EBU4375: SIGNALS AND SYSTEMS

LECTURE 10

Queen Mary
University of London

Convolution (CT Systems)

Convolution (CT Systems) – Recap

The impulse-response (IR) $h(t)$ of a continuous-time LTI system is defined to be the response following excitation by the signal $\delta(t)$ i.e.

$$h(t) = T\{\delta(t)\}$$

system response to the impulse

"impulse" excitation

Transfer Function

Convolution (CT Systems) – Recap

Recall the earlier result that a general signal could be expressed by

$$x(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t-\tau)$$

Since the system is **linear**, the response $y(t)$ to an excitation $x(t)$ can be written as

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t-\tau)\right\}$$

$$= \int_{-\infty}^{\infty} d\tau \, x(\tau) T\{\delta(t-\tau)\} \qquad (1)$$

Time-invariance implies

$$h(t-\tau) = T\{\delta(t-\tau)\} \qquad (2)$$

$$(2) \rightarrow (1) \Rightarrow y(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau) h(t-\tau) \qquad (3)$$

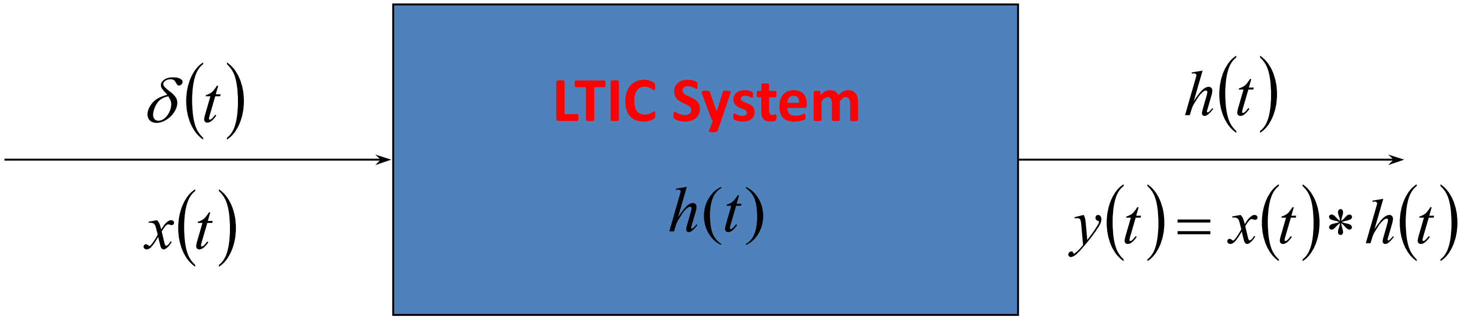
(3) says that the continuous-time response of an LTI system is entirely characterised by it's impulse response $h(t)$.

Convolution (CT Systems) – Recap

Equation (3) ***<u>defines</u>*** the convolution operation, i.e.

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} d\tau \, x(\tau) h(t-\tau) \qquad (4)$$

so that (4) is the *convolution integral*.



Convolution (CT Systems) – General Formula

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution (CT Systems) – Example 1

To find the output of the system with impulse response $h(t) = e^{-2t}u(t)$

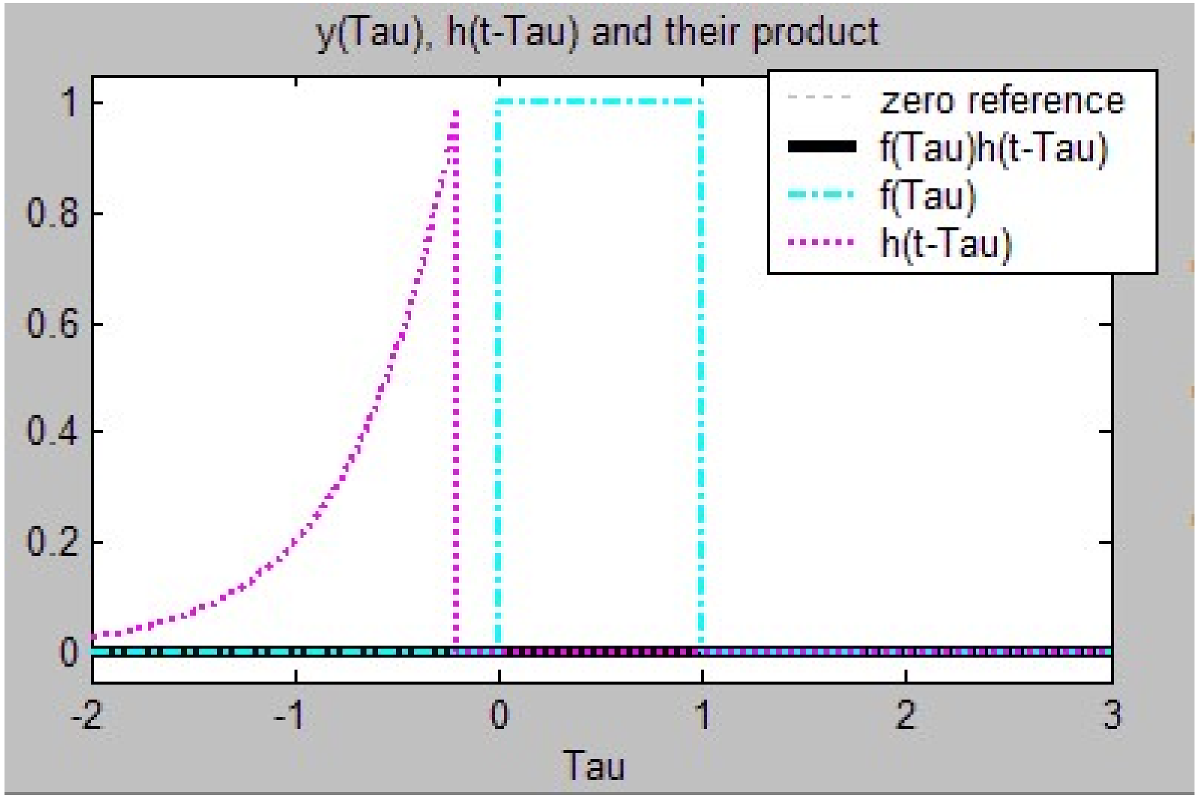
to the input $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & otherwise \end{cases}$

