



EBU4375: SIGNALS AND SYSTEMS

LECTURE 15



Queen Mary
University of London

Fourier Transform

Fourier Transform

- The **Fourier Series (FS)** can only be applied to periodic signals
- Non-periodic signals cannot be analysed using Fourier series, the **Fourier Transform (FT)** is required

The Fourier Transform (FT) is defined as: In this course, the **Fourier Transform** will always be denoted by an **uppercase** letter or symbol, whereas **time signals** will be denoted by **lowercase** letters or symbols.

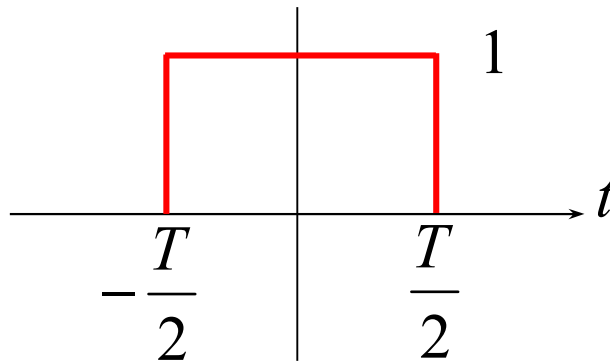
$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t) dt$$

$X(f)$ is the frequency signal

$x(t)$ is the time signal

Fourier Transform – Rectangular Pulse (also known as Rect Function)

$$x(t) = \begin{cases} 1, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{all other } t \end{cases}$$



Fourier Transform – Rectangular Pulse (also known as Rect Function)

Method 1:

$x(t)$ is an even signal

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t) dt = \int_{t=-\infty}^{t=\infty} (\underbrace{\cos(\omega t)}_{\text{even}} - j \underbrace{\sin(\omega t)}_{\text{odd}}) \cdot x(t) dt$$

$$X(f) = 2 \int_0^{T/2} (1) \cos(\omega t) dt = \frac{2}{\omega} \left[\sin(\omega t) \right]_{t=0}^{t=T/2} = \frac{2}{\omega} \sin\left(\frac{\omega T}{2}\right) = \frac{1}{\pi f} \sin(\pi f T)$$

Let's recall the sinc function $\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$. After some mathematical

Manipulation, we have $X(f) = T \frac{\sin(\pi f T)}{\pi f T}$. Therefore,

$$X(f) = T \text{sinc}(f T) = T \text{sinc}\left(\frac{T\omega}{2\pi}\right)$$

Fourier Transform – Rectangular Pulse (also known as Rect Function)

Method 2:

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t) dt = \int_{t=-T/2}^{t=T/2} e^{-j\omega t} \cdot 1 dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{t=-T/2}^{t=T/2}$$

Therefore

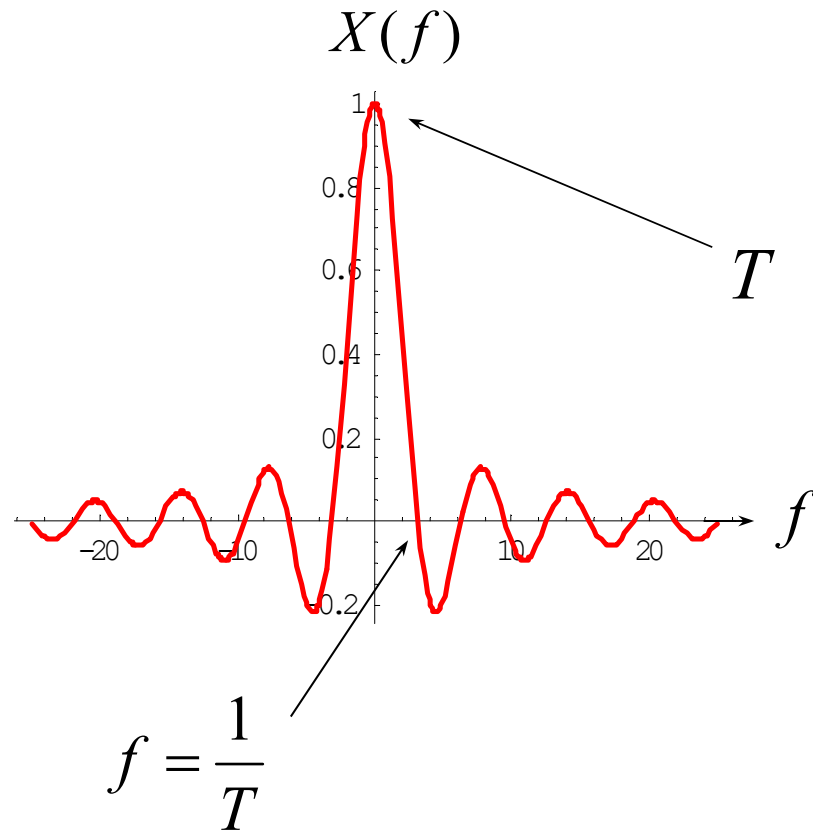
$$X(f) = \frac{1}{-j\omega} [e^{-j\omega T/2} - e^{+j\omega T/2}] = \frac{1}{j\omega} [e^{+j\omega T/2} - e^{-j\omega T/2}] = \frac{1}{j2\pi f} [e^{+j\pi f T} - e^{-j\pi f T}]$$

