Chapter 8 Graph Theory 图论

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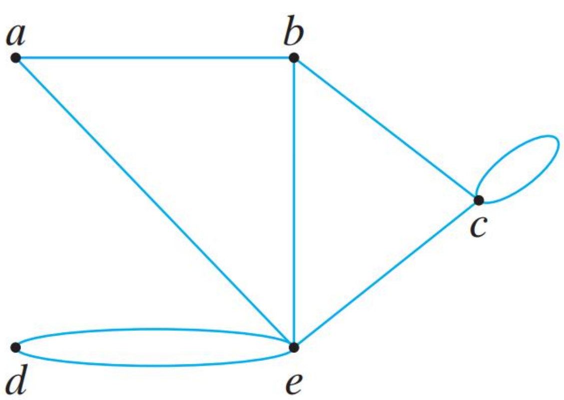
8.5 Representations of Graphs 图的表示

Adjacency Matrix (邻接矩阵)

- Select an ordering of the vertices.
- Label the rows and columns of a matrix with the ordered vertices.
- The entry in this matrix in row i, column j:

  if $i \neq j$, is the number of edges incident on i and j;

  if $i = j$, is twice the number of loops incident on i.



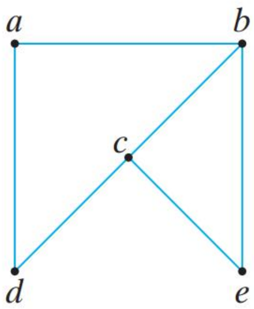
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

The degree of a vertex v in a graph G can be obtained by summing row v or column v in G's adjacency matrix.

8.5 Representations of Graphs 图的表示

Adjacency Matrix (邻接矩阵)

True or False?
The entry on the main diagonal of A^2 give the degrees of the vertices (when the graph is a simple graph).



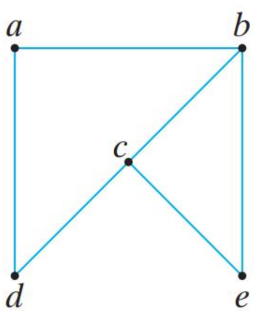
$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

If A is the adjacency matrix of a simple graph, the ijth entry of A^2 is equal to the number of paths of length 2 from vertex i to vertex j.

8.5 Representations of Graphs 图的表示

Adjacency Matrix (邻接矩阵)

Theorem 8.5.3 If A is the adjacency matrix of a simple graph, the ijth entry of A^n is equal to the number of paths of length n from vertex i to vertex j.



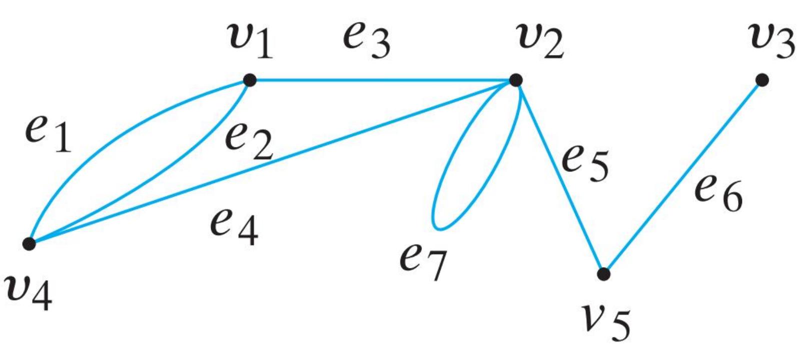
$$A^2=\begin{pmatrix}0&1&0&1&0\\1&0&1&0&1\\0&1&0&1&1\\1&0&1&0&0\\0&1&1&0&0\end{pmatrix}\begin{pmatrix}0&1&0&1&0\\1&0&1&0&1\\0&1&0&1&1\\1&0&1&0&0\\0&1&1&0&0\end{pmatrix}=\begin{array}{c c}& \begin{array}{ccccc}a&b&c&d&e\end{array}\\ \begin{array}{c}a\\b\\c\\d\\e\end{array} & \begin{pmatrix}2&0&2&0&1\\0&3&1&2&1\\2&1&3&0&1\\0&2&0&2&1\\1&1&1&1&2\end{pmatrix}\end{array}$$

If A is the adjacency matrix of a simple graph, the ijth entry of A^2 is equal to the number of paths of length 2 from vertex i to vertex j.

8.5 Representations of Graphs 图的表示

Incidence Matrix (关联矩阵)

- Label the rows with the vertices. Label the columns with the edges.
- If e is incident on v, the entry for row v and column e is 1, otherwise it is 0.



$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

True or False? In a graph without loops
each column has two 1's.
and the sum of a row gives the degree of the vertex identified with that row.

8.5 Representations of Graphs 图的表示

Exercise 1 Draw the graph represented by the 7×7 adjacency matrix whose ij th entry is 1 if $i+1$ divides $j+1$ or $j+1$ divides $i+1$, $i \neq j$; whose ij th entry is 2 if $i=j$; and whose ij th entry is 0 otherwise.

	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							

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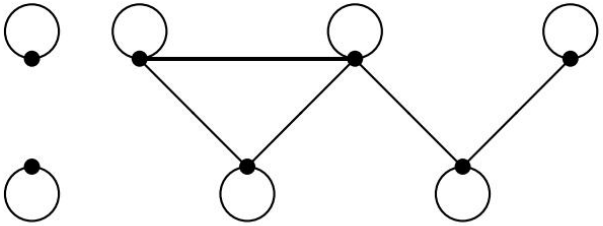
	1	2	3	4	5	6	7
1	2		1		1		1
2		2			1		
3	1		2				1
4				2			
5	1	1			2		
6						2	
7	1		1				2



8.5 Representations of Graphs 图的表示

Exercise 2 What must a graph look like if some row of its incidence matrix consists only of 0’s?

8.6 Isomorphisms of Graphs 图的同构

Definition 8.6.1 G_1 and G_2 are **isomorphic (同构的)** if there exist a one-to-one, onto functions f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2. The pair of functions f and g is called an **isomorphism of G_1 onto G_2 (G_1到G_2上的同构映射)**.

one-to-one, onto function？

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

Definition 3.1.29 A function f from X to Y is said to be **onto Y (or surjective) (满射的)** if **for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$**.

