EBU4375: SIGNALS AND SYSTEMS

TOPIC 4-2: SAMPLING THEORY

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ACKNOWLEDGMENT

These slides are partially from lectures prepared by

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AGENDA

1. A strange world!
2. Introduction to sampling
3. Analog to digital conversion
4. Digital to analog conversion

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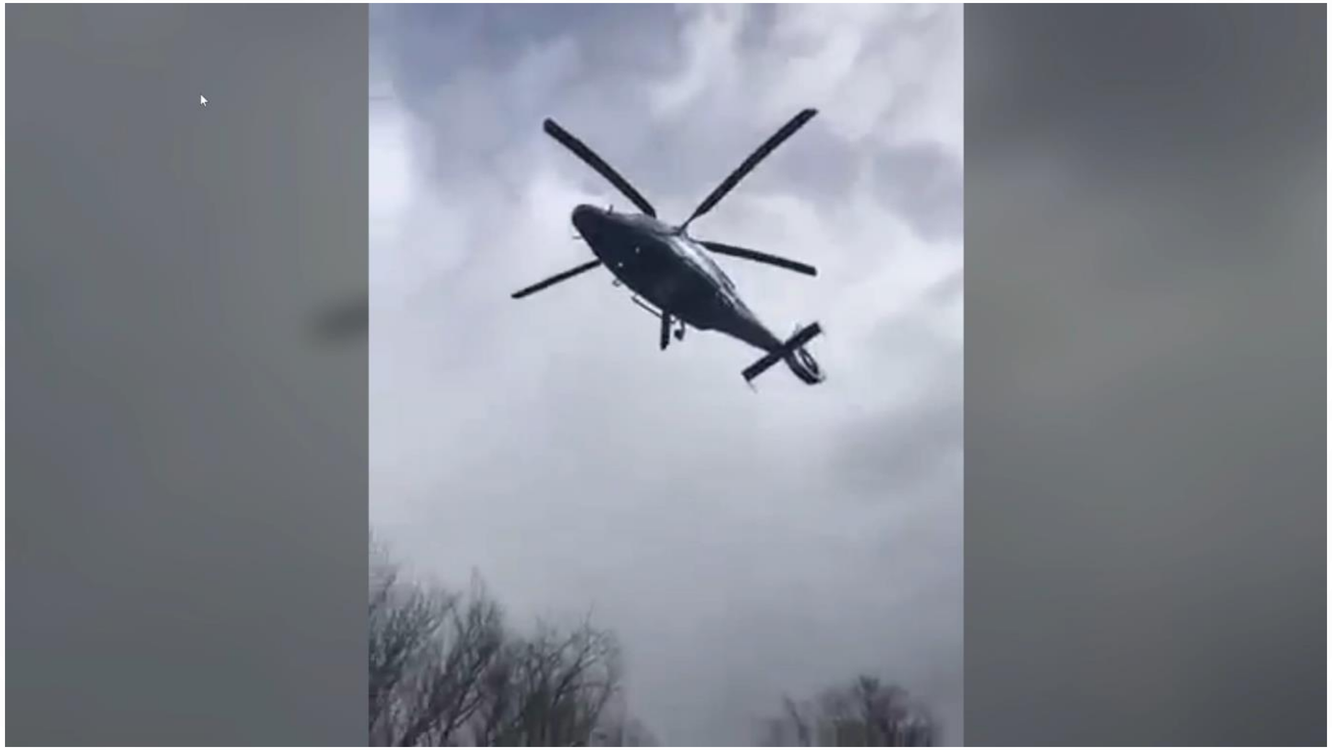
1- A STRANGE WORLD



1: STRANGE WORLD

- After watching the video clip of the chopper,
- Please put the recording on hold and login to QM+
- Go to Topic 4
- Take 10 minutes to answer the questions in T4-Q2
- You can retry as many times as you wish.

1: LET'S INVESTIGATE!

These bizarre phenomena can be explained by studying Sampling Theory and Interpolation.

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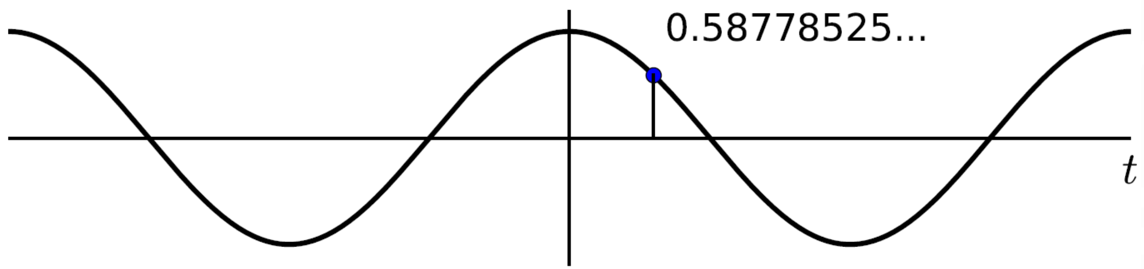
4. Digital to analog conversion

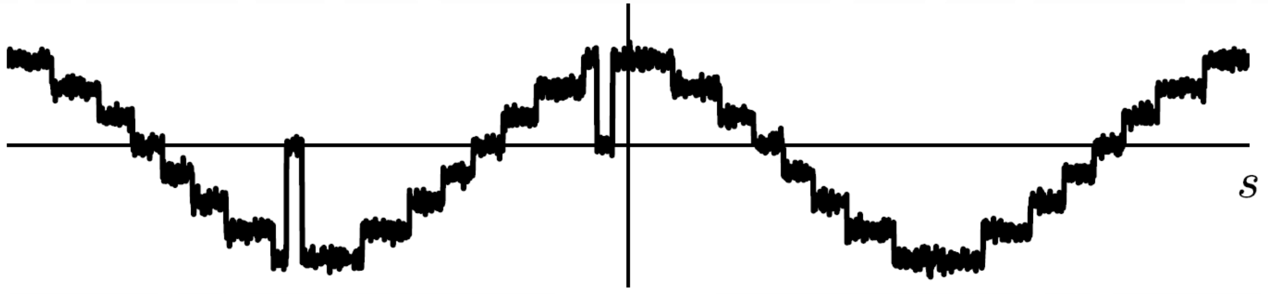
2: RUNNING OUT OF STORAGE SPACE!

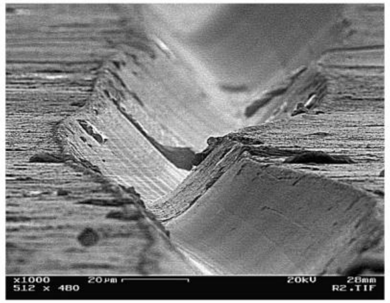
How can we store a signal from the physical world? Storage media copy values that are contiguous in time into contiguous spatial locations.



Physical signals are **continuous both in time and in amplitude,** which means that in order for us to store them we need to store an **infinite number of values described by an infinite number of digits**!

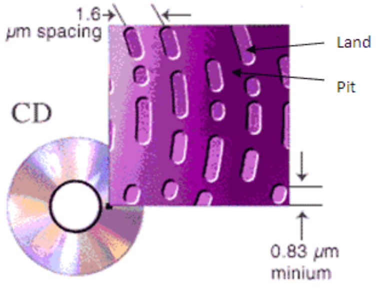
2: ANALOG VS DIGITAL STORAGE MEDIA





- **Analog** approaches use physical media that are **continuous in space and in amplitude**. However, the **quality** of the stored signal is **poor** due to a limited precision and media imperfections.

- **Digital** approaches store physical signals as binary sequences (0's and 1's) and hence are **discrete in space and in amplitude.** But how? And how are binary sequences converted back into physical signals? **What is the quality** of the stored signal?



2: ANALOG VS DIGITAL STORAGE MEDIA

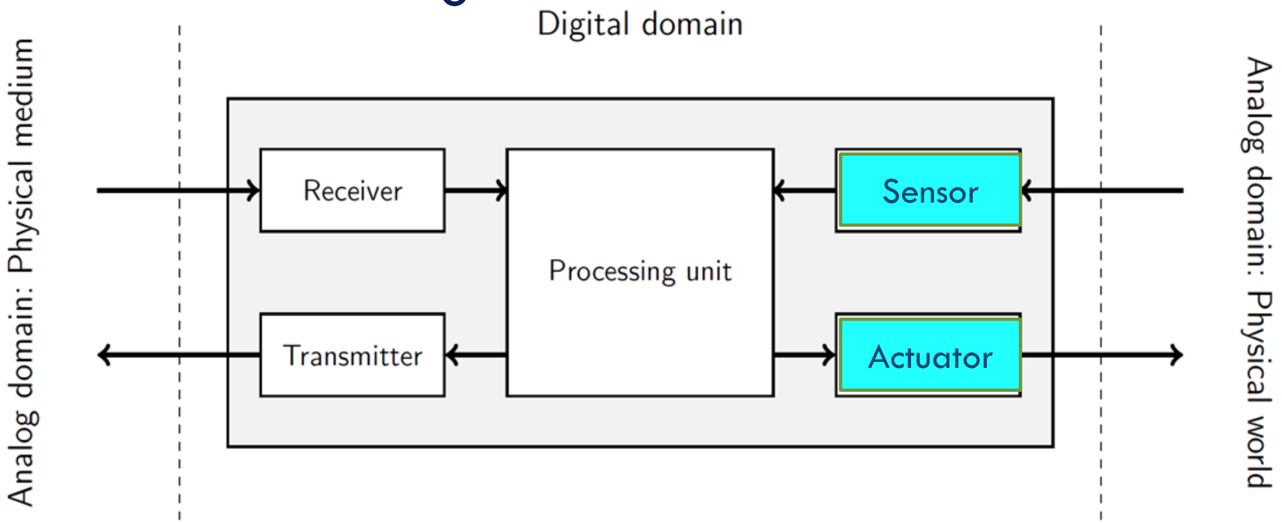
Nowadays, most of the systems that interface with the physical world (for instance, for storing, analysing and transmitting signals produced in the physical world) use digital approaches. This is made possible by two format-conversion techniques:

- AD conversion or digitisation, which converts signals in an analog format into signals in a digital format.
- DA conversion, which converts signals in a digital format into signals in an analog format.

2: IMPLICATION TO IoT DESIGN

IoT devices interact with the physical world through sensors and actuators.

- **Sensors** provide **inputs** from the physical world. Therefore, they must be associated to an **AD conversion** stage.
- **Actuators** provide **outputs** to the physical world. Therefore, they must be associated to a **DA conversion** stage.



