Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

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3.1 Functions 函数

Definition 3.1.1 Let X and Y be sets. A *function f* from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

We sometimes denote a function f from X to Y as $f: X \to Y$.

- The set X is called the **domain (定义域)** of f.
- The set Y is called the **codomain (陪域)** of f.
- The set $\{Y \mid (x, y) \in f\}$ (which is a subset of the codomain Y) is called the **range (值域)** of f.

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Example 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.

Example 3.1.4 The set $f = \{(1, a), (2, b), (3, a)\}$ is not a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$.

Example 3.1.5 The set $f = \{(1, a), (2, b), (3, c), (1, b)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.

Arrow Diagram 箭头图

Example 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Yes.

Example 3.1.4 The set $f = \{(1, a), (2, a), (3, b)\}$ is not a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$?
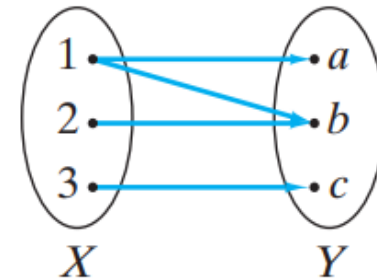
No.

Example 3.1.5 The set $f = \{(1, a), (2, b), (3, c), (1, b)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

No.

Arrow Diagram 箭头图

Example 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

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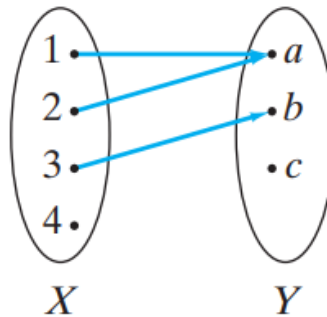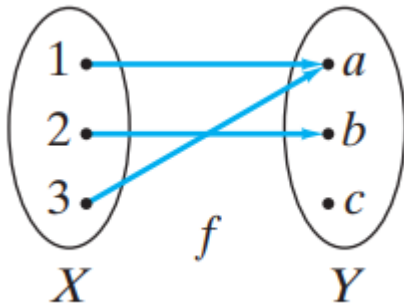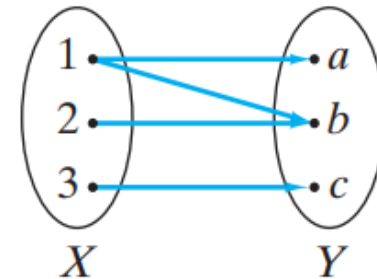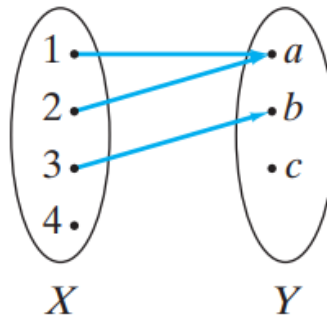
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No.

Arrow Diagram 箭头图

> **A function: There is exactly one arrow from each element in X.**

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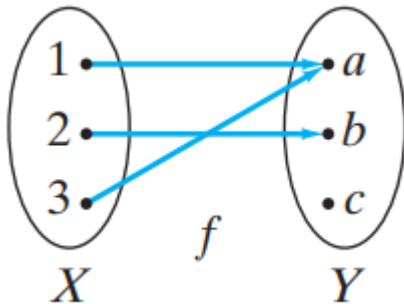
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3.1 Functions 函数

domain   **range**

$$f(x) = e^x$$

$$f(x) = \log(x)$$

$$f(x) = \sin x$$

$$f(x) = \sqrt{x}$$

3.1 Functions 函数

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A set may contain any kind of element.
Examples:
{1, 2, Jason}
{1, 5, {3.5, 17}, Jason }

3.1 Functions 函数

$$f(S) = |S|$$

$$f(\text{string}) = \text{length}(\text{string})$$

$$f(\text{student}) = \text{student's ID}$$

$$f(p) = \text{is-prime}(p)$$

3.1 Functions 函数

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

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$X \qquad\qquad Y$

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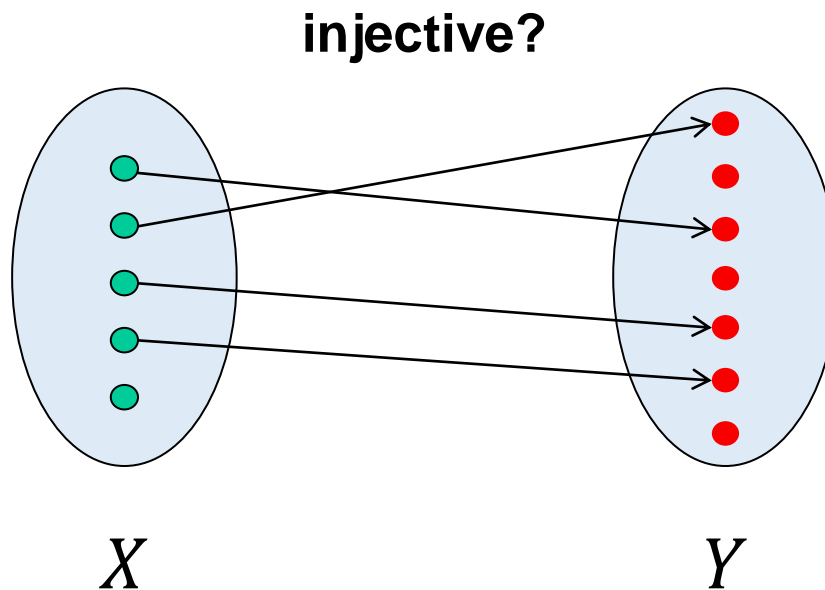
injective   ≤ 1 arrow in