we will use the convolution integral $y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$

Because the input function has three distinct regions **t<0, 0<t<1 and t>1**, we will need to split up the integral into three parts.

Convolution (CT Systems) – Example 1

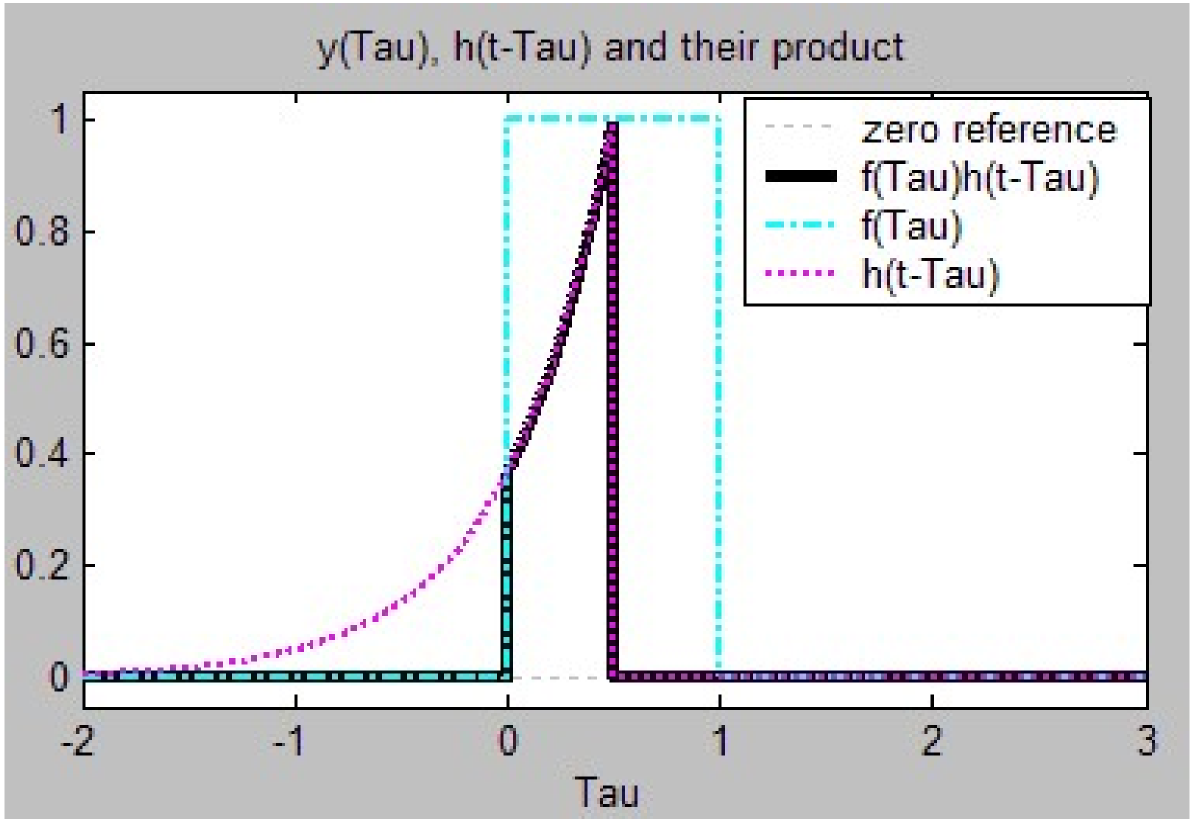
Part I: t<0



Therefore, the result for the first part of our solution is $y(t) = 0 \qquad t < 0$

