Ch3:

3.27 Suppose that a box has 12 balls labeled $1, 2, \cdots, 12$. Two independent repetitions are made of the experiment of selecting a ball at random from the box. Let X denote the max numbers on the balls selected. Compute the probability function of X.

X_1 = number of the 1st ball, X_2 = number of the 2nd ball

$X = \max(X_1, X_2)$

$P(X=1) = \frac{1}{12\times 12} = \frac{1}{144}$ $\quad$ $P(X=7) = \frac{13}{144}$

$P(X=2) = \frac{3}{144}$ $\quad$ $P(X=8) = \frac{15}{144}$

$P(X=3) = \frac{5}{144}$ $\quad$ $P(X=9) = \frac{17}{144}$

$P(X=4) = \frac{7}{144}$ $\quad$ $P(X=10) = \frac{19}{144}$

$P(X=5) = \frac{9}{144}$ $\quad$ $P(X=11) = \frac{21}{144}$

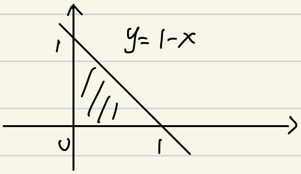
$P(X=6) = \frac{11}{144}$ $\quad$ $P(X=12) = \frac{23}{144}$

3.31 The joint p.d.f. of X and Y is defined as

$$f(x,y) = \begin{cases} 6x & \text{for } x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $U = X + Y$ and $Z = X - Y$. Find $f_U(u)$ and $f_Z(z)$.

$$f_U(u) = \int_{-\infty}^{\infty} f(x, u-x)\,dx$$



$$\begin{cases} x \geqslant 0 \\ u - x \geqslant 0 \\ u \leq 1 \end{cases} \Rightarrow 0 \leq x \leq u$$

$$\Rightarrow f_U(u) = \int_0^u f(x, u-x)\,dx$$

$$= \int_0^u 6x\,dx$$

$$= 3u^2$$

3.36 Suppose that X and Y are independent Poisson random variables such that $Var(X)+Var(Y)=5$. Evaluate $P(X+Y<2)$.

method 1° $X \sim Poi(\lambda_X)$ $Y \sim Poi(\lambda_Y)$ $X \perp Y$

$Var(X)+Var(Y)=\lambda_X+\lambda_Y=5 \Rightarrow \lambda_Y=5-\lambda_X$

$X \sim Poi(\lambda_X)$ $Y \sim Poi(5-\lambda_X)$

$p_X(x)=\frac{\lambda_x^x}{x!}e^{-\lambda_x}$ $p_Y(y)=\frac{(5-\lambda_x)^y}{y!}e^{-(5-\lambda_x)}$

$$P(X+Y<2)=P(X=0,Y=0)+P(X=0,Y=1)+P(X=1,Y=0)$$

$$\overset{X\perp Y}{=\!=} P(X=0)\cdot P(Y=0)+P(X=0)\cdot P(Y=1)+P(X=1)\cdot P(Y=0)$$

$$=e^{-\lambda x}\cdot e^{-(5-\lambda x)}+e^{-\lambda x}\cdot(5-\lambda x)e^{-(5-\lambda x)}+\lambda x\cdot e^{-\lambda x}\cdot e^{-(5-\lambda x)}$$

$$=e^{-\lambda x}\cdot e^{-(5-\lambda x)}(1+5-\lambda x+\lambda x)$$

$$=6\cdot e^{-5}$$

method 2° Let $Z=X+Y \sim Poi(5)$

$$P(X+Y<2)=P(Z<2)$$

$$=P(Z=0)+P(Z=1)$$

$$=e^{-5}+5e^{-5}$$

$$=6e^{-5}$$

3.37 Suppose that X, Y and Z are independent and identically distributed random variables, and each has a standard normal distribution. Evaluate $P(3X+2Y<6Z-7)$.

$X, Y, Z \sim N(0,1)$

$$P(3X+2Y<6Z-7)=P(3X+2Y-6Z<-7)=P(-3X-2Y+6Z>7)$$

Let $W=-3X-2Y+6Z \sim N(0,49)$

$$P(3X+2Y<6Z-7)=P(-3X-2Y+6Z>7)$$

$$=1-P(-3X-2Y+6Z\le 7)$$

$$=1-P(W\le 7)$$

$$=1-P\left(\frac{W}{7}\le\frac{7}{7}\right)$$

$$=1-\Phi(1)$$

$$=1-0.84134$$

$$=0.15866$$

3.39 Show that $f(x, y) = 1/x$, $0 < y < x < 1$, is a joint density function. Assuming that f is the joint density function of X, Y, find

(a) the marginal density of Y,

(b) the marginal density of X,

(c) $E(X)$,

(d) $E(Y)$.

