Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

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3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

A relation on a set

- draw its **digraph (有向图)**
- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**

if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Negation

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

not reflexive: if there exists $x \in X$, such that $(x, x) \notin R$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

not symmetric: if there exists x and y, such that $(x, y) \in R$ and $(y, x) \notin R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

not antisymmetric: if there exists x and y, $x \neq y$, such that $(x, y) \in R$ and $(y, x) \in R$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**

if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

not transitive: if there exists $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \notin R$.

3.3 Relations 关系

Exercise Give examples of relations on $\{1, 2, 3, 4\}$ having the properties specified as follows.

(a) Reflexive, symmetric, and not transitive

(b) Reflexive, not symmetric, and not transitive

(c) Reflexive, antisymmetric, and not transitive

(d) Not reflexive, symmetric, not antisymmetric, and transitive

(e) Not reflexive, not symmetric, and transitive

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

We say that

- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \npreceq y$ or $y \npreceq x$.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

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Example ?

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.
Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

Whether the ralation of $R = X \times X$ is a toal order on X?

3.3 Relations 关系

Definition 3.3.23 Let R be a relation from X to Y. The **inverse of R (R的逆)**, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

$R^{-1} =$

3.3 Relations 关系

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 =$

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$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

(2) $R \cap S$

(3) $R \circ S$

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(1) $R \cup S$

$$R = \{(1, 2)\}, S = \{(2, 3)\}, R \cup S = \{(1, 2), (2, 3)\}.$$

(2) $R \cap S$

If $\{x, y\}, \{y, z\} \in R \cap S$, then $\{x, z\} \in R \cap S$.

(3) $R \circ S$

$$R = \{(5, 2), (6, 3)\}, S = \{(1,5), (2, 6)\}, R \circ S = \{(1, 2), (2, 3)\}.$$

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Example Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

3.4 Equivalence Relations 等价关系

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Obviously, these three properties are necessary for a reasonable definition of equivalence.

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Theorem 3.4.1 Let $\mathcal{S}$ be a partition of a set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

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- A **partition (划分)**:

- A **collection of sets (集族)**：A set $\mathcal{S}$ whose elements are sets.

Example: $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$.

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- A **partition (划分)**：A collection $\mathcal{S}$ of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of $\mathcal{S}$.
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- A **collection of sets (集族)**： A set $\mathcal{S}$ whose elements are sets.

Example: $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$.

In other words, the collection of subsets $A_i, i \in I$, forms a **partition** of X if and only if (i) $A_i \neq \emptyset$ for $i \in I$ (ii) $A_i \cap A_j = \emptyset$, if $i \neq j$ (iii) $\cup_{i \in I} A_i = X$

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- A **partition (划分)**: A collection $\mathcal{S}$ of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of $\mathcal{S}$.

Example
$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is a partition of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

In other words, the collection of subsets A_i, $i \in I$, forms a **partition** of X if and only if (i) $A_i \neq \emptyset$ for $i \in I$ (ii) $A_i \cap A_j = \emptyset$, if $i \neq j$ (iii) $\bigcup_{i \in I} A_i = X$

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- reflexive
- symmetric
- transitive

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Example 3.4.2 Consider the partition $\mathcal{S} = \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$ of $X = \{1, 2, 3, 4, 5, 6\}$. The relation R on X is given by Theorem 3.4.1. Then

$R =$

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Example Consider the relation $R = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$ on $\{1, 2, 3, 4, 5\}$. Is it an equivalence relation?

Diagraph?

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Exercise Which of the following relation is an equivalence relation?

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$.

The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$.

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Exercise Which of the following relation is an equivalence relation?

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$. No.

The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$. No.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Definition 3.4.9 Let R be an equivalence relation on a set X. For $\forall\ a \in X$, let $[a] = \{x \in X \mid xRa\}$. The sets $[a]$ is called the **equivalence classes (等价类)** of X given by the relation R.

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Theorem 3.4.8 Let R be an equivalence relation on a set X.
Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.

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Theorem 3.4.8 Let R be an equivalence relation on a set X.
Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.

Example 3.4.2 Consider the relation
$R = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 6), (6, 2), (6, 6), (4, 4)\}$
on $\{1, 2, 3, 4, 5, 6\}$.

