} \in \mathbb{S}$ and $\frac{1}{x} \oplus x = 1$.
6. $a \odot x = x^n > 0$ since $x > 0$ so $a \odot x \in \mathbb{S}$.
7. $a \odot (b \odot x) = a \odot x^b = (x^b)^a = x^{ab} = (ab) \odot x$
8. $(a+b) \odot x = x^{a+b} = x^a x^b = x^a \oplus x^b = a \odot x \oplus b \odot x$
9. $a \odot (x \oplus y) = a \odot (xy) = (xy)^a = x^a y^a = x^a \oplus y^a = a \odot x \oplus a \odot y$
10. $1x = x^1 = x$

Therefore $\mathbb{S}$ is a vector space.

Example 4.4. $\mathbb{V} = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1, x_2 \in \mathbb{R}\}$ with standard scalar multiplication, but addition defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ y_1 + x_2 \end{bmatrix}$$

Solution. This is not a vector space because $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
This does not satisfy V3.

Example 4.5. Show that the set $\mathbb{Z}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1, x_2 \in \mathbb{Z} \right\}$ is not a vector space under standard addition and scalar multiplication of vectors.

Solution. Observe that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{Z}^2$, but $\sqrt{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix} \notin \mathbb{Z}^2$. Hence this does not satisfy V6 and is not a vector space.

Theorem 4.1.1. Let $\mathbb{V}$ be a vector space with addition defined by $\vec{x}+\vec{y}$ and scalar multiplication defined by $c\vec{x}$ for all $\vec{x},\vec{y} \in \mathbb{V}$, and $c \in \mathbb{R}$, Then

- $0\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{V}$
- $-\vec{x} = (-1)\vec{x}$ for all $\vec{x} \in \mathbb{V}$

4.1.1 Subspaces

Definition 4.2. Let $\mathbb{V}$ be a vector space. If $\mathbb{S}$ is a subset of $\mathbb{V}$ and $\mathbb{S}$ is a vector space under the same operations as $\mathbb{V}$, then $\mathbb{S}$ is called a **subspace** of $\mathbb{V}$.

Theorem 4.1.2. Let $\mathbb{S}$ be a non-empty subset of $\mathbb{V}$. If $\vec{x}+\vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x},\vec{y} \in \mathbb{S}$, and $c \in \mathbb{R}$ under the operations of $\mathbb{V}$, then $\mathbb{S}$ is a subspace of $\mathbb{V}$

Example 4.6. Is $\mathbb{W} = \{p(x) \in P_2(\mathbb{R}) | p(2) = 0\}$ a subspace of $P_2(\mathbb{R})$?

Solution. In $P_2(\mathbb{R})$ the zero vector is the polynomial that satisfies $z(x) = 0$ for all x. Hence $z(x) \in \mathbb{W}$ since $z(2) = 0$. Therefore $\mathbb{W}$ is non-empty.

Let $p(x), q(x) \in \mathbb{W}$. Then $p(2) = 0, q(2) = 0, (p+q)(2) = p(2)+q(2) = 0+0 = 0$. Hence $(p+q) \in \mathbb{W}$ and $\mathbb{W}$ is closed under addition.

Similarly, $(cp)(2) = cp(2) = c0 = 0$ for all $c \in \mathbb{R}$ so $(cp) \in \mathbb{W}$. Thus, it is also closed unders calar multiplication. Therefore $\mathbb{W}$ is a subspace of $P_2(\mathbb{R})$ by the Subspace Test.

Example 4.7. IS $T = \{a + bx + cx^2 \in P_3(\mathbb{R}) | a^2 - b^2 = 0\}$ a subspace of $P_2(\mathbb{R})$?

Solution. Observe $-1 + 4x \notin T$. Therefore this is not a subspace.

4.1.2 Spanning

Definition 4.3. Let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. Then we define the **span** of B by

$$\operatorname{span}(B) = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k | c_1, \ldots, c_k \in \mathbb{R}\}$$

$\operatorname{span} B$ is **spanned** by B and B is a **spanning set** for $\operatorname{span} B$.

Theorem 4.1.3. If If $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a span of vectors in a vector space $\mathbb{V}$, then $\operatorname{span} B$ is a subspace of $\mathbb{V}$.

Theorem 4.1.4. Let $\mathbb{V}$ be a vector space and $\vec{v}_1, \ldots, vv_k \in \mathbb{V}$. Then $v_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$.

Example 4.8. Determine if $p(x) = 3 - 4x + 2x^2$ is in $\operatorname{span}\{1 + 2x, 1 - x + 3x^2, 2 - x + x^2\}$ in $P_2(\mathbb{R})$.

Solution. Must determine if there exists coefficients such that

$$\begin{aligned} 3 - 2x + 2x^2 &= c_1(1 + 2x) + c_2(1 - x + 3x^2) + c_3(2 - x + x^2) \\ &= (c_1 + c_2 + 3c_3) + (2c_1 - c_2 - c_3)x + (3c_2 + c_3)x^2 \end{aligned}$$

Collect like coefficients

$$\begin{aligned} c_1 + c_2 + 3c_3 &= 3 \\ 2c_1 - c_2 - c_3 &= 4 \\ 3c_2 + c_3 &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 3 & -1 & -1 & 4 \\ & 3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore $p(x) \in \operatorname{span}\{1 + 2x, 1 - x + 2x^2, 2 - x + x^2\}$

Definition 4.4. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ is **linearly dependent** if there exists at least one non-zero coefficient that satisfies

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

The set is **linearly independent** if the only solution is the trivial solution.

