1.2 Propositions 命题

Logic is a system based on **propositions**.
A sentence that is either true or false, but not both, is called a **proposition**.

Truth Value of a proposition

True: T

False: F

1.2 Propositions

Logic is a system based on **propositions**.
A sentence that is either true or false, but not both, is called a **proposition**.

We use variables, such as $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$, to represent propositions.

We will also use the notation

$$\boldsymbol{p}\text{: } \mathbf{1+1=3}$$

to define *p* to be the proposition 1+1=3.

1.2 Propositions

Definition 1.2.1 Let p and q be propositions.

The conjunction（合取）of p and q, denote $\boldsymbol{p \wedge q}$, is the proposition

p **and** q.

The disjunction（析取）of p and q, denote $\boldsymbol{p \vee q}$, is the proposition

p **or** q.

Example: If p: It is raining, and q: It is cold, then

$p \wedge q$: ?

$p \vee q$: ?

1.2 Propositions

Definition 1.2.9 The negation of p, denote $\neg \boldsymbol{p}$, is the proposition **not** p.

Example: Tim is a boy.

Logic Operators: Exclusive-Or (异或)

$\wedge$ ::= AND $\vee$ ::= OR (兼或) $\neg$ ::= NOT

$\oplus$ exclusive-or（异或）

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.3 Conditional Propositions and Logical Equivalence

Definition 1.3.1 A **conditional proposition (条件命题)** is of the form

"If p then q"

In symbols: $p \to q$.

1.3 Conditional Propositions and Logical Equivalence

Definition 1.3.1 A **conditional proposition (条件命题)** is of the form

"If p then q"

In symbols: $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional proposition that is true because the hypothesis is false is said to be **true by default** （默认为真） or **vacuously true**（空虚真）.

1.3 Conditional Propositions and Logical Equivalence

Your parents say: “If your got at least 85 in the this course, then I will buy you a gift.”

When is the above sentence false?

- It is false when you get an 85 but your parents do not buy you a gift.
- In particular, it is not false if your score is below 85.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	**T**	**T**
F	**F**	**T**

$\wedge$ ::= AND $\quad \vee$ ::= OR $\quad \neg$ ::= NOT

$\rightarrow$ ::= IMPLIES

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT $\rightarrow$ ::= IMPLIES

Operator Precedence 操作符的优先级

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT $\rightarrow$ ::= IMPLIES

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
then $\vee$,
and then $\rightarrow$.

Example: $p \vee q \longrightarrow \neg r$

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \land \neg q) \lor (\neg p \land q)$$

$$\neg(p \land q) \land \neg(\neg p \land \neg q)$$

$$\vdots$$

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$\boxed{(p \wedge \neg q) \vee (\neg p \wedge q)}$$

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

Idea 1: Look at the true rows

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

$$\boxed{\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)}$$

Idea 2: Look at the false rows

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 1:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 1:

$$(p \wedge q \wedge \neg r)$$
$$\vee (p \wedge \neg q \wedge r)$$
$$\vee (\neg p \wedge q \wedge r)$$
$$\vee (\neg p \wedge q \wedge \neg r)$$
$$\vee (\neg p \wedge \neg q \wedge r)$$

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 2:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 2:

$$\neg(p \wedge q \wedge r)$$

$$\wedge \neg(p \wedge \neg q \wedge \neg r)$$

$$\wedge \neg(\neg p \wedge \neg q \wedge \neg r)$$

$$p \rightarrow q \equiv ?$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 1:

Idea 2:

Example 1.3.13

Which proposition is logically equivalent to the negation of $p \longrightarrow q$?

Example 1.3.13

Show that the negation of $p \longrightarrow q$ is logically equivalent to $p \land \neg q$.

De Morgan’s Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Statement: The number 6 is divisible by 2 or 5.

Negation: The number 6 is not divisible by 2 and not divisible by 5.

De Morgan's Laws for Logic

$$\boxed{\neg(p \wedge q) \equiv \neg p \vee \neg q}$$

Truth Table

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

Are these two propositions logically equivalent?

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Are these two propositions logically equivalent?

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

Solution: Let p: Jerry receives a scholarship, and q: Jerry goes to college. The given proposition can be written symbolically as $p \longrightarrow q$. The coverse of the proposition is

If Jerry goes to college, then he receives a scholarship.

The converse can be written as $q \longrightarrow p$.

Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

The original proposition is true and its converse is false.

Biconditional Proposition 双条件命题

Definition 1.3.8

If p and q are propositions, the proposition

$$p \text{ if and only if } q,$$

is called a biconditional proposition and is denoted

$$p \leftrightarrow q.$$

$\neg$

Biconditional Proposition 双条件命题

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If p and q are propositions, the proposition

$$p \text{ if and only if } q,$$

is called a biconditional proposition and is denoted

$$p \leftrightarrow q.$$

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Biconditional Proposition 双条件命题

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p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Proposition 双条件命题

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If p and q are propositions, the proposition

$$p \text{ if and only if } q,$$

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$$p \leftrightarrow q.$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$$\leftrightarrow \ ::= \mathrm{IFF}$$

Contrapositive (Transposition) Proposition
逆否命题（转换命题）

Definition 1.3.16

The contrapositive (or transposition) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \longrightarrow \neg p$.

Contrapositive (Transposition) Proposition
逆否命题（转换命题）

Definition 1.3.16

The contrapositive (or transposition) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \longrightarrow \neg p$.

The conditional proposition $p \rightarrow q$ and its contrapositive $\neg q \longrightarrow \neg p$ are logically equivalent.

Proof by Truth Table

If p, then q.