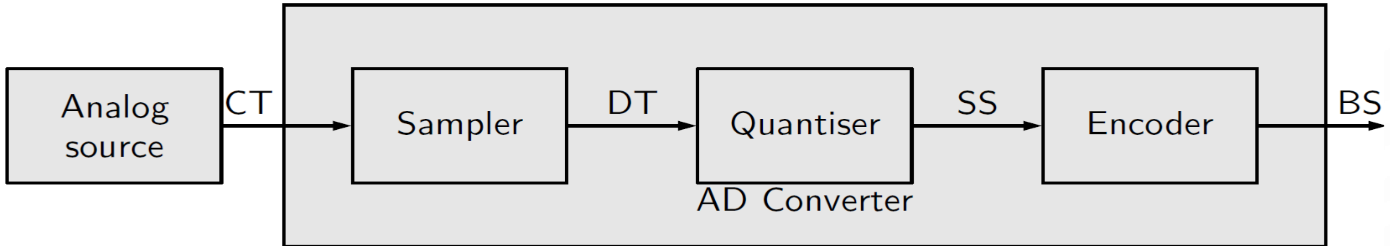
AGENDA

1. A strange world!
2. Introduction to sampling
3. Analog to digital conversion
4. Digital to analog conversion

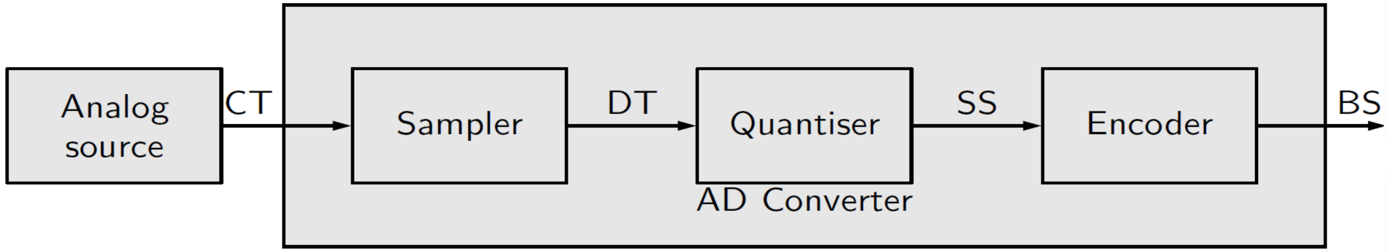
3: THE DIGITISATION PIPELINE

The process of digitisation or AD conversion converts a signal which is continuous in time and in amplitude into a binary sequence (discrete in time and in amplitude) and involves several steps:

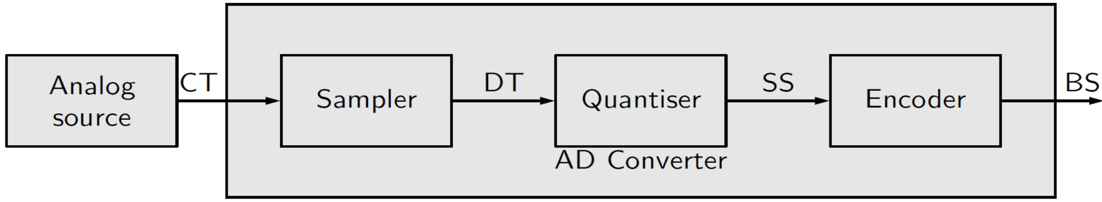
- **Sampling** converts a CT signal into a DT signal.
- **Quantisation** converts each sample of a DT signal into one out of an infinite number of values called symbols. The result is a symbol sequence (SS).
- **Encoding** converts a SS into a bit sequence (BS).



3: THE DIGITISATION PIPELINE



- Signals produced by the analog source are CT and their amplitude can take a continuous range of values.
- Sampled signals are DT and their amplitude can take a continuous range of values.
- Symbol sequences are DT and their amplitude can take a finite range of values.
- Binary sequences are DT and their amplitude can take either one of two values (0 and 1).

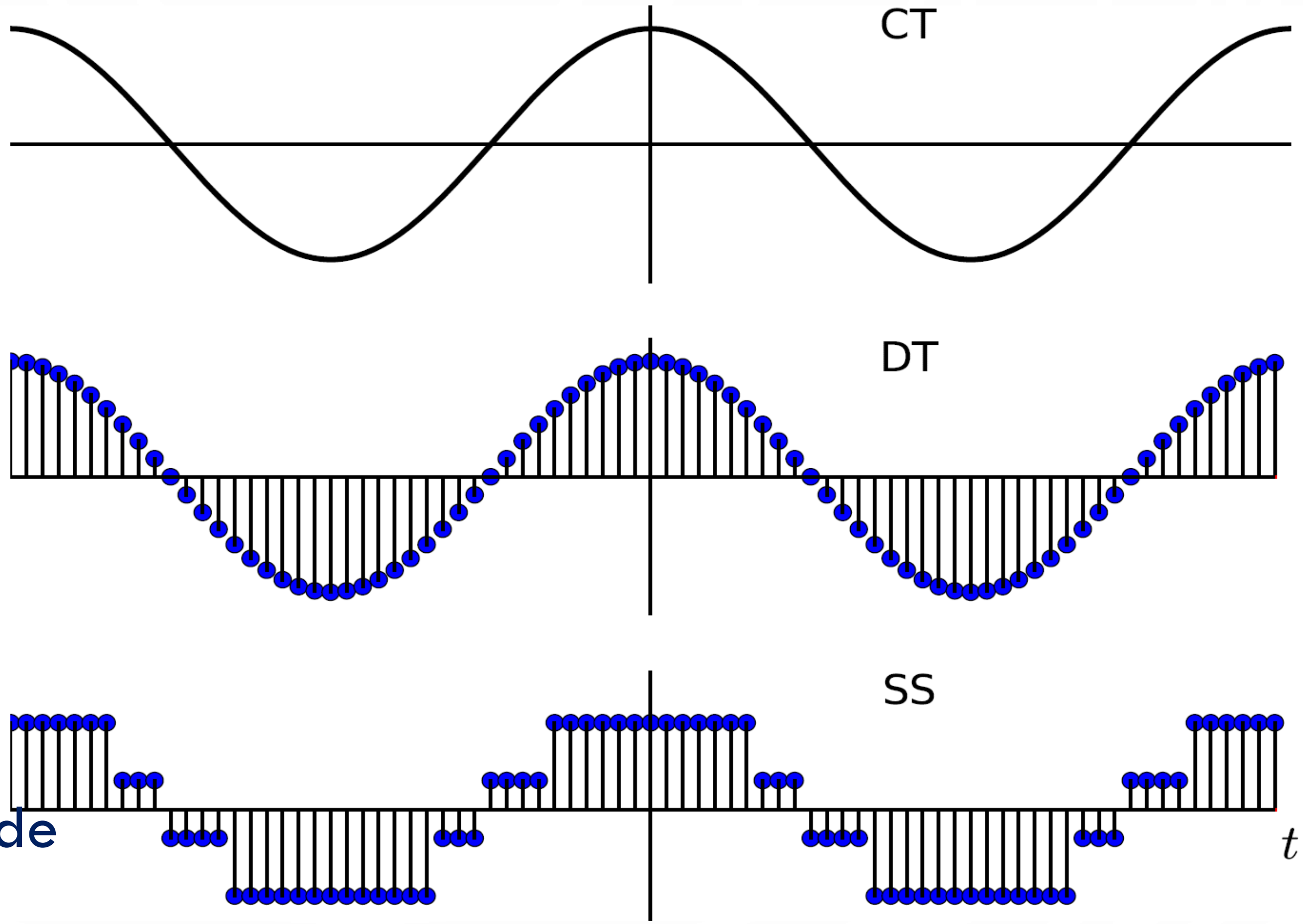


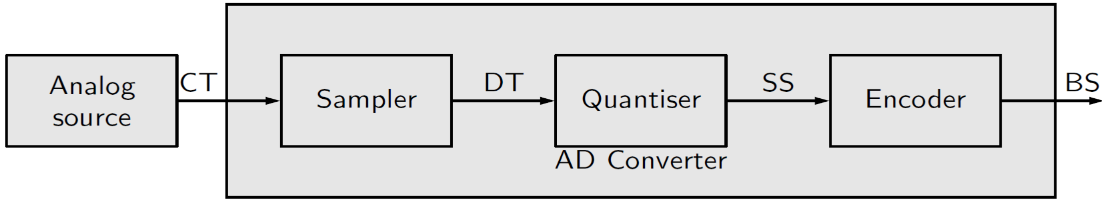
3: SAMPLING & QUANTISATION

Analog signal:
continuous in time
continuous in amplitude

Sampled signal:
Discrete in time
continuous in amplitude

Quantised signal:
Discrete in time
Finite range of amplitude





3: ENCODING

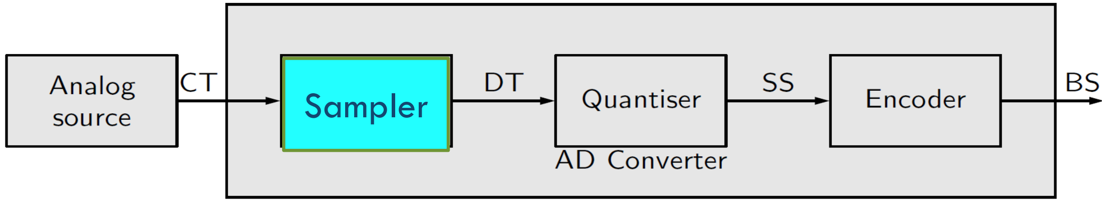
Encoding is the last step in the digitisation pipeline and consists of generating a binary sequence from a sequence of symbols. The easiest form of encoding translates each symbol into a binary sequence (binary code).

For instance, assigning the following binary codes to each symbol -0.75, -0.25, 0.25 and 0.75:

$$-0.75 \rightarrow 00$$
$$-0.25 \rightarrow 01$$
$$0.25 \rightarrow 10$$
$$0.75 \rightarrow 11$$

We produce the following conversion from SS to BS:

$$[-0.75; -0.75; 0.25; -0.25; 0.75; 0.25] \rightarrow 000010011110$$

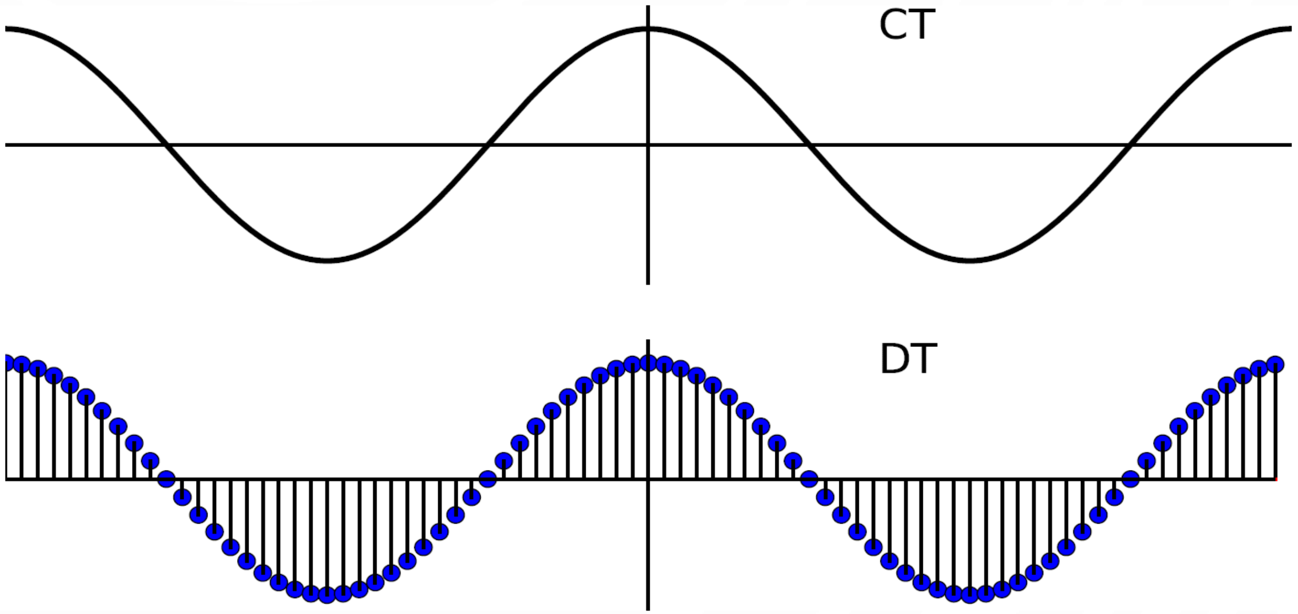


3: SAMPLING PRINCIPLE

Sampling is the first state of the digitisation pipeline and creates a DT signal from a CT signal.