Theorem 4.1.5. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ is linearly dependent if and only if there exists $1 \leq i \leq k$ such that

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$$

Theorem 4.1.6. A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ which contains the zero vector is linearly dependent.

Example 4.9. Determine if the set $\{1 + x + 2x^2, x - x^2, -2x^2\}$ is linearly independent.

Solution. A set is linear independnet if and only if the only solution to

$$0 = c_1(1 + x + 2x^2) + c(x - x^2) + c_3(-2 + x^2)$$

is $c_1 = c_2 = c_3 = 0$. Rearranging,

$$(c_1 - 2c_3) + (c_1 + c_2)x + (2c_1 - c_2 + c_3)x^2 = 0$$

Solve the homogeneous system,

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This system has a unique solution, thus the set is linearly independent.

Example 4.10. Determine if $\{1 + 2x + x^2, 3 + 3x + 2x^2, 5 + x + 3x^2\}$ is linearly independent in $P_2(\mathbb{R})$.

Solution.

$$
\begin{aligned}
0 &= c_1(1+2x+x^2) + c_2(3+3x+2x^2) + c_3(5+x+3x^2) \\
&= (c_1+3c_2+5c_3) + (2c_1+3c_2+c_3)x + (c_1+2c_2+2c_3)x^2
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}
$$

Since there are infinitely many solutions, the system is linearly dependent.

4.2 Bases and Dimension

Definition 4.5. Let $\mathbb{V}$ be a vector space. The set B is called a basis for $\mathbb{V}$ if B is linearly independent spanning set for $\mathbb{V}$.

Example 4.11. Find the standard basis for $P_n(\mathbb{R})$.

Solution. Every vector in $P_n(\mathbb{R})$ has teh form

$$P(x) = a_0 + a_1x + \cdots + a_nx^n$$

Thus the set $\{1, x, \ldots, x^n\}$ spans $P_n(\mathbb{R})$. In addition

$$0 + 0x + \cdots + 0x^n = a_0 + a_1x + \cdots + a_nx^n$$

By equating like powers of x, the only solution is the trivial solution. Therefore $\{1, x, \ldots, x^n\}$ is a linearly independent spanning set for $P_n(\mathbb{R})$ and is its standard basis.

Example 4.12. Prove that $B = \{1, (1-x), (1-x)^2\}$ is a basis for $P_2(\mathbb{R})$.

Solution. Let $p(x) = a + bx + cx^2$.

$$
\begin{aligned}
a + bx + cx^2 &= c_1 + c_2(1-x) + c_3(1-x)^2 \\
&= (c_0 + c_1 + c_2) + (-c_1 - 2c_2)x + c_2x^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_0 + c_1 + c_2 &= a \\
-c_1 - 2c_2 &= b \\
c_2 &= c
\end{aligned}
$$

If the system is row reduced, we . Thus B is a linearly independent spanning set for $P_2(\mathbb{R})$.

Example 4.13. Find a basis for the space of $M_{2\times 2}(\mathbb{R})$ defined by

$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a - b = 2c \right\}$$

Solution. Every vector has the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & a-2c \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$$

Thus $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \right\}$ spans S, and is clearly linearly independent. Therefore it is a basis for S.

Theorem 4.2.1. Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for a vector space $\mathbb{V}$ and let $C = \{\vec{w}_1, \dots, \vec{w}_k\}$ be a set in $\mathbb{V}$. If $k > n$ (rank) then C is linearly dependent.

Proof. Consider $0 = c_1\vec{w}_1 + \cdots + c_k\vec{w}_k$.

Since B is a basis for $\mathbb{V}$, we can write every vector $\vec{w}$ as a linear combination of the vectors in B.

$$w_i = a_{i1}\vec{v}_1 + \cdots + a_{in}\vec{v}_n, \text{ for } 1 \le i \le k$$

Substituting,

$$0 = (c_1a_{11} + \cdots + c_ka_{k1})\vec{v}_1 + \cdots + (c_1a_{1n} + \cdots + c_ka_{kn})\vec{v}_n$$

Since B is a basis, it is linearly independent, and the only solution is when

$$c_1a_{11} + \cdots + c_ka_{k1} = 0$$

$$\vdots$$

$$c_1a_{1n} + \cdots + c_ka_{kn} = 0$$

Since $k > n$, the system has infinitely many solutions by the system rank theorem, so the equation has infinitely many solutions, and hence C is linearly independent. □

Theorem 4.2.2. If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $C = \{\vec{w}_1, \dots, \vec{w}_k\}$ are bases for the vector space $\mathbb{V}$, then $k = n$.

Proof. Since B is a basis and C is linearly independent, $k \le n$ by the previous theorem. Similary, since C is a basis and B is linearly independent, $n \neq k$. Hence $n = k$. □

Definition 4.6. Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for the vector space $\mathbb{V}$. The **dimension** of $\mathbb{V}$ is n (number of elements in the basis) and we write

$$\dim(V) = n$$

If $V = \{\vec{0}\}$ then $\dim(V) = 0$. If $\mathbb{V}$ does not have a basis with a finite number of vectors in it, then $\mathbb{V}$ is **infinite dimensional**.

Example 4.14. Some common dimensions

- The dimension of $\mathbb{R}^n$ is n.
- The dimension of $P_n(\mathbb{R})$ is $n+1$.

- The dimension if $M_{m\times n}(\mathbb{R})$ is mn
- The vector space $P(\mathbb{R})$ of all polynomials with eral coefficients is infinite dimensional since the basis is $\{1, x, x^2, \dots\}$.

Example 4.15. The basis of $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \right\}$ is 2 because there are two elements in the set.

