Chapter 1 Sets and Logic

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1.1 Sets

Set = a collection of objects

Example: A = {1, 2, 3, 4}

Elements

A set is determined by its elements and not by any particular order.

A = {1, 2, 4, 3}

1.1 Sets

Set = a collection of objects

Example: $B = \{x \mid x = 2k + 1, 0 < k < 3\}$

1.1 Sets

Set = a collection of objects

How to determine a set

- **Listing** $\longrightarrow$ a set is finite and not too large

  $A = \{1, 2, 3, 4\}$

- **Describing Property** $\longrightarrow$ a set is a large finite set or an infinte set

  $B = \{x \mid x = 2k + 1, 0< k < 3\}$

1.1 Sets

Finite sets

Examples:

A = {1, 2, 3, 4}

B = {x | x is an integer, $1 \leq x \leq 4$}

Infinite sets

Examples:

Z = {integers} = {…, -3, -2, -1, 0, 1, 2, 3,…}

S={x| x is a real number and $1 \leq x \leq 5$} = [1, 5]

1.1 Sets

A set may contain any kind of element

Examples:

{1, 2, Jason}

{1, 5, {3.5, 17}, Jason }

Sets of Numbers

Examples:

Z : Integers

Q: Rational numbers

R: Real numbers

Guess: What are $\mathbf{Z}^{+}$, $\mathbf{Z}^{-}$, and $\mathbf{Z}^{nonneg}$**?**

1.1 Sets

Cardinality of a set A (in symbols $|A|$) 集合的势

$|A|$=the number of elements in A

Examples:

If $A = \{1, 2, 3\}$ then $|A| = 3$

If $B = \{x \mid x$ is a natural number and $1 \le x \le 9\}$, then $|B| = 9$

1.1 Sets

Cardinality of a set A (in symbols |A|)

|A|=the number of elements in A

An element x **is in a set** X**:** $x \in X$

An element x **is not in a set** X**:** $x \notin X$

Empty Set $\varnothing$

The set with no elements is call the empty set

or null set
or void set

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

- For every x, if $x \in A$, then $x \in B$,

and

- For every x, if $x \in B$, then $x \in A$.

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

Examples:

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2}
- A = {dog, cat, horse},
  B = {cat, horse, squirrel, dog}
- A = {dog, cat, horse},
  B = {cat, horse, dog, dog}

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

Examples1.1.3

- A = $\{x \mid x^2 + x - 6 = 0\}$, B = {2, -3}, A=B?

Subsets

A is a subset of B if every element of A is also contained in B. (in symbols $A \subseteq B$)

- Equality: A = B if $A \subseteq B$ and $B \subseteq A$

Subsets

A is a subset of B if every element of A is also contained in B. (in symbols $A \subseteq B$)

- Equality: A = B if $A \subseteq B$ and $B \subseteq A$
- A is a **proper subset** of B if $A \subseteq B$ but does not equal B and write $A \subset B$
- Observation: $\varnothing$ is a subset of every set

Power set of A 集合A的幂集

The set of all subsets (proper or not) of a set A.
(in symbols $\mathcal{P}(\mathrm{A})$)

Examples1.1.14

If A={a, b, c}, the members of $\mathcal{P}(\mathrm{A})$ are

$\varnothing$, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}

The power set of a set with n elements has ? elements

Set Operations

Given two sets X and Y

- The union (并集) of X and Y is defined as the set

$$\mathbf{X} \cup \mathbf{Y} = \{ \mathbf{x} \mid \mathbf{x} \in \mathbf{X} \text{ or } \mathbf{x} \in \mathbf{Y} \}$$

- The intersection (交集) of X and Y

$$\mathbf{X} \cap \mathbf{Y} = \{ \mathbf{x} \mid \mathbf{x} \in \mathbf{X} \text{ and } \mathbf{x} \in \mathbf{Y} \}$$

- The difference (or relative complement) (差集) of X and Y

$$\mathbf{X} - \mathbf{Y} = \{ \mathbf{x} \mid \mathbf{x} \in \mathbf{X} \text{ and } \mathbf{x} \notin \mathbf{Y} \}$$

Set Operations

Example 1.1.15 If A={1, 3, 5} and B={4, 5, 6} then

$A \cup B =$

$A \cap B =$

$A - B=$

Set Operations

Sets X and Y are called **disjoint** (不相交) if their intersection is empty, that is, they share no elements:

$$X \cap Y = \varnothing$$

A collection of sets S (集族) is said to be **pairwise disjoint** （两两不相交） if, whenever X and Y are distinct sets in S, X and Y are disjoint

Set Operations

Sets X and Y are called **disjoint** (不相交) if their intersection is empty, that is, they share no elements.

$$X \cap Y = \varnothing$$

A collection of sets S is said to be **pairwise disjoint** （两两不相交） if, whenever X and Y are distinct sets in S, X and Y are disjoint.