Analog signal:
continuous in time
continuous in amplitude

Sampled signal:
Discrete in time
continuous in amplitude



How can this be done? Can you think of an LTE system?

3: SAMPLING PRINCIPLE

Mathematically, sampling can be described as a two-step process:

1. **Sample extraction:** multiplication by a CT impulse train.

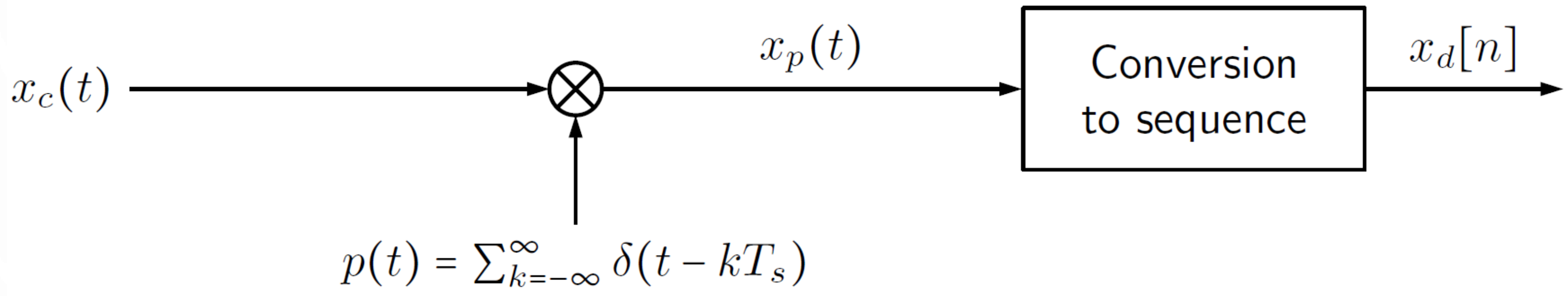
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

where T_s is known as the sampling period, $\omega_s = 2\pi f_s = 2\pi/T_s$ is the sampling frequency

2. **DT sequence generation:** conversion to a DT impulse train..

3: SAMPLING PRINCIPLE

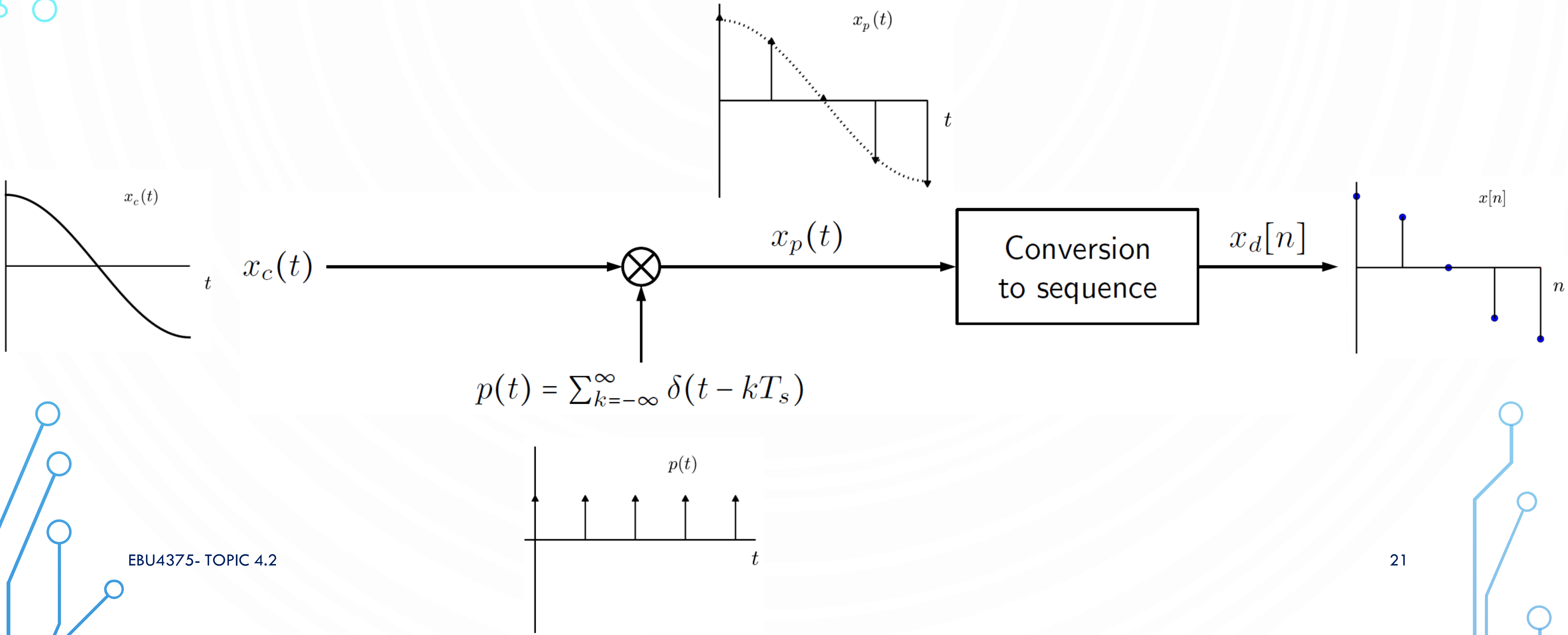
Schematically:



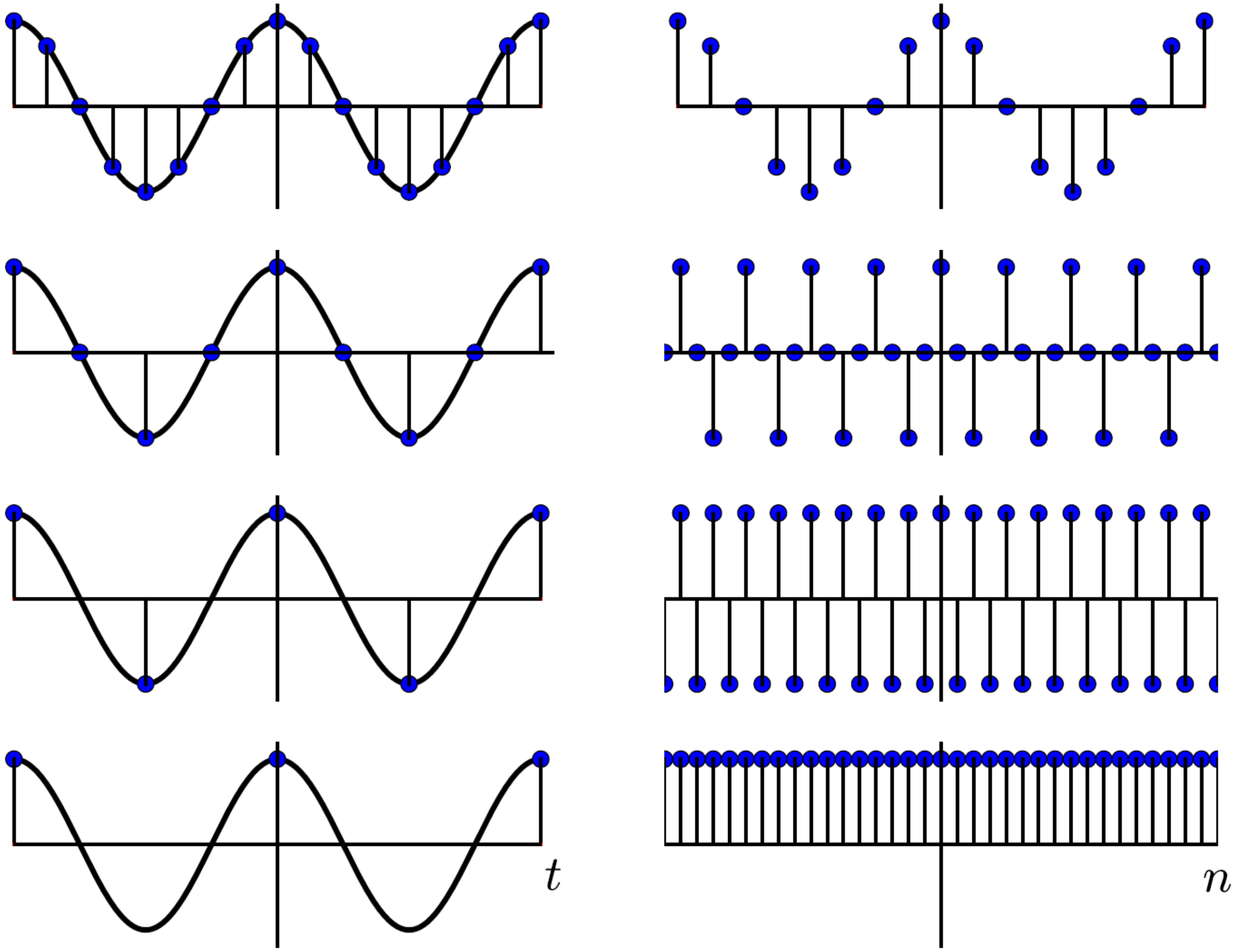
$$x_p(t) = x_c(t)p(t) = x_c(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_S) = \sum_{k=-\infty}^{\infty} x_c(kT_s)\delta(t - kT_s)$$