Theorem 4.2.3. Let $\mathbb{V}$ be an n-dimensional vector space. Then

1. A set of more than n vectors in $\mathbb{V}$ must be inearly dependent.
2. A set of fewer than n vectors in $\mathbb{V}$ cannot span $\mathbb{V}$.
3. A set of n vectors in $\mathbb{V}$ is linearly independent if and only if it spans $\mathbb{V}$.

Theorem 4.2.4. If $\mathbb{V}$ is an n-dimensional vector space and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{V}$ with $k < n$, then there exists vectors $\vec{w}_{k+1}, \dots, \vec{w}_n$ in $\mathbb{V}$ such that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ is a basis for $\mathbb{V}$.

Proof. By Theorem 4.2.3, $\{\vec{v}_1, \dots, \vec{v}_k\}$ does not span $\mathbb{V}$.Let $\vec{w}_{k+1}$ be a vector in $\mathbb{V}$ such that $\vec{w}_{k+1} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\}$. If $k + 1 = n$, then by Theorem 4.2.3, $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}\}$ is a basis. Else, repeat the procedure until it is true, and the set will be

$$\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$$

□

Example 4.16. Find a basis for the hyperplane with the equation $2x_1 + x_2 - x_3 - x_4 = 0$ and extend the basis to be a basis for $\mathbb{R}^4$.

Solution. Pick three vectors that are linearly independent and satisfy the hyperplane.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Clearly the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, so it is a basis by Theorem 4.2.3 since the dimension of a hyperplanein $\mathbb{R}^4$ is 3. To extend the basis to $\mathbb{R}^4$, we pick

$$\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

(Observe that this vector is not spanned by the hyperplane). By Theorem 4.2.4, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{n}\}$ is a basis for $\mathbb{R}^4$.

Corollary 4.1. If S is a subspace of a finite dimensional vector space $\mathbb{V}$, then $\dim(S) \leq \dim(V)$.

4.3 Coordinates

Theorem 4.3.1. If $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector space $\mathbb{V}$, then every vector $\vec{v} \in \mathbb{V}$ can be represented as a **unique** linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

Proof. Since B is a basis, it a spanning set. Ths for every vector $\vec{v} \in \mathbb{V}$ there exists constants such that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{v}$$

Assume that there also exists constants such that $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = \vec{v}$. Then

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = \vec{v}$$

$$(c_1 - d_1)\vec{v}_1 + \cdots + (c_n - d_n)\vec{v}_n = \vec{0}$$

But this implies $c_i = d_i$ for all $1 \leq i \leq n$ since B is linearly independent. Thus there exists only one linear combination of the vectors in B that equals $\vec{v}$. $\square$

Definition 4.7. Let $\mathbb{V}$ be a vector space with basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$. For any $\vec{v} \in \mathbb{V}$, the **coordinate vector** of $\vec{v}$ with respect to B is

$$[\vec{v}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

where $\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$.

Example 4.17. Given that $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right\}$ is a basis for a subspace S of $M_{2\times 2}(\mathbb{R})$ and $[A]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, what is A?

Solution. We have $A = 2\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 6 & 6 \end{bmatrix}$

Example 4.18. Consider the basis $B = \{1, (x-1), (x-1)^2\}$ for $P_2(\mathbb{R})$. Find the B coordinate vectors of $p(x) = 3 - 5x + 4x^2$ and $q(x) = x$.

Solution. Must find constants such that

$$\begin{aligned} 3 - 5x + 4x^2 &= c_1 + c_2(x-1) + c_3(x-1)^2 \\ &= (c_1 - c_2 + c_3) + (c_2 - 2c_3)x + c_3x^2 \end{aligned}$$

Similarly, we need to find

$$\begin{aligned} x &= d_1 + d_2(x-1) + d_3(x-1)^2 \\ &= (d_1 - d_2 + d_3) + (d_2 - 2d_3)x + d_3x^2 \end{aligned}$$

The coefficients for both these augmented matrices are the same, so a double augmented matrix can be created.

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 1 & 3 & 0 \\ 0 & 1 & -2 & -5 & 1 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$$

Therefore $[3-5x+4x^2]_B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $[x]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Theorem 4.3.2. If $\mathbb{V}$ is a vector space with $B = \{\vec{v}_1, \dots, \vec{v}_n\}$, then for any $\vec{v}, \vec{w} \in \mathbb{V}$, and $s, t \in \mathbb{R}$, we have

$$[s\vec{v} + t\vec{w}]_B = s[\vec{v}]_b + t[\vec{w}]_B$$

Proof. Let $\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$ and $w = c_1\vec{v}_1 + \cdots c_n\vec{v}_n$. Then we have

$$s\vec{v} + t\vec{w} = (sb_1 + tc_1)\vec{v}_1 + \cdots + (sb_n + tc_n)\vec{v}_n$$

Therefore,

$$[s\vec{v} + t\vec{w}]_B = \begin{bmatrix} sb_1 + tc_1 \\ \vdots \\ sb_n + tc_n \end{bmatrix} = s\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + t\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = s[\vec{v}]_B + t[\vec{w}]_B$$

□

4.3.1 Change of Coordinates

Example 4.19. Let B be any basis for $\mathbb{R}^3$ and let $\vec{x} \in \mathbb{R}^3$.