If $\neg q$, then $\neg p$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg q$	$\neg p$	$\neg q \to \neg p$
F	F	T
T	F	F
F	T	T
T	T	T

Propositional Equivalent

$$\neg(\neg A) \equiv A$$

$$A \vee A \equiv A$$

$$A \wedge A \equiv A$$

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$

$$(A \vee B) \vee C \equiv A \vee (B \vee C)$$

$$A \vee B \equiv B \vee A$$

$$A \wedge B \equiv B \wedge A$$

Propositional Equivalent

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

$\neg (A \vee B) \equiv \neg A \wedge \neg B$

$\neg (A \wedge B) \equiv \neg A \vee \neg B$

$A \wedge (A \vee B) \equiv A$

$A \vee (A \wedge B) \equiv A$

$A \wedge T \equiv$?

$A \vee F \equiv$?

Propositional Equivalent

$$A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$$

$$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$$

$$\neg (A \lor B) \equiv \neg A \land \neg B$$

$$\neg (A \land B) \equiv \neg A \lor \neg B$$

$$A \land (A \lor B) \equiv A$$

$$A \lor (A \land B) \equiv A$$

$$A \land T \equiv A$$

$$A \lor F \equiv A$$

Propositional Equivalent

$A \land F \equiv$?

$A \lor T \equiv$?

$A \lor (\neg A) \equiv$?

$A \land (\neg A) \equiv$?

$A \to B \equiv \neg A \lor B$

$A \leftrightarrow B \equiv (A \to B) \land (B \to A)$

$A \leftrightarrow B \equiv (A \land B) \lor (\neg A \land \neg B)$? Use Truth table to prove it?

Propositional Equivalent

$$A \wedge F \equiv F$$

$$A \vee T \equiv T$$

$$A \vee (\neg A) \equiv T$$

$$A \wedge (\neg A) \equiv F$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$$

Idea 1

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$?

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \equiv \neg A$$

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \equiv \neg A \ ?$$

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$

$$(A \rightarrow B) \land (A \rightarrow \neg B) \equiv \neg A \ ?$$

① **Use truth table.**

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \equiv \neg A$$?

② **Simplify.**

Simplifying Statement

$$\neg(\neg p \land q) \land (p \lor q)$$

Tautology, Contradiction 重言式/永真式，矛盾式

A tautology is a statement that is always true.

$$p \vee \neg p$$

$$(p \wedge q) \vee (\neg q \wedge p) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q)$$

- A compound proposition is called tautology if and only if it is true for all possible truth values of its propositional variables.
- It contains only T (Truth) in last column of its truth table.

Tautology, Contradiction 重言式/永真式，矛盾式

Is proposition $p \rightarrow (p \vee q)$ a tautology?

Tautology, Contradiction 重言式/永真式，矛盾式

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Tautology, Contradiction 重言式/永真式，矛盾式

A contradiction is a statement that is always false. (negation of a tautology)

$$p \wedge \neg p$$

$$(p \vee q) \wedge (\neg q \vee p) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee q)$$

- A compound proposition is called contradiction if and only if it is false for all possible truth values of its propositional variables.
- It contains only F (False) in last column of its truth table.

Contingency

- A compound proposition is called contingency if and only if it is neither a tautology nor a contradiction.
- It contains both T (True) and F (False) in last column of its truth table.

Contingency

$$((p \wedge r) \vee (q \wedge r)) \wedge (\neg(p \vee q) \vee r)$$

(A) It is a tautology.
(B) It is a contradiction.
(C) It is not sure.

Solution: (C)

- A compound proposition is called contingency if and only if it is neither a tautology nor a contradiction.
- It contains both T (True) and F (False) in last column of its truth table.

If, Only-If

- You will succeed if you work hand.
- You will succeed only if you work hard.

r if s means "if s then r"
We also say r is a necessary condition for s.

r only if s means "if r then s"
We also say r is a sufficient condition for s.

Math vs Language

Parent: if you don't clean your room, then you can't watch a DVD.

$$\underbrace{\text{clean your room}}_{C} \qquad \underbrace{\text{watch a DVD}}_{D}$$

This sentence says $\neg C \rightarrow \neg D$

In real life it also means $C \rightarrow D$

So $C \leftrightarrow D$

Math vs Language

Parent: if you don't $\underbrace{\text{clean your room}}_{\text{C}}$, then you can't $\underbrace{\text{watch a DVD}}_{\text{D}}$.

This sentence says $\neg C \rightarrow \neg D$

In real life it also means $C \rightarrow D$

So $C \leftrightarrow D$

Mathematician: if a number x greater than 2 is not an odd number, then x is not a prime number.

This sentence says $\neg O \rightarrow \neg P$

But of course it doesn't mean $O \rightarrow P$

Problem-Solving Tips

- In formal logic, "if" and "if and only if" are quite different. The conditional proposition $p \to q$ (if p then q) is true except when p is true and q is false. On the other hand, the biconditional proposition $p \leftrightarrow q$ (p if and only if q) is true precisely when p and q are both true or both false.
- To determine whether propositions P and Q, made up of the propositions $p_1, \ldots,$ p_n, are logically equivalent, write the truth tables for P and Q. If all of the entries for P and Q are always both true or both false, then P and Q *are* equivalent. If some entry is true for one of P or Q and false for the other, then P and Q are *not* equivalent.
- De Morgan's laws for logic

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

give formulas for negating "or" ($\vee$) and negating "and" ($\wedge$). Roughly speaking, negating "or" results in "and," and negating "and" results in "or."

Problem-Solving Tips

- Example 1.3.13 states a very important equivalence

$$\neg(p \to q) \equiv p \wedge \neg q,$$

which we will meet throughout this book. This equivalence shows that the negation of the conditional proposition can be written using the "and" ($\wedge$) operator. Notice that there is no conditional operator on the right-hand side of the equation.