Example 1.1.17

{1, 4, 3} and {2, 6}

S={{1, 3, 5}, {2, 8}, {4}}

Univeral Set and Complement

Universal set (全集/域): When all of the consided sets are subset of a set U, we called U a universal set or a universe

Complement of X(余/补集): U-X

Venn Diagrams

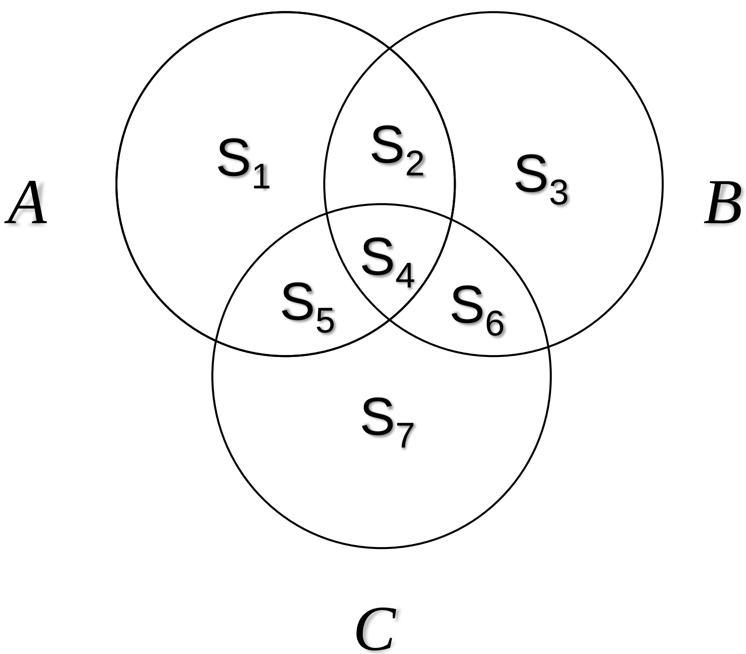
- A Venn diagram provides a graphic view of sets.
- Set union, intersection, difference, symmetric difference and complements can be identified.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive Law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

We can also verify this law more carefully



L.H.S

$$A = S_1 \cup S_2 \cup S_4 \cup S_5$$

$$B \cap C = S_4 \cup S_6$$

$$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

R.H.S.

$$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$$

$$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$$

$$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

There are formal proofs in the textbook, but we don't do that.

Venn Diagrams

- A Venn diagram provides a graphic view of sets.
- Set union, intersection, difference, symmetric difference and complements can be identified.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Venn Diagrams

Example 1.1.21

Among a group of 165 students,

- 8 are taking calculus, psychology, and computers science;
- 33 are taking calculus and computer science;
- 20 are taking calculus and psychology
- 24 are taking psychology and computer science：
- 79 are taking calculus;
- 83 are taking psychology;
- 63 are taking computer science.

How many are taking none of the three subjects?

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

a) Associative law (结合律）:

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

b) Commutative law (交换律)

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

c) Distributive laws (分配律)

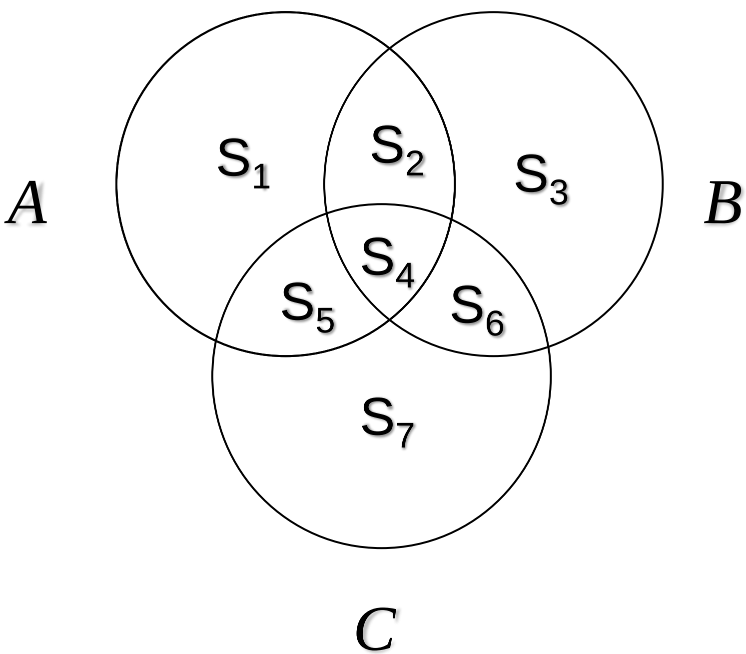
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distributive Law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

We can also verify this law more carefully



L.H.S

$$A = S_1 \cup S_2 \cup S_4 \cup S_5$$

$$B \cap C = S_4 \cup S_6$$

$$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

R.H.S.

$$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$$

$$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$$

$$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

There are formal proofs in the textbook, but we don't do that.

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

c) Distributive laws (分配律)

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$$
$$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$$

d) Identity laws (同一律)

$$A\cap U=A$$
$$A\cup\varnothing = A$$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

e) Complement laws (补余律)

$A \cup A^c = U$

$A \cap A^c = \varnothing$

f) Idempotent laws (等幂律)

$A \cup A = A$

$A \cap A = A$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

g) Bound laws (零律)

$$A \cup U = U$$

$$A \cap \varnothing = \varnothing$$

h) Absorption laws (吸收律)

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

i) Involution law (对合律)

$$(A^c)^c = A$$

j) 0/1 laws (0/1律)

$$\varnothing^c = U,\ U^c = \varnothing$$

k) De Morgan's laws for sets (德•摩根定律)

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

We define the union of a collection of sets S to be those elements x belonging to at least one set X in S. Formally,

$\bigcup S = \{x \mid x \in X \text{ for some } X \in S\}$.

Similarly, we define the intersection of a collection of sets S to be those elements x belonging to every set X in S. Formally,

$\bigcap S = \{x \mid x \in X \text{ for all } X \in S\}$.

If $S = \{A_1, A_2, \ldots, A_n\}$, we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{n} A_i,$$

If $S = \{A_1, A_2, \ldots A_n \ldots, \}$, we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Set partition

A partition of a set *X* divides *X* into nonoverlapping subsets.

More formally, a collection *S* of nonempty subsets of *X* is said to be a **partition** of the set *X* if every element in *X* belongs to exactly one member of *S*.