X Y $|X| \leq |Y|$

3.1 Functions 函数

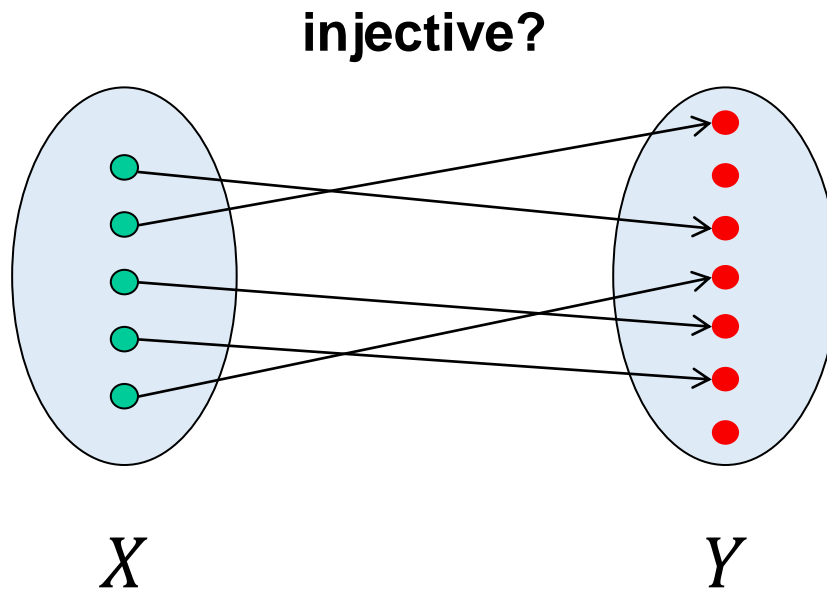**Definition 3.1.21** A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

Example 3.1.23 The set $f = \{(1, b), (3, a), (2, c)\}$ is a one-to-one function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$?

Example 3.1.24 The set $f = \{(1, a), (2, b), (3, a)\}$ is a one-to-one function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

3.1 Functions 函数

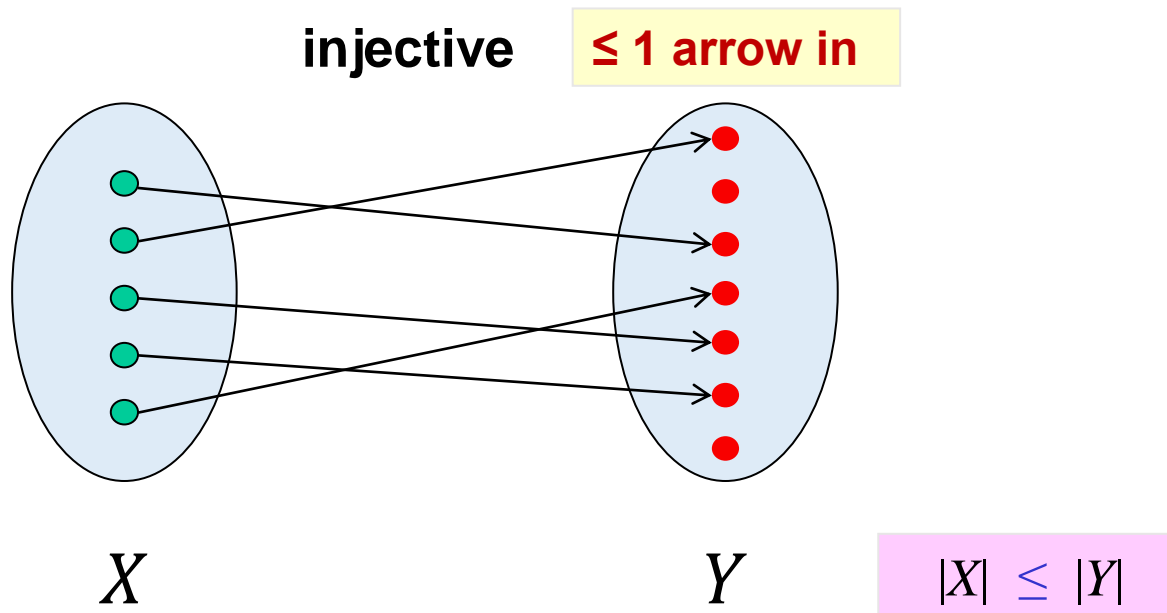**Definition 3.1.21** A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

Example 3.1.27 Prove that the function $f(n) = 2n + 1$ from the set of positive integers to the set of positive integers is one-to-one.

3.1 Functions 函数

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

Example 3.1.28 Prove that the function $f(n) = 2^n - n^2$ from the set of positive integers to the set of integers is not one-to-one.

3.1 Functions 函数

Definition 3.1.29 A function f from X to Y is said to be **onto** Y **(or surjective) (**满射的**)** if **for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$**.

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surjective?