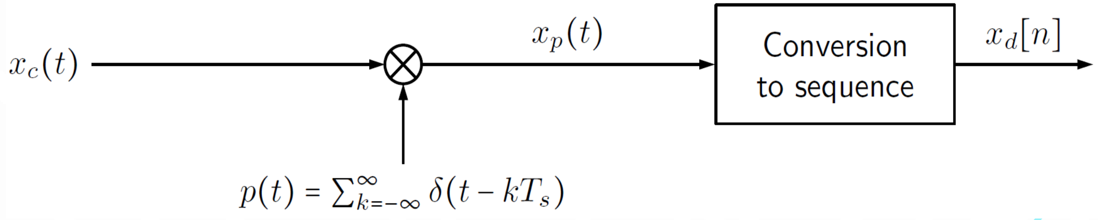
$$x_d[n] = \sum_{k=-\infty}^{\infty} x_c(kT_s)\delta[n - k] \longrightarrow x_d[n] = x_c(nT_s)$$

3: SAMPLING PRINCIPLE



3: SAMPLING: SINUSOIDS





3: SAMPLING: FREQUENCY DOMAIN

What happens in the frequency domain? We know that: $x_p(t) = x_c(t) \times p(t)$

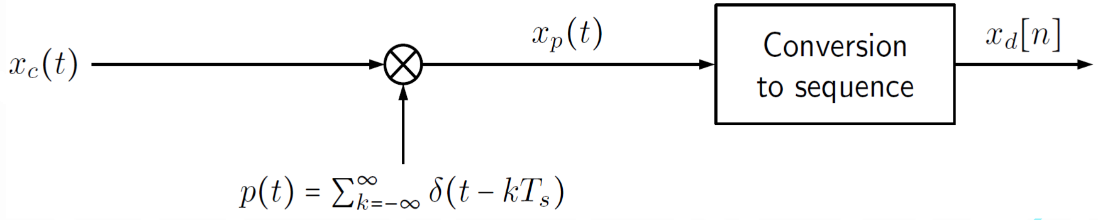
hence, $X_p(\omega) = \frac{1}{2\pi} X_c(\omega) * P(\omega)$

Where $X_c(\omega)$, $P(\omega)$, and $X_p(\omega)$ are the Fourier Transforms of $x_c(t), p(t)$, and $x_p(t)$.

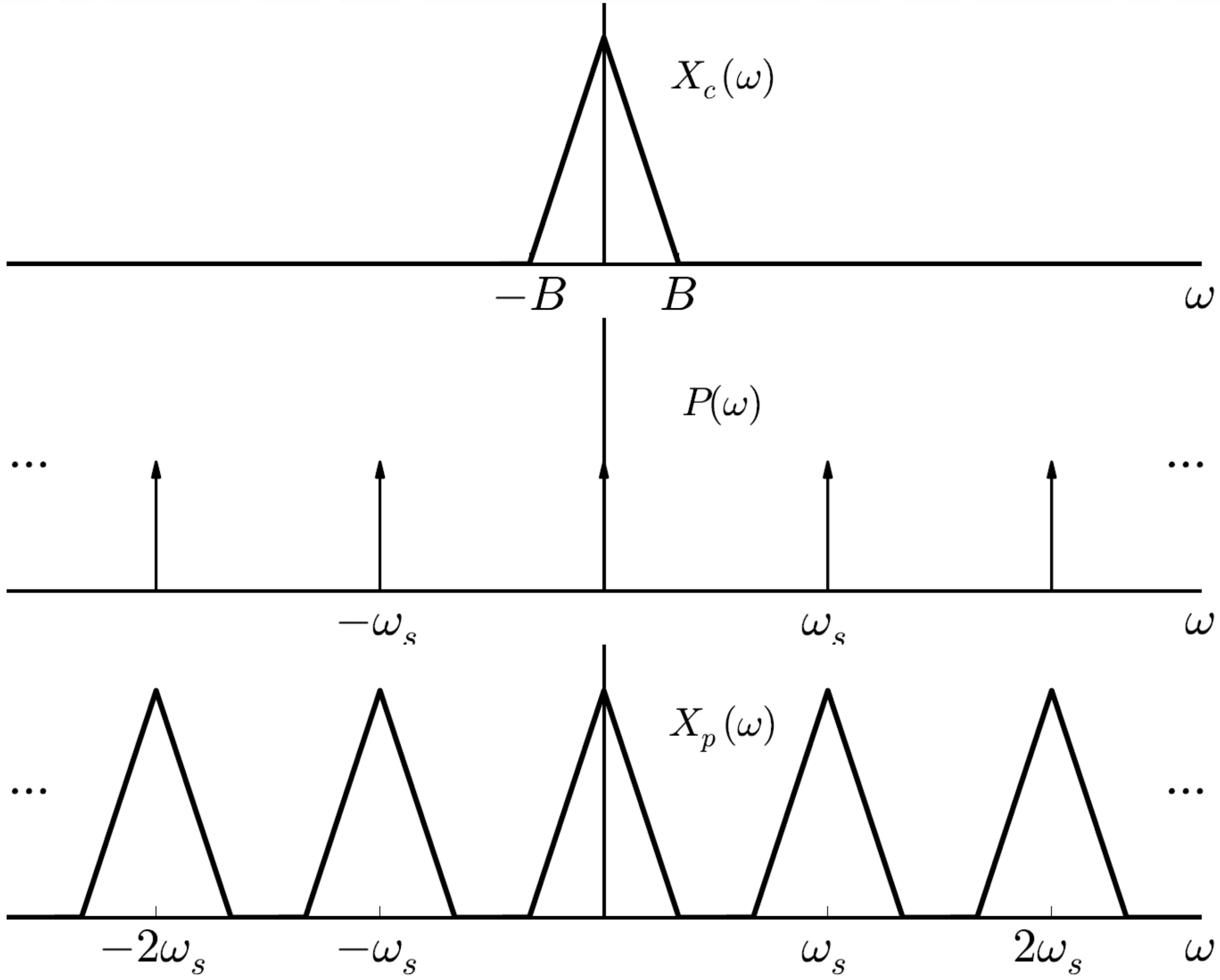
We also know that: $P(\omega) = \frac{2\pi}{T_s} \sum \delta(\omega - k\omega_s)$

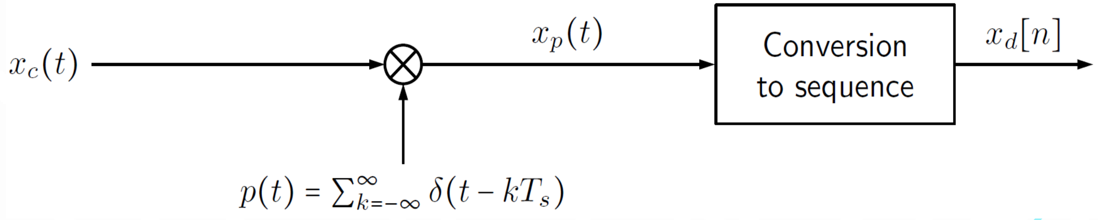
Hence, $X_p(\omega) = \frac{1}{2\pi} X_c(\omega) * \frac{2\pi}{T_s} \sum \delta(\omega - k\omega_s) = \frac{1}{T_s} \sum X_c(\omega - k\omega_s)$

$X_p(\omega)$ is **periodic** and consists of **frequency-shifted replicas** of $X(\omega)$