Example 1.1.25

Since each element of X={1,2,3,4,5,6,7,8} is in exactly one member of S={{1, 4, 5}, {2, 6}, {3, 8}, {7}}, S is a partition of X

Notice that if *S* is a partition of *X*, *S* is pairwise disjoint and $\cup S = X$.

Cartesian Product 笛卡尔积

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Example: $A = \{x, y\}$, $B = \{a, b, c\}$

$A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product 笛卡尔积

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

$|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } 1 \leq i \leq n\}$

Problem-Solving Tips

To verify that two sets A and B are equal, written $A = B$, show that for every x, if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.

To verify that two sets A and B are not equal, written $A \neq B$, find at least one element that is in A but not in B, or find at least one element that is in B but not in A. One or the other conditions suffices; you need not (and may not be able to) show both conditions.

Problem-Solving Tips

To verify that A is a subset of B, written $A \subseteq B$, show that for every x, if $x \in A$, then $x \in B$. Notice that if A is a subset of B, it is possible that $A = B$.

To verify that A is not a subset of B, find at least one element that is in A but not in B.

Problem-Solving Tips

To verify that A is a proper subset of B, written $A \subset B$, verify that A is a subset of B as described previously, and that $A = B$, that is, that there is at least one element that is in B but not in A.

To visualize relationships among sets, use a Venn diagram. A Venn diagram can suggest whether a statement about sets is true or false.

A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n-tuples take order into account.

1.2 Propositions

Logic is a system based on propositions.
A proposition is a statement or sentence that can be determined to be either true or false , but not both.

Two factors:

- a statement or sentence
- can be determined to be either true or false

1.2 Propositions

A proposition is a statement or sentence that can be determined to be either true or false, but not both.

Truth Value of a proposition

True: T

False: F

1.2 Propositions

A proposition is a statement or sentence that can be determined to be either true or false, but not both.

Truth Value of a proposition

True: T

False: F

"Elephants are bigger than mice."

It is a proposition and it is True!

1.2 Propositions

A proposition is a statement or sentence that can be determined to be either true or false, but not both.

We use variables, such as p,q and r, to represent propositions.

We will also use the notation

$$p\text{: } 1+1=3$$

to define p to be the proposition $1+1=3$.

Combining Propositions

Definition 1.2.1 Let p and q be propositions.

The **conjunction (合取)** of p and q, denote $\boldsymbol{p} \wedge \boldsymbol{q}$, is the proposition p and q.

The **disjunction (析取)** of p and q, denote $\boldsymbol{p} \vee \boldsymbol{q}$, is the proposition p or q.

Example: If p: It is raining, and q: It is cold, then

Combining Propositions

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Example: If p: It is raining, and q: It is cold, then

$$p \wedge q: ?$$
$$p \vee q: ?$$

1.2 Propositions

Definition 1.2.9 The negation of p, denote $\neg \boldsymbol{p}$, is the proposition

not p.

Example: Tim is a boy.

Logic Operators

$\wedge ::= \text{AND} \vee ::= \text{OR} \neg ::= \text{NOT}$

Example:

p: It is hot. q: It is sunny.

It is hot and sunny.

It is not hot but sunny

It is neither hot nor sunny

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logic Operators

$$\wedge ::= \text{AND} \vee ::= \text{OR} \neg ::= \text{NOT}$$

Truth Table

p	p
T	F
F	T

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

Truth Table

Think about a general version of a truth table.

p	$\neg p$
T	F
F	T

p_1 p_2 p_3 P

Truth table of (p∨q)^r

p	q	r	p ∨ q	(p ∨ q) ^ r

Truth table of (p∨q)^r

p	q	r	p ∨ q	(p ∨ q) ^ r
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

Truth Table

Think about a general version of a truth table.

p	$\neg p$
T	F
F	T

p_1 p_2 $p_3 \ldots p_n$ P

Logic Operators

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
and then $\vee$.

Example：Given that proposition p is false, proposition q is true, and proposition r is false, determine whe ther the proposition $\neg p \vee q \wedge r$ is true or false.

Solution: True

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
and then $\vee$.

Example：

Given that proposition p is false, proposition q is true, and proposition r is false, determine whether each proposition in Exercises 17–22 is true or false.

17. $p \vee q$

18. $\neg p \vee \neg q$

19. $\neg p \vee q$

20. $\neg p \vee \neg(q \wedge r)$

21. $\neg(p \vee q) \wedge (\neg p \vee r)$

22. $(p \vee \neg r) \wedge \neg((q \vee r) \vee \neg(r \vee p))$

Solution:
17-20 are True
21 and 22 are False

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
and then $\vee$.

Logic Operators: Exclusive-Or (异或)

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

(兼或)

exclusive-or

p	q	$p \vee q$

p	q	$p \oplus q$

Logic Operators: Exclusive-Or (异或)

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

(兼或)

Does this definition make sense?

exclusive-or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p ? q
T	T	F
T	F	T
F	T	T
F	F	F

Logic Operators: Exclusive-Or (异或)

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

(兼或)

coffee “or” tea ← exclusive-or (异或)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