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

If we find the coordinates of the standard basis vectors with respect to the basis B, then calculating $[x]_B$ will be easy

$$\begin{aligned} \left[\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right]_B &= [x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3]_B \\ &= x_1[\vec{e}_1]_B + x_2[\vec{e}_2]_B + x_2[\vec{e}_2]_B \\ &= [[\vec{e}_1]_B + [\vec{e}_2]_B + [\vec{e}_3]_B] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

We call ${}_BP_S = [\vec{e}_1]_B + [\vec{e}_2]_B + [\vec{e}_3]_B$ the change of coordinates matrix from the standard basis S to the basis B.

Example 4.20. Let $B = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

Find $[\vec{x}]_B$ for any $\vec{x} \in \mathbb{R}^3$.

Solution.

$$\vec{e}_1 = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$$
$$\vec{e}_2 = d_1\vec{b}_1 + d_2\vec{b}_2 + d_3\vec{b}_3$$
$$\vec{e}_3 = f_1\vec{b}_1 + f_2\vec{b}_2 + f_3\vec{b}_3$$

Use a triple augmented matrix to find the coefficients

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & -\frac{1}{5} & -1 \\ 0 & 1 & 0 & \frac{7}{5} & -\frac{4}{5} & -1 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{3}{5} & 1 \end{array}\right]$$

$$[\vec{x}]_B =_B P_S = [[\vec{e}_1]_B + [\vec{e}_2]_B + [\vec{e}_3]_B] = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & 1 \\ \frac{7}{5} & -\frac{4}{5} & -1 \\ -\frac{4}{5} & \frac{3}{5} & 1 \end{bmatrix}$$

$$[\vec{x}]_B =_B P_S\vec{x} = \begin{bmatrix} \frac{3}{5}x_1 - \frac{1}{5}x_2 + x_3 \\ \frac{7}{5}x_1 - \frac{4}{5}x_2 - x_3 \\ -\frac{4}{5}x_1 + \frac{3}{5}x_2 + x_3 \end{bmatrix}$$

Definition 4.8. Let $B = \{\vec{v}+1, \dots, \vec{v}_n\}$ and C both be basis for a vector space $\mathbb{V}$. The **change of coordinate matrix** from B-coordinate to C-coordinate is defined by

$$_CP_B = [[\vec{v}_1]_C \cdots [\vec{v}_n]_C]$$

and $\forall \vec{x} \in \mathbb{V}$, we have

$$[\vec{x}]_C =_C P_B[\vec{x}]_B$$

Example 4.21. Let $B = \{1+3x, 2+x\}$ and $C = \{-1+x, 5-4x\}$ both be basis of $P_1(\mathbb{R})$. Find $_CP_B$ and $_BP_C$.

Solution. To find $_CP_B$, must find the C-coordinate of the vectors in B,

$$1 + 3x = c_1(-1+x) + c_2(5-4x)$$

$$2 + x = d_1(-1+x) + d_2(5-4x)$$

Create a double augmented matrix and row reduce

$$\left[\begin{array}{cc|cc} -1 & 5 & 1 & 2 \\ 1 & -4 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 19 & 13 \\ 0 & 1 & 4 & 3 \end{array}\right]$$

Therefore $_CP_B = \begin{bmatrix} 19 & 13 \\ 4 & 3 \end{bmatrix}$. To find $_BP_C$,

$$-1 + x = c_1(1+3x) + c_2(2+x)$$

$$5 - 4x = d_1(1+3x) + d_2(2+x)$$

Creating a double augmented matrix and row reducing gives

$$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 5 \\ 3 & 1 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{19}{15} \end{array}\right]$$

Therefore $_BP_C = \begin{bmatrix} \frac{3}{5} & -\frac{13}{5} \\ -\frac{4}{5} & \frac{19}{5} \end{bmatrix}$.

Theorem 4.3.3. If B and C are bases for an n-dimensional vector space $\mathbb{V}$, then the change of coordinate matrices ${}_CP_B$ and ${}_BP_C$ satisfy

$${}_CP_B\,{}_BP_C = I = {}_BP_C\,{}_CP_B$$

5 Inverses and Determinants

5.1 Matrix Inverses

Definition 5.1. Let A be an $m \times n$ matrix. If B is an $n \times m$ matrix such that $AB = I_m$, then B is called the **right inverse** of A. IF C is an $n \times m$ matrix such that $CA = I_n$, then C is called the **left inverse** of A.

Example 5.1.

$$AB = \begin{bmatrix} -2 & -3 & 4 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The left inverse of B is A. The right inverse of A is B.

Theorem 5.1.1. If A is an $m \times n$ matrix with $n > m$, then A cannot have a right inverse.

Proof.

$$I = AB = [A\vec{b}_1 \cdots A\vec{b}_m] = [\vec{e}_1 \cdots \vec{e}_m]$$

Need to find $\vec{b}_i$ such that $A\vec{b}_i = \vec{e}_i$. But this is just solving m systems of linear equations with the same coefficient matrix. If we row reduce $[A|I_m]$ to RREF, we can find a solution to each equation. For this to be reduced to the identity matrix, we require rank$(A) = M$. Therefore we require that $n \geq m$. □

Theorem 5.1.2. If A is an $m \times n$ matrix with $n > m$, then A cannot have a left inverse.

Proof. If A has a left inverse C, then C is an $n \times n$ matrix with $n > m$ with a right inverse, contradicting the previous theorem. □

Definition 5.2. An $n \times n$ matrix is called a square matrix.

Definition 5.3. Let A be an $n \times n$ matrix. If B is a matrix such that $AB = I = BA$ then B is called the **inverse** of A. We write $B = A^{-1}$ and A is **invertible**.

Note. If $B = A^{-1}$ then $A = B^{-1}$.

Theorem 5.1.3. The inverse of a matrix is unique.