Using identity $\sin \theta = \frac{1}{2j} [e^{+j\theta} - e^{-j\theta}]$, we get $X(f) = \frac{1}{j2\pi f} [2j \sin(\pi f T)]$.

And after some mathematical manipulation, we get $X(f) = T \left[\frac{\sin(\pi f T)}{\pi f T} \right]$.

Using identity $\text{sinc}(\theta) = \left[\frac{\sin(\pi \theta)}{\pi \theta} \right]$, we get $X(f) = T \text{sinc}(f T)$.

Fourier Transform – Rectangular Pulse (also known as Rect Function)

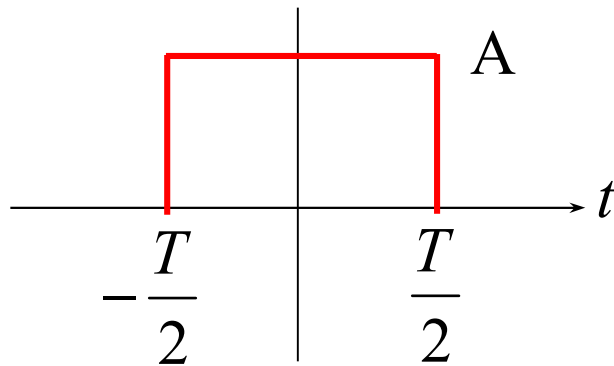


$$X(f) = 0, \quad f = \frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \dots$$

Fourier Transform – Rectangular Pulse (also known as Rect Function – General Formula)

The Fourier transform of a rectangular pulse is given as

$$A \operatorname{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT \operatorname{sinc}(f T) = AT \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right)$$



Inverse Fourier Transform

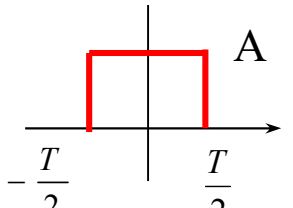
Give a signal $x(t)$ with Fourier transform $X(f)$, $x(t)$ can be recomputed from $X(f)$ by application of the inverse Fourier transform give by