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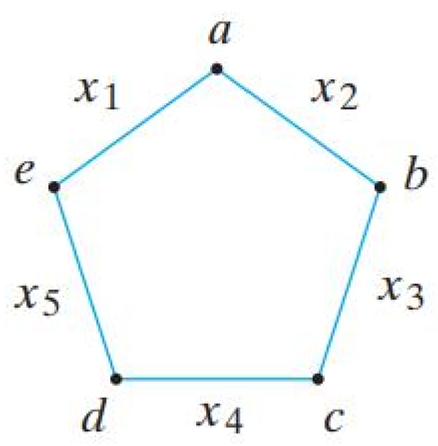
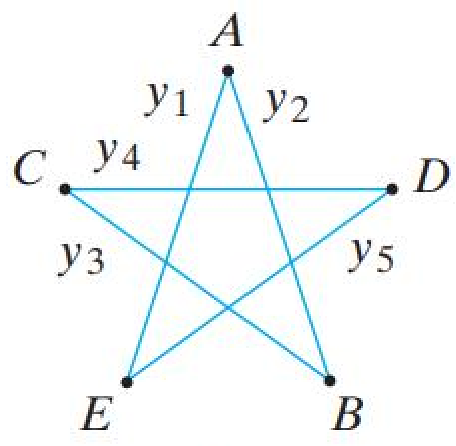


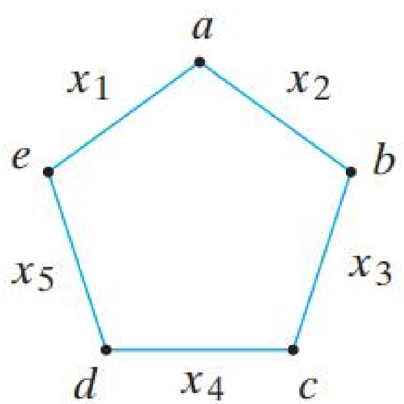
G_1

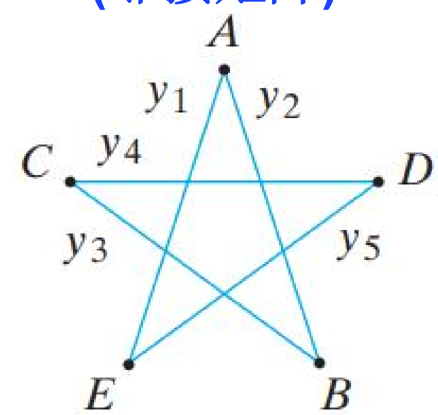


G_2

8.6 Isomorphisms of Graphs 图的同构

Theorem 8.6.4 Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their **adjacency matrices (邻接矩阵)** are equal.





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↓

incidence matrices (关联矩阵)
for some ordering of their vertices and edges

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Corollary 8.6.5 Let G_1 and G_2 be **simple graphs**. The following are equivalent:
(a) G_1 and G_2 are isomorphic.
(b) There is a one-to-one, onto function f from the vertex set of G_1 to the vertex set of G_2 satisfying the following: Vertices v and w are adjacent in G_1 if and only if the vertices $f(v)$ and $f(w)$ are adjacent in G_2.

8.6 Isomorphisms of Graphs 图的同构

How to prove that two simple graphs Graphs G_1 and G_2 are not isomorphic?

Find a property of G_1 that G_2 does not have but that G_2 would have if G_1 and G_2 were isomorphic. Such a property is called an **invariant (不变量)**.

A property P is an invariant if whenever G_1 and G_2 are isomorphic graphs:

If G_1 has property P, G_2 also has property P.

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If G_1 has property P, G_2 also has property P.

- If G_1 and G_2 are isomorphic, then G_1 and G_2 have the same number of edges and the same number of vertices.
- If k is a positive integer, "has a vertex of degree k" is an invariant.
- If l is a positive integer, "has a simple cycle of length l" is an invariant.

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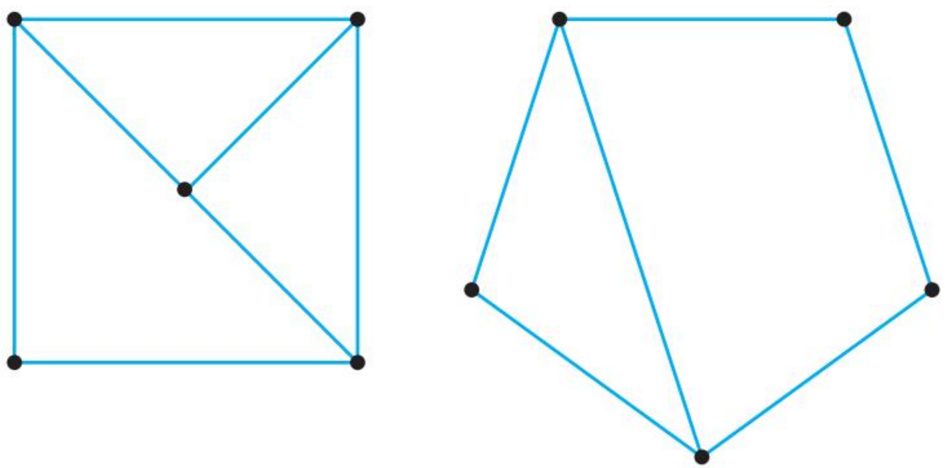
Example 8.6.7