Proof. Assume B and C are both inverses of A, then

$$B = BI = B(AC) = (BA)C = C$$

Therefore $B = C$, and the inverse is unique. □

Theorem 5.1.4. If A and B are $n \times n$ matrices such that $AB = I$, then A and B are invertible and $\text{rank}(A) = \text{rank}(B) = n$.

Proof. Assume that $AB = I$ and consider the homogeneous system

$$B(\vec{x}) = \vec{0}$$
$$A(B\vec{x}) = A\vec{0}$$
$$(AB)\vec{x} = \vec{0}$$
$$I\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0}$$

So the system has a unique solution and by the system rank theorem, the coefficient matrix B has rank(n). This implies that $B\vec{x} = \vec{y}$ is consistent for all $y \in \mathbb{R}^n$.

$$BA\vec{y} = BA(B\vec{x}) = B(AB)\vec{x} = BI(vx) = B\vec{x} = \vec{y} = I\vec{y}$$

By the matrices equal theorem, $BA = I$, and we can repeat the previous procedure to obtain $\text{rank}(A) = n$, $\square$

Example 5.2. Determine if $n = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ is invertible.

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{8} & \frac{9}{16} & -\frac{1}{16} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{3}{16} & \frac{5}{16} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{9}{16} & -\frac{1}{16} \\ -\frac{3}{5} & \frac{3}{16} & \frac{5}{16} \\ \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Theorem 5.1.5. IF A and B are invertible matrices, and $c \in \mathbb{R}$ with $c \neq 0$, then

- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 5.1.6. If A is an $n \times n$ matrix such that $\text{rank}(A) = n$, then A is invertible.

Proof. If $\text{rank}(A) = n$, then the system of equations $A\vec{b}_i = \vec{e}_i$, $1 \leq i \leq n$ are all consistent by Theorem 2.2.3. Let $B = [\vec{b}_1 \cdots \vec{b}_n]$, then we get

$$AB = A[\vec{b}_1 \cdots \vec{b}_n] = [A\vec{b}_1 \cdots A\vec{b}_n] = I$$

This by Theorem 5.1.4, A is invertible. $\square$

Method 5.1. Assume that A is invertible, to find A^{-1}, row reduce the multiple augmented matrix to the identity matrix.

$$[A \mid I] \sim [I \mid A^{-1}]$$

Example 5.3. Find the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution. Consider $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$.

Require that rank$(A) = 2$ for A to be invertible, Therefore both a and c must be non-zero in the RREF form.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \tfrac{1}{c}R_2 \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 1 & \frac{d}{c} & 0 & \frac{1}{c} \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{cc|cc} 1 & \frac{d}{c} & 1 & 0 \\ a & b & 0 & \frac{1}{c} \end{array}\right] R_2 - aR_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & \frac{bc-ad}{c} & 1 & -\frac{a}{c} \end{array}\right] (-c)R_2 \sim \left[\begin{array}{cc|cc} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & bc-ad & -c & a \end{array}\right]$$

Since rank$(A) = 2$, it is required that $bc - ad \neq 0$. Continuing to row reduce we get

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

This we get that A is invertible if and only if $ad - bc \neq 0$ and if A is invertible, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 5.1.7. Invertible Matrix Theorem: For an $n \times n$ matrix A, the following are equivalent

1. A is invertible
2. THE RREF of A is I
3. rank$(A) = n$
4. The system of equations $A\vec{x} = \vec{b}$ is consistent with a unique solution for all $\vec{b} \in \mathbb{R}^n$
5. The nullspace of A is $\vec{0}$
6. The columns of A form a basis for $\mathbb{R}^n$
7. The rows of A for ma basis for $\mathbb{R}^n$
8. A^T is invertible

To solve a system $A\vec{x} = \vec{b}$, we can simply rearrange to get $\vec{x} = A^{-1}\vec{b}$.

Example 5.4. Solve the system of linear equations:

$$2x_1 + 4x_2 = 3$$

$$-x_1 - 5x_2 = 5$$

The system has coefficient matrix $\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Since $ad - bc \neq 0$ we know that A is invertible

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -5 & -4 \\ 1 & 2 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = A^{-1} = \frac{1}{6}\begin{bmatrix} -5 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{35}{6} \\ -\frac{13}{6} \end{bmatrix}$$

5.2 Elementary Matrices

Note. Observe the correlation between matrix multiplication and row operation

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} kR_1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - sc & b - sd \\ c & d \end{bmatrix} R_1 - sR_2$$

Definition 5.4. An $n \times n$ matrix E is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix by performing exactly one elementary row operation.

Theorem 5.2.1. If E is an elementary matrix, then E is invertible. Moreover E^{-1} is the elementary matrix corresponding to the reverse elementary row operations of E.

Theorem 5.2.2. If A is an $m \times m$ matrix, and E is an $m \times m$ matrix corresponding to the row operation of $R_i + cR_j$, for $i \neq j$, then EA is the matrix obtained from A by performing the row operation $R_i + cR_j$ on A.

Theorem 5.2.3. Let A be an $m \times n$ matrix and let E be the $m \times m$ elementary matrix corresponding to the row operation of cR_i. Then EA is the matrix obtained from A by performing the row operation cR_i on A.

Theorem 5.2.4. Let A be an $m \times n$ matrix and let E be the $m \times m$ elementary matrix corresponding to the row operation $R_i \leftrightarrow R_j$, for $i \neq j$. Then EA is teh matrix obtained from A by performing the row operation $R_i \leftrightarrow R_j$ on A.