(a) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 \frac{1}{x} dx = -\ln y$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x \frac{1}{x} dy = 1$

(c) $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x dx = \frac{1}{2}$

(d) $E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y \cdot (-\ln y) dy = \frac{1}{4}$

3.41 Let X be a random variable uniformly distributed in $[0, b]$. Compute $Cov(X, X^2)$ and the correlation coefficient $\rho(X, X^2)$.

$X \sim U[0, b]$

$$f(x) = \begin{cases} \frac{1}{b}, & 0 \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$Cov(X, X^2) = E(X \cdot X^2) - E(X) \cdot E(X^2)$

$Var(X) = E(X^2) - E^2(X) \Rightarrow E(X^2) = Var(X) + E^2(X) = \frac{b^2}{12} + (\frac{b}{2})^2 = \frac{1}{3} b^2$

$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x) dx = \int_0^b x^3 \cdot \frac{1}{b} dx = \frac{1}{4} b^3$

$\Rightarrow Cov(X, X^2) = \frac{1}{4} b^3 - \frac{b}{2} \cdot \frac{1}{3} b^2 = \frac{1}{12} b^3$

$$\rho(X, X^2) = \frac{Cov(X, X^2)}{\sqrt{Var(X)} \cdot \sqrt{Var(X^2)}}$$

$Var(X) = \frac{b^2}{12}$

$Var(X^2) = E(X^4) - E^2(X^2)$

$E(X^4) = \int_{-\infty}^{\infty} x^4 f(x) dx = \int_0^b x^4 \cdot \frac{1}{b} dx = \frac{1}{5} b^4$

$Var(X^2) = \frac{1}{5} b^4 - \frac{1}{9} b^4 = \frac{4}{45} b^4$

$$\Rightarrow \rho(X, X^2) = \frac{\frac{1}{12} \cancel{b^3}}{\frac{\cancel{b}}{2\sqrt{3}} \cdot \frac{2\cancel{b^2}}{\sqrt{45}}} = \frac{\sqrt{15}}{4}$$

3.42 You roll two fair dice, and they show X and Y, respectively. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$. Write down the joint distributions of:

(a) $\{U, X\}$

(b) $\{U, V\}$

Find $Cov(U, V)$.

~~X/Y~~	~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~
$p(\cdot)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

~~Let F be the d.f of X&Y~~

$\cancel{F_U(u) = 1-(1-F(u))^2}$

$\cancel{F_V(v) = F(v)^2}$

(a)

U \ X	1	2	3	4	5	6
1	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2		$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3			$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4				$\frac{1}{12}$	$\frac{1}{36}$	$\frac{1}{36}$
5					$\frac{1}{18}$	$\frac{1}{36}$
6						$\frac{1}{36}$
marginal X	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(b)

U \ V	1	2	3	4	5	6	marginal U
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{11}{36}$
2		$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{4}$
3			$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{7}{36}$
4				$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{5}{36}$
5					$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$
6						$\frac{1}{36}$	$\frac{1}{36}$
marginal V	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$	

$$Cov(U,V) = E(U \cdot V) - E(U) \cdot E(V)$$

$$= 12.25 - \frac{91}{36} \times \frac{161}{36}$$

$$= \frac{1225}{1296}$$

3.43 Let X and Y be independent random variables following the standard normal distribution $N(0,1)$;

(a) Compute $Cov(3X+2Y, X+5Y+10)$.

(b) Compute $P(X+4Y \geqslant 2)$.

(c) Compute $P((X-Y)^2 > 9)$.