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surjective   ≥ 1 arrow in
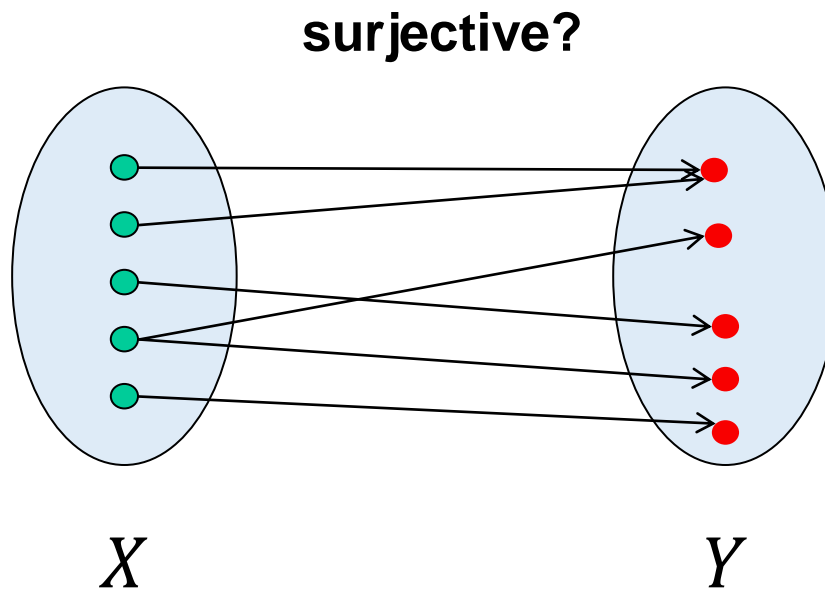

X Y $|X| \geq |Y|$

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Example 3.1.30 The set $f = \{(1, a), (2, c), (3, b)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Example 3.1.31 The set $f = \{(1, b), (3, a), (2, c)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$?
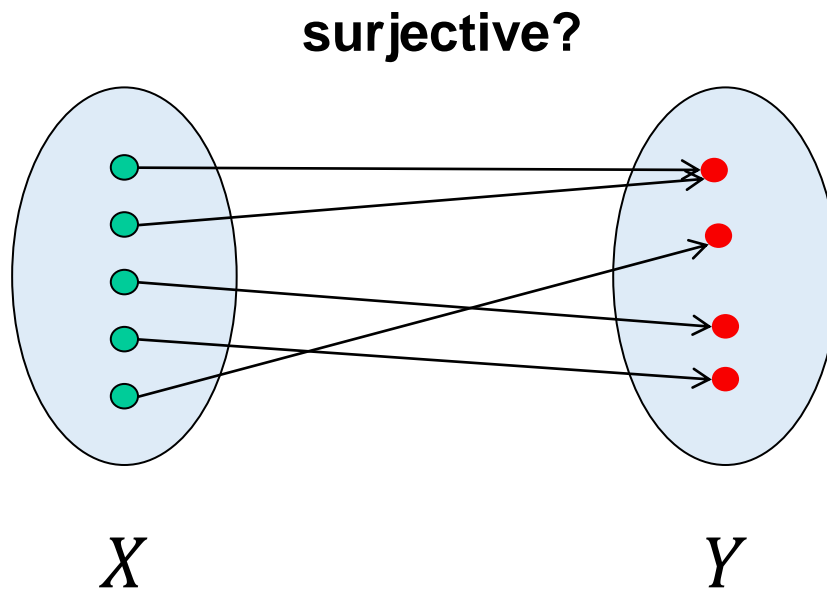
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Example 3.1.33 Prove that the function $f(x) = \frac{1}{x^2}$ from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.
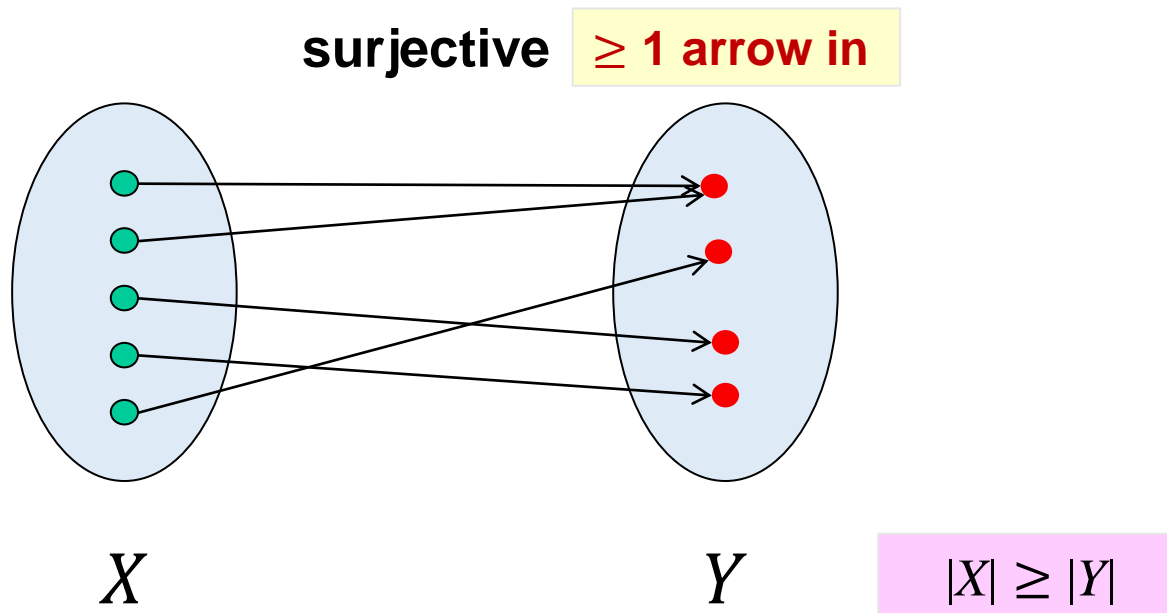
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Example 3.1.34 Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is not onto Y.