Theorem 5.2.5. Let A be an $m \times n$ matrix, and let E be an $m \times m$ elementary matrix. Then

$$\text{rank}(EA) = \text{rank}(A)$$

Note. To find something like $E_3E_2E_1A$ without multiplying, where A, E are all $m \times m$ matrices, simply determine the row operation to obtain the elementary matrices, and apply them to the matrix A.

Theorem 5.2.6. If A is an $m \times n$ matrix in its RREF form, then there exists a sequence $E_k, \ldots, E_2, E_1$, of $m \times m$ matrices such that $E_k \cdots E_2E_1A = R$. In particular

$$A = E_1^{-1} \cdots E_k^{-1}R$$

Proof. We know that A can be reduced to its RREF, R with a sequence of elementary row operations. Let E_1 be the first operation, E_2 be teh second row operation, and so on. By the previous few theorems,

$$E_k \cdots E_2E_1A = R$$

□

Quote. Matrix multiplication begins from the right. that is $E_2E_1A = E_2(E_1A)$.

Theorem 5.2.7. If A is an invertible matrix, then A and A^{-1} can be written as a product of elementary matrices.

5.3 Determinants

Definition 5.5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of A is defined as

$$\det(A) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 5.5. Calculate $\det \begin{bmatrix} 1 & 7 \\ -3 & 2 \end{bmatrix}$.

Solution.

$$\det \begin{vmatrix} 1 & 7 \\ -3 & 2 \end{vmatrix} = (1)(2) - (7)(-3) = 23$$

Definition 5.6. Let A be an $n \times n$ matrix with $n \geq 2$. Let $A(i, j)$ be the $n - 1 \times n - 1$ matrix obtained from A by deleting the i-th row and the j-th column. The **cofactor** of a_{ij} is

$$C_{ij} = (-1)^{i+j} \det A(i, j)$$

Definition 5.7. Let A be an $n \times n$ matrix with $n \geq 2$. Then, the **determinant** if A is defined as

$$\det A = \sum_{i=1}^{n} a_{i1}C_{i1}$$

where the determinant of a 1×1 matrix is defined to be $\det[c] = c$

Note. The determinant of a matrix is denoted with square lines instead of square brackets.

Theorem 5.3.1. Let A be an $n \times n$ matrix.

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik}$$

$$\det A = \sum_{k=1}^{n} a_{ij} C_{kj}$$

The cofactor expansion can be called upon any row or column to obtain the determinant.

Note. Since determinants can be calculated with a cofactor expansion along any row or column, it is best to use the row/col with the most zeros.

Definition 5.8. An $m \times n$ matrix is said to be **upper triangular** if $M_{ij} = 0$ whenever $i > j$. It is **lower triangular** if $M_{ij} = 0$ whenever $i < j$.

Theorem 5.3.2. If a matrix is upper or lower triangular, the determinant is just the product of the numbers along the diagonal

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

Proof. If A is a 2×2 upper triangular matrix, then det $A = a_{11}a_{22}$ as required. Assume that if B is an $(n-1) \times (n-1)$ upper triangular matrix, then det $B = b_{11} \cdots b_{(n-1)(n-1)}$. Let A be an $n \times n$ upper triangular matrix. Expanding the determinant along the first column gives

$$\det A = a_{11}(-1)^{1+1} C_{11} + 0 + \cdots + 0 = a_{11} \det A(1,1)$$

But $A(1,1)$ is the $(n-1) \times (n-1)$ upper triangular matrix formed by deeting the first row and first column of A. Thus, det $A(1,1) = a_{11}a_{22} \cdots a_{(n-1)(n-1)}$ by the inductive hypothesis. Thus det $A = a_{11}a_{22} \cdots a_{nn}$ as required. $\square$

Theorem 5.3.3. If B is the matrix obtained by multiplying one row of A by $c \in \mathbb{R}$, then det $B = c$ det A.

Theorem 5.3.4. If B is the matrix obtained from A by swapping two rows of A. Then det $B =$ $-$det A.

Theorem 5.3.5. If a matrix A has two identical rows, then det $A = 0$.

Theorem 5.3.6. If B is the matrix obtained from A by adding a multiple of one row of A to another, then det $B =$det A.

Theorem 5.3.7. Let A be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then det $EA =$det E det A.

Theorem 5.3.8. An $n \times n$ matrix is invertible if and only if its determinant is not 0.

Theorem 5.3.9. If A and B are $n \times n$ matrices, then det $(AB) =$det A det B.

Theorem 5.3.10. If A is an invertible matrix, then det $A^{-1} = \frac{1}{\det A}$.

Theorem 5.3.11. If A is an $n \times n$ matrix, then det $A =$ det A^T.

6 Diagonalization

Alot of stuff here

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Example 6.1. Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Example 6.2. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not an eigenvector of A because

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \\ 16 \end{bmatrix}$$

and this is not a scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example 6.3. Is $\lambda = 2$ an eigenvalue of A? Is there a non-zero vector $\vec{v}$ such that $A\vec{v} = 2\vec{v}$?

Solution.

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$3v_1 + 6v_2 + 7v_3 = 2v_1$$

$$3v_1 + 3v_2 + 7v_3 = 2v_2$$

$$5v_1 + 6v_2 + 5v_3 = 2v_3$$

so

$$1v_1 + 6v_2 + 7v_3 = 0$$

$$3v_1 + 1v_2 + 7v_3 = 0$$

$$5v_1 + 6v_2 + 3v_3 = 0$$

Put this into a matrix and row reduce.

$$\begin{bmatrix} 1 & 6 & 7 \\ 3 & 1 & 7 \\ 5 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The only solution is $\vec{v} = 0$ therefore, there is no non-zero vector that satisfies $A\vec{v} = 2\vec{v}$ so $\lambda = 2$ is not an eigenvalue.