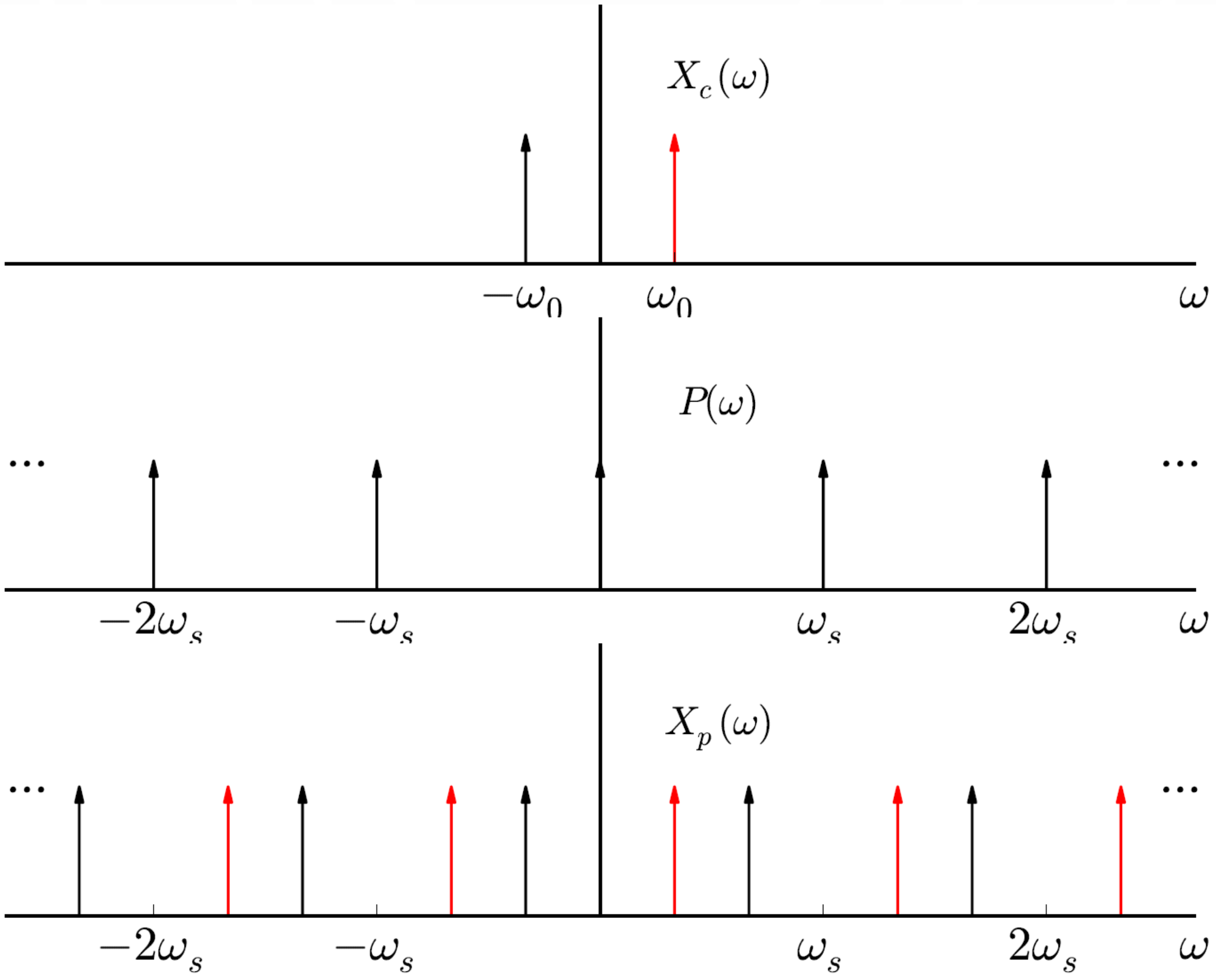


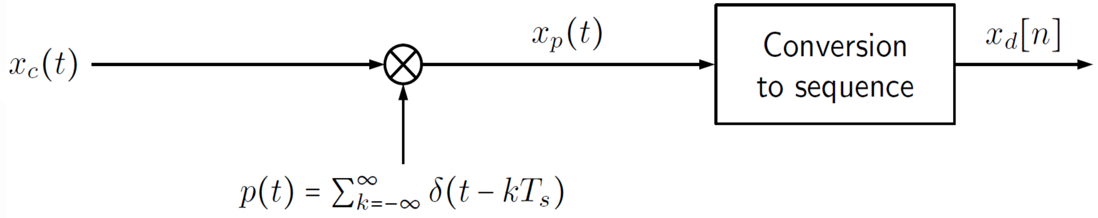
3: SAMPLING: FREQUENCY DOMAIN





3: SAMPLING: SINUSOIDS





3: SAMPLING: FREQUENCY DOMAIN

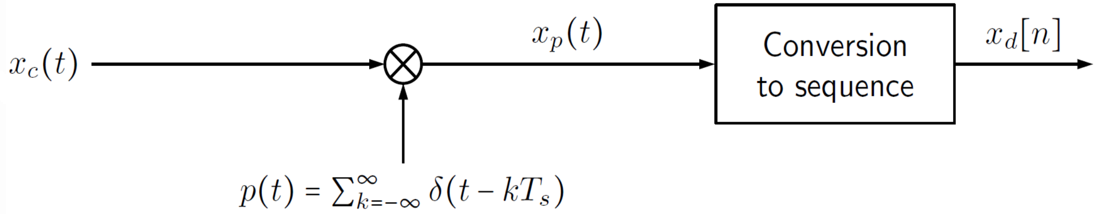
Signal $x_p(t)$ can be expressed in terms of $x[n]$ as follows:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Hence, its Fourier transform $X_p(\omega)$ is

$$\begin{aligned} X_p(\omega) &= \int_{-\infty}^{\infty} x_p(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT_s)e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT_s} = X_d(\omega T_s) \end{aligned}$$

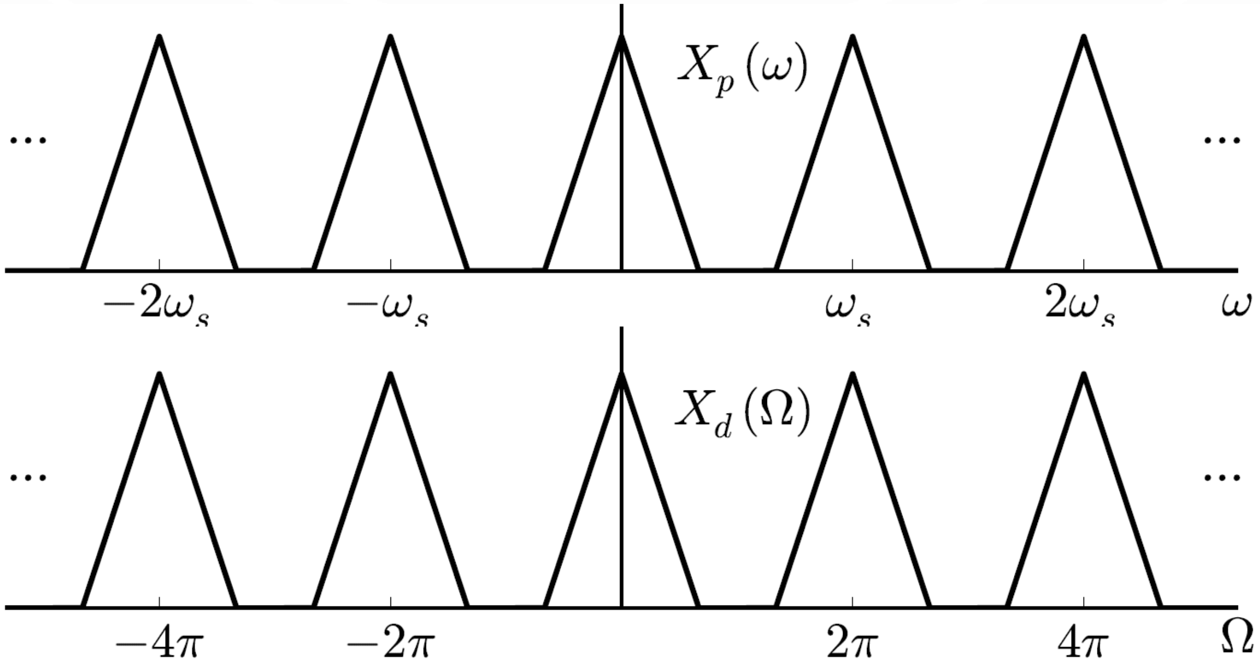
where $X_d(\Omega)$ is the Fourier transform of $x[n]$.

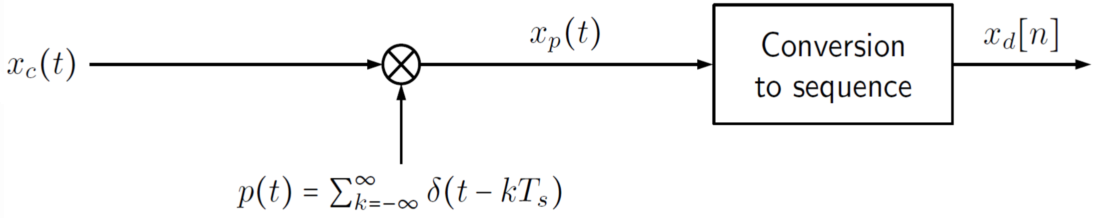


3: SAMPLING: FREQUENCY DOMAIN

In summary, $X_p(\omega) = X_d(\omega T_s)$ and $X_d(\Omega) = X_p(\Omega/T_s)$. Specifically,

$$X_p(\omega_s) = X_p\left(\frac{2\pi}{T_s}\right) = X_d\left(\frac{2\pi}{T_s}T_s\right) = X_d(2\pi)$$





3: SAMPLING: FREQUENCY DOMAIN

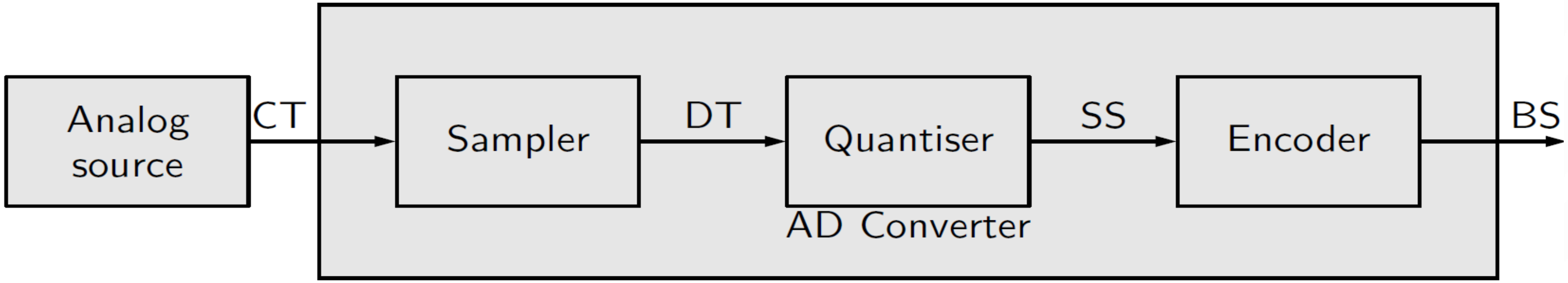
In summary, if $x_d[n]$ is the DT signal obtained by sampling a CT signal x_c(t) with a sampling rate $f_s = \frac{1}{T_s}$, then its Fourier Transform $X_d(\Omega)$ can be expressed as:

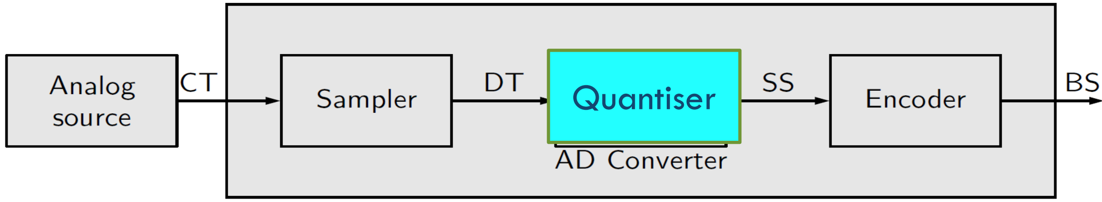
$X_p(\omega) = X_d(\omega T_s)$ and $X_d(\Omega) = X_p(\Omega/T_s)$. Specifically,

$$X_d(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\Omega}{T_s} - k\frac{2\pi}{T_s}\right)$$

Where $X_c(\omega)$ is the Fourier transform of x_c(t).

3: DIGITISATION PIPELINE



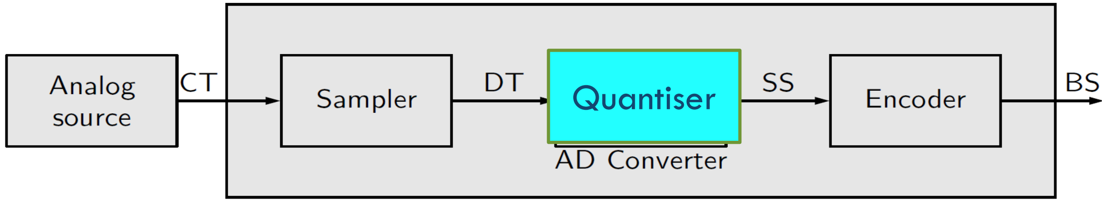


3: QUANTISATION

By sampling an analog signal we obtain a **sequence of continuous values.** For representing one of such values we might need an **infinite sequence of digits**, for instance, 3.14159265359….