组合电路

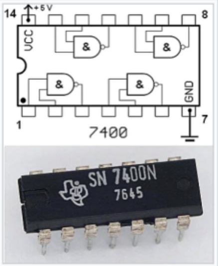
T: has power

F: do not has power

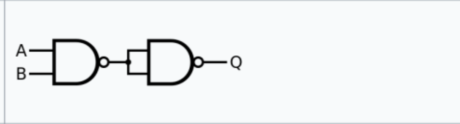
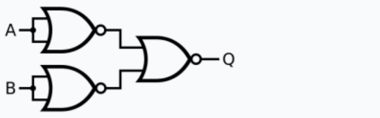
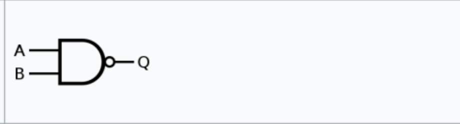
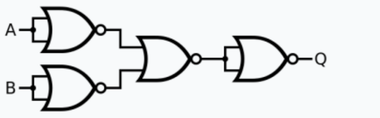
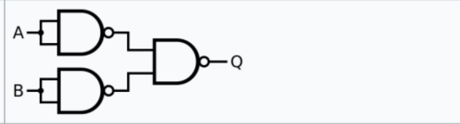
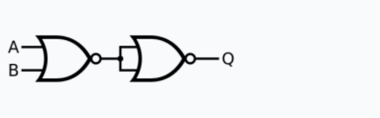
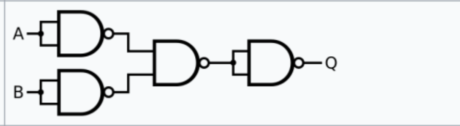
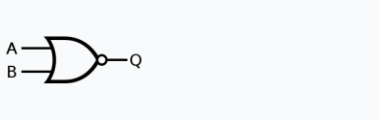
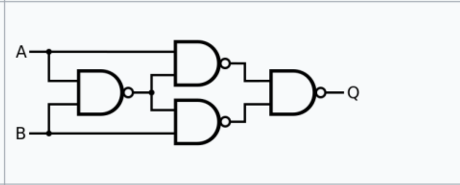
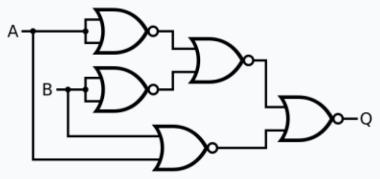
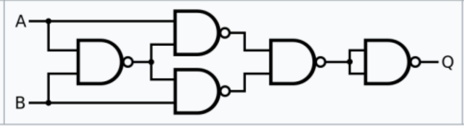
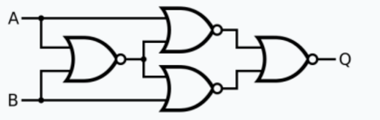
Universal logic gates [edit]

Further information on the theoretical basis: Functional completeness

Charles Sanders Peirce (during 1880–81) showed that NOR gates alone (or alternatively NAND gates alone) can be used to reproduce the functions of all the other logic gates, but his work on it was unpublished until 1933.[15] The first published proof was by Henry M. Sheffer in 1913, so the NAND logical operation is sometimes called Sheffer stroke; the logical NOR is sometimes called *Peirce's arrow*.[16] Consequently, these gates are sometimes called *universal logic gates*.[17]



The 7400 chip, containing four NANDs. The two additional pins supply power (+5 V) and connect the ground.

type	NAND construction	NOR construction
NOT		
AND		
NAND		
OR		
NOR		
XOR		
XNOR		

How to construct a compound proposition for exclusive-or?

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

How to construct a compound proposition for exclusive-or?

Definition 1.3.10 Suppose that the propositions P and Q are made up of the propositions $p_1, p_2, p_3, \ldots, p_n$. We said that P and Q are **logically equivalent (逻辑等价)**, and write

$$P \equiv Q,$$

provided that given any truth value of $p_1, p_2, p_3, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

How to construct a compound proposition for exclusive-or?

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

How to construct a compound proposition for exclusive-or?

p	q				$p \oplus q$
T	T				F
T	F				T
F	T				T
F	F				F

Solution: $(p \vee q) \wedge \neg (p \wedge q)$

Problem-Solving Tips

Although there may be a shorter way to determine the truth values of a proposition P formed by combining propositions $p_1, \ldots, p_n$ using operators such as $\neg$ and $\vee$, a truth table will always supply all possible truth values of P for various truth values of the constituent propositions $p_1, \ldots, p_n$.

1.3 Conditional Propositions and Logical Equivalence

Definition 1.3.1 A **conditional proposition (条件命题)** is of the form

"If p then q"

In symbols: $p \rightarrow q$.

1.3 Conditional Propositions and Logical Equivalence

Definition 1.3.1 A **conditional proposition (条件命题)** is of the form

"If p then q"

In symbols: $p \rightarrow q$.

p : hypothesis (or antecedent) 假设（前件）

q : conclusion (or consequent) 结论（后件）

1.3 Conditional Propositions and Logical Equivalence

Definition 1.3.1 A **conditional proposition (条件命题)** is of the form

"If p then q"

In symbols: $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional proposition that is true because the hypothesis is false is said to be **true by default (默认为真)** or **vacuously true (空虚真)**.

1.3 Conditional Propositions and Logical Equivalence

Your parents say: “If your got at least 85 in the this course, then I will buy you an iPad.”

When is the above sentence false?

- It is false when you get an 85 but your parents do not buy you an iPad.
- In particular, it is not false if your score is below 85.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	**T**	**T**
F	**F**	**T**

$\wedge$ ::= AND $\vee$ ::= OR $\neg$ ::= NOT

$\rightarrow$ ::= IMPLIES

1.3 Conditional Propositions and Logical Equivalence

Some statements may be rephrased as conditional propositions.

Example 1.3.6

(a) Mary will be a good student if she studies hard.

(b) John takes calculus only if he has sophomore, junior, or senior standing.

(c) When you sing, my ears hurt.

(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.

(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

1.3 Conditional Propositions and Logical Equivalence

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1.3 Conditional Propositions and Logical Equivalence

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1.3 Conditional Propositions and Logical Equivalence

Some statements may be rephrased as conditional propositions.

Example 1.3.6

(c) When you sing, my ears hurt.

1.3 Conditional Propositions and Logical Equivalence

Some statements may be rephrased as conditional propositions.

Example 1.3.6

(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.

1.3 Conditional Propositions and Logical Equivalence

Some statements may be rephrased as conditional propositions.

Example 1.3.6

(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

Logic Operators

$\wedge ::= \text{AND} \quad \vee ::= \text{OR} \quad \neg ::= \text{NOT} \quad \rightarrow ::= \text{IMPLIES}$

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
then $\vee$,
and then $\rightarrow$.

Logic Operators

$\wedge ::= \text{AND} \vee ::= \text{OR} \; \neg ::= \text{NOT} \; \rightarrow ::= \text{IMPLIES}$

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
then $\vee$,
and then $\rightarrow$.

Example: $p \vee q \longrightarrow \neg r$

Logic Operators

$$\wedge ::= \text{AND} \quad \vee ::= \text{OR} \quad \neg ::= \text{NOT} \quad \rightarrow ::= \text{IMPLIES}$$

Example 1.3.5

Assume that p is true, q is false, and r is true, find the truth value of each proposition.

(a) $p \wedge q \longrightarrow r$

(b) $p \wedge (q \longrightarrow r)$

(c) $p \rightarrow (q \longrightarrow r)$

(d) $p \vee q \longrightarrow \neg r$

Definition 1.3.10 Suppose that the propositions P and Q are made up of the propositions $p_1, p_2, p_3, \ldots, p_n$. We said that P and Q are **logically equivalent (逻辑等价)**, and write

$$P \equiv Q,$$

provided that given any truth value of $p_1, p_2, p_3, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

$$p \rightarrow q \equiv ?$$

- If you don't give me all your money, then you will be killed.
- Either you give me all your money or you will be killed (or both).

p	q		

Solution: $p \rightarrow q \equiv \neg p \vee q$

Exclusive-Or

p	q	$p\ q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

$$\vdots$$

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$\boxed{(p \wedge \neg q) \vee (\neg p \wedge q)}$$

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

Idea 1: Look at the true rows

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

$$\boxed{\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)}$$

Idea 2: Look at the false rows

$$p \rightarrow q \equiv ?$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 1:

Idea 2:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 1:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 2:

Example 1.3.13

Which proposition is logically equivalent to the negation of $p \longrightarrow q$?

Example 1.3.13

Show that the negation of $p \longrightarrow q$ is logically equivalent to $p \wedge \neg q$.

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Statement: The number 6 is divisible by 2 or 5.

Negation: The number 6 is not divisible by 2 and not divisible by 5.

De Morgan's Laws for Logic

$$\boxed{\neg(p \wedge q) \equiv \neg p \vee \neg q}$$

Truth Table

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

Are these two propositions logically equivalent?

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

p	q	$p \rightarrow q$	$q \rightarrow p$

Are these two propositions logically equivalent?

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

$\neg$

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

¬

Biconditional Proposition 双条件命题

Definition 1.3.8

If p and q are propositions, the proposition

$$p \text{ if and only if } q,$$

is called a biconditional proposition and is denoted

$$p \leftrightarrow q.$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$$\leftrightarrow ::= \text{IFF}$$

Contrapositive (Transposition) Proposition 逆否命题（转换命题）

Definition 1.3.16

The contrapositive (or transposition) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \longrightarrow \neg p$.

The conditional proposition $p \rightarrow q$ and its contrapositive $\neg q \longrightarrow \neg p$ are logically equivalent.

Proof by Truth Table

If p, then q.

If $\neg q$, then $\neg p$.

p	q	$p \rightarrow q$

$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

Propositional Equivalent

De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$p \rightarrow q \equiv \neg p \vee q$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Others

Propositional Equivalent

$$\neg(\neg A) \equiv A$$

$$A \vee A \equiv A$$

$$A \wedge A \equiv A$$

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$

$$(A \vee B) \vee C \equiv A \vee (B \vee C)$$

$$A \vee B \equiv B \vee A$$

$$A \wedge B \equiv B \wedge A$$

Propositional Equivalent

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$
$$\neg (A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg (A \wedge B) \equiv \neg A \vee \neg B$$
$$A \wedge (A \vee B) \equiv A$$
$$A \vee (A \wedge B) \equiv A$$
$$A \wedge T \equiv A$$
$$A \vee F \equiv A$$

Propositional Equivalent

$$A \wedge F \equiv F$$
$$A \vee T \equiv T$$
$$A \vee (\neg A) \equiv T$$
$$A \wedge (\neg A) \equiv F$$
$$A \rightarrow B \equiv \neg A \vee B$$
$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$
$$A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$$

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \equiv \neg A$$

Simplifying Statement

$$\neg(\neg p \land q) \land (p \lor q)$$

Tautology, Contradiction

A tautology is a statement that is always true.

$$p \vee \neg p$$

$$(p \wedge q) \vee (\neg q \wedge p) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q)$$

A contradiction is a statement that is always false. (negation of a tautology)

$$p \wedge \neg p$$

$$(p \vee q) \wedge (\neg q \vee p) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee q)$$

$$((p \wedge r) \vee (q \wedge r)) \wedge (\neg(p \vee q) \vee r)$$

If, Only-If

- You will succeed if you work hand.
- You will succeed only if you work hard.

r if s means "if s then r"

We also say r is a necessary condition for s.

r only if s means "if r then s"

We also say r is a sufficient condition for s.