Example 6.4. Is $\lambda = 15$ an eigenvalue of A? $A\vec{v} = 15\vec{v}$ exists?

Solution.

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 15 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$3v_1 + 6v_2 + 7v_3 = 15v_1$$

$$3v_1 + 3v_2 + 7v_3 = 15v_2$$

$$5v_1 + 6v_2 + 5v_3 = 15v_3$$

so

$$-12v_1 + 6v_2 + 7v_3 = 0$$

$$3v_1 - 12v_2 + 7v_3 = 0$$

$$5v_1 + 6v_2 - 10v_3 = 0$$

$$\begin{bmatrix} -12 & 6 & 7 \\ 3 & -12 & 7 \\ 5 & 6 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the solution set is $\vec{v} = t \begin{bmatrix} 1 \\ \frac{5}{6} \\ 1 \end{bmatrix}$, $t \neq 0$, and $\lambda = 15$ is an eigenvalue.

Let A be an $n \times n$ matrix. Determine an easy way of determining if a scalar λ is an eigenvalue of A. If λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$, then

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

λ is an eigenvalue of A if there exists a non-trivial solution to this equation. This is only possible if $A - \lambda I$ is not invertible. Need to find det$(A - \lambda I) = 0$.

Example 6.5. Find all eigenvalues of $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$.

Solution.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix}$$

Use the formula $ad - bc$.

$$= \lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6)$$

Therefore the eigenvalues are 3 and 6.

Example 6.6. We can find the eigenvectors corresponding to the eigenvalues by solving the homogeneous system $(A-\lambda I)\vec{v}=\vec{0}$ for each eigenvalue.

For $\lambda_1 = 3$, we have $A-3I = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$. Thus all eigenvectors corresponding to λ_1 are $\vec{v}=t\begin{bmatrix}-2\\ 1\end{bmatrix}, t\neq 0$.

For $\lambda_2 = 6$, we have $A-6I = \begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$. So all eigenectors corresponding to λ_2 are $\vec{v}=t\begin{bmatrix}1\\ 1\end{bmatrix}, t\neq 0$.

Example 6.7. Find all eigenvalues of

$$a=\begin{bmatrix}-4 & 2 & -6\\ 6 & 7 & 3\\ 12 & -3 & 14\end{bmatrix}$$

Solution.

$$0=\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & 2 & -6\\ 6 & 7-\lambda & 3\\ 12 & -2 & 14-\lambda\end{vmatrix}=\begin{vmatrix}-4-\lambda & 2 & -6\\ 6 & 7-\lambda & 3\\ 8-\lambda & 0 & 8-\lambda\end{vmatrix}$$

$$=\begin{vmatrix}2-\lambda & 2 & -6\\ 3 & 7-\lambda & 3\\ 0 & 0 & 8-\lambda\end{vmatrix}=(8-\lambda)(-1)^{3+3}(\lambda^2-9\lambda+8)=-(\lambda-8)(\lambda-8)(\lambda-1)$$

Therefore the eigenvalues are $\lambda_1=8, \lambda_2=8, \lambda_3=1$

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6.1 temp

Definition 6.1. Let A be an $n\times n$ matrix. The **characteristic polynomial** of A is the nth degree polynomial

$$C(\lambda)=\det(A-\lambda)$$

Theorem 6.1.1. A scalar λ is an eigenvalue of an $n\times n$ matrix A if and only if $C(\lambda)=0$

Definition 6.2. Let A be an $n\times n$ matrix with eigenvalue λ. The nullspace of $A-\lambda I$ is the **eigenspace** of λ. The eigenspace is denoted as E_λ

Definition 6.3. Let A be an $n\times n$ matrix with eigenvalue λ_1. The **algebraic multiplicity** of λ_2, denoted a_{λ_1}, is the number of times that the λ_1 is a root of the characteristic polynomial $C(\lambda)$. If $C(\lambda)=(\lambda-\lambda_1)^k C_1(\lambda)$, where $C_1(\lambda_1)\neq 0$. Then $a_{\lambda_k}=k$

Definition 6.4. The **geometric multiplicity** of λ, denoted g_{λ_1} is the dimension of its eigenspace. So $g_{\lambda_1} = \dim(E_{\lambda_1})$

Example 6.8. Find teh algebraic and geometric multiplicity of all eigenvalues of $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & 03 \end{bmatrix}$

Solution. We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ -2 & 5-\lambda & -2 \\ -6 & 6 & -3\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 0 & -2 \\ -2 & 3-\lambda & -2 \\ -6 & 3-\lambda & -3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & -2 \\ -2 & 3-\lambda & -2 \\ -4 & 0 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2 - 9) = -(\lambda - 3)^2(\lambda + 3)$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$.

$$a_{\lambda_1} = 2 \qquad a_{\lambda_2} = 1$$

For $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} -2 & 2 & -2 \\ -2 & 2 & -2 \\ -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$g_{\lambda_2} = 2$$

For $\lambda_2 = -3$

$$A + 3I = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & -2 \\ -6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for the eigenspace E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

$$g_{\lambda_2} = 1$$

Example 6.9. Find the algebraic and geometric multiplicity of all eigenvalues of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution. The determinant is just $(1-\lambda)^3$ because it is a triangular matrix (multiply along diagonal). Thus the only eigenvalue is $\lambda_1 = 1$, with $a_{\lambda_1} = 3$.