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j\omega t} df$$

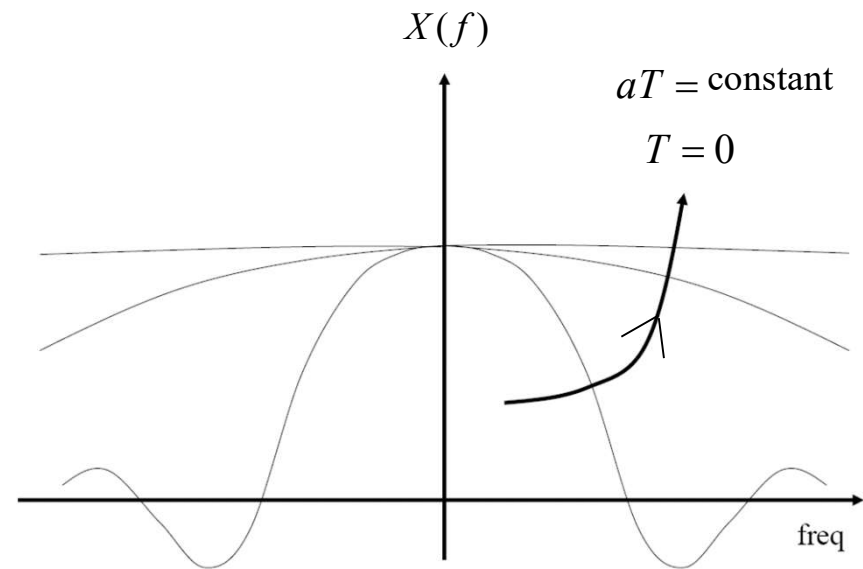
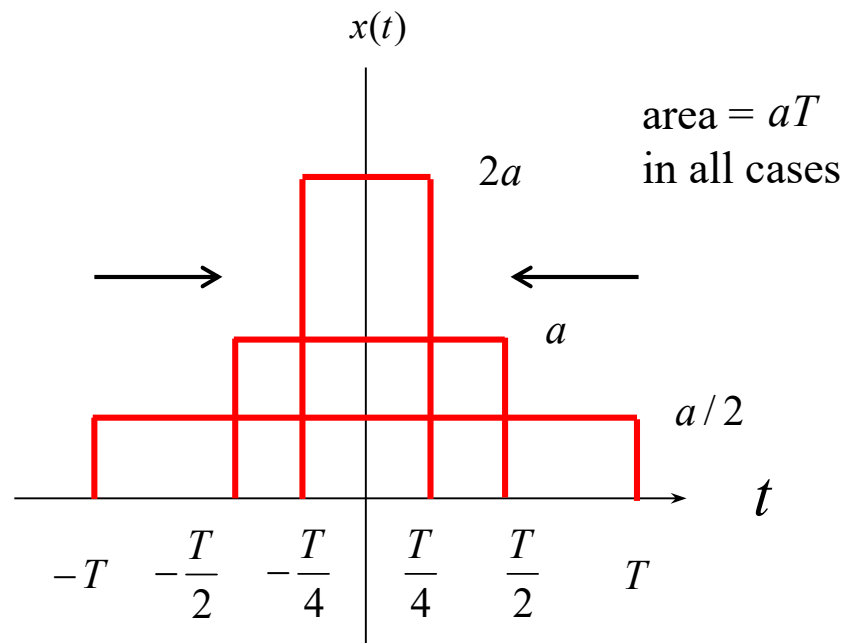
To denote the fact that $X(f)$ is the Fourier transform of $x(t)$, or that $X(f)$ is the inverse Fourier transform of $x(t)$, the transform pair notation

$$x(t) \leftrightarrow X(f)$$

will sometimes be used. One of most fundamental transform pairs in the Fourier theory is the pair

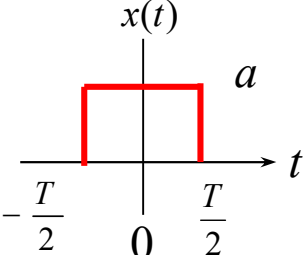

$$\leftrightarrow AT \operatorname{sinc}\left(\frac{T\omega}{2\pi}\right) = AT \operatorname{sinc}(T f)$$