A function f from X to Y is not onto Y if for some $y \in Y$, for every $x \in X$, $f(x) \neq y$.

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Definition 3.1.35 A function that is **both one-to-one and onto** is called a **bijection (**双射**)**.

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bijective

X Y

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bijective exactly 1 arrow in

X Y $|X| = |Y|$
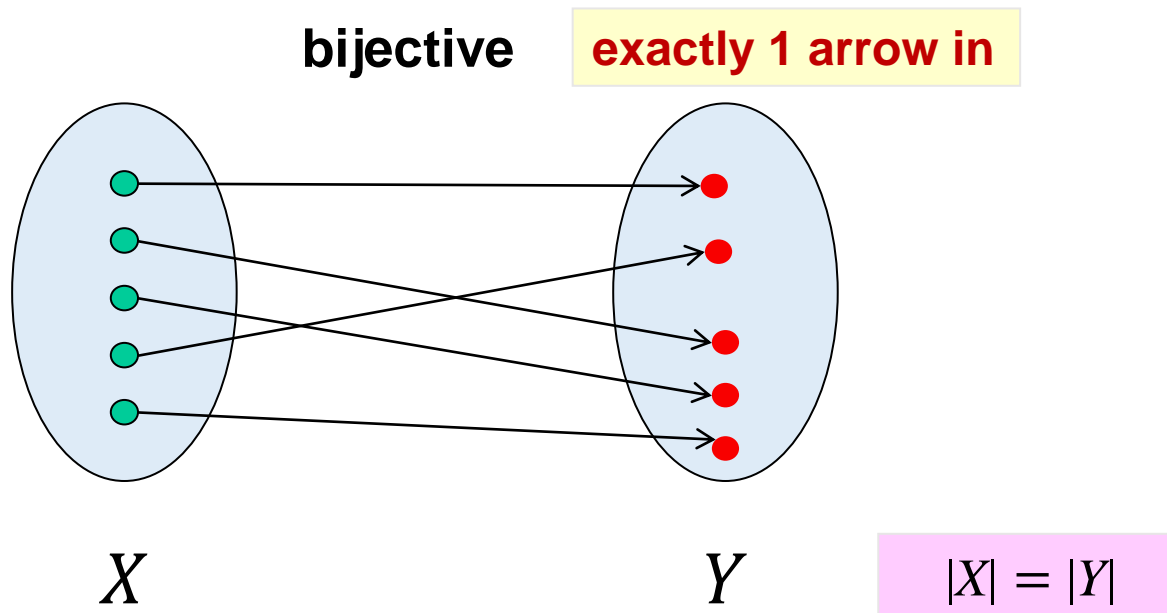
3.1 Functions 函数

Exercise

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x) = \sin(x)$	$\mathbf{R}$	$\mathbf{R}$			
$f(x) = 2^x$	$\mathbf{R}$	$\mathbf{R}^+$			
$f(x) = x^2$	$\mathbf{R}$	$\mathbf{R}^{nonneg}$			
Reverse String	Bit Strings of length n	Bit Strings of length n			

3.1 Functions 函数

Inverse Function (反函数)

Suppose that f is one-to-one, onto function from X to Y. It can be shown that $\{(y, x) \mid (x, y) \in f\}$ is a one-to-one, onto function from Y to X. This new function, denote f^{-1}, is called f **inverse (逆)**.

3.1 Functions 函数

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Example 3.1.38 For function $f = \{(1, a), (2, c), (3, b)\}$, we have

$$f^{-1} = ?$$

3.1 Functions 函数

Definition 3.1.41 Let g be a function X to Y and let f be a function from Y to Z. The **composition of f with g (f 与 g 的复合函数)**, denoted $f \circ g$, is the function

$$(f \circ g)(x) = f(g(x))$$

from X to Z.

3.1 Functions 函数

Example 3.1.42 Given $X = \{1, 2, 3\}, Y = \{a, b, c\},\ Z = \{x, y, z\}$.
The function $g = \{(1, a), (2, a), (3, c)\}$ from X to Y.
The function $f = \{(a, y), (b, x), (c, z)\}$ from Y to Z.

The composition function from X to Z is $f \circ g =$?

3.1 Functions 函数

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Example 3.1.43 Draw the arrow diagram of the function $f \circ g$.

3.1 Functions 函数

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Example 3.1.43 Draw the arrow diagram of the function $f \circ g$.

3.1 Functions 函数

Example 3.1.44 If $f(x) = \log_3 x$ and $g(x) = x^4$,

then $(f \circ g)(x) = ?$

and $(g \circ f)(x) = ?$

3.1 Functions 函数

Exercise: $g \circ f$

$f: X \to Y$	$g: Y \to Z$	Injective?	Surjective?	Bijective?
f is injective	g is injective			
f is surjective	g is surjective			
f is injective	g is surjective			
f is surjective	g is injective			
f is bijective	g is bijective			