Convolution (CT Systems) – Example 1

Part II: 0<t<1



Convolution (CT Systems) – Example 1

Part II: 0<t<1

We can now evaluate the integral of the solid black line:

$$y(t) = \int_0^{\infty} f(\tau)h(t-\tau)d\tau = \int_0^{t} f(\tau)h(t-\tau)d\tau$$

$$= \int_0^{t} (1)(e^{-2(t-\tau)})d\tau = e^{-2t}\int_0^{t} e^{2\tau}d\tau$$

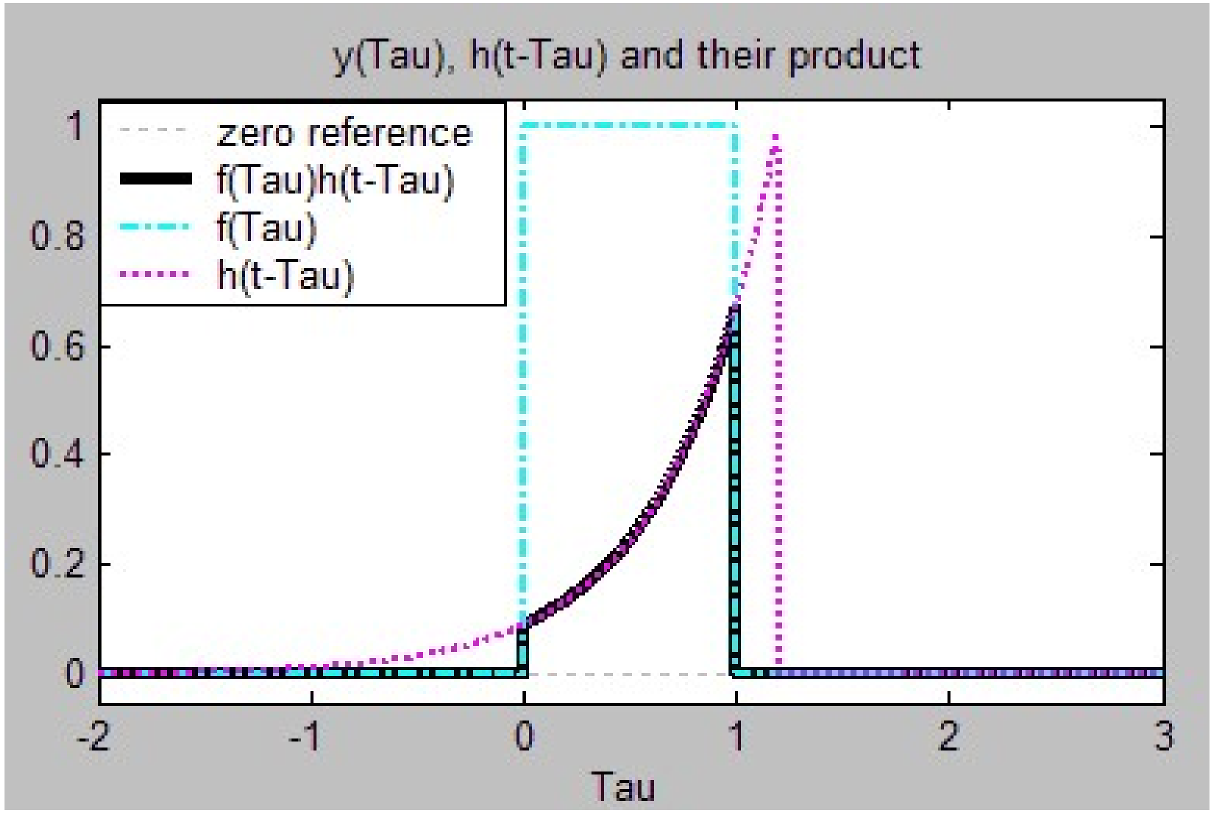
$$= e^{-2t}\left(\frac{1}{2}e^{2\tau}\Big|_0^t\right) = \frac{1}{2}\left(1-e^{-2t}\right)$$

As such, the result for the second part of our solution is

$$y(t) = \frac{1}{2}\left(1-e^{-2t}\right) \qquad 0 < t < 1$$

Convolution (CT Systems) – Example 1

Part III: t>1



Convolution (CT Systems) – Example 1

Part III: t>1

We can now evaluate the integral:

$$y(t)=\int_0^\infty f(\tau)h(t-\tau)d\tau=\int_0^1 f(\tau)h(t-\tau)d\tau$$

$$=\int_0^1 (1)(e^{-2(t-\tau)})d\tau=e^{-2t}\int_0^1 e^{2\tau}d\tau$$

$$=e^{-2t}\left(\frac{1}{2}e^{2\tau}\bigg|_0^1\right)=e^{-2(t-1)}\frac{1}{2}\left(1-e^{-2}\right)$$