For $\lambda_1 = 1$

$$A - 1I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Therefore $g_{\lambda_1} = 1$

Theorem 6.1.2. If A is an $n \times n$ upper or lower trianglar matrix, then the eigenvalues of A are the diagonal entries of A.

Theorem 6.1.3. If A and B are similar matrices, then A and B have the same characteristic polynomial, and hence the same eigenvalues.

Theorem 6.1.4. IF A is an $n \times n$ matrix, with eigenvalue λ_1, then

$$1 \leq g_{\lambda_1} \leq a_{\lambda_1}$$

6.2 Diagonalization

Definition 6.5. An $n \times n$ matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix D. If $P^{-1}AP = D$, then we say that P **diagonalizes** A.

Theorem 6.2.1. A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there exists a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of A.

Proof. Assume that A is diagonalizable. By definition, there exists an invertible matrix $P = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ such that

$$P^{-1}AP = D$$

where $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$ is diagonal. Since D is diagonal, the eigenvalues of D are $\lambda_1, \dots, \lambda_n$. Therefore, by Lemma 6.2.3, $\lambda_1, \dots, \lambda_n$ are also the eigenvalues of A.

Thus

$$AP = PD$$

$$A[\vec{v}_1 \cdots \vec{v}_n] = P[\lambda_1 \vec{e}_1 \cdots \lambda_n \vec{e}_n]$$

$$[A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1 P\vec{e}_1 \cdots \lambda_n P\vec{e}_n]$$

$$[A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1 \vec{v}_1 \cdots \lambda_n \vec{v}_n]$$

Thus $A\vec{v}_1 = \lambda_1 \vec{v}_1$ for $1 \leq i \leq n$. Moreover, since P is invertible, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem, and this implies that $\vec{v}_i \neq \vec{0}$ for $1 \leq i \leq n$.

On the other hand, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ of eigenvectors, then the matrix $P = [\vec{v}_1 \cdots \vec{v}_n]$ is invertible by the INvertible Matrix Theorem and we have

$$P^{-1}AP = P^{-1}[A\vec{v}_1 \cdots \vec{v}_n]$$

$$= [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n]$$

$$= [\vec{e}_1 \cdots \vec{e}_n]$$

dots

$$= \text{diag}(\lambda_1, \ldots, \lambda_n)$$

□

Prove the theorem

Theorem 6.2.2. If A is an $n \times n$ matrix with eigenpairs, $(\lambda_1, \vec{v}_1), \ldots, (\lambda_k, \vec{v}_k)$ where $\lambda_i \neq \lambda_j$, for $i \neq j$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Theorem 6.2.3. Of A is an $n \times n$ matrix with distinct eigenvaleus, and B is a basis for the eigenspace of λ_1 for $i \leq i \leq k$, then $B_1 \cup \cdots \cup B_k$ is a linearly independent set.

Theorem 6.2.4. If A is an $n \times n$ matrix with distinct eigenvallues $\lambda_1. \ldots, \lambda_k$, then A is diagonalizable if and only if $g_{\lambda_i} = g_{\lambda_k}$ for $1 \leq i \leq k$.

Note. If λ is an eigenvalue of A such that $g_\lambda < a_\lambda$, then λ is said to be **deficient**.

Theorem 6.2.5. If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Example 6.10. For each of the matrices, diagonalize them

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

Solution.

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{vmatrix}$$

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 8 & 16 \\ 0 & -3-\lambda & -3-\lambda \\ -4 & -4 & -11-\lambda \end{vmatrix}$$

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 8 & 8 \\ 0 & -3-\lambda & 0 \\ -4 & -4 & -7-\lambda \end{vmatrix}$$

$$= -(\lambda+3)(\lambda^2+2\lambda-3) = -(\lambda+3)^2(\lambda-1)$$

Therefore the eigenvalues are
$lambda_1 = -3$ with an algebraic multiplicity of 2, and $\lambda_2 = 1$ with an algebraic multiplicity of 1.

For $\lambda_1 = -3$,

$$A - (-3)I = \begin{bmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$, hence $g_{\lambda_1} = 2$.

For $\lambda_2 = 1$, we get

$$A - I = \begin{bmatrix} 4 & 8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Then the change of basis matrix which diagonalizes A is

$$P = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 6.11.

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

Solution. The characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 1-\lambda & -2 \\ -1 & 0 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 1-\lambda & -2 \\ -3-\lambda & 0 & -2\lambda \end{vmatrix}$$

$$= (3-\lambda)(\lambda-1)(\lambda+2) - (\lambda+3)(-2-(1-\lambda))$$

$$= -(\lambda-3)(\lambda^2+2\lambda+1) = -(\lambda-3)(\lambda+1)^2$$

Hence the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$ with $a_{\lambda_1} = 1, a_{\lambda_2} = 2$ For λ_2

$$B - \lambda_2 I = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$. This $g_{\lambda_2} = 1 < a_{\lambda_2}$, so B is not diagonalizable.

Example 6.12.

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution.

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

Since C has a non-real eignvalue, it is not diagonalizable over $\mathbb{R}$.

Example 6.13.

$$A \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Solution.

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

So the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 4$ with $a_{\lambda_1} = 1 = a_{\lambda_2}$.

A basis for $E_{\lambda_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $E_{\lambda_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

Theorem 6.2.6. If A is an $n \times n$ matrix, then

- the determinant of A equals the product of the eigenvalues
- λ is an eigenvalue of A if and only if $\det(A) = 0$. If it is, then the geometric multiplicity of 0 equals $1 - \text{rank}(A)$.
- The trace of A equals the sum of the eigenvalues.