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

If more pigeons
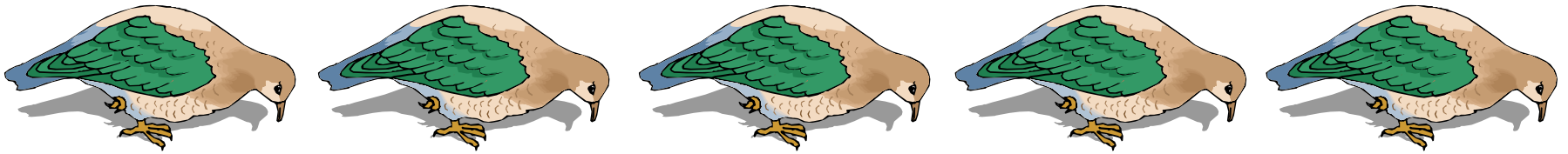
than pigeonholes,

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

then some hole must have at least two pigeons!
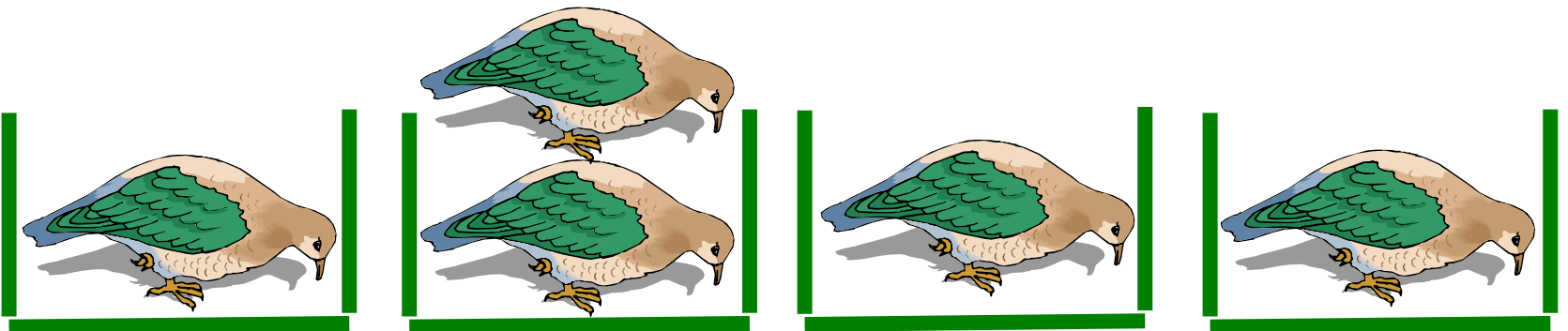
Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Pigeonhole Principle (First Form)
If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

then some hole must have at least two pigeons!

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Pigeonhole Principle (First Form)
If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

Pigeonhole Principle (Second Form)
If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then ...

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If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that have

the same image in the codomain.)

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Example
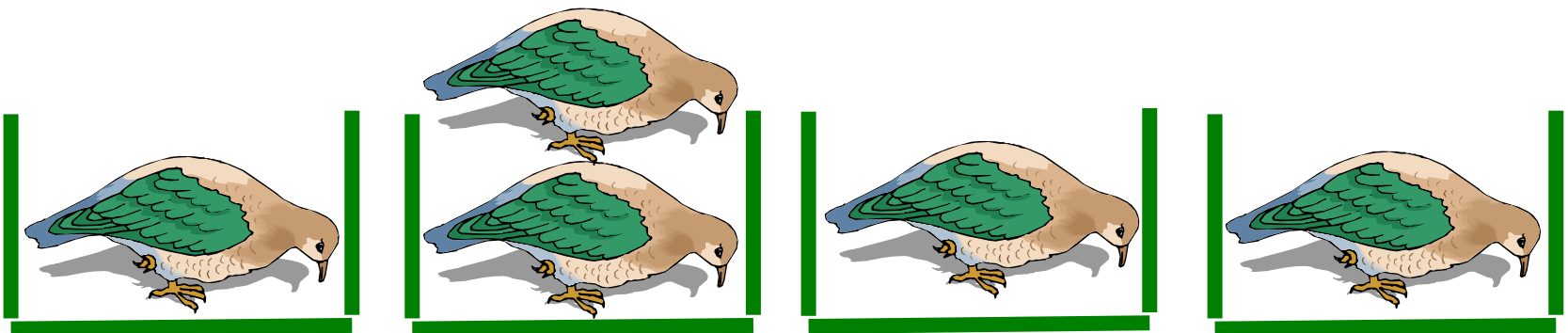In a group of 366 people, there must be two people having the same birthday.

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Extension select 5 numbers from 1-8, there exists two number that their sum is exactly 9.

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $x \bmod y$ to be the remainder when x is divided by y.
● **mod** is called the **modulus operator(模算子)**

3.1 Functions 函数

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Example 3.1.12 We have
$11 \bmod 7 = 4, -11 \bmod 7 =$?

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Quotient-Remainder Theorem 商和余数定理

If d and n are integers, $d > 0$, there exist integers q (quotient) and r (remainder) satisfying $n = dq + r$ **$(0 \leq r < d)$**

Furthermore, q and r are unique; that is, if

$n = dq_1 + r_1$ $(0 \leq r_1 < d)$ and $n = dq_2 + r_2$ $(0 \leq r_2 < d)$,

then $q_1 = q_2$ and $r_1 = r_2$.

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $\boldsymbol{x \bmod y}$ to be the remainder when x is divided by y.
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$11 \bmod 7 = 4, -11 \bmod 7 = 3.$

Example 3.1.14 What day of the week will it be 365 days from Wednesday?

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Exercise Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by $f(x) = 4x \bmod 5$. Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

3.1 Functions 函数

Definition 3.1.47 A function from $X \times X$ to X is called a **binary operator on X**.