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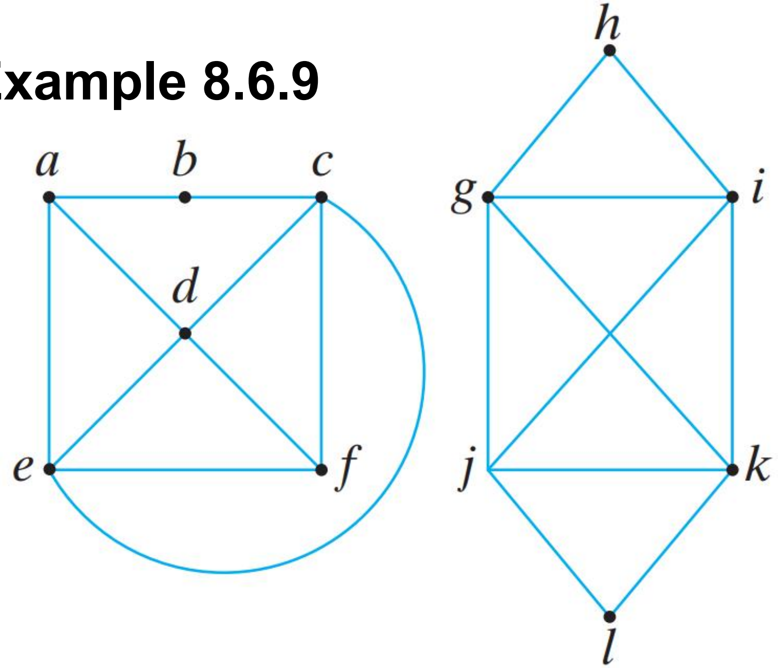

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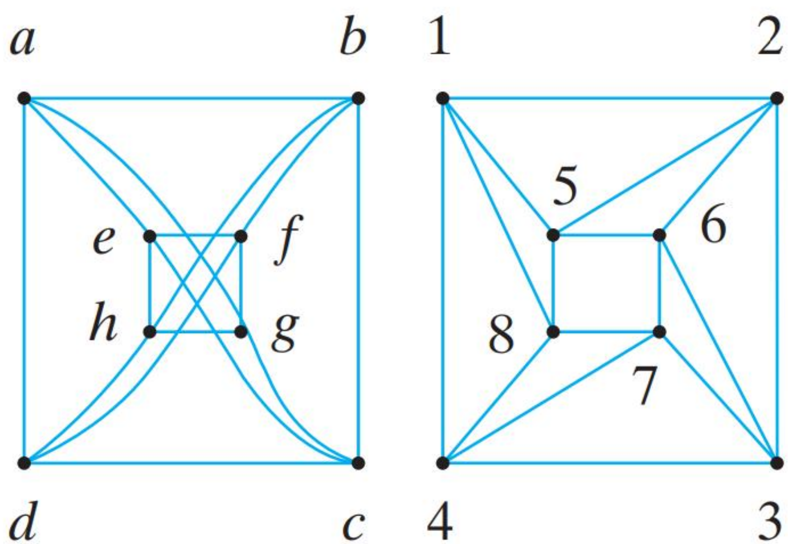
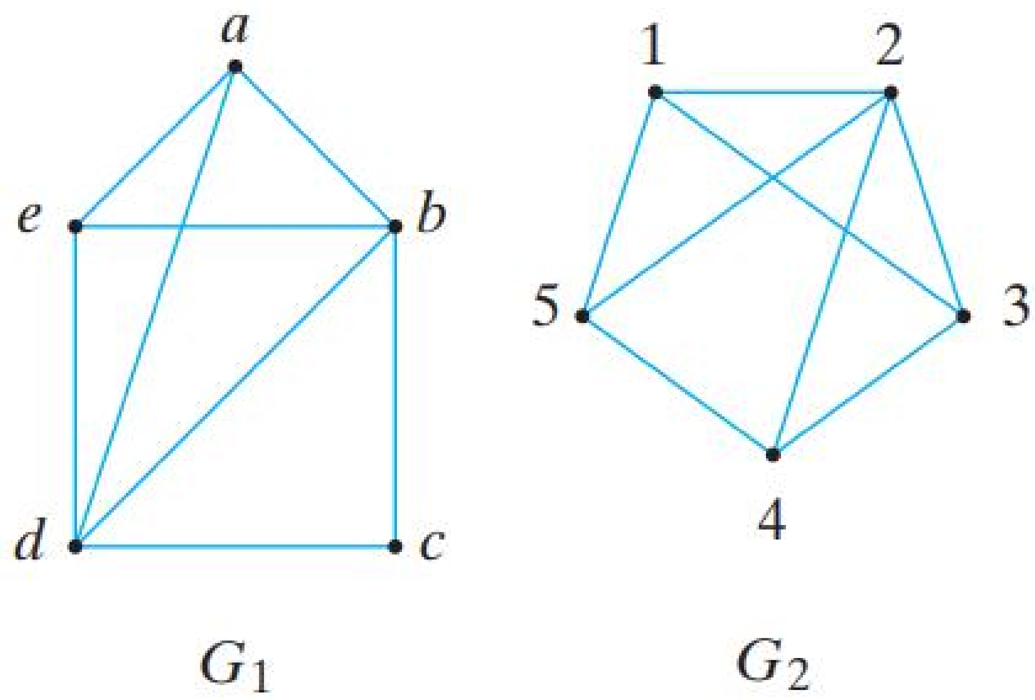
Example 8.6.10



8.6 Isomorphisms of Graphs 图的同构

Determine whether the graphs G_1 and G_2 are isomorphic.