Fourier Transform – Delta Function

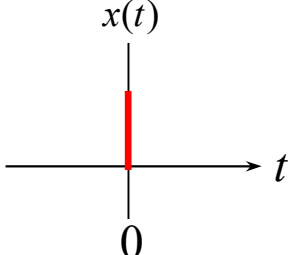


Fourier Transform – Delta Function

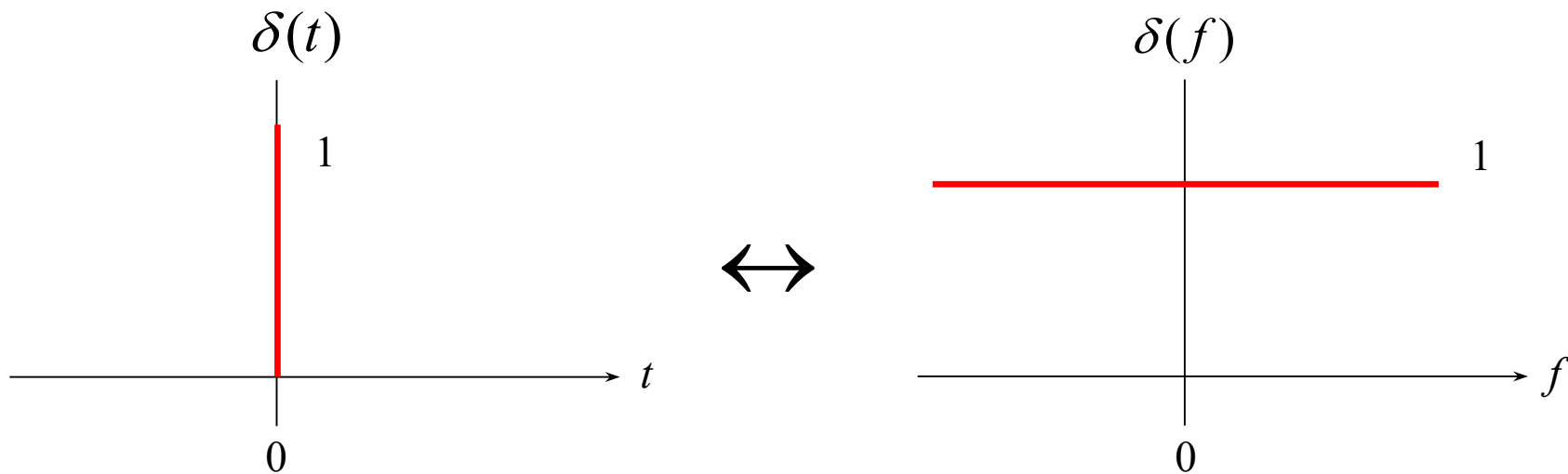
We know that


$$\leftrightarrow aT \operatorname{sinc}(f T)$$

Note that $\operatorname{sinc}(0) = 1$. We also know that the area of a delta function is $aT = 1$, therefore


$$\leftrightarrow 1$$

Fourier Transform – Delta Function



Fourier Transform Table

Fourier-Transform Pairs

<i>Time Function</i>	<i>Fourier Transform</i>
$\text{rect}\left(\frac{t}{T}\right)$	$T \text{ sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W} \text{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a > 0$	$\frac{1}{a + j2\pi f}$
$\exp(-a t), \quad a > 0$	$\frac{2a}{a^2 + (2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$

Fourier Transform Table

Fourier-Transform Pairs

<i>Time Function</i>	<i>Fourier Transform</i>
$\begin{cases} 1 - \frac{ t }{T}, & t < T \\ 0, & t \geq T \end{cases}$	$T \operatorname{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$

Fourier Transform Table

Fourier-Transform Pairs

<i>Time Function</i>	<i>Fourier Transform</i>
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j \text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty} \delta(t - iT_0)$	$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$

Trigonometric Identities

Some useful trigonometric identities:

$$\exp(\pm j\theta) = \cos \theta \pm j \sin \theta$$

$$\cos \theta = \frac{1}{2}[\exp(j\theta) + \exp(-j\theta)]$$

$$\sin \theta = \frac{1}{2j}[\exp(j\theta) - \exp(-j\theta)]$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos^2 \theta - \sin^2 \theta = \cos(2\theta)$$

$$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)]$$

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

Fourier Transform Properties

Part I:

- Linearity
- Time Shift
- Time Scaling
- Multiplication by a Complex Exponential (Frequency Shift)
- Tutorial

Part II:

- Multiplication by a Sinusoid
- Differentiation in the Time Domain
- Integration in the Time Domain
- Convolution in the Time Domain
- Multiplication in the Time Domain
- Tutorial

Fourier Transform Properties – Linearity

The Fourier transform is a linear operation; that is, if $x(t) \leftrightarrow X(f)$ and $v(t) \leftrightarrow V(f)$, then for any real or complex scalars a, b

$$ax(t) + bv(t) \leftrightarrow aX(f) + bV(f)$$

The properties of linearity can be proved by computing the Fourier transform of $ax(t) + bv(t)$: By definition of the Fourier transform,

$$ax(t) + bv(t) \leftrightarrow \int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t} dt$$

By linearity of integration,

$$\int_{-\infty}^{\infty} [ax(t) + bv(t)] e^{-j\omega t} dt = a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt$$

and thus $ax(t) + bv(t) \leftrightarrow aX(f) + bV(f)$

Fourier Transform Properties – Time Shift

If $x(t) \leftrightarrow X(f)$, then for any positive or negative real number c ,

$$x(t - c) \leftrightarrow X(f)e^{-j\omega c}$$

Note that if $c > 0$, then $x(t - c)$ is c -second right shift of $x(t)$;

if $c < 0$, then $x(t - c)$ is c -second left shift of $x(t)$.

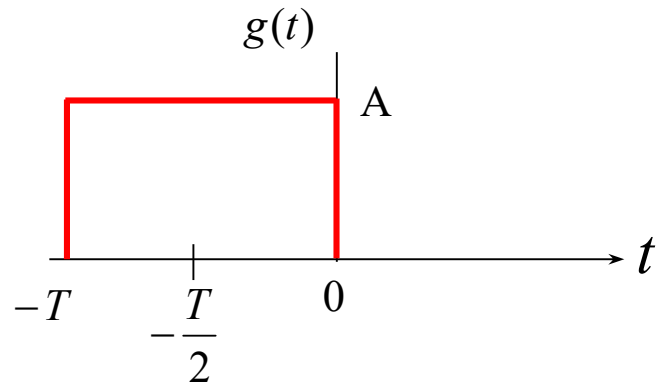
Thus the above transform pair is valid for both left and right shifts of $x(t)$.