(X上的二元操作符)

Definition 3.1.50 A function from X to X is called a **unary operator on X**.

(X上的一元操作符)

3.1 Functions 函数

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$6 \bmod 2 = 0$, $199673 \bmod 2 = 1$.

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3.1 Functions 函数

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Figure 3.1.7 The graphs of the floor (left graph) and ceiling (right graph) functions.

3.1 Functions 函数

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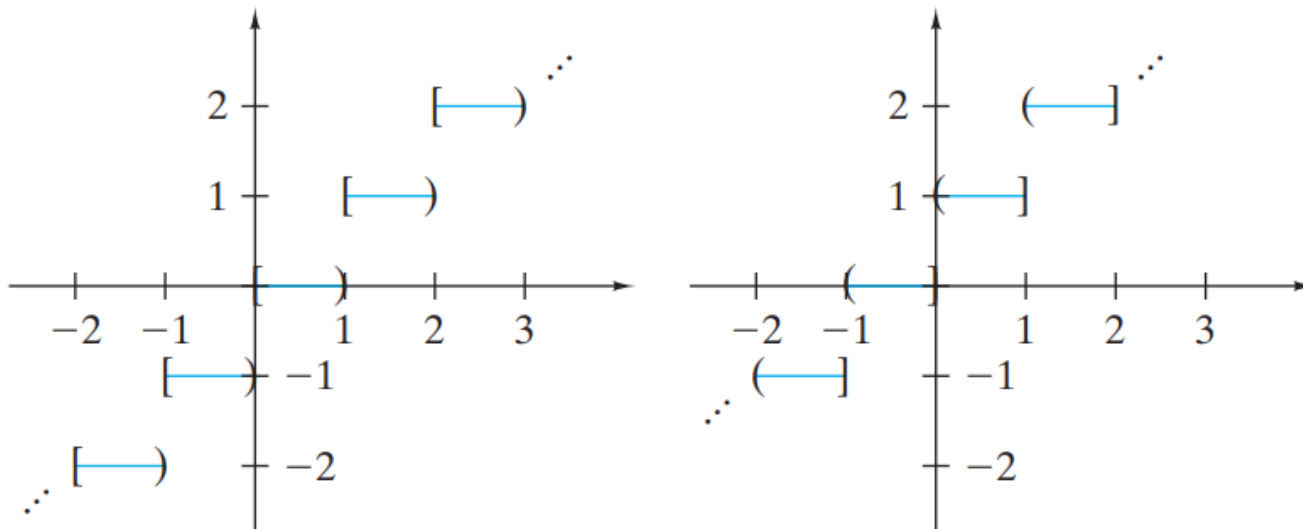# 3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $x \bmod y$ to be the remainder when x is divided by y.
● **mod** is called the **modulus operator(**模算子**)**

Exercise Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by $f(x) = 4x \bmod 5$. Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

3.1 Functions 函数

Definition 3.1.47 A function from $X \times X$ to X is called a **binary operator on X**.
(X上的二元操作符)

Example Let $X = \mathbf{R}.$ If we define $f: X \to X$ the function defined by $f(x, y) = x + y,$ then f is a binary operator on X.

Definition 3.1.50 A function from X to X is called a **unary operator on X**.
(X上的一元操作符)

Example Let U be a universal set. If we define $f: \mathcal{P}(U) \to \mathcal{P}(U)$ the function defined by $f(X) = \bar{X},$ then f is a unary operator on $\mathcal{P}(U)$.

3.2 Sequences and Strings 序列和串

Definition 3.2.1 A **sequence (**序列**)** s is a function whose domain D is a subset of integers.

The notation s_n is typically used instead of the more general function notation $s(n)$. The term n is called the **index(**下标**)** of the sequence.

If D is a finite set, we call s a finite sequence **(**有限序列**)**;

otherwise, s is an infinite sequence **(**无限序列**)**.

3.2 Sequences and Strings 序列和串

A sequence s is denoted s or $\{s_n\}$ if n is the index of the sequence.

● Notation s_n denotes the single element of the sequence s at index n.

> If s is a sequence $\{s_n\}$, where $n = 1, 2, 3, \ldots$
> - s_1 denotes the first element,
> - s_2 the second element,
> - s_n the nth element…

3.2 Sequences and Strings 序列和串

A sequence s is denoted s or $\{s_n\}$ if n is the index of the sequence.

● Notation s_n denotes the single element of the sequence s at index n.

● We will frequently use $\mathbf{Z}^+$ or $\mathbf{Z}^{nonneg}$ as the domain of a sequence.

If the domain of a sequence s is $\{k, k+1, k+2, \dots\}$ and the index of s is n, we can denote the sequence s as $\{s_n\}_{n=k}^{\infty}$.

A sequence s whose domain is $\mathbf{Z}^{nonneg}$:

A sequence s whose domain is $\{i, i+1, \dots, j\}$:

A sequence s whose domain is $\{-1, 0, 1, 2, 3\}$:

3.2 Sequences and Strings 序列和串

Example 3.2.5 Define a sequence b by the rule b_n is the nth letter in the word digital. If the domain of b is $\{1, 2, \ldots, 7\}$,

then $b_1 =?$, $b_2 =?$, $b_4 =?$ and $b_7 =?$

3.2 Sequences and Strings 序列和串

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then $b_1 =?$, $b_2 =?$, $b_4 =?$ and $b_7 =?$

Example 3.2.6 If x is the sequence defined by

$$x^2 = \frac{1}{2^n} \quad -1 \leq n \leq 4,$$

the elements of x are ?

3.2 Sequences and Strings 序列和串

Example 3.2.7 Define a sequence s as
$$s_n = 2^n + 4 \times 3^n \quad n \geq 0.$$

(a) Find s_0.

(b) Find s_1.

(c) Find a formula for s_i.

(d) Find a formula for s_{n-1}.

(e) Find a formula for s_{n-2}.

(f) Prove that $\{s_n\}$ satisties
$$s_n = 5s_{n-1} - 6s_{n-2} \text{ for all } n \geq 2.$$

Important Types of Sequences

- **Increasing Sequences (递增序列)**

A sequence s is **increasing** if for all i and j in the domain of s,

if $i < j$, then $s_i < s_j$.

- **Decreasing Sequences (递减序列)**

A sequence s is **decreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i > s_j$.

- **Nonincreasing Sequences (非增序列)**

A sequence s is **nonincreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \geq s_j$.

- **Nondecreasing Sequences (非减序列)**

A sequence s is **nondecreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \leq s_j$.

Important Types of Sequences

- **Increasing Sequences (**递增序列**)**

- **Decreasing Sequences (**递减序列**)**

- **Nonincreasing Sequences (**非增序列**)**

- **Nondecreasing Sequences (**非减序列**)**

Examples

1. The sequence $100, 90, 90, 74, 74, 74, 30$.

2. The sequence $\{s_n\}$ defined by the rule $s_n = 2n - 1$, for all $n \geq 1$.

Subsequences 子序列

Definition 3.2.12 Let s be **a sequence. A subsequence of** s is a sequence obtained from s by choosing certain terms of s in the same order in which they appear in s.

Example

Let $s = \{s_n = n \mid n = 1, 2, 3, \dots\}$.

Let $t = \{t_n = 2n \mid n = 1, 2, 3, \dots\}$.

t is a subsequence of s

Sigma and Pi Notation

Definition 3.2.17 If $\{a_i\}_{i=m}^{n}$ is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n,$$

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \ldots a_n.$$

The formalism $\sum_{i=m}^{n} a_i$ is called the **sum (or sigma) notation (求和符号)** and

$\prod_{i=m}^{n} a_i$ is called the **product (or pi) notation (乘积符号)**.

Sigma and Pi Notation

Exercise 1 For the sequence w defined by

$$w_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1.$$

(1) Find $\sum_{i=1}^{3} w_i$.

(2) Find $\sum_{i=1}^{10} w_i$.

(3) Find a formula for the sequence c defined by $c_n = \sum_{i=1}^{n} w_i$.

(4) Find a formular for the sequence d defined by $d_n = \prod_{i=1}^{n} w_i$.

(5) Is w increasing?

(6) Is w decreasing?

(7) Is w nonincreasing?

(8) Is w nondecreasing?

Sigma and Pi Notation

Exercise 2 For the sequence a defined by

$$a_n = \frac{n-1}{n^2(n-2)^2}, n \geq 3.$$

and the sequence z defined by $z_n = \sum_{i=3}^{n} a_i$

(1) Find a_3.

(2) Find a_4.

(3) Find z_3.

(4) Find z_4.

(5) Find z_{100}.

(6) Is z increasing, decreasing, nondecreasing and nonincreasing?

Sigma and Pi Notation

Exercise 2 For the sequence a defined by

$$a_n = \frac{n-1}{n^2(n-2)^2}, n \geq 3.$$

and the sequence z defined by $z_n = \sum_{i=3}^{n} a_i$

(1) Find a_3.

(2) Find a_4.

(3) Find z_3.

(4) Find z_4.

(5) Find z_{100}.

Hint: Show that $a_n = \frac{1}{4}[\frac{1}{(n-2)^2} - \frac{1}{n^2}]$

and use this form in the sum. Write out $a_3 + a_4 + a_5 + a_6$ to see what is going on.

(6) Is z increasing, decreasing, nondecreasing and nonincreasing?

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

- Finite sequences are also called strings.
- The string with no elements is called **null string (空串)** and is denoted λ.
- Let X^* denote the set of all strings over X.
- Let X^+ denote the set of all nonnull strings over X.

String

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- The string with no elements is called **null string (空串)** and is denoted λ.
- Let X^* denote the set of all strings over X.
- Let X^+ denote the set of all nonnull strings over X.

Example 3.2.5 Let $X = \{a, b\}$. Some elements in X^* are λ, a, b, $abab$ and $b^2 a^{50} ba$.

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

• The **length (长度)** of a string α is the number of elements in α. The length of α is denoted $|\alpha|$.

Example 3.2.26 If $\alpha = aabab$ and $\beta = a^3 b^4 a^{32}$, then $|\alpha| = 5$ and $|\beta| = 39$.

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (**二元关系**)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y.

If $X = Y$, we call R a (binary) relation on X.

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• **The relationship between Function, Sequence and Relation**

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Special case

sequence ⟵ **function**

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If $X = Y$, we call R a (binary) relation on X.

• **The relationship between Function, Sequence and Relation**

A function f from X to Y is a relation from X to Y having the properties:

(a) The domain of f is equal to X.
(b) For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$.