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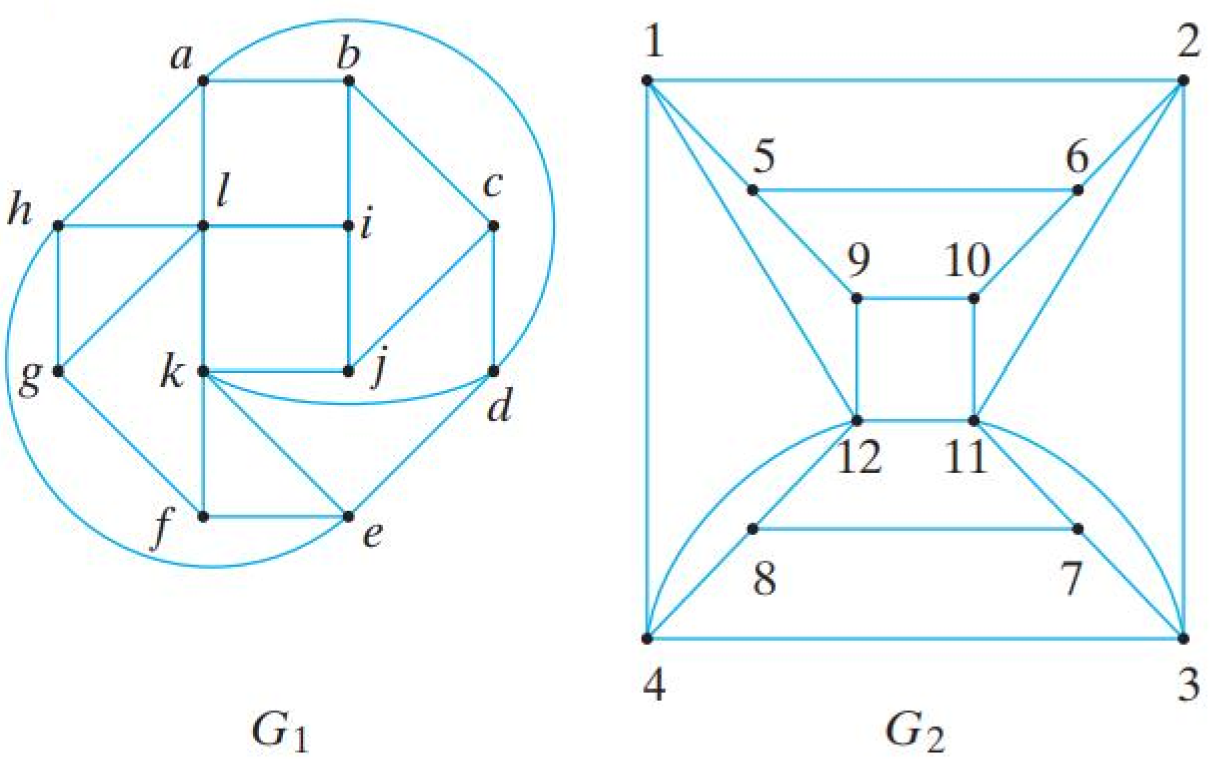

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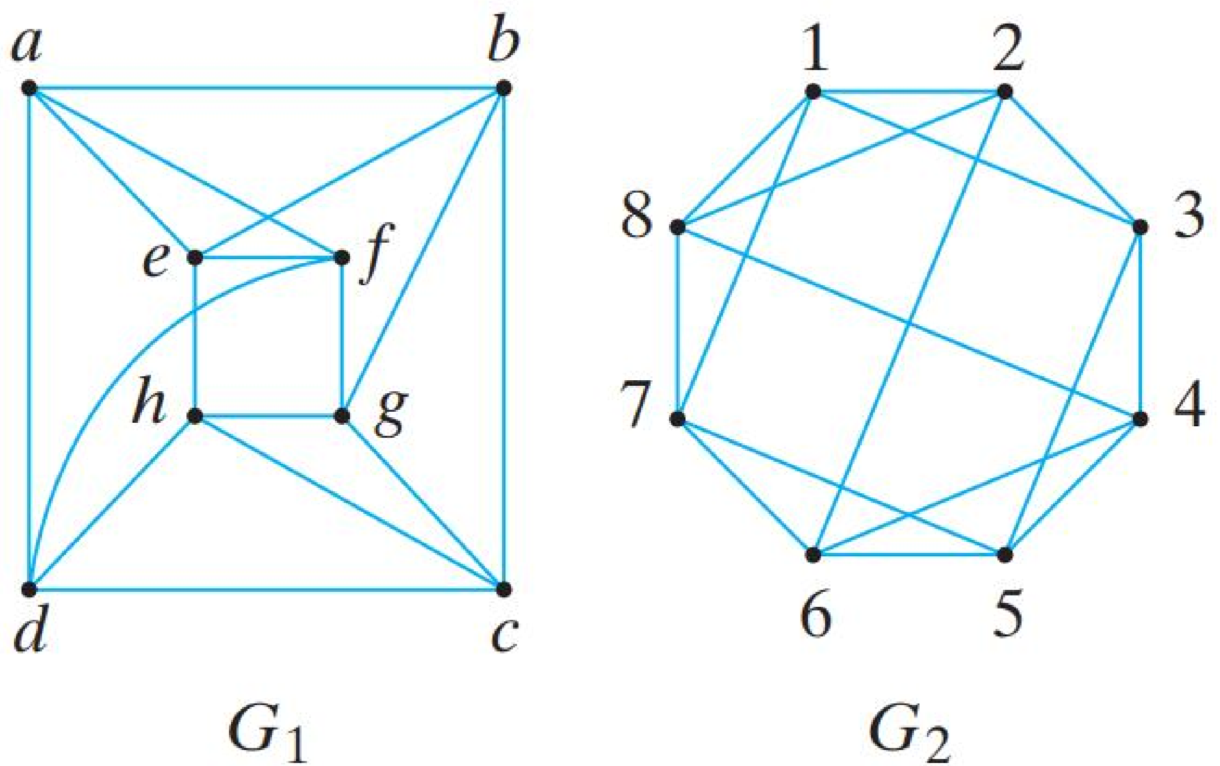

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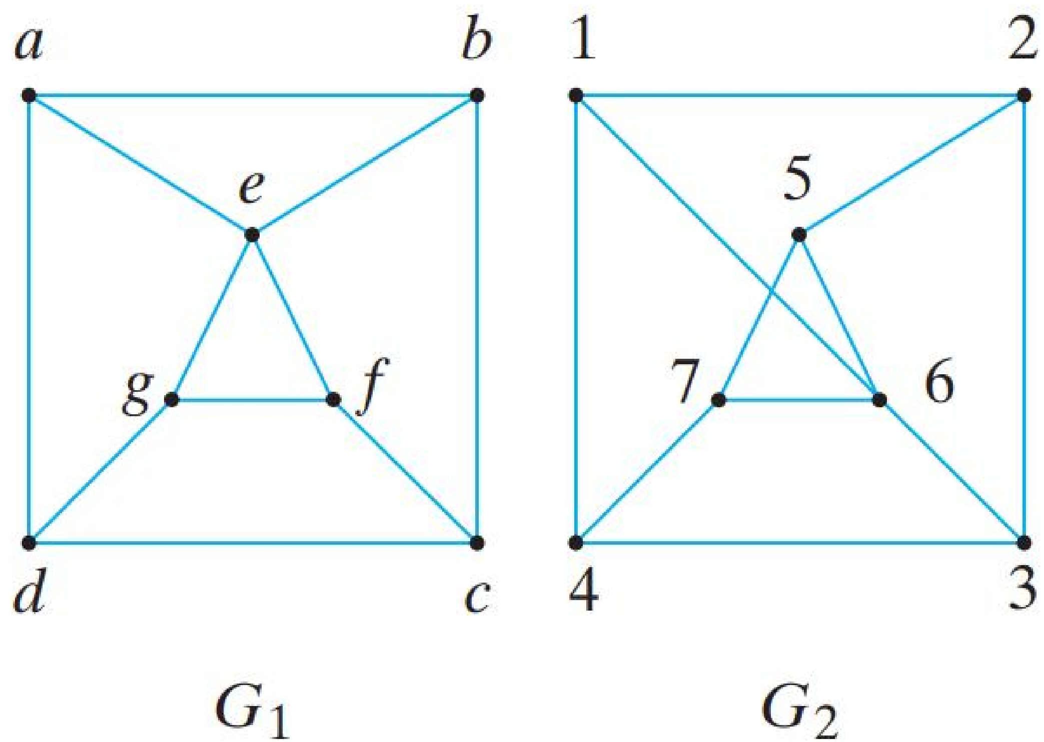