You will succeed if and only if you work hard.

p if and only if (iff) q means p and q are logically equivalent.

Math vs Language

Parent: if you $\underbrace{\text{don't clean your room}}_{\text{C}}$, then you $\underbrace{\text{can't watch a DVD}}_{\text{D}}$.

This sentence says $\neg C \longrightarrow \neg D$

In real life it also means $C \longrightarrow D$

So $C \leftrightarrow D$

Mathematician: if a number x greater than 2 is not an odd number, then x is not a prime number.

This sentence says $\neg O \longrightarrow \neg P$

But of course it doesn't mean $O \longrightarrow P$

Knights and Knaves

Knights always tell the truth.

Knaves always lie.

A says: B is a knight.

B says: A and I are of opposite type.

Problem-Solving Tips

- In formal logic, "if" and "if and only if" are quite different. The conditional proposition $p \to q$ (if p then q) is true except when p is true and q is false. On the other hand, the biconditional proposition $p \leftrightarrow q$ (p if and only if q) is true precisely when p and q are both true or both false.
- To determine whether propositions P and Q, made up of the propositions $p_1, \ldots,$ p_n, are logically equivalent, write the truth tables for P and Q. If all of the entries for P and Q are always both true or both false, then P and Q *are* equivalent. If some entry is true for one of P or Q and false for the other, then P and Q are *not* equivalent.
- De Morgan's laws for logic

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

give formulas for negating "or" ($\vee$) and negating "and" ($\wedge$). Roughly speaking, negating "or" results in "and," and negating "and" results in "or."

Problem-Solving Tips

- Example 1.3.13 states a very important equivalence

$$\neg(p \to q) \equiv p \wedge \neg q,$$

which we will meet throughout this book. This equivalence shows that the negation of the conditional proposition can be written using the "and" ($\wedge$) operator. Notice that there is no conditional operator on the right-hand side of the equation.

1.4 Arguments and Rules of Inference 论证和推理规则

Consider the following sequence of propositions.

- The bug is either in module 17 or in module 81.
- The bug is a numerical error.
- Module 81 has no numerical error.

“ The bug is in module 17”

This process of drawing a conclusion from a sequence of propositions is called **deductive reasoning (演绎推理)**.

Argument 论证

Definition1.4.1 An argument is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

or

$$p_1, p_2, \ldots, p_n / \therefore q$$

Argument 论证

Definition1.4.1 An argument is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

hypotheses (假设)
or premises (前提)

conclusion (结论)

or

$$p_1, p_2, \ldots, p_n / \therefore q$$

$\therefore$ means therefore

Argument 论证

Definition1.4.1 An argument is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

or

$$p_1, p_2, \ldots, p_n / \therefore q$$

Argument is said to be **valid (有效的)** if the conclusion follows from the hypotheses; that is, if p_1 and p_2 and … and p_n are true, then q must also be true. otherwise, the argument is invalid (or a fallacy).

Rules of inference 推理规则

1. **Modus ponens rule of inference** or **law of detachment** (假言推理或分离定律)

$$\begin{array}{l} \quad p \rightarrow q \\ \quad p \\ \hline \therefore q \end{array}$$

Rules of inference 推理规则

2. **Modus tollens**
(拒取)

$$\begin{array}{l} p \longrightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Rules of inference 推理规则

3. **Addtion**
(附加)

$$\frac{p}{\therefore p \vee q}$$

Rules of inference 推理规则

4. **Simplification**
(化简)

$$\frac{p \wedge q}{\therefore p}$$

Rules of inference 推理规则

5. **Conjunction**
(合取)

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

Rules of inference 推理规则

6. **Hypothetical syllogism**
(假言三段论)

$$\begin{array}{c} p \longrightarrow q \\ q \longrightarrow r \\ \hline \therefore p \longrightarrow r \end{array}$$

Rules of inference 推理规则

7. **Disjunctive syllogism**
(析取三段论)

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Problem-Solving Tips

The validity of a very short argument or proof might be verified using a truth table. In practice, arguments and proofs use rules of inference.

Problem-Solving Tips

TABLE 1.4.1 ■ Rules of Inference for Propositions

Rule of Inference	*Name*	*Rule of Inference*	*Name*
$p \to q$ p $\therefore q$	Modus ponens	p q $\therefore p \land q$	Conjunction
$p \to q$ $\neg q$ $\therefore \neg p$	Modus tollens	$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
p $\therefore p \lor q$	Addition	$p \lor q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \land q$ $\therefore p$	Simplification		

A game

Imagine there are 3 coins on the table. Gold, silver and copper.

If you say a truthful sentence, you will get one coin. If you say a false sentence, you get nothing.

Which sentence can guarantee gaining the gold coin?

A game

Imagine there are 3 coins on the table. Gold, silver and copper.

If you say a truthful sentence, you will get one coin. If you say a false sentence, you get nothing.

Which sentence can guarantee gaining the gold coin?

(A) You will give me the gold coin.
(B) You will give me all the coins.
(C) You will not give me any of the coins.
(D) You will give me either silver or copper coin.
(E) You will give me neither silver nor copper coin.

1.5 Quantifiers 量词

Consider the following statement

n is an odd integer

Is this a proposition?

1.5 Quantifiers 量词

Consider the following statement

n is an odd integer

Is this a proposition?

NO: Its truth value is based on the value of n.

1.5 Quantifiers 量词

Consider the following statement

n is an odd integer

Is this a proposition?

NO: Its truth value is based on the value of n.

An argument is a sequence of **propositions** written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore \ q \end{array}$$

1.5 Quantifiers 量词

Definition 1.5.1 Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a propositional function (命题函数) or predicate (谓词) (with respect to D) if for each $x \in D$， $P(x)$ is a proposition.

We call D the domain of discourse (论域) of P.

The domain of discourse specifies the alllowable values for x.

1.5 Quantifiers 量词

Example 1.5.3 Explain why the following are propositional functions. (命题函数)

(a) $n^2 + 2n$ is an odd integer (domain of discourse = $\mathbf{Z}^+$).

(b) $x^2 + x - 6 = 0$ (domain of discourse = $\mathbf{R}$).

(c) The baseball player hit over $.300$ in 2015 (domain of discourse = set of baseball players).

(d) The film is rated over 20% by Rotten Tamatoes (domain of discourse = set of films rated by Rotten Tomatoes).

Practice

Let $D = \{2, 3, 4, 5, 6, 7, 8\}$

For each of the following propositional function (or predicates), list those elements of D that make the statement true:

1. $P(x)$ is "$x^2 + 3$ is evenly divisible by 5".

2. $Q(x)$ is "$x > 1$ and $2x \leq 10$".

3. $R(x)$ is "x is even or prime".

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句)**.

It may be written

$$\forall x\ P(x).$$

The symbol ∀ means "for every", and is called a **universal quantifier (全称量词)**.

The statement is true if
The statement is false if
?

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句)**.

It may be written

$$\forall x\ P(x).$$

The symbol ∀ means "for every", and is called a **universal quantifier (全称量词)**.

The statement is true if $P(x)$ is true for every x in D.
The statement is false if $P(x)$ is false for at least one x.

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句)**.

for all x, $P(x)$
for any x,
$P(x)$

Alternative ways to write $\forall x\, P(x)$.

for all x in D, $P(x)$

Specify the domain of discourse.

Universal Quantifiers 全称量词

Example 1.5.8 Whether the universally quantified statement for every real number x, if $x > 1$, then $x + 1 > 1$ is true for every real number x?

The statement is true if $P(x)$ is true for every x in D.
The statement is false if $P(x)$ is false for at least one x.

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an **existentially quantified statement (存在量词语句)**.

It may be written

$$\exists x\ P(x).$$

The symbol ∃ means “there exists”, and is called a **existential quantifier (存在量词)**.

The statement is true if
The statement is false if

?

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an **existentially quantified statement (存在量词语句)**.

It may be written

$$\exists x\, P(x).$$

The symbol ∃ means "there exists", and is called a **existential quantifier (存在量词)**.

The statement is true if $P(x)$ is true for at least one x in D.
The statement is false if $P(x)$ is false for every x in D.

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an **existentially quantified statement (存在量词语句)**.

there exists x such that, $P(x)$
for some x, $P(x)$
for at least one x, $P(x)$

Alternative ways to write $\exists x\ P(x)$.

Existential Quantifiers 存在量词

Example 1.5.11 Verify that the existentially quantified statement

$$\exists x \in \boldsymbol{R} \left(\frac{x}{x^2+1} > 1\right)$$

is false.

Existential Quantifiers 存在量词

Example 1.5.13 Whether the following existentially quantified statement is true or false?

For some n, if n is prime, then $n+1$, $n+2$, $n+3$, $n+4$ are not prime.

Generalized De Morgan's Laws for Logic 广义德•摩根定律

De Morgan's Laws for Logic

?

Generalized De Morgan's Laws for Logic 广义德•摩根定律

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\,\forall x\, P(x)\,) \equiv ?$$

De Morgan's Laws for Logic

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\forall x\, P(x)) \equiv ?$$

Everyone likes football.

What is the negation of this statement?

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\,\forall x\, P(x)\,) \equiv \exists x\, \neg P(x)$$

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\,\forall x\, P(x)\,) \equiv \exists x\, \neg P(x)$$

$$\neg(\,\exists x\, P(x)\,) \equiv \text{?}$$

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan’s Laws for Logic 广义德•摩根定律

$$\neg(\,\forall x\, P(x)\,) \equiv \exists x\, \neg P(x)$$

$$\neg(\,\exists x\, P(x)\,) \equiv ?$$

There is a plant that can fly.

What is the negation of this statement?

De Morgan’s Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\ P(x)\ ) \equiv \exists x\ \neg P(x)$$

$$\neg(\ \exists x\ P(x)\ ) \equiv \forall x\ \neg P(x)$$

De Morgan's Laws for Logic

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function P is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function P is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-2) \wedge P(0) \wedge P(5)$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function P is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-2) \wedge P(0) \wedge P(5)$$

The propositional function $\exists x\, P(x)$ is equivalent to

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Suppose that the domain of discourse of the propositional function P is $\{-2, 0, 5\}$.

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The propositional function $\exists x\, P(x)$ is equivalent to

$$P(-2) \lor P(0) \lor P(5)$$

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$$\neg(\,\forall x\, P(x)\,) \equiv \exists x\, \neg P(x)$$

$$\neg(\,\exists x\, P(x)\,) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

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$$\neg(\ \forall x\ P(x)\ )\equiv \exists x\ \neg P(x)$$

$$\neg(\ \exists x\ P(x)\ )\equiv \forall x\ \neg P(x)$$

Truth Table

Exercise

"All lions are fierce"

$P(x)$: x is a lion

$Q(x)$: x is fierce

(A) $\forall x\ (P(x) \land Q(x))$

(B) $\forall x\ (P(x) \rightarrow Q(x))$

Exercise

"All lions are fierce"

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(A) $\forall x\ (P(x) \land Q(x))$

(B) $\forall x\ (P(x) \rightarrow Q(x))$

$P(x)$	$Q(x)$	$P(x) \land Q(x)$

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$

Exercise

"All lions are fierce"

