7.5 (a) The probability of A gets B's penny is $0.6^2+0.4^2=0.52$.

The probability of B gets A's penny is $0.6\times0.4\times2=0.48$.

The probability space is $\{0,1,\cdots,4\}$, means the number of pennies of A.

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.48 & 0 & 0.52 & 0 & 0 \\ 0 & 0.48 & 0 & 0.52 & 0 \\ 0 & 0 & 0.48 & 0 & 0.52 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $$P^2=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.48 & 0.2496 & 0 & 0.2704 & 0 \\ 0.2304 & 0 & 0.4992 & 0 & 0.2704 \\ 0 & 0.2304 & 0 & 0.2496 & 0.52 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The probability that A will have four pennies after the second tosses is $1\times P_{24}(2)=0.2704$

(c) The probability that B will be broke after three tosses is

$1\times P_{24}(3)=P_{21}\times P_{14}(2)+P_{23}\times P_{34}(2)=0.52\times0.52=0.2704$

(d) The probability that the game will be over before the third toss is

$P_{24}(2)+P_{20}(2)=0.2704+0.2304=0.5008$

(e) The answer is $4-1\times(0+0+2\times0.4992+0+4\times0.2704)=4-2.08=1.92$

(f) $$P_2(n)=\begin{cases} P_2(n-2)\,P_{22}^2, & n=2,4,6,\cdots \\ 1, & n=0 \\ 0, & n=1,3,5,\cdots \end{cases}$$

So for $n\equiv0 \pmod 2$, $P_2(n)=0.4992^{\frac{n}{2}}$

The answer $=$ the probability that A will have 4 pennies $=\sum_{n=0}^{+\infty}P_4(n)$

$=\sum_{n=0}^{+\infty}P_2(2n)P_{24}(2)=0.2704\sum_{n=0}^{+\infty}0.4992^n=\frac{0.2704}{1-0.4992}=0.54$

(g) The probability that A will broke is $=\sum_{n=0}^{+\infty}P_0(n)=\sum_{n=0}^{+\infty}P_2(2n)P_{20}(2)=0.46$

$\because \sum_{n=0}^{+\infty}P_0(n)+\sum_{n=0}^{+\infty}P_4(n)=1$

~~$\therefore$ A will finally get 4 pennies or broke~~.

~~The answer is $0.54\times4=2.16$~~

$$\sum_{n=0}^{\infty}P_2(2n)\,P_{24}(2)\cdot2(n+1)+\sum_{n=0}^{\infty}P_2(2n)\,P_{20}(2)\,2(n+1)$$

$$=2+1.0016\sum_{n=0}^{\infty}n\cdot0.4992^n=3.993184$$

7.9 (a) $P(X_2=0 \mid X_1=2) = P_{20} = 0$

(b) $P(X_2=0 \mid X_1=2, X_0=0) = P(X_2=0 \mid X_1=2) = 0$

(c) $P^2 = \begin{pmatrix} 0.17 & 0.53 & 0.3 \\ 0.22 & 0.72 & 0.06 \\ 0.06 & 0.45 & 0.49 \end{pmatrix}$, $P(X_{35}=0 \mid X_{33}=2) = P_{20}(2) = 0.06$

(d) $\vec{p}(2) = \vec{p}(0)P(2) = \left(0.15 \quad \frac{1.7}{3} \quad \frac{0.85}{3}\right)$, $P(X_2=0) = p_0(2) = 0.15$

(e) $\vec{p}(1) = \vec{p}(0)P = \left(\frac{0.5}{3} \quad 0.5 \quad \frac{1}{3}\right)$, $E(X_1) = 0 + 0.5\times 1 + \frac{1}{3}\times 2 = \frac{7}{6}$

$E(X_1X_2) = \sum_{i=0}^{2}\sum_{j=0}^{2} ij\, p_i(1) P_{ij} = 0.8\times 1\times 1\times 0.5 + 0.3\times 2\times 1\times \frac{1}{3} + 0.7\times 2\times 2\times \frac{1}{3} = \frac{4.6}{3}$

$E(X_2) = 0 + \frac{1.7}{3}\times 1 + \frac{0.85}{3}\times 2 = \frac{3.4}{3}$, $\mathrm{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{1.9}{9}$

$\pi = \pi P = (0.3\pi_0 + 0.2\pi_1,\ 0.4\pi_0 + 0.8\pi_1 + 0.3\pi_2,\ 0.3\pi_0 + 0.7\pi_2)$

$\begin{cases} 0.3\pi_0 + 0.2\pi_1 = \pi_0 \\ 0.3\pi_0 + 0.7\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \pi = \left(\frac{2}{11} \quad \frac{7}{11} \quad \frac{2}{11}\right)$