8.7 Planar Graphs 平面图

Definition 8.7.1 A graph is **planar (平面图)** if it can be drawn in the plane without its edges crossing.

Application: In designing printed circuits it is desirable to have as few lines cross as possible; thus the designer of printed circuits faces the problem of planarity.

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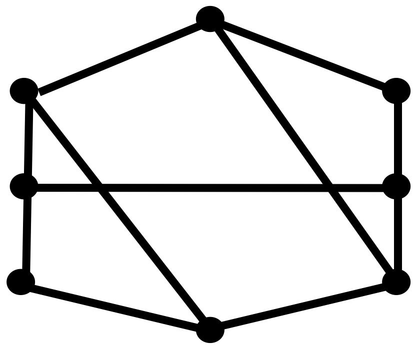
Exercise 1 Show that the following graph is planar by redrawing it so that no edges cross.



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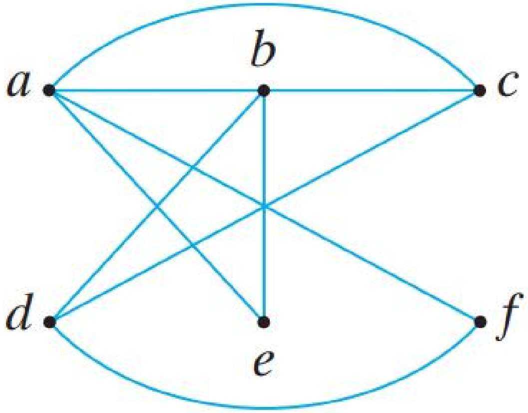

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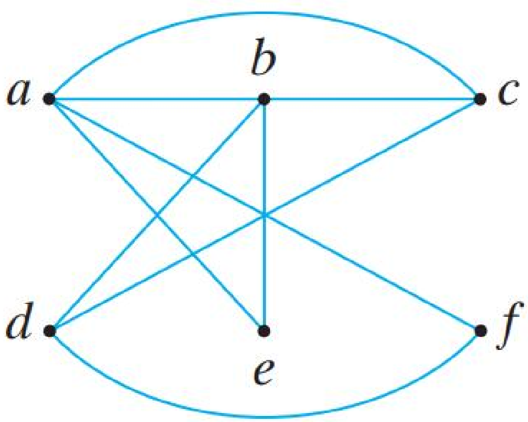

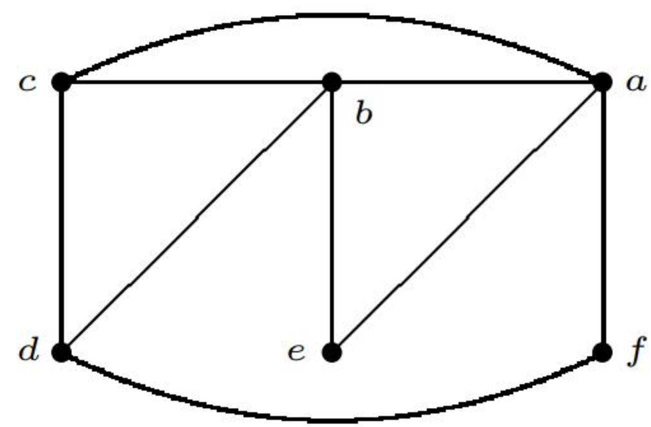

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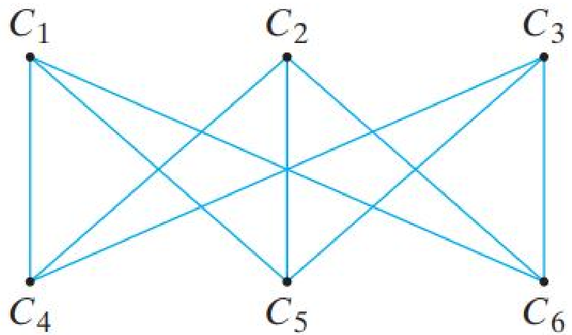
Whether graph $K_{3,3}$ is planar?



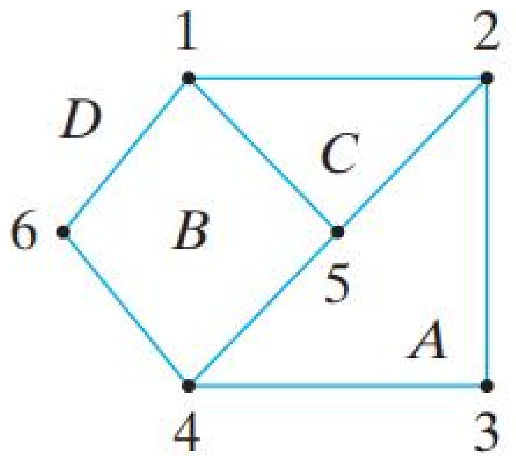
$K_{3,3}$

8.7 Planar Graphs 平面图

Definition 8.7.1 A graph is **planar (平面图)** if it can be drawn in the plane without its edges crossing.

If a connected, planar graph is drawn in the plane, the plane is divided into contiguous regions called **faces (面)**.