Special case   Special case

sequence ⟵ **function** ⟵ **relation**

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

A relation can be defined by

● simply specifying which ordered pairs belong to the relation

$$R = \{(\text{Bill}, CompSci), (Mary, Math), (Bill, Art), (Beth, History),$$
$$(Beth, CompSci), (Dave, Math)\}$$

TABLE 3.3.1 ■ Relation of Students to Courses

Student	Course
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

3.3 Relations 关系

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$$R = \{(\text{Bill}, CompSci), (Mary, Math), (Bill, Art), (Beth, History),$$
$$(Beth, CompSci), (Dave, Math)\}$$

● defining a relation by giving a rule for membership in the relation

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R =$?

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by $(x, y) \in R$ if $x \le y$, $x, y \in X$.

Then $R = ?$

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

A relation on a set

- draw its **digraph (有向图)**

Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by $(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (**二元关系**)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

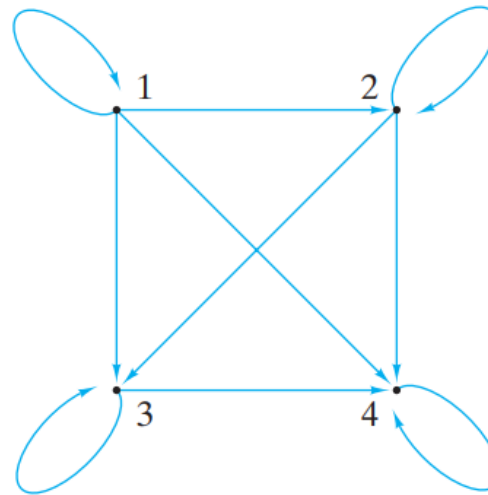Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

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A relation on a set

- draw its **digraph (**有向图**)**

- reflexive 自反的

- symmetric 对称的

- antisymmetric 反对称的

- transitive 传递的

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How many different relations can we define on a set X with n elements?

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A relation on a set

- draw its **digraph (**有向图**)**

- reflexive 自反的

- symmetric 对称的

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- transitive 传递的

How many different relations can we define on a set X with n elements?

A relation on a set X is a subset of X×X.
How many elements are in X×X ?
There are n^2 elements in X×X, so how many subsets (= relations on X) does X×X have?
Therefore, 2^{n^2} subsets can be formed out of X×X.

Answer: We can define 2^{n^2} different relations on X.

3.3 Relations 关系

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$

$R = \{(1, 1), (2, 2), (3, 3)\}$

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Exercise Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

$R = \{(1, 1), (2, 2), (3, 3)\}$ No.

3.3 Relations 关系

Definition 3.3.9 A relation R on a set X is **symmetric (**对称的**)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (**反对称的**)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

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Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by $(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

symmetric? antisymmetric?

3.3 Relations 关系

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for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

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not symmetric = antisymmetric?

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not symmetric = antisymmetric?

$$R = \{(a, a), (b, b), (c, c)\} \text{ on } X = \{a, b, c\}.$$

Both symmetric and antisymmetric!

3.3 Relations 关系

Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$

$R = \{(1, 3), (3, 2), (2, 1)\}$

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$

3.3 Relations 关系

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**
if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Exercise Are the following relations
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$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ Yes.

$R = \{(1, 3), (3, 2), (2, 1)\}$ No.

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ No.

Negation

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Negation

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

> not reflexive: if there exists $x \in X$, such that $(x, x) \notin R$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

> not symmetric: if there exists x and y, such that $(x, y) \in R$ and $(y, x) \notin R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

> not antisymmetric: if there exists x and y, $x \neq y$, such that $(x, y) \in R$ and $(y, x) \in R$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**

if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

> not transitive: if there exists $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \notin R$.

3.3 Relations 关系

Exercise Give examples of relations on $\{1, 2, 3, 4\}$ having the properties specified as follows.

(a) Reflexive, symmetric, and not transitive

(b) Reflexive, not symmetric, and not transitive

(c) Reflexive, antisymmetric, and not transitive

(d) Not reflexive, symmetric, not antisymmetric, and transitive

(e) Not reflexive, not symmetric, and transitive

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is reflexive, antisymmetric, and transitive.

3.3 Relations 关系

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If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

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Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is reflexive, antisymmetric, and transitive.

If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

We say that

• x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.

• x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \npreceq y$ or $y \npreceq x$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is reflexive, antisymmetric, and transitive.

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We say that
- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

Example ?

3.3 Relations 关系

> **Definition 3.3.20** A relation R on a set X is a **partial order (**偏序**)** if R is **reflexive, antisymmetric, and transitive**.
>
> If every pair of elements in X is comparable, we call R a **total order (**全序**)**.

Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by $(x, y) \in R$ if $x \leq y$, $x, y \in X$.
Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

3.3 Relations 关系

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by
$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$
we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

3.3 Relations 关系

Definition 3.3.23 Let R be a relation from X to Y. The **inverse of R (R的逆)**, denoted R^{-1}, is the relation from Y to X defined by
$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by
$$(x, y) \in R \ \text{ if } x \text{ divides } y,$$
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$R^{-1} =$

3.3 Relations 关系

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 3.3.26 The composition of the relations
$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$
and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

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3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

(2) $R \cap S$

(3) $R \circ S$

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

$$R = \{(1, 2)\}, S = \{(2, 3)\}, R \cup S = \{(1, 2), (2, 3)\}.$$

(2) $R \cap S$

$$\text{If } \{x, y\}, \{y, z\} \in R \cap S, \text{ then } \{x, z\} \in R \cap S.$$

(3) $R \circ S$

$$R = \{(5, 2), (6, 3)\}, S = \{(1, 5), (2, 6)\}, R \circ S = \{(1, 2), (2, 3)\}.$$