To verify this property, first apply the definition of the Fourier transform

$$\begin{aligned} x(t - c) &\leftrightarrow \int_{-\infty}^{\infty} x(t - c)e^{-j\omega t} dt & x(t - c) &\leftrightarrow \int_{-\infty}^{\infty} x\left(\bar{t}\right)e^{-j\omega\left(\bar{t}+c\right)} d\bar{t} \\ & & &\leftrightarrow \left[\int_{-\infty}^{\infty} x\left(\bar{t}\right)e^{-j\omega\bar{t}} d\bar{t} \right] e^{-j\omega c} \\ & & &\leftrightarrow X(f)e^{-j\omega c} \end{aligned}$$

Let $\bar{t} = t - c$, then $t = \bar{t} + c$ and $dt = d\bar{t}$

Fourier Transform Properties – Time Shift



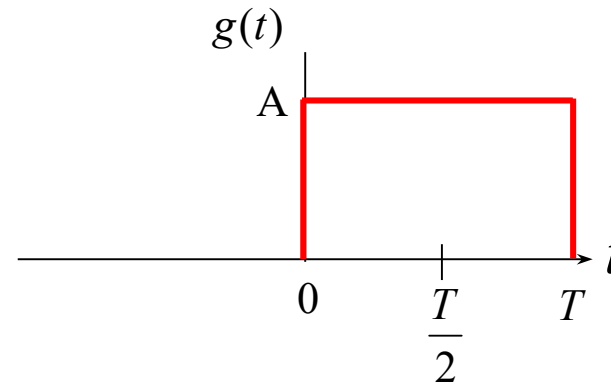
$$g(t) = A \operatorname{rect}\left(\frac{t + T/2}{T}\right)$$

$$A \operatorname{rect}\left(\frac{t}{T}\right) \leftrightarrow AT \operatorname{sinc}(f T)$$

$$\begin{aligned} x(t) &\leftrightarrow X(f) \\ x(t - t_0) &\leftrightarrow X(f) e^{-j\omega t_0} \end{aligned}$$

Therefore,

$$G(f) = AT \operatorname{sinc}(f T) e^{j2\pi f \frac{T}{2}}$$



$$g(t) = A \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

$$A \operatorname{rect}\left(\frac{t}{T}\right) \leftrightarrow AT \operatorname{sinc}(f T)$$

$$\begin{aligned} x(t) &\leftrightarrow X(f) \\ x(t - t_0) &\leftrightarrow X(f) e^{-j\omega t_0} \end{aligned}$$

Therefore,

$$G(f) = AT \operatorname{sinc}(f T) e^{-j2\pi f \frac{T}{2}}$$

Fourier Transform Properties – Time Scaling

If $x(t) \leftrightarrow X(f)$, for any positive real number a , $x(at) \leftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right)$

To verify this property, first apply the definition of the Fourier transform

$$x(at) \leftrightarrow \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

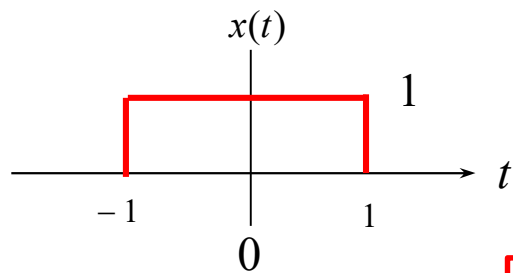
Let $\bar{t} = at$, then $t = \bar{t}/a$ and $d\bar{t} = a dt$,

$$\begin{aligned} x(at) &\leftrightarrow \int_{-\infty}^{\infty} x\left(\frac{\bar{t}}{a}\right) e^{-j\frac{\omega}{a}\bar{t}} \frac{1}{a} d\bar{t} \\ &\leftrightarrow \frac{1}{a} \int_{-\infty}^{\infty} x\left(\frac{\bar{t}}{a}\right) e^{-j\frac{\omega}{a}\bar{t}} d\bar{t} \\ &\leftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right) \end{aligned}$$

Fourier Transform Properties – Time Scaling

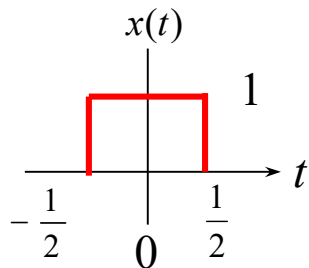
If $0 < a < 1$, $x(at)$ is a **time expansion** of $x(t)$ and $X(\frac{f}{a})$ is a **frequency compression** of $X(f)$

If $a > 1$, $x(at)$ is a **time compression** of $x(t)$ and $X(\frac{f}{a})$ is a **frequency expansion** of $X(f)$

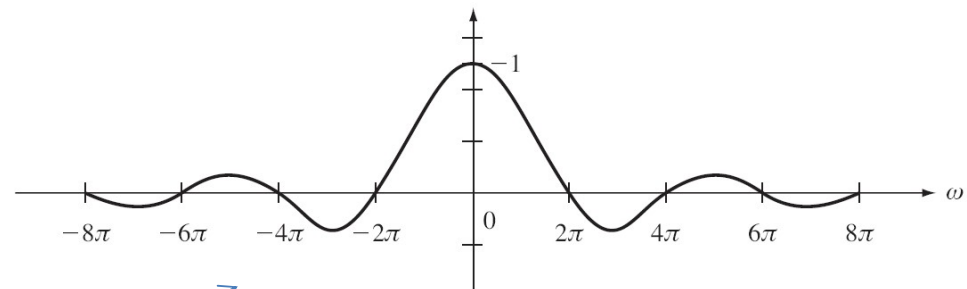
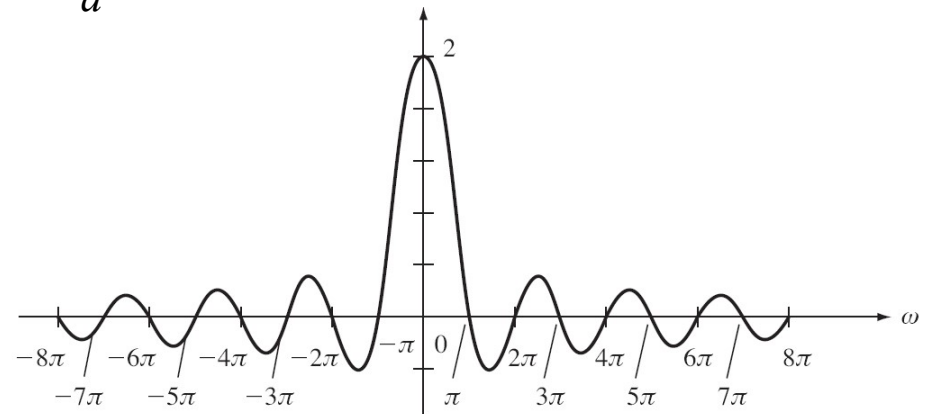


$$x(t) \leftrightarrow 2\text{sinc}(2f)$$

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{f}{a}\right)$$



$$x(2t) \leftrightarrow \text{sinc}(f)$$



time compression

frequency expansion

Fourier Transform Properties – Multiplication by a Complex Exponential (Frequency Shift)

If $x(t) \leftrightarrow X(f)$, then

$$x(t)e^{j\omega_0 t} \leftrightarrow X(f - f_0)$$

So, multiplication a **complex exponential** in the **time domain** corresponds to a **frequency shift** in the **frequency domain**.

The proof of this properties follows directly from the definition of the Fourier transform. You can do the verification after the lecture.

Fourier Transform Properties – Tutorial

1) Find the Fourier transform of

i) $g(t) = e^{-t} \sin(2\pi f_o t) u(t)$

ii) $g(t) = 8 \operatorname{rect}(t/4) \cos(2\pi 10^6 t)$

2) Find the inverse Fourier transform of

$$G(f) = 12 \operatorname{sinc}(4f) \sin(4\pi f)$$

Fourier Transform Properties

Part I:

- Linearity
- Time Shift
- Time Scaling
- Multiplication by a Complex Exponential (Frequency Shift)
- Tutorial

Part II:

- Multiplication by a Sinusoid
- Differentiation in the Time Domain
- Integration in the Time Domain
- Convolution in the Time Domain
- Multiplication in the Time Domain
- Tutorial

Fourier Transform Properties – Multiplication by a Sinusoid

If $x(t) \leftrightarrow X(f)$, for any real number f_0 , where $\omega_0 = 2\pi f_0$

$$\begin{aligned}x(t)\sin(\omega_0 t) &\leftrightarrow \frac{1}{2j}[X(f - f_0) - X(f + f_0)] \\x(t)\cos(\omega_0 t) &\leftrightarrow \frac{1}{2}[X(f - f_0) + X(f + f_0)]\end{aligned}$$

The proof of this property follows directly from the definition of the Fourier transform and Euler's identity.

The signals $x(t)\sin(\omega_0 t)$ and $x(t)\cos(\omega_0 t)$ can be viewed as amplitude-modulated signals. More precisely, they are called the **modulation theorems** of the Fourier transform.

The above relationships show that modulation of a carrier by a signal $x(t)$ results in the frequency translations $X(f + f_0)$, $X(f - f_0)$ of Fourier transform $X(f)$.

Fourier Transform Properties – Differentiation in the Time Domain

If $x(t) \leftrightarrow X(f)$, then

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(f)$$

It follows from the above equation that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. To prove this property, observe that the Fourier transform of $dx(t)/dt$ is

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt$$

The integral can be computed “by parts” as follows: with $v = e^{-j\omega t}$ and $\omega = x(t)$ $dv = -j\omega e^{-j\omega t}$ and $d\omega = [dx(t)/dt]$. Then,

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt = v\omega \Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} \omega dv$$

Then, $x(t) \rightarrow 0$ as $t \rightarrow \infty$,

$$= e^{-j\omega t} x(t) \Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega) e^{-j\omega t} dt = j\omega \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = j\omega X(f)$$

Fourier Transform Properties – Integration in the Time Domain

Suppose that $x(t)$ has the Fourier transform $X(f)$. Then the integration of a time function $x(t)$ results in the following generalized transform in the frequency

$$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(f) + \pi X(0) \delta(f)$$

where $\delta(f)$ is the impulse function in the frequency domain.

Note that if the signal $x(t)$ has no dc component (e.g. $X(0)=0$), then the above equation reduces to

$$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(f)$$

This means the integration of a time function has the effect of dividing its Fourier transform by the factor of $j\omega$.

Fourier Transform Properties – Convolution in the Time Domain

Given two signals $x(t)$ and $v(t)$ with Fourier transforms $X(f)$ and $V(f)$, the Fourier transform of the convolution $x(t)*v(t)$ is equal to the product $X(f)V(f)$ which results in the transform pair

$$x(t)*v(t) \leftrightarrow X(f)V(f)$$

This means that **convolution** in the **time domain** corresponds to **multiplication** in the **frequency domain**.

To prove it, first recall that by definition of convolution, $x(t)*v(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda$
Hence, the Fourier transform of $x(t)*v(t)$ is given by

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda \right] e^{-j\omega t} dt \quad \text{rewritten} \quad \int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} v(t-\lambda)e^{-j\omega t} dt \right] d\lambda$$

Let $\bar{t} = t - \lambda$ in the second integral,

$$\int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} v(\bar{t})e^{-j\omega(\bar{t}+\lambda)} d\bar{t} \right] d\lambda \quad \text{rewritten} \quad \left[\int_{-\infty}^{\infty} x(\lambda)e^{-j\omega\lambda} d\lambda \right] \left[\int_{-\infty}^{\infty} v(\bar{t})e^{-j\omega\bar{t}} d\bar{t} \right]$$

Fourier Transform Properties – Multiplication in the Time Domain

If $x(t) \leftrightarrow X(f)$ and $v(t) \leftrightarrow V(f)$, then

$$x(t)v(t) \leftrightarrow [X(f)*V(f)] = \int_{-\infty}^{\infty} X(\lambda)V(f-\lambda)d\lambda$$

It is seen that **multiplication** in the **time domain** corresponds to **convolution** in the **Fourier transform domain**.

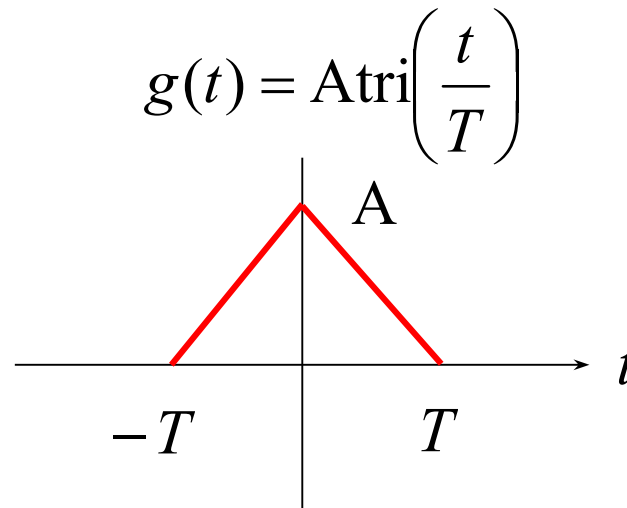
The proof of this property follows from the definition of the Fourier transform and the manipulation of integrals.

Fourier Transform Properties – Tutorial

1) Find the Fourier transform of a triangular pulse

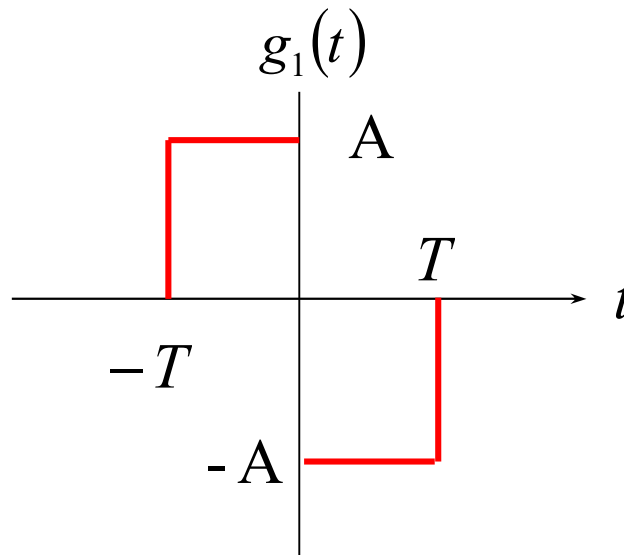
$$g(t) = \begin{cases} A\left[1 - \frac{|t|}{T}\right], & |t| < T \\ 0, & |t| \geq T \end{cases}$$

Let's draw it



Fourier Transform Properties – Tutorial

Now let's define a *doublet pulse* which looks like this



It is the superposition of 2 shifted rectangular functions, i.e.,

$$g_1(t) = A \operatorname{rect}\left(\frac{t + T/2}{T}\right) - A \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

Fourier Transform Properties – Tutorial

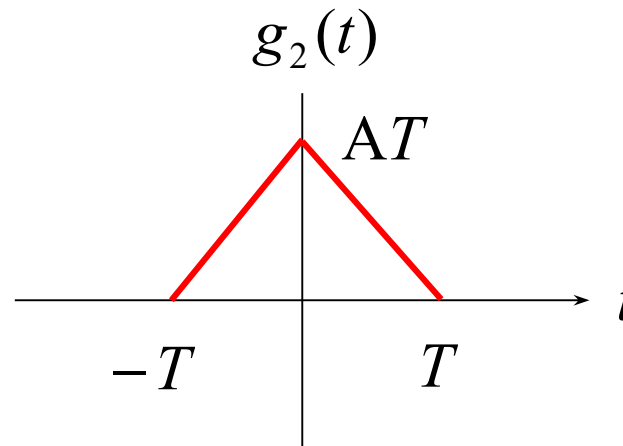
$$g_1(t) = A \operatorname{rect}\left(\frac{t + T/2}{T}\right) - A \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

What is the Fourier transform of this doublet pulse?

$$\begin{aligned} G_1(f) &= AT \operatorname{sinc}(f T) \left\{ e^{j\pi f T} - e^{-j\pi f T} \right\} \\ &= j2AT \operatorname{sinc}(f T) \sin(\pi f T) \end{aligned}$$

Fourier Transform Properties – Tutorial

Integrating $g_1(t)$, we get



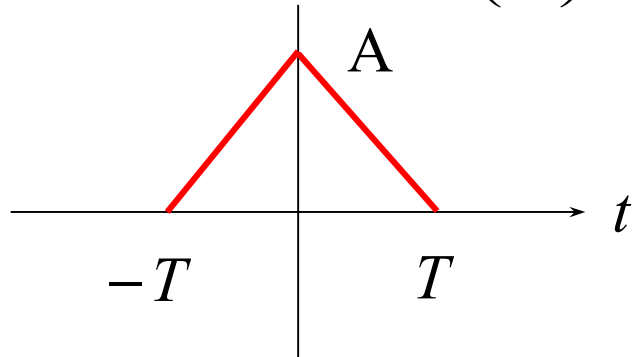
Therefore, using the integration property, we get the Fourier Transform of $g_2(t)$

$$\begin{aligned} G_2(f) &= \frac{1}{j\omega} G_1(f) \\ &= AT \operatorname{sinc}(f T) \frac{\sin(\pi f T)}{\pi f} \\ &= AT^2 \operatorname{sinc}^2(f T) \end{aligned}$$

Fourier Transform Properties – Tutorial

Keep in mind that $g(t) = g_2(t)/T$, and therefore $G(f) = G_2(f)/T$.

The Fourier transform of a triangular pulse is given as

$$A \operatorname{tri}\left(\frac{t}{T}\right) \leftrightarrow AT \operatorname{sinc}^2(fT)$$


Fourier Transform Properties – Tutorial

2) Find the inverse Fourier transform of

$$G(f) = 16\text{sinc}^2(4(f-10^6)) + 16\text{sinc}^2(4(f+10^6))$$

3) Find the Fourier transform of

i) $g(t) = 10 \text{tri}(2t - 1/2)$

ii) $g(t) = 8 \text{tri}(t/2) \cos(2\pi 10^6 t)$