(a) $Cov(3X+2Y, X+5Y+10) = Cov(3X, X+5Y+10) + Cov(2Y, X+5Y+10)$

$= Cov(3X, X+10) + Cov(3X, 5Y) + Cov(2Y, X+10) + Cov(2Y, 5Y)$

$= 3\,Cov(X, X+10) + 15\,Cov(X,Y) + 2\,Cov(Y, X+10) + 10\,Cov(Y,Y)$

$\overset{X\perp Y}{=} 3\,Var(X) + 3\,Cov(X,10) + 2\,Cov(Y,10) + 10\,Var(Y)$

$= 3+0+0+10$

$= 13$

(b) Let $Z = X+4Y \sim N(0,17)$

$P(X+4Y \geqslant 2) = P(Z \geqslant 2) = 1-P(Z<2)$

$= 1-P\left(\frac{Z}{\sqrt{17}} < \frac{2}{\sqrt{17}}\right)$

$= 1-\Phi\left(\frac{2}{\sqrt{17}}\right) \approx 1-\Phi(0.49)$

$= 1-0.68793$

$= 0.31207$

(c) Let $W = X-Y \sim N(0,2)$

$P((X-Y)^2 > 9) = P(X-Y>3) + P(X-Y<-3)$

$= P(W>3) + P(W<-3)$

$= 1-P(W\leq 3) + P(W<-3)$

$= 1-\Phi\left(\frac{3}{\sqrt{2}}\right) + \Phi\left(-\frac{3}{\sqrt{2}}\right)$

$= 1-\Phi\left(\frac{3}{\sqrt{2}}\right) + \left(1-\Phi\left(\frac{3}{\sqrt{2}}\right)\right)$

$= 2-2\Phi\left(\frac{3}{\sqrt{2}}\right)$

$\approx 2-2\times\Phi(2.12)$

$= 2-2\times 0.98300$

$= 0.034$

CH4

4.12 A fair coin is tossed 900 times. Find the probability that the number of heads is between 420 and 465.

$X_1, X_2, \cdots$ iid $\sim B(1, \frac{1}{2})$

$X \sim B(900, \frac{1}{2})$ $\quad \mu = np = 450 \quad \sigma^2 = np(1-p) = 225$

$X \sim N(450, 225)$

$$P(420 \le X \le 465) = P\left(\frac{420-450}{15} \le \frac{X-450}{15} \le \frac{465-450}{15}\right)$$

$$= \Phi(1) - \Phi(-2)$$

$$= \Phi(1) - (1 - \Phi(2))$$

$$= \Phi(1) - 1 + \Phi(2)$$

$$= 0.84134 - 1 + 0.97725$$

$$= 0.81859$$

4.13 A fair coin is tossed n times. Find n such that the probability that the number of heads is between $0.49n$ and $0.51n$ is at least 0.9.

$X_1, X_2, \cdots$ idd $\sim B(1, \frac{1}{2})$

$X \sim B(n, \frac{1}{2})$ $\quad \mu = np = \frac{n}{2} \quad \sigma^2 = np(1-p) = \frac{n}{4}$

$X \sim N(\frac{n}{2}, \frac{n}{4})$

$P(0.49n \le X \le 0.51n) = 0.9$

$$P\left(\frac{0.49n - 0.5n}{0.5\sqrt{n}} \le \frac{X - 0.5n}{\cancel{0.25n}\ 0.5\sqrt{n}} \le \frac{0.51n - 0.5n}{0.5\sqrt{n}}\right) = 0.9$$

$\Phi(0.02\sqrt{n}) - \Phi(-0.02\sqrt{n}) = 0.9$

$2\Phi(0.02\sqrt{n}) - 1 = 0.9$

$\Phi(0.02\sqrt{n}) = 0.95$

$0.02\sqrt{n} \doteq 1.65$

$n \doteq 6806.25$

and n is an integer

$\Rightarrow n = 6807$

4.15 Throw 100 fair dice and compute, approximately the probability that the sum of the values of the dice is in [325, 375].

$$E(X) = (1+2+3+4+5+6) \times \frac{1}{6} = \frac{7}{2}, \quad Var(X) = E(X^2) - E^2(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$$S_n^* = \frac{S_n - 100 \times \frac{7}{2}}{10 \cdot \sqrt{\frac{35}{12}}}$$

$$P(325 \le S_n \le 375) = P\left(\frac{325 - 100 \times \frac{7}{2}}{10 \cdot \sqrt{\frac{35}{12}}} \le S_n^* \le \frac{325 - 100 \times \frac{7}{2}}{10 \cdot \sqrt{\frac{35}{12}}} \right)$$

$$= \Phi\left(\frac{\sqrt{105}}{7}\right) - \left(1 - \Phi\left(\frac{\sqrt{105}}{7}\right)\right)$$

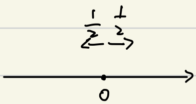
$$= 2\Phi\left(\frac{\sqrt{105}}{7}\right) - 1$$

$$\approx 2\Phi(1.46) - 1$$

$$= 2 \times 0.92785 - 1$$

$$= 0.8557$$

4.23 A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 1/2. Estimate the probability that, after 100 steps, the walker is more than 10 steps from the starting position.