3.4 Equivalence Relations 等价关系

Theorem 3.4.16 Let R be an equivalence relation on a finite set X. If each equivalence class has r elements, there are $|X|/r$ equivalence classes.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

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Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is ?

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$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The matrix of a relation from X to Y is dependent on the orderings of X and Y.

3.5 Matrices of Relations 关系矩阵

Example 3.5.1 The relation $R = \{(1,b),(1,d),(2,c),(3,c),(3,b),(4,a)\}$ from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$ relative to the orderings $1,2,3,4$ and a,b,c,d is

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Example 3.5.2 The matrix of the above relation R relative to the orderings $2,3,4,1$ and d,b,a,c is ?

The matrix of a relation from X to Y is dependent on the orderings of X and Y.

3.5 Matrices of Relations 关系矩阵

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{array}$$

Example 3.5.2 The matrix of the above relation R relative to the orderings $2, 3, 4, 1$ and d, b, a, c is

$$\begin{array}{c c} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \end{array}$$

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The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

Example 3.5.4 The relation $R = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$ on $\{a,b,c,d\}$ relative to the ordering a, b, c, d is ?

When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

A relation on a set

- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is A_1 =?

The matrix of R_2 relative to the orderings a, b and x, y, z is A_2 =?

The product of these matrices is A_1A_2 =?

设矩阵 $\boldsymbol{A} = (\boldsymbol{a}_{ij})_{m\times s}$, $\boldsymbol{B} = (\boldsymbol{b}_{ij})_{s\times n}$, 令 $\boldsymbol{C} = (\boldsymbol{c}_{ij})_{m\times n}$ 是由下面的 $\boldsymbol{m} \times \boldsymbol{n}$ 个元素 $\boldsymbol{c}_{ij} = \boldsymbol{a}_{i1}\boldsymbol{b}_{1j} + \boldsymbol{a}_{i2}\boldsymbol{b}_{2j} + \ldots + \boldsymbol{a}_{is}\boldsymbol{b}_{sj}$ $(\boldsymbol{i} = 1, 2, \ldots, \boldsymbol{m}; \boldsymbol{j} = 1, 2, \ldots, \boldsymbol{n})$ 构成的 $\boldsymbol{m}$ 行 $\boldsymbol{n}$ 列矩阵. 称矩阵 $\boldsymbol{C}$ 为矩阵 $\boldsymbol{A}$ 与矩阵 $\boldsymbol{B}$ 的乘积，记为 $\boldsymbol{C} = \boldsymbol{AB}$.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_2 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is $A_1 A_2 =$?

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

设矩阵 $\boldsymbol{A} = (\boldsymbol{a_{ij}})_{\boldsymbol{m \times s}}$，$\boldsymbol{B} = (\boldsymbol{b_{ij}})_{\boldsymbol{s \times n}}$，令 $\boldsymbol{C} = (\boldsymbol{c_{ij}})_{\boldsymbol{m \times n}}$ 是由下面的 $\boldsymbol{m \times n}$ 个元素 $\boldsymbol{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{is}b_{sj}}$ $(\boldsymbol{i = 1, 2, \ldots, m; j = 1, 2, \ldots, n})$ 构成的 $\boldsymbol{m}$ 行 $\boldsymbol{n}$ 列矩阵. 称矩阵 $\boldsymbol{C}$ 为矩阵 $\boldsymbol{A}$ 与矩阵 $\boldsymbol{B}$ 的乘积，记为 $\boldsymbol{C = AB}$.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

$$A_1 = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

$$A_2 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is $A_1 A_2 =$?

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $\boldsymbol{R_2 \circ R_1}$, is the relation from X to Z defined by
$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by 1.

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $R_2 \circ R_1$, is the relation from X to Z defined by $R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by 1.

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a,a), (b,b), (c,c), (d,d), (b,c), (c,b)\}$ on $\{a,b,c,d\}$ is transitive?

The relation R is transitive if and only if whenever entry i,j in A^2 is nonzero, entry i,j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, b)\}$ on $\{a, b, c, d\}$ is transitive?

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many relations are there on X?

- How many reflexive relations are there on X?

- How many symmetric relations are there on X?

- How many antisymmetric relations are there on X?

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many reflexive and symmetric relations are there on X?

- How many reflexive and antisymmetric relations are there on X?

- How many aymmetric and antisymmetric relations are there on X?

- How many reflexive, symmetric, and antisymmetric relations are there on X?