7.10 (a) $P(X_2=2 \mid X_1=1) = P_{12} = 0.7$

(b) $P^2 = \begin{pmatrix} 0.09 & 0.63 & 0.28 \\ 0.16 & 0.4 & 0.44 \\ 0.32 & 0.39 & 0.29 \end{pmatrix}$, $P(X_3=2 \mid X_1=1) = P_{12}(2) = 0.63$

(c) $P(X_3=2 \mid X_1=1, X_0=2) = P(X_3=2 \mid X_1=1) = 0.63$

(d) $P(X_2=1) = p_1(0)P_{1\cancel{2}}(2) + p_2(0)P_{2\cancel{2}}(2) + p_3(0)P_{3\cancel{2}}(2) = 0.1\times \overset{0.09}{\cancel{0.63}} + 0.3\times \overset{0.16}{\cancel{0.4}} + 0.6\times \overset{0.32}{\cancel{0.39}} = \overset{0.249}{\cancel{0.417}}$

(e) $P(X_1=2, X_2=1 \mid X_0=3) = P(X_1=2 \mid X_0=3)P(X_2=1 \mid X_1=2, X_0=3) = P_{32}P_{21} = 0$

$\pi = \pi P = (0.3\pi_1 + 0.4\pi_3 \quad 0.7\pi_1 + 0.6\pi_2 + 0.1\pi_3 \quad 0.4\pi_2 + 0.5\pi_3)$

$\begin{cases} 0.3\pi_1 + 0.4\pi_3 = \pi_1 \\ 0.4\pi_2 + 0.5\pi_3 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} 4\pi_3 = 7\pi_1 \\ 4\pi_2 = 5\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 : \pi_2 : \pi_3 = 16 : 35 : 28 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \pi = \left(\frac{16}{79} \quad \frac{35}{79} \quad \frac{28}{79}\right)$

7.12 (a)

$$P^2 = \begin{pmatrix} 0.3 & 0.25 & 0.45 \\ 0.61 & 0.09 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

$$P(X_0=1, X_2=3, X_3=3) = P(X_0=1)P(X_2=3|X_0=1)P(X_3=3|X_0=1, X_2=3)$$

$$= p_1(0)\,P_{13}(2)\,P_{33} = 0.3 \times 0.45 \times 0.2 = 0.027$$

(b) $P(X_2=1, X_4=2, X_5=3) = P(X_2=1)P(X_4=2|X_2=1)P(X_5=3|X_2=1, X_4=2)$

$$= \left(p_1(0)P_{11}(2) + p_2(0)P_{21}(2) + p_3(0)P_{31}(2)\right) P_{12}(2)\,P_{23} = 0.475 \times 0.25 \times 0.7 = 0.083125$$

$$\pi = \pi P = (0.3\pi_1 + 0.1\pi_2 + 0.8\pi_3 \quad 0.5\pi_1 + 0.2\pi_2 \quad 0.2\pi_1 + 0.7\pi_2 + 0.2\pi_3)$$

$$\begin{cases} 0.3\pi_1 + 0.1\pi_2 + 0.8\pi_3 = \pi_1 \\ 0.5\pi_1 + 0.2\pi_2 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} \pi_2 + 8\pi_3 = 7\pi_1 \\ 5\pi_1 = 8\pi_2 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 : \pi_2 : \pi_3 = 64 : 40 : 51 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \pi = \left(\frac{64}{155} \;\; \frac{8}{31} \;\; \frac{51}{155}\right)$$

8.3 $P(N_1=0, N_2=2, N_3=3) = P(N_1=0)\,P(N_2=2 \mid N_1=0)\,P(N_3=3 \mid N_2=2, N_1=0)$

$= e^{-\lambda} \cdot e^{-\lambda}\frac{\lambda^2}{2} \cdot e^{-\lambda} \cancel{\frac{\lambda^3}{6}}\, \lambda = \frac{\lambda^3}{2} e^{-3\lambda}$

8.5 (a) $E(N_2) = 2\lambda = 4$

(b) $E(N_1^2) = Var(N_1) + E(N_1)^2 = \lambda + \lambda^2 = 6$

(c) $E(N_1 N_2) = C_N(1,2) + E(N_1)E(N_2) = \cancel{\sigma_N^2(1) + 2\lambda^2} = \lambda + 2\lambda^2 = 10$

8.15 (a) $W_4 - W_0 \sim N(0,4)$, $P(W_4 \le 3 \mid W_0 = 1) = F_{W_4}(2) = \Phi(1) = 0.84134$

(b) $P(W_9 > c \mid W_0 = 1) = 1 - \Phi\left(\frac{c-1}{3}\right) = 0.1 \Rightarrow \Phi\left(\frac{c-1}{3}\right) = 0.9$