$$E(X) = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0 \quad Var(X) = E(X^2) - E^2(X) = 1$$

$$S_n^* = \frac{S_n - 0}{10} = \frac{S_n}{10}$$

$$P(-10 \le S_n \le 10) = P(-1 \le S_n^* \le 1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2 \times 0.84134 - 1$$

$$= 0.68268$$

$$P(|S_n| > 10) = 1 - P(-10 \le S_n \le 10)$$

$$= 1 - 0.68268$$

$$= 0.31732$$

5.3 Consider the process $X_t = X\cos\omega_0 t$, where ω_0 is a constant and $X \sim N(2,4)$. Find the one dimensional probability density function of X_t.

$$X_t \sim N(2\cos w_0 t,\ 4\cos^2 w_0 t)$$

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi}\cdot 2\cos w_0 t} \cdot e^{-\frac{(x-2\cos w_0 t)^2}{2\times 4\cos^2 w_0 t}}$$

$$= \frac{1}{2\sqrt{2\pi}\cos w_0 t} \cdot e^{-\frac{(x-2\cos w_0 t)^2}{8\cos^2 w_0 t}}$$

5.6 Consider the process

$$X_t = At, \quad t \in \mathbb{R}$$

Suppose that $A \sim N(0,4)$. Find

(a) the one dimensional p.d.f. of X_t.

(b) $\mu_X(t)$, $R_X(t_1,t_2)$, $C_X(s,t)$ and $\sigma_X^2(t)$ of X_t.

$$X_t \sim N(0, 4t^2)$$

(a) $$f_{X_t}(x) = \frac{1}{\sqrt{2\pi}\cdot 2t} \cdot e^{-\frac{x^2}{2\cdot 4t^2}}$$

$$= \frac{1}{2\sqrt{2\pi}\, t} \cdot e^{\frac{-x^2}{8t^2}}$$

(b) $\mu_X(t) = E(X(t)) = E(At) = tE(A) = 0$

$R_X(t_1,t_2) = E(X_{t_1}\cdot X_{t_2})$

$= E(At_1 \cdot At_2)$ $\qquad Var(A) = E(A^2) - E^2(A)$

$= t_1 t_2 E(A^2)$ $\qquad E(A^2) = Var(A) + E^2(A) = 4$

$= 4t_1 t_2$

$C_X(s,t) = Cov(X_s, X_t)$

$= R_X(s,t) - \mu_X(s)\cdot\mu_X(t)$

$= 4st$

$\sigma_X^2(t) = R_X(t,t) - \mu_X^2(t)$

$= 4t^2$

5.7 Let $X_n = \{1, 2, \cdots\}$ be a sequence of independent random variables with $S_{X_n} = \{0, 1\}$, $P(X_n = 0) = 2/3$, $P(X_n = 1) = 1/3$. Let $Y_n = \sum_{i=1}^{n} X_i$.

(a) Find the first order p.f. of Y_n.

(b) Find $\mu_Y(n)$, $R_Y(n, n+k)$ and $C_Y(n, n+k)$.

(a) $P(Y_n = \sum_{i=1}^{n} X_i) = C_n^x (\frac{1}{3})^x (\frac{2}{3})^{n-x}$

(b) $\mu_Y(n) = E(\sum_{i=1}^{n} X_i) = \frac{n}{3}$

$R_Y(n, n+k) = E(Y_n \cdot Y_{n+k}) = E[Y_n^2 + (X_1 + \cdots + X_n)(X_{n+1} + \cdots + X_{n+k})]$

$= \cancel{E(Y_n^2 + Y_{n+k})}$ $\qquad E(Y_n^2) = Var(Y_n) + E^2(Y_n)$

$= \frac{n^2}{9} + \frac{2n}{9} + \frac{n}{3} \cdot \frac{k}{3}$ $\qquad = \frac{2n}{9} + \frac{n^2}{9}$

$= \frac{n^2 + 2n + nk}{9}$

$C_Y(n, n+k) = Cov(Y_n, Y_{n+k})$

$= E(Y_n \cdot Y_{n+k}) - E(Y_n) \cdot E(Y_{n+k})$

$= \frac{n^2 + 2n + nk}{9} - \frac{n}{3} \cdot \frac{n+k}{3}$

$= \frac{2n}{9}$