2.2 Let X have distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < -1, \\ 1-p & \text{for } -1 \leqslant x < 0, \\ 1-p+\frac{1}{2}xp & \text{for } 0 \leqslant x \leqslant 2, \\ 1 & \text{for } x > 2. \end{cases}$$

Sketch this function and find (a) $P(X=-1)$, (b) $P(X=0)$, (C) $P(X>1)$.

Solution: (a) $P(X=-1) = F(-1) - F(-1^-) = 1-p-0 = 1-p$

(b) $P(X=0) = F(0) - F(0^-) = 1-p-1+p = 0$

(c) $P(X>1) = 1-F(1) = 1-1+p-\frac{1}{2}p = \frac{p}{2}$

2.4 Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ x/3 & \text{for } 0 \leqslant x < 1, \\ x/2 & \text{for } 1 \leqslant x < 2, \\ 1 & \text{for } x \geqslant 2. \end{cases}$$

Find

(a) $P(1/2 \leqslant X \leqslant 3/2)$,

(b) $P(1/2 \leqslant X \leqslant 1)$,

(c) $P(1/2 \leqslant X < 1)$,

(d) $P(1 \leqslant X \leqslant 3/2)$,

(e) $P(1 < X < 2)$.

Solution: (a) $P(\frac{1}{2} \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}^-) = \frac{3}{4} - \frac{1}{6} = \frac{7}{12}$

(b) $P(\frac{1}{2} \le X \le 1) = F(1) - F(\frac{1}{2}^-) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

(c) $P(\frac{1}{2} \le X < 1) = F(1^-) - F(\frac{1}{2}^-) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

(d) $P(1 \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(1^-) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$

(e) $P(1 < X < 2) = F(2^-) - F(1) = 1 - \frac{1}{2} = \frac{1}{2}$

2.6 A coin having probability p of coming up heads is successively flipped until the rth head appears. Argue that X, the number of flips required, will be $n, n \geqslant r$, with probability

$$P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n \geqslant r.$$

This is known as the *negative binomial distribution.*

Hint: How many successes must there be in the first $n-1$ trials?

Solution: When the number of coin tosses in n, in the first $n-1$ tosses, heads must have appeared $r-1$ times, and tails $n-r$ times.

$\therefore P(X=n) = C_{n-1}^{r-1} p^{r-1} (1-p)^{n-r} \cdot p = C_{n-1}^{r-1} p^r (1-p)^{n-r}$

2.7 Suppose that a coin having probability 0.7 of coming up heads is tossed three times. Let X denote the number of heads that appear in the three tosses. Determine the probability mass function of X.

Solution: when $x=0$, $P(X)=0.3^3=0.027$
when $x=1$, $P(X)=C_3^1\times0.7\times0.3^2=0.189$
when $x=2$, $P(X)=C_3^2\times0.7^2\times0.3=0.441$
When $x=3$, $P(X)=0.7^3=0.343$

$$F(x)=P(X=x)=\begin{cases} 0 & , x<0 \\ 0.027 & 0\le x<1 \\ 0.189 & 1\le x<2 \\ 0.441 & 2\le x<3 \\ 0.343 & 3\le x<4 \\ 1 & x\ge 4 \end{cases}$$

2.8 Let

$$p(x)=\begin{cases} a/8 & \text{when } x=-1, \\ a/4 & \text{when } x=0, \\ a/8 & \text{when } x=1, \end{cases}$$

where $a>0$. Find the constant a.

Solution: $\frac{a}{8}+\frac{a}{4}+\frac{a}{8}=1$
$a=2$

2.10 Suppose that the distribution function of X is given by

$$F(x)=\begin{cases} 0 & \text{for } x<0, \\ 1/2 & \text{for } 0\leqslant x<1, \\ 3/5 & \text{for } 1\leqslant x<2, \\ 4/5 & \text{for } 2\leqslant x<3, \\ 9/10 & \text{for } 3\leqslant x<3.5, \\ 1 & \text{for } x\geqslant 3.5. \end{cases}$$

What is the p.f. of X?

Solution: $P(X=0)=F(0)-F(0^-)=\frac{1}{2}$
$P(X=1)=F(1)-F(1^-)=\frac{3}{5}-\frac{1}{2}=\frac{1}{10}$
$P(X=2)=F(2)-F(2^-)=\frac{4}{5}-\frac{3}{5}=\frac{1}{5}$
$P(X=3)=F(3)-F(3^-)=\frac{9}{10}-\frac{4}{5}=\frac{1}{10}$
$P(X\geqslant 3.5)=1-F(3.5^-)=1-\frac{9}{10}=\frac{1}{10}$
$P(X<0)=F(0^-)=0$

2.11 Let

$$f(x) = \frac{3}{8}(1-x)^2, \quad \text{if } -1 < x < 1.$$

Calculate $F(0)$.

Solution: $F(0) = P(X \leq 0) = \int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} \frac{3}{8}(1-x)^2 dx = \frac{3}{8}x + \frac{1}{8}x^3 - \frac{3}{8}x^2 \Big|_{-1}^{0} = \frac{7}{8}$

2.13 For some constant c, the random variable X has the p.d.f.

$$f(x) = \begin{cases} cx^n & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find (a) the value of c, and (b) $P(X > x), 0 < x < 1$.

Solution: (a) $\int_0^1 f(x)\,dx = \int_0^1 cx^n\,dx = \frac{c}{n+1}x^{n+1} \Big|_0^1 = \frac{c}{n+1} = 1$

$\therefore c = n+1$

(b) $P(X > x) = \int_x^1 f(x)\,dx = 1 - x^{n+1}$