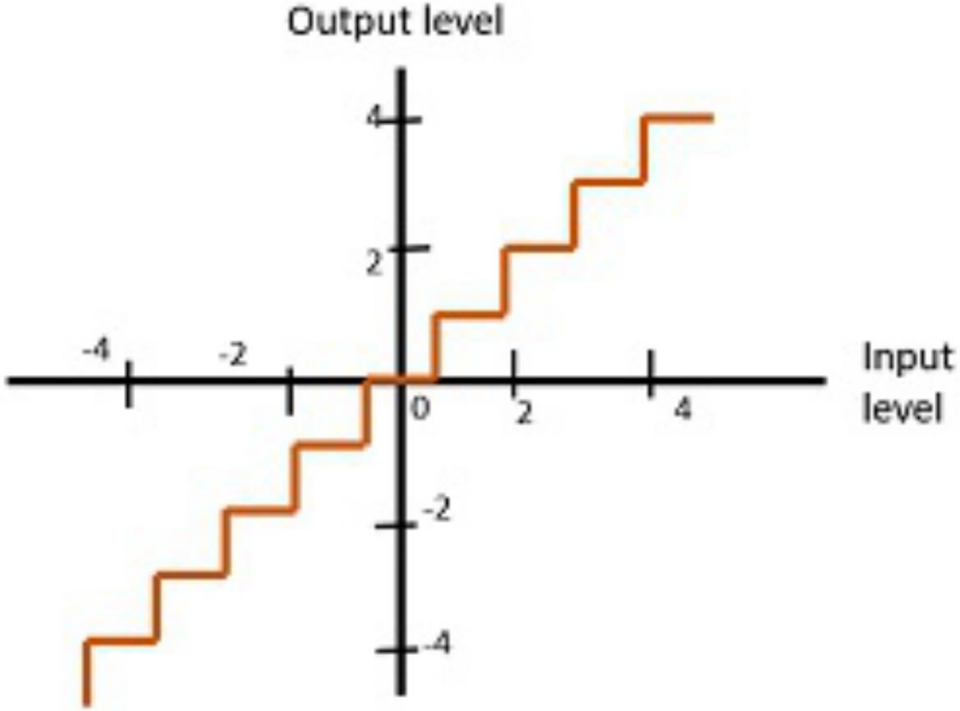
If we want to obtain a digital representation of an analog waveform, after sampling the original waveform we need to **quantise the resulting amplitudes.**

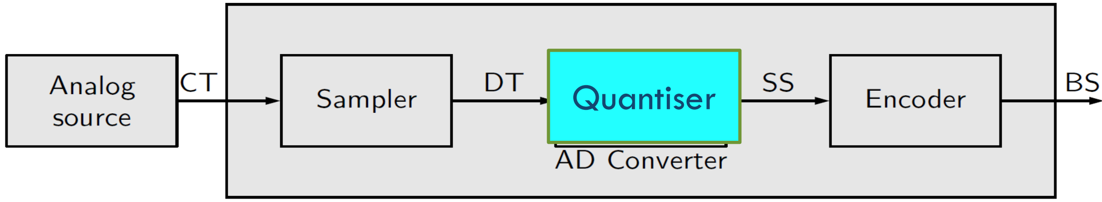
Quantisation consist of mapping continuous values into discrete ones that can be represented digitally, i.e. that can be represented as symbols. This leads to errors known as **quantisation errors.**



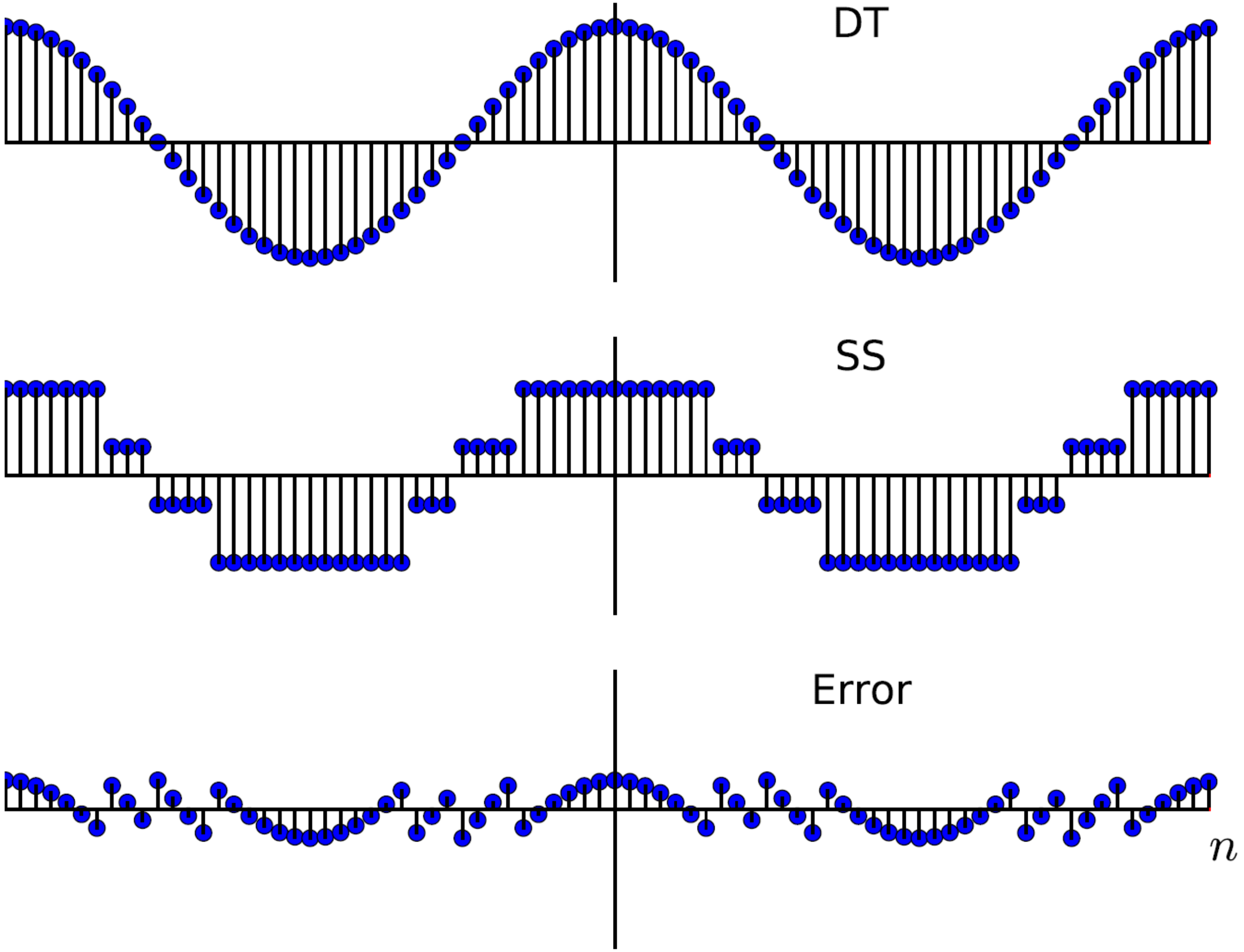
3: UNIFORM QUANTISATION

In uniform quantisation the amplitude range is divided into equally-sized intervals known as quantisation levels. If the size of a quantisation level is q, the error for each sample is no larger than $\pm q/2$.





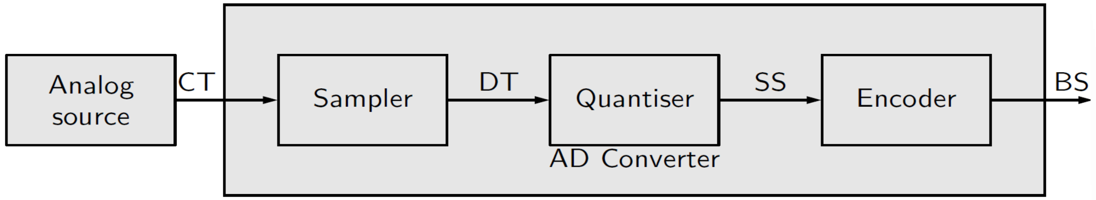
3: UNIFORM QUANTISATION



- The quantisation error is the difference of the DT signal and the SS signal.
- The smaller the quantisation interval q, the smaller the quantisation error.

3: SAMPLING, QUANTISATION, AND STORAGE

- Please put the recording on hold and login to QM+
- Go to Topic 4
- Take 10 minutes to answer the questions in T4-Q3
- You can retry as many times as you wish (this is not graded).
- Take 5 minutes to discuss with your friends if you're not sure.
- Let's think together…

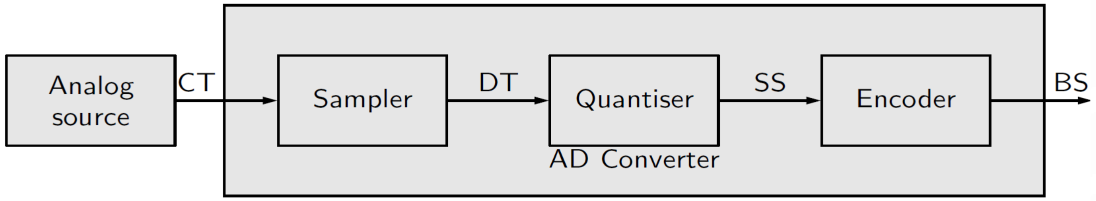


3: SAMPLING, QUANTISATION, AND STORAGE

A signal x(t) with a duration of 2sec is sampled at 1KHz. Each sample is quantised by a uniform quantiser with 256 levels. How can we determine the size of the memory that we need to store the digital version of x(t)?

3: SAMPLING, QUANTISATION, AND STORAGE

- The higher the sampling rate the higher the required storage space.
- The higher the sampling rate the better the quality.
- The higher the number of quantisation levels the better the quality.
- The higher the number of quantisation levels the higher the required storage space.



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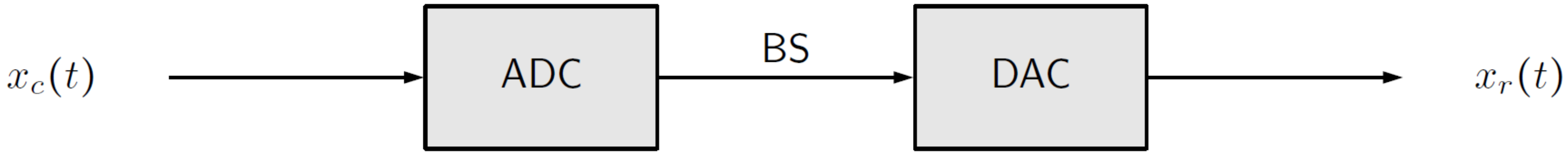
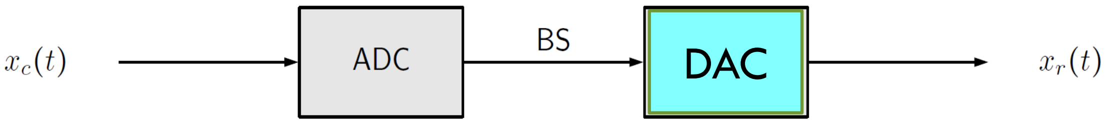
4: DA CONVERSION

The process of DA conversion converts a binary sequence (discrete in time and in amplitude) into a signal which is continuous in time and in amplitude.

If we assume that the binary sequence is the result of digitising a signal $x_c(t)$ then our aim is for the result of the process of DA conversion, $x_r(t)$, to be as close to $x_c(t)$ as possible.