Thus, the result for the third part of our solution is

$$y(t)=e^{-2(t-1)}\frac{1}{2}\left(1-e^{-2}\right) \qquad t>1$$

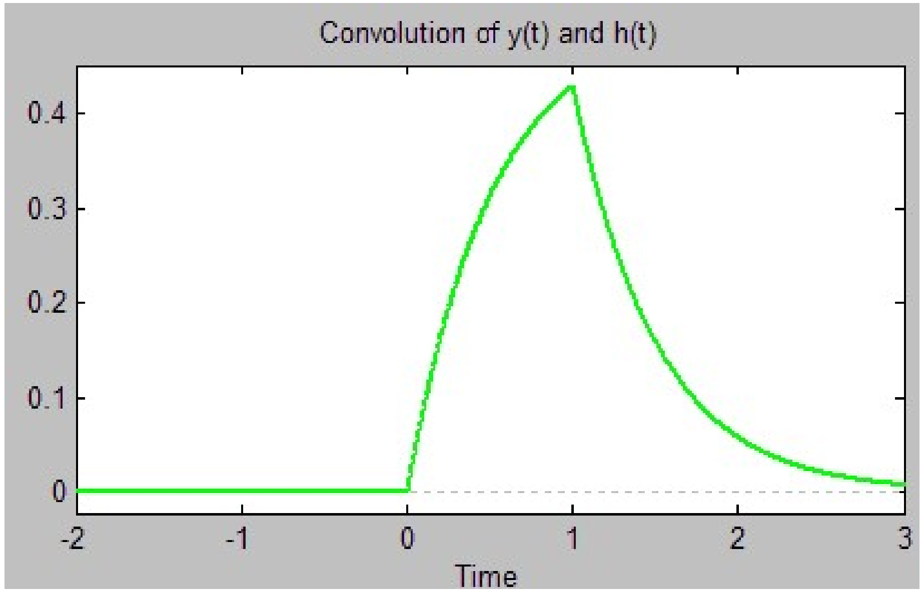
Convolution (CT Systems) – Example 1

The complete answer:

$$y(t) = 0 \qquad t < 0$$

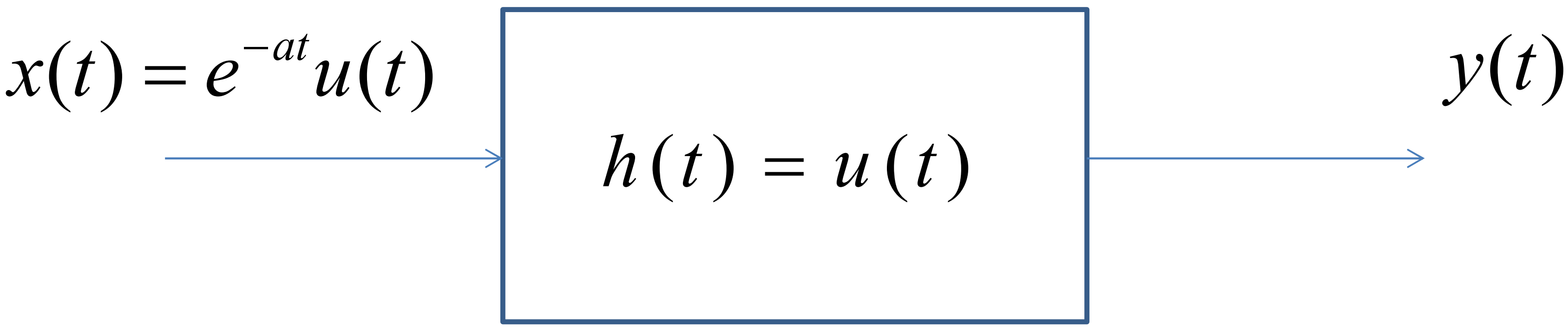
$$y(t) = \frac{1}{2}\left(1 - e^{-2t}\right) \qquad 0 < t < 1$$

$$y(t) = e^{-2(t-1)}\,\frac{1}{2}\left(1 - e^{-2}\right) \qquad t > 1$$



Convolution (CT Systems) – Example 2

Find the output of the following system:



Convolution (CT Systems) – Example 3

Find the output of the following system:

$$x(t) = e^{-2t}u(t) \longrightarrow \boxed{h(t) = e^{-t}u(t)} \longrightarrow y(t)$$

Convolution (CT Systems) – Example 4

Find the output of the following system: