HW 11

Eg 3.3.3 $X \sim Poi(\lambda)$

Let Y be the number of customers who purchase the certain goods.

$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad k=0,1,2,\cdots$

$\therefore P(Y=y \mid X=x) = C_x^y \cdot p^y (1-p)^{x-y} \quad y=0,1,2,\cdots,x$

$\therefore P(Y=y) = \sum_{x=y}^{\infty} P(X=x)\cdot P(Y=y \mid X=x)$

$= \sum_{x=y}^{+\infty} \frac{\lambda^x}{x!} e^{-\lambda} \cdot C_x^y p^y (1-p)^{x-y}$

$= e^{-\lambda} p^y \frac{\lambda^y}{y!} \sum_{x=y}^{\infty} \frac{[(1-p)\cdot\lambda]^{x-y}}{(x-y)!}$

$= e^{-\lambda} p^y \frac{\lambda^y}{y!} \cdot e^{\lambda(1-p)} = e^{\lambda p} \frac{(\lambda p)^y}{y!}$

$\therefore Y \sim Poi(\lambda p)$

Eg 3.3.4

$\iint_{R^2} f(x,y)\,dx\,dy = \int_0^1 dy \int_0^y k\,dx = \int_0^1 ky\,dy = \frac{k}{2} = 1 \Rightarrow k=2$

$f_X(x) = \int_x^1 2\,dy = 2(1-x), \quad 0<x<1$

$f_Y(y) = \cancel{\int_x^1 2\,dx} = \int_0^y 2\,dy = 2y, \quad 0<y<1$

$\therefore f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \begin{cases} \frac{1}{y} & 0<x<y \\ 0 & \text{other} \end{cases}$

$f_{Y|X}(y|x) = \begin{cases} \frac{1}{1-x}, & x<y<1 \\ 0, & \text{other} \end{cases}$

Eg 3.3.5

(a) if $0<x<2$, then

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_0^{4-2x} \frac{3}{16}(4-2x-y)\,dy = \frac{3}{8}(2-x)^2$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{4-2x-y}{2(2-x)^2}, & 0<y<4-2x \\ 0, & \text{other} \end{cases}$

(b) $P(Y \geq 2 \mid X \leq \frac{1}{2}) = \frac{P(Y\geq 2, X \leq \frac{1}{2})}{P(X \leq \frac{1}{2})}$

$= \frac{\int_0^{\frac{1}{2}} dx \int_2^{4-2x} \frac{3}{16}(4-2x-y)\,dy}{\int_0^{\frac{1}{2}} \frac{3}{8}(2-x)^2\,dx} = \frac{\frac{3}{16}\int_0^{\frac{1}{2}}(2x^2-4x+2)\,dx}{\frac{3}{8}\int_0^{\frac{1}{2}}(x^2-4x+4)\,dx} = \frac{7}{64}$

(c) $f_{Y|X}(y|\frac{1}{2}) = f_{Y|X}(y|x)\big|_{x=\frac{1}{2}} = \begin{cases} \frac{2}{9}(3-y), & 0<y<3 \\ 0, & \text{other} \end{cases}$

$\therefore P(Y\geq 2 \mid X=\frac{1}{2}) = \int_2^3 \frac{2}{9}(3-y)\,dy = \frac{1}{9}$

Eg 3.4.3 $\quad X, Y \sim Geom(p)$

$$P(X+Y=n) = \sum_{x=1}^{n-1} P(X=x, Y=n-x)$$
$$= \sum_{x=1}^{n-1} P(X=x)\cdot P(Y=n-x)$$
$$= \sum_{x=1}^{n-1} p(1-p)^{x-1} \cdot p(1-p)^{n-x-1}$$
$$= (n-1)p^2(1-p)^{n-2} \qquad n=2,3,\cdots$$

Eg 3.4.4

$$P(X+Y=k) = \sum_{x_1} P(X=x_1)\cdot P(Y=k-x_1)$$
$$= \sum_{x_1=0}^{k} C_{n_1}^{x_1} p^{x_1}(1-p)^{n_1-x_1} \cdot C_{n_2}^{k-x_1} p^{k-x_1}(1-p)^{n_2-k+x_1}$$
$$= p^k(1-p)^{n_1+n_2-k} \sum_{x_1=0}^{k} C_{n_1}^{x_1} C_{n_2}^{k-x_1}$$
$$= C_{n_1+n_2}^{k} \cdot p^k (1-p)^{n_1+n_2-k} \qquad k=0,1,\cdots,n_1+n_2$$

$\therefore X+Y \sim B(n_1+n_2, p)$

Eg 3.4.5

$X \sim P(\lambda) \quad Y \sim P(\mu) \quad \Rightarrow X+Y \sim Poi(\lambda+\mu)$

$$P(X+Y=n) = \sum_{x=0}^{n} P(X=x, Y=n-x)$$
$$= \sum_{x=0}^{n} e^{-\lambda}\frac{\lambda^x}{x!}\cdot e^{-\mu}\frac{\mu^{n-x}}{(n-x)!}$$
$$= e^{-(\lambda+\mu)} \cdot \sum_{x=0}^{n} \frac{\lambda^x \mu^{n-x}}{x!(n-x)!}$$
$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{x=0}^{n} \frac{n!}{x!(n-x)!}\lambda^x \mu^{n-x} = \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n \qquad n=1,2,\cdots$$

$\therefore X+Y \sim Poi(\lambda+\mu)$

Ex 3.17

(a) $\quad 1 \quad 2 \quad 3$

$\cancel{P(X=x, Y=y) = C_2^x \cdot (\tfrac{1}{2})^x (\tfrac{1}{2})^{2-x} C_2^y}$

1	2	3	X	Y
H	H	H	2	0
H	H	T	2	1
H	T	H	1	1
T	H	H	1	0
H	T	T	1	2
T	H	T	1	1
T	T	H	0	1
T	T	T	0	2

Y \ X	0	1	2
0	0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0

(b) $P(Y=y \mid X=1)$

$$= \frac{P(Y=y, X=1)}{P(X=1)}$$

$$= \begin{cases} \frac{1}{4} & y=0 \\ \frac{1}{2} & y=1 \end{cases}$$

3.18

(a)

V \ U	1	2	3	4	5	6
1	1					
2	2/3	1/3				
3	2/5	2/5	1/5			
4	2/7	2/7	2/7	1/7		
5	2/9	2/9	2/9	2/9	1/9	
6	2/11	2/11	2/11	2/11	2/11	1/11

(b)

U \ V	1	2	3	4	5	6
1	1/11	2/11	2/11	2/11	2/11	2/11
2		1/9	2/9	2/9	2/9	2/9
3			1/7	2/7	2/7	2/7
4			0	1/5	2/5	2/5
5					1/3	2/3
6						1

3.20 $N \sim Poi(\lambda=1)$ $\quad N=n, \ Y \sim B(n,p)$

(a) $P_N(n) = \frac{e^{-1}}{n!}$ $\quad P_{Y|N}(y) = C_n^y \cdot p^y (1-p)^{n-y}$

$P(N,Y) = P(N=n) \cdot P(Y=y|N=n)$

$p(n,y) = P_N(n) P_{Y|N}(y) = \frac{e^{-1}}{n!} \cdot \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} = \frac{e^{-1}}{y!(n-y)!} p^y (1-p)^{n-y}$

(b) $P(Y=y) = \sum_{n=y}^{\infty} p(n,y) = \sum_{n=y}^{\infty} \frac{e^{-1}}{y!} \cdot p^y \frac{(1-p)^{n-y}}{(n-y)!}$

$= \frac{e^{-1}}{y!} p^y \sum_{n=y}^{\infty} \frac{(1-p)^{n-y}}{(n-y)!} = \frac{e^{-1}}{y!} p^y \cdot e^{1-p} = \frac{e^{-p}}{y!} p^y$

(c) $P(N|Y=k) = \frac{P(N=n, Y=k)}{P(Y=k)} = \frac{\frac{e^{-1}}{k!(n-k)!} p^k (1-p)^{n-k}}{\frac{e^{-p}}{k!} p^k} = \frac{e^{p-1}}{(n-k)!} (1-p)^{n-k}$

3.21

(a) $f_Y(y) = \int_0^1 \frac{12}{5} x(2-x-y)\,dx = \frac{2}{5}(4-3y)$ for $0<y<1$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{6x(2-x-y)}{4-3y}, & 0<x<1 \\ 0 & \text{other} \end{cases}$

(b) $P(X>\frac{1}{2}|Y=\frac{1}{3}) = \int_{\frac{1}{2}}^1 \frac{6x(2-x-\frac{1}{3})}{3} dx = \frac{2}{3}$

3.23 $f(x,y) = 15x^2y$ $\quad 0 \le x \le y \le 1$

(a) $\cancel{P(X)}$ $f_Y(y) = \int_0^y 15x^2y\,dx = 5y^4$ $\quad x \le y \le 1$

$\therefore f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{3x^2}{y^3}$ $\quad 0 \le x \le y$

(b) $f_X(x) = \int_x^1 15x^2y\,dy = \frac{15}{2}x^2(1-x^2)$ $\quad 0 \le x \le y$

$f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{1-x^2}$, $\quad x \le y \le 1$

(c) $f_{X|Y}(x) \cdot f_{Y|X}(y) = 15x^2y = f(x,y)$

$\therefore$ X and Y are independent.

3.25 $X \sim U(0,1)$ $X=x$ $Y \sim U(0,x)$

(a) $f_X(x)=1$ $f_{Y|X}(y|x)=\frac{1}{x}$

$\therefore f(x,y)=f_X(x)\cdot f_{Y|X}(y|x)=\frac{1}{x}$ $0<y<x<1$

(b) $f_Y(y)=\int_y^1 \frac{1}{x}\,dx=-\ln y$ $0<y<1$

(c) $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=-\frac{1}{x\ln y}$ $0<y<x<1$

3.26

(a) $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$

$f_Y(y)=\int_{-\infty}^{+\infty}\frac{1}{12\pi}e^{-(\frac{x^2}{8}+\frac{y^2}{18})}dx=\frac{e^{-\frac{y^2}{18}}}{12\pi}\underbrace{\int_{-\infty}^{+\infty}e^{-\frac{x^2}{8}}dx}_{A}$ Let $t=2x$

$A=\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2}}\frac{1}{2}dt=\frac{\sqrt{2\pi}}{2}$

$\therefore f_Y(y)=\cancel{\frac{1}{24\sqrt{\pi}}e^{-\frac{y^2}{18}}}\ \frac{e^{-\frac{y^2}{18}}}{12\pi}\cdot\frac{\sqrt{2\pi}}{2}$

$\therefore f_{X|Y}(x|y)=\sqrt{\frac{2}{\pi}}\cdot e^{-\frac{x^2}{8}}$ $x\in R$

(b) $f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$

$f_X(x)=\int_{-\infty}^{\infty}\frac{1}{12\pi}e^{-(\frac{x^2}{8}+\frac{y^2}{18})}dy=\frac{1}{12\pi}e^{-\frac{x^2}{8}}\cdot\int_{-\infty}^{\infty}e^{-\frac{y^2}{18}}dy$ $t=3y$

$=\frac{1}{12\pi}e^{-\frac{x^2}{8}}\cdot\frac{\sqrt{2\pi}}{3}$

$\therefore f_{Y|X}(y|x)=\frac{3}{\sqrt{2\pi}}e^{-\frac{y^2}{18}}$ $y\in R$

(c) $f_{X|Y}(x|y)\cdot f_{Y|X}(y|x)=\frac{3}{\pi}e^{-(\frac{x^2}{8}+\frac{y^2}{18})}\neq f(x,y)$

$\therefore X, Y$ are dependent.

3.27

Let X_1 = number of first X_2 = number of second.

$X=\max\{X_1,X_2\}$

$P(X=1)=\frac{1}{12\times12}=\frac{1}{144}$ $P(X=7)=\frac{13}{144}$

$P(X=2)=\frac{3}{144}$ $P(X=8)=\frac{15}{144}$

$P(X=3)=\frac{5}{144}$ $P(X=9)=\frac{17}{144}$

$P(X=4)=\frac{7}{144}$ $P(X=10)=\frac{19}{144}$

$P(X=5)=\frac{9}{144}$ $P(X=11)=\frac{21}{144}$

$P(X=6)=\frac{11}{144}$ $P(X=12)=\frac{23}{144}$

3.28 $X \sim Geom(p)$ $\qquad$ $Y = \min\{X, M\}$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$P(Y=y) = \begin{cases} (1-p)^{m-1} \cdot p & y \geq m \\ (1-p)^{y-1} \cdot p & y \leq m \end{cases}$$

3.36 $X, \cancel{Y} \sim Poi(\lambda)$ $Y \sim Poi(\mu)$ $\qquad$ Let $Z = X+Y \sim Poi(5)$

$$Var(X) + Var(Y) = \lambda + \mu = 5$$

$P(\cancel{X+Y<2}) =$

$$P(X+Y<2) = P(Z<2) = P(Z=0) + P(Z=1) = e^{-5} + 5e^{-5} = 6e^{-5}$$