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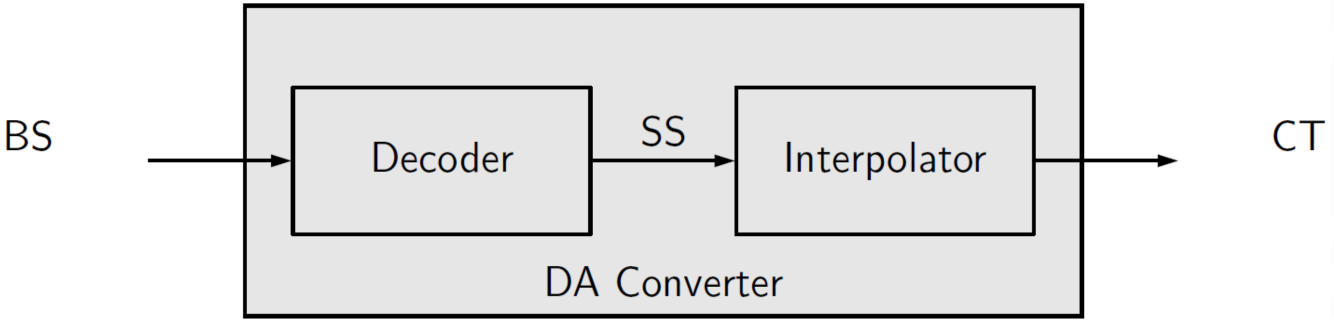
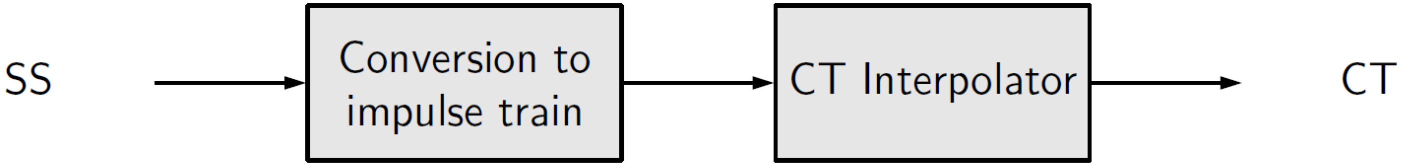
The process of DA conversion converts a binary sequence (discrete in time and in amplitude) into a signal which is continuous in time and in amplitude.

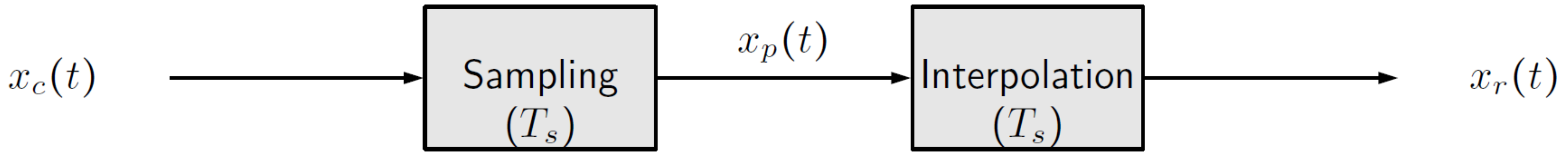


The interpolator first converts the symbol sequence into a CT impulse train and then proceeds to interpolate between the impulses.



4: DA CONVERSION

In our analysis of DA conversion, we will not consider the effects of quantisation, only sampling. The ADC will be modelled as a sampler and the DAC as an interpolator.

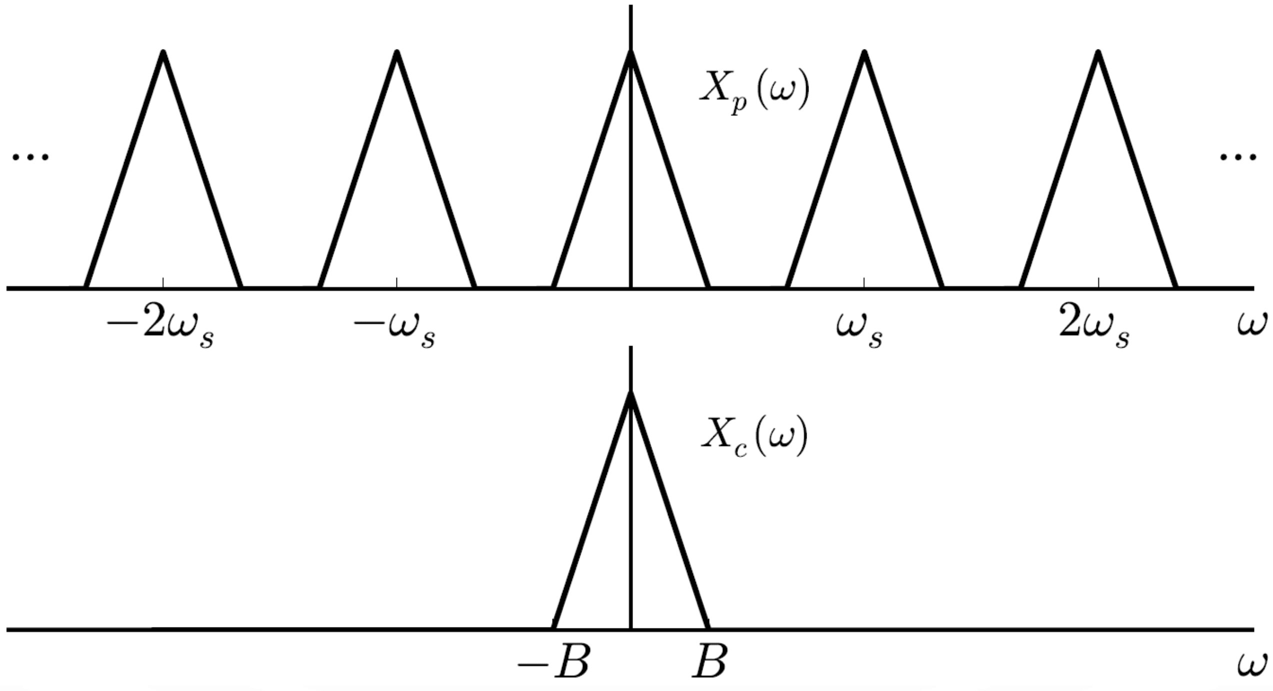


We will ask ourselves the following question:

How can we design the sampling and interpolation stages so that $x_r(t) = x_c(t)$?

4: INTERPOLATION IN THE FREQUENCY DOMAIN

Given $x_p(t)$, how can we obtain $x_c(t)$?
Hint: Look at the frequency domain, i.e. at $X_p(\omega)$ and $X_c(\omega)$

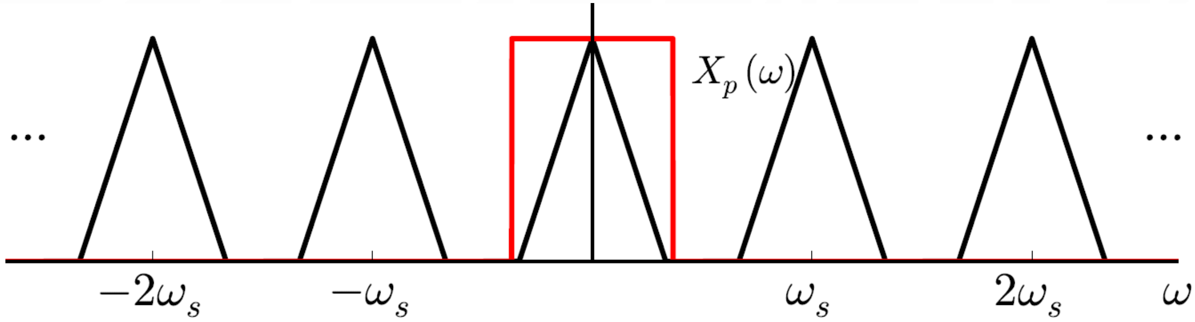


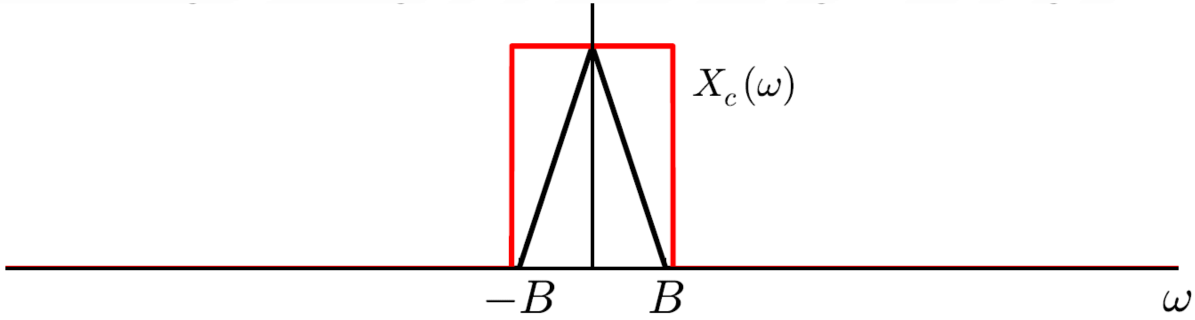
4: INTERPOLATION IN THE FREQUENCY DOMAIN

By looking at the frequency domain, we can conclude that by low-pass filtering the sampled signal $x_p(t)$ we could obtain $x_c(t)$.

Since $X_p(\omega) = (1/T_s)\sum X_c(\omega - k\omega_s)$, the interpolator will be a lowpass filter with frequency response $H_I(\omega) = T_s$ for $|\omega| \leq \omega_s/2$, and 0 elsewhere.

What is the relation between B and ω_s?

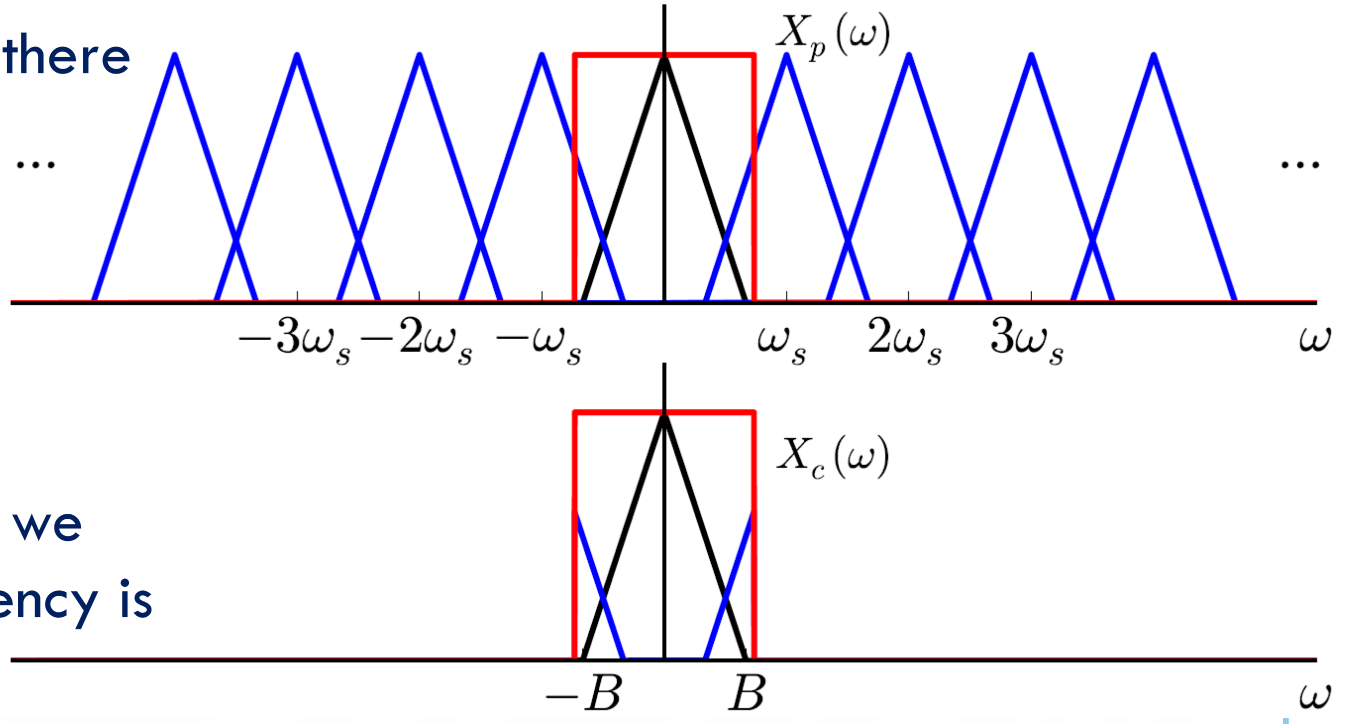




4: Interpolation: Undersampling and Aliasing

If we undersample a signal (ω_s is too low), there is overlap between the replicas of $X(\omega)$ in $X_p(\omega)$. This is called aliasing.

If there is aliasing, $X_c(\omega)$ will contain both $X(\omega)$ and part of the replicas $X(\omega + \omega_s)$ and $X(\omega - \omega_s)$. In order for us to avoid it, we need to make sure that the sampling frequency is high enough, specifically $\omega_s \geq 2\text{B}$.



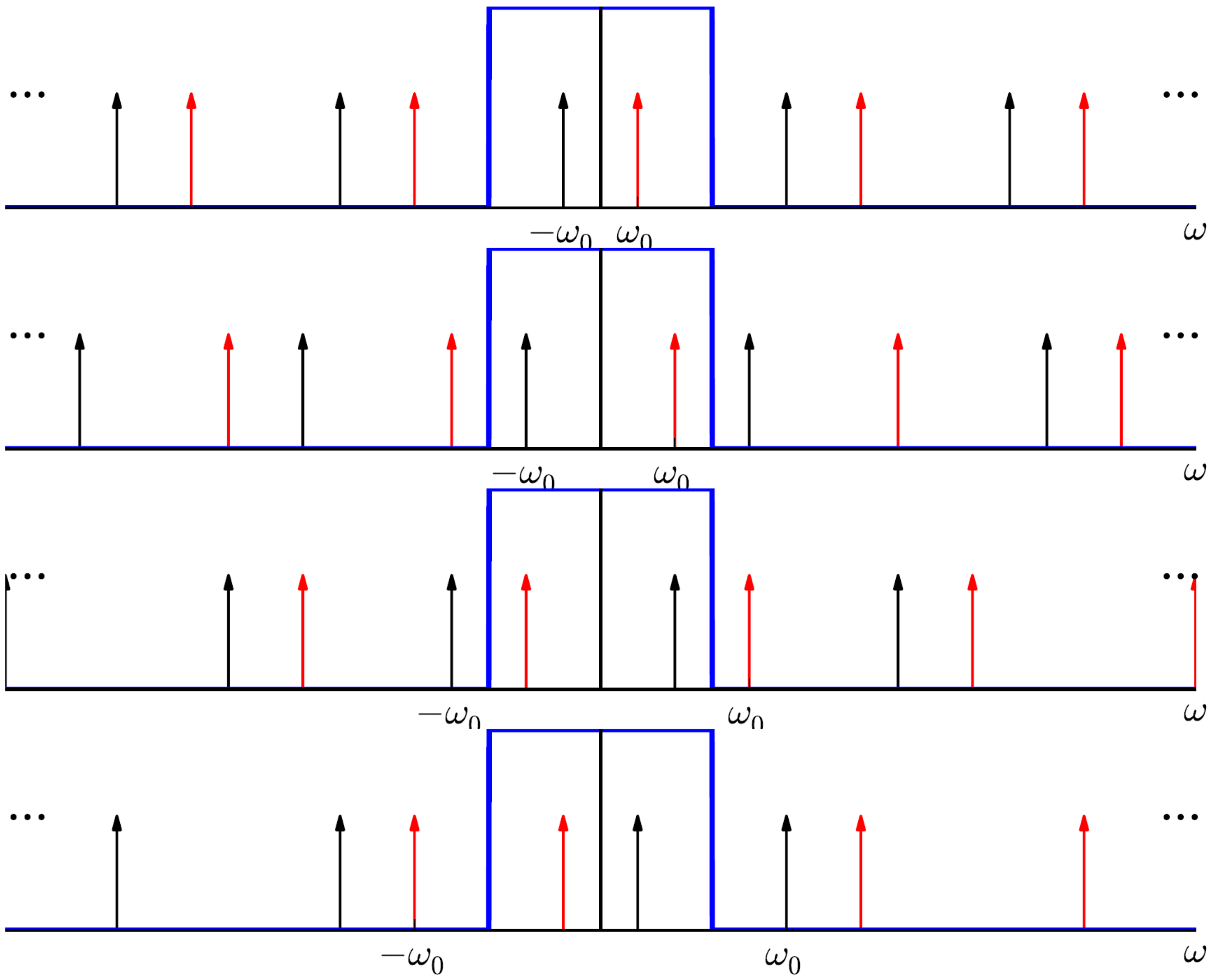
4: What are the rules then?

Let x(t) be a band-limited signal with $X(\omega) = 0$ for $|\omega| > B$. Then x(t) is **uniquely** determined by its samples $x(nT_s)$; $n = 0; \pm1; \pm2; \pm3; \ldots$
if $\omega_s \geq 2B$, where $\omega_s = 2\pi/T_s$.

Given these samples, we can reconstruct x(t) by generating an impulse train in which successive impulses are separated by T_s units of time and have the amplitudes of successive samples. If this impulse train is processing it through a lowpass filter with gain Ts and cutoff frequency ω_c such that $B < \omega_c < \omega_s - B$, the resulting signal is **exactly** x(t).

This theorem is also called Nyquist or Nyquist-Shannon Theorem and 2B is known as Nyquist frequency.

4: Undersampling and Aliasing: Sinusoid



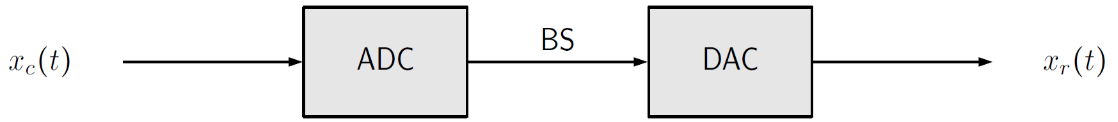
4: INTERPOLATION: Time Domain

In the time domain, an ideal lowpass filter is characterised by a sinc impulse response. Let $\mathrm{H}(\omega) = T_s$ for $|\omega| \le B$, and 0 elsewhere, then:

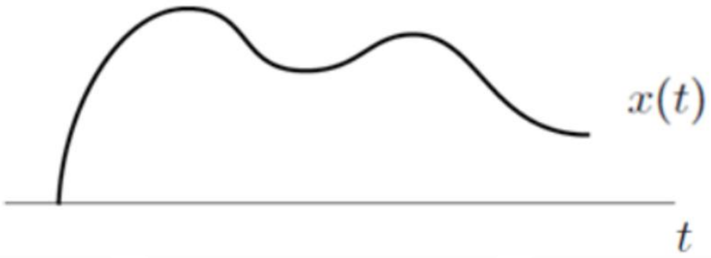
$$h(t) = \frac{T_s \sin(\pi t / T_s)}{\pi t}$$

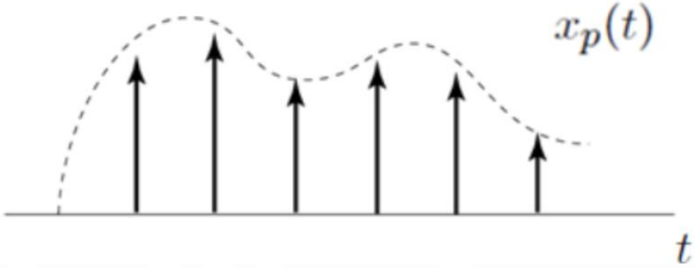
and $x_c(t)$ can be obtained in the time domain from $x_p(t)$ as follows:

$$\begin{aligned}
x_c(t) &= x_p(t) \star h(t) \\
&= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \right) \star h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} x_d[nT_s] \frac{T_s \sin(\pi(t/T_s - n))}{\pi(t - nT_s)}
\end{aligned}$$



4: INTERPOLATION: Time Domain





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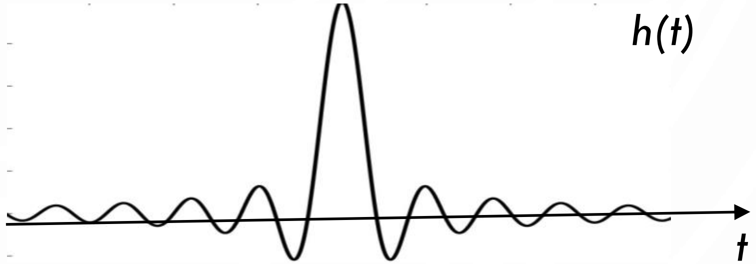
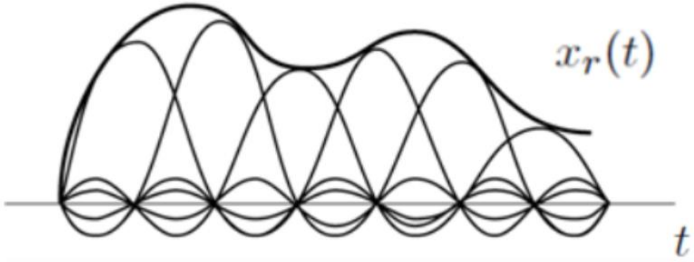




AGENDA-SUMMARY

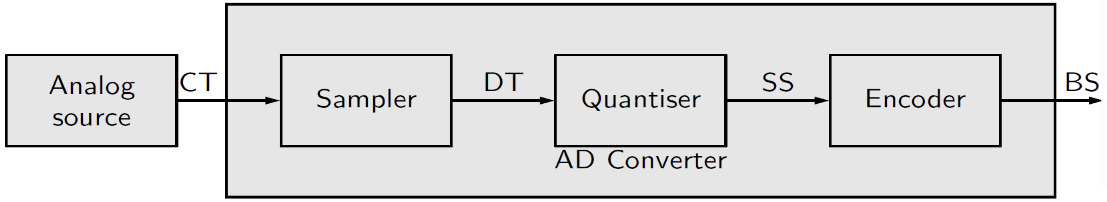
1. A strange world!

It can be if Nyquist-Shannon theorem is not respected!

2. Introduction to sampling.

Multiplication by impulse train.

3. Analog to digital conversion



4. Digital to analog conversion