Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

Lu Han

hl@bupt.edu.cn

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

A relation on a set

- draw its **digraph (有向图)**
- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

Negation

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

not reflexive: if there exists $x \in X$, such that $(x, x) \notin R$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

not symmetric: if there exists x and y, such that $(x, y) \in R$ and $(y, x) \notin R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

not antisymmetric: if there exists x and y, $x \neq y$, such that $(x, y) \in R$ and $(y, x) \in R$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**

if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

not transitive: if there exists $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \notin R$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

We say that

- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

3.3 Relations 关系

Definition 3.3.23 Let R be a relation from X to Y. The **inverse of R (R的逆)**, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \text{ if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

$R^{-1} =$

3.3 Relations 关系

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Definition 3.4.9 Let R be an equivalence relation on a set X. For $\forall\ a \in X$, let $[a] = \{x \in X \mid xRa\}$. The sets $[a]$ is called the **equivalence classes (等价类)** of X given by the relation R.

Theorem 3.4.8 Let R be an equivalence relation on a set X.
Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Example Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Exercise Determine whether the relation R defined on the collection of all nonempty subsets of real numbers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.

$(A, B) \in R$ if for every $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ with $|a - b| < \varepsilon$.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The matrix of a relation from X to Y is dependent on the orderings of X and Y.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

A relation on a set

- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is $A_1 A_2 =$?

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $R_2 \circ R_1$, is the relation from X to Z defined by
$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product A_1A_2 by 1.

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad A_1A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $R_2 \circ R_1$, is the relation from X to Z defined by
$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by 1.

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$ on $\{a, b, c, d\}$ is transitive?

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, b)\}$ on $\{a, b, c, d\}$ is transitive?

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many relations are there on X?

- How many reflexive relations are there on X?

- How many symmetric relations are there on X?

- How many antisymmetric relations are there on X?

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many reflexive and symmetric relations are there on X?
- How many reflexive and antisymmetric relations are there on X?
- How many aymmetric and antisymmetric relations are there on X?
- How many reflexive, symmetric, and antisymmetric relations are there on X?

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many relations are there on X?

$$2^{n^2}$$

- How many reflexive relations are there on X?

$$2^{n(n-1)}$$

- How many symmetric relations are there on X?

$$2^{\frac{n(n+1)}{2}}$$

- How many antisymmetric relations are there on X?

$$2^n 3^{\frac{n(n-1)}{2}}$$

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many reflexive and symmetric relations are there on X?

$$2^{\frac{n(n-1)}{2}}$$

- How many reflexive and antisymmetric relations are there on X?

$$3^{\frac{n(n-1)}{2}}$$

- How many aymmetric and antisymmetric relations are there on X?

$$2^{n}$$

- How many reflexive, symmetric, and antisymmetric relations are there on X?

$$1$$