- A face is characterized by the cycle that forms its boundary.



Face A is bounded by the cycle (5, 2, 3, 4, 5).

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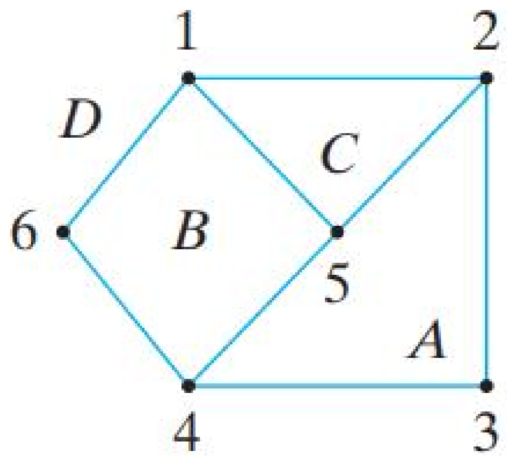


The graph has $f = 4$ faces, $e = 8$ edges, and $v = 6$ vertices.

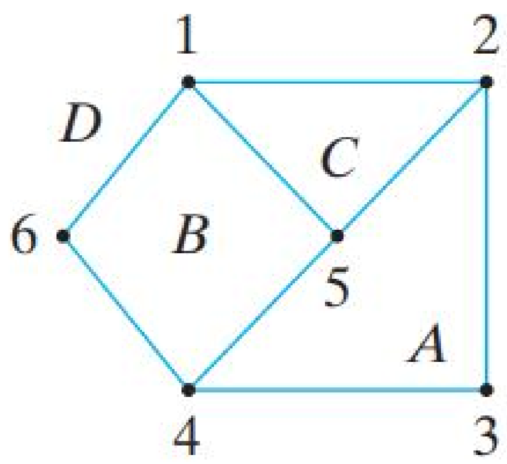
$$f = e - v + 2.$$

Face A is bounded by the cycle (5, 2, 3, 4, 5).

8.7 Planar Graphs 平面图

Theorem 8.7.9 Euler's Formula for Graphs (图的欧拉公式)
If G is a connected, planar graph with e edges, v vertices, and f faces, then

$$f = e - v + 2.$$



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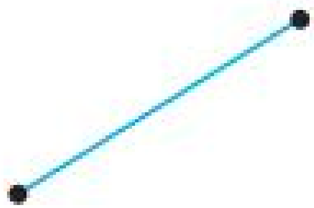
Proof We will use induction on the number of edges.

Basic Step $e = 1$



8.7 Planar Graphs 平面图

Theorem 8.7.9 Euler's Formula for Graphs (图的欧拉公式)
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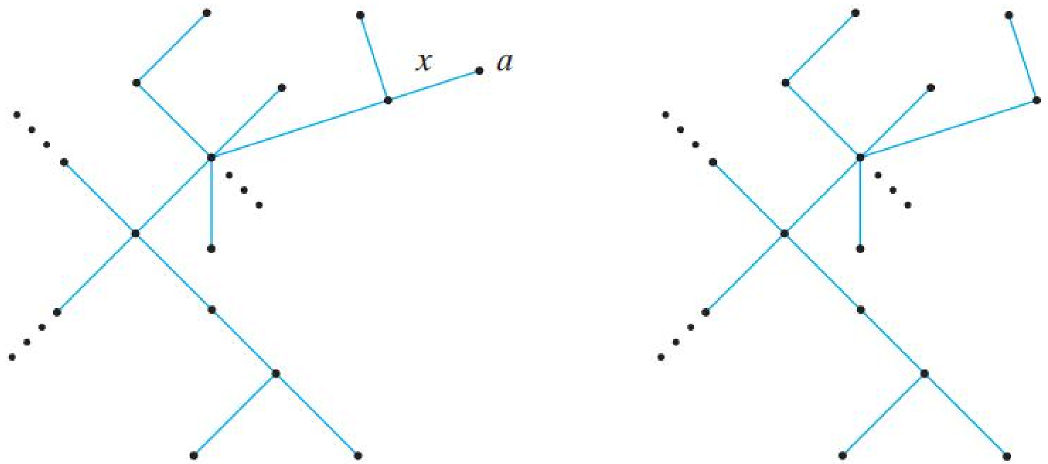
Inductive Step Suppose that the formula holds for connected, planar graphs with n edges.



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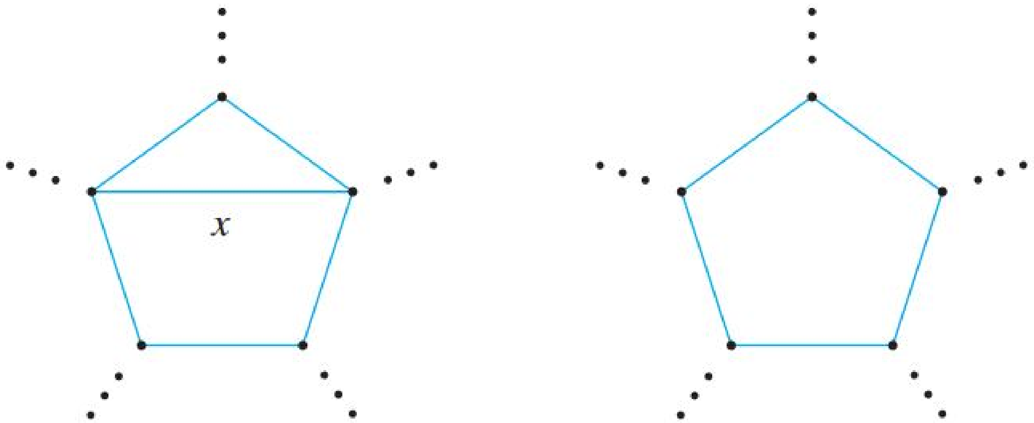


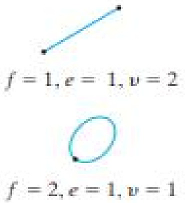


Figure 8.7.8 The Basis Step of Theorem 8.7.9.

Proof We will use induction on the number of edges.

Suppose that $e = 1$. Then G is one of the two graphs shown in Figure 8.7.8. In either case, the formula holds. We have verified the Basis Step.

Suppose that the formula holds for connected, planar graphs with n edges. Let G be a graph with $n + 1$ edges. First, suppose that G contains no cycles. Pick a vertex v and trace a path starting at v. Since G is cycle-free, every time we trace an edge, we arrive at a new vertex. Eventually, we will reach a vertex a, with degree 1, that we cannot leave (see Figure 8.7.9). We delete a and the edge x incident on a from the graph G. The resulting graph G' has n edges; hence, by the inductive assumption, (8.7.3) holds for G'. Since G has one more edge than G', one more vertex than G', and the same number of faces as G', it follows that (8.7.3) also holds for G.

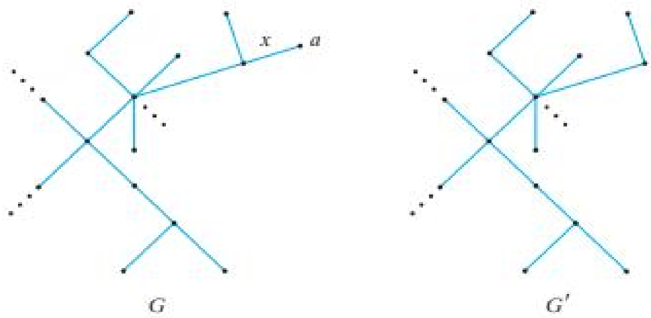


Figure 8.7.9 The proof of Theorem 8.7.9 for the case that G has no cycles. We find a vertex a of degree 1 and delete a and the edge x incident on it.

Now suppose that G contains a cycle. Let x be an edge in a cycle (see Figure 8.7.10). Now x is part of a boundary for two faces. This time we delete the edge x but no vertices to obtain the graph G' (see Figure 8.7.10). Again G' has n edges; hence,

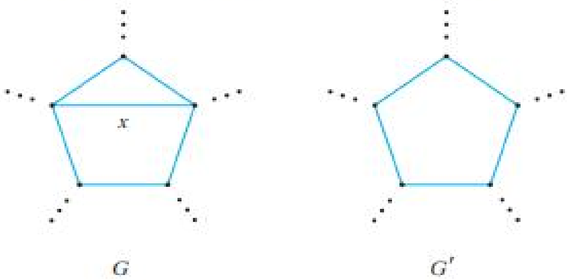


Figure 8.7.10 The proof of Theorem 8.7.9 for the case that G has a cycle. We delete edge x in a cycle.

by the inductive assumption, (8.7.3) holds for G'. Since G has one more face than G', one more edge than G', and the same number of vertices as G', it follows that (8.7.3) also holds for G.

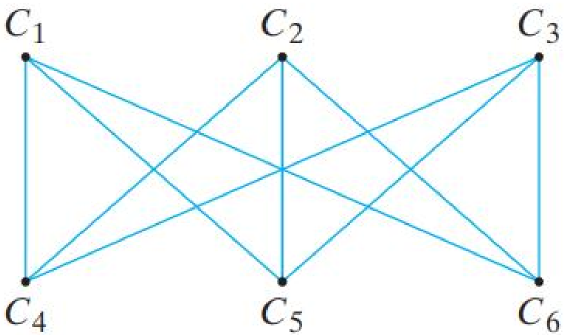
Since we have verified the Inductive Step, by the Principle of Mathematical Induction, the theorem is proved. ◀

8.7 Planar Graphs 平面图

Theorem 8.7.9 Euler's Formula for Graphs (图的欧拉公式)
If G is a connected, planar graph with e edges, v vertices, and f faces, then

$$f = e - v + 2.$$

Whether graph $K_{3,3}$ is planar?



$K_{3,3}$

$\boldsymbol{2e \geq 4f}$

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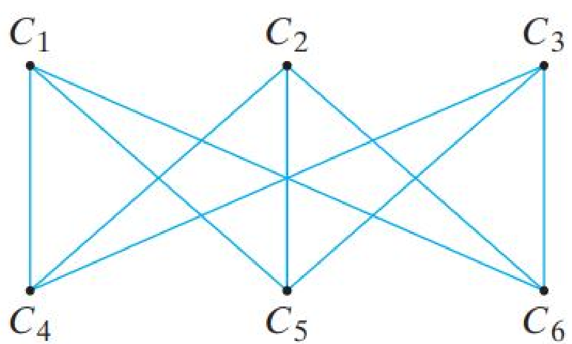

$K_{3,3}$

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8.7 Planar Graphs 平面图

Definition 8.7.3 If a graph G has a vertex v of degree 2 and edges (v, v_1) and (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in **series (串联的)**. A **series reduction (串联约减)** consists of deleting the vertex v from the graph G and replacing the edges (v, v_1) and (v, v_2) by the edge (v_1, v_2). The resulting graph G' is said to be obtained from G by a series reduction. By convention, G is said to be obainable from itself by a series reduction.

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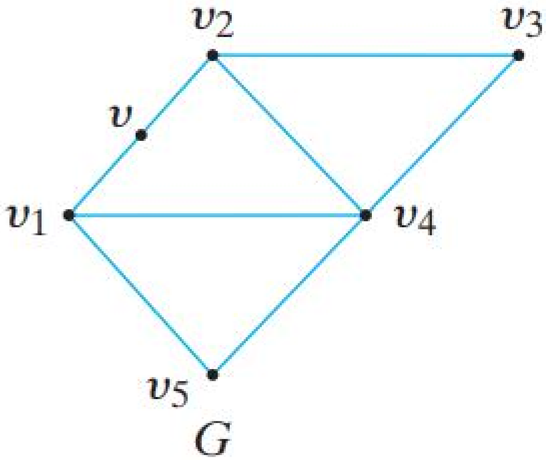

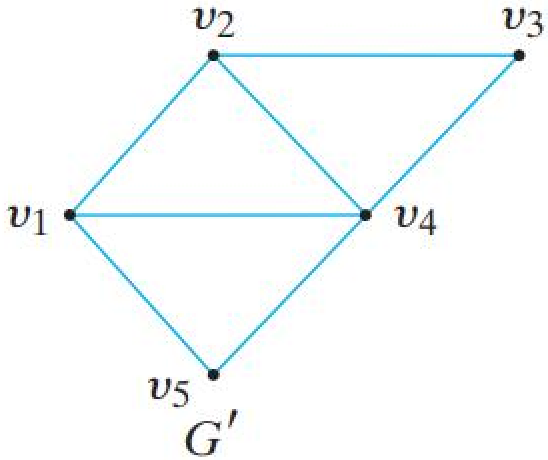

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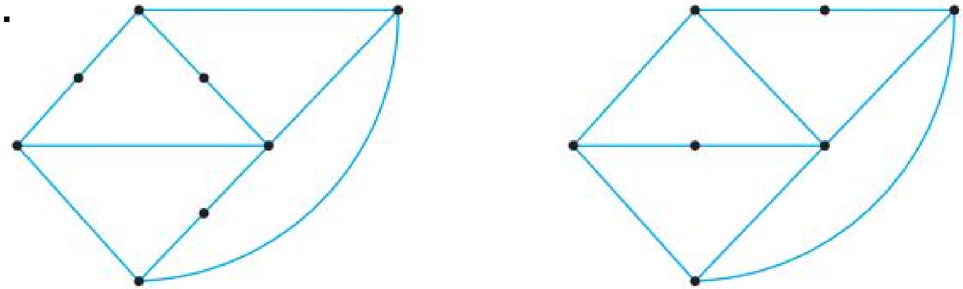
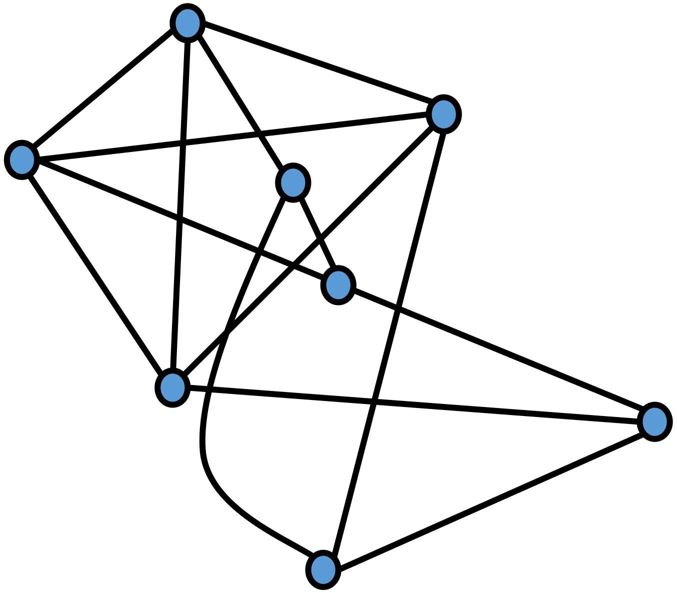
Definition 8.7.5 Graph G_1 and G_2 are **homeomorphic (同胚的)** if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.



8.7 Planar Graphs 平面图

Theorem 8.7.7 Kuratowski's Theorem 库拉托夫斯基定理
A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 and $K_{3,3}$.

Example 8.7.8



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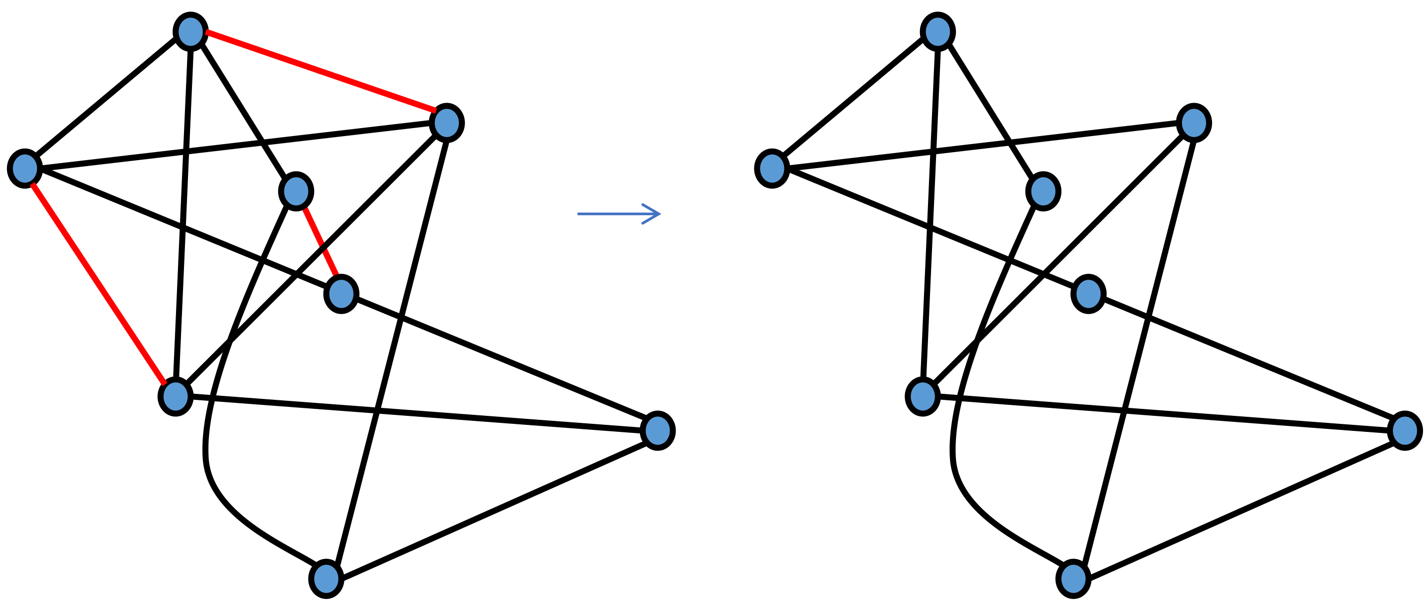

8.7 Planar Graphs 平面图

Exercise 1 Show that in any simple, connected, planar graph, $e \leq 3v - 6$.

8.7 Planar Graphs 平面图

Exercise 2 Give an example of a simple, connected, nonplanar graph for which $e \leq 3v - 6$.

8.7 Planar Graphs 平面图

Show that each graph is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$.