$P(x)$: x is a lion

$Q(x)$: x is fierce

(A) $\forall x\ (P(x) \wedge Q(x))$

(B) $\forall x\ (P(x) \rightarrow Q(x))$

$P(x)$	$Q(x)$	$P(x) \wedge Q(x)$
T	T	T
T	F	F
F	T	F
F	F	F

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise

"Some lions do not drink coffee"

$P(x)$: x is a lion

$R(x)$: x drinks coffee

(A) $\exists\, x\ (\ P(x) \land \neg Q(x)\ )$

(B) $\exists\, x\ (P(x) \rightarrow \neg Q(x)\ )$

Exercise

"Some lions do not drink coffee"

$P(x)$: x is a lion

$R(x)$: x drinks coffee

(A) $\exists\, x\ (\ P(x) \land \neg Q(x)\ )$

(B) $\exists\, x\ (P(x) \rightarrow \neg Q(x)\ )$

$P(x)$	$R(x)$	$\neg R(x)$	$P(x)\ \land\ \neg R(x)$

$P(x)$	$R(x)$	$\neg R(x)$	$P(x) \rightarrow\ \neg R(x)$

Exercise

"Some lions do not drink coffee"

$P(x)$: x is a lion

$R(x)$: x drinks coffee

(A) $\exists x\ (\ P(x) \wedge \neg Q(x)\ )$

(B) $\exists x\ (P(x) \rightarrow \neg Q(x)\ )$

$P(x)$	$R(x)$	$\neg R(x)$	$P(x)\ \wedge\ \neg R(x)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

$P(x)$	$R(x)$	$\neg R(x)$	$P(x) \rightarrow\ \neg R(x)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Rules of Inference for Quantified Statements 量词推理规则

TABLE 1.5.1 Rules of Inference for Quantified Statements[†]

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore P(d) \text{ if } d \in D}$	Universal instantiation
$\dfrac{P(d) \text{ for every } d \in D}{\therefore \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore P(d) \text{ for some } d \in D}$	Existential instantiation
$\dfrac{P(d) \text{ for some } d \in D}{\therefore \exists x\, P(x)}$	Existential generalization

[†] The domain of discourse is D.

Universal Instantiation 全称例化

$$\frac{\forall x\, P(x)}{\therefore\ P(d) \text{ if } d \in D}$$

Universal Generalization 全称一般例化

$$\frac{P(d) \text{ for every } d \in D}{\therefore\ \forall x\, P(x)}$$

Existential Instantiation 存在例化

$$\exists x\, P(x)$$

$$\therefore\; P(d) \text{ for some } d \in D$$

Existential Generalization 存在一般例化

$$\frac{P(d) \text{ for some } d \in D}{\therefore\ \exists\, x\, P(x)}$$

Rules of Inference for Quantified Statements 量词推理规则

Practice Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

Every BUPT student is a genius. Zhang is a BUPT student. Therefore, Zhang is a genius.

Rules of Inference for Quantified Statements 量词推理规则

Example 1.5.23 Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

For every real number x, if x is an integer, then x is a rational number. The number $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

Arguments with Quantified Statements

Universal instantiation:

$$\forall x, P(x)$$
$$\therefore P(a)$$

Universal modus ponens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$P(a)$$
$$\therefore\ Q(a)$$

Universal modus tollens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$\neg Q(a)$$
$$\therefore\ \neg P(a)$$

Practice

$$\forall x\left(p(x)\vee q(x)\right)$$

$$\forall x\left(\left(\neg p(x)\wedge q(x)\right)\rightarrow r(x)\right)$$

$$\therefore \forall x\left(\neg r(x)\rightarrow p(x)\right)$$

$$\begin{array}{l} \forall x\left(p(x)\vee q(x)\right) \\ \forall x\left(\left(\neg p(x)\wedge q(x)\right)\rightarrow r(x)\right) \\ \hline \therefore \forall x\left(\neg r(x)\rightarrow p(x)\right) \end{array}$$

(1)$\forall x\left(p(x)\vee q(x)\right)$ premise

(2)$p(c)\vee q(c)$ step1+rule of universal specification

(3)$\forall x\left(\left(\neg p(x)\wedge q(x)\right)\rightarrow r(x)\right)$ premise

(4)$\left(\neg p(c)\wedge q(c)\right)\rightarrow r(c)$ step3+rule of univ. specif.

(5)$\neg r(c)\rightarrow\neg\left(\neg p(c)\wedge q(c)\right)$ step4+Contrapositive

(6)$\neg r(c)\rightarrow\left(p(c)\vee\neg q(c)\right)$DeMorgan's law+Double Negation

(7)$\neg r(c)$ premise assumed

(8)$p(c)\vee\neg q(c)$ Step 7+6+Modus ponens

(9)$\left(p(c)\vee q(c)\right)\wedge\left(p(c)\vee\neg q(c)\right)$ Step2+8+Rule Conjunction

(10)$p(c)\vee\left(q(c)\wedge\neg q(c)\right)$ Step 9+ Distrutive law

(11)$p(c)$ Step 10+$q(c)\wedge\neg q(c)\Leftrightarrow F+P(c)\vee F=P(c)$

(12)$\therefore\forall x\left(\neg r(x)\rightarrow p(x)\right)$ Step 7+11+rule univ generalization

Problem-Solving Tips

- To prove that the universally quantified statement $\forall x P(x)$ is true, show that for *every* x in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for a *particular* value x does *not* prove that $\forall x P(x)$ is true.
- To prove that the existentially quantified statement $\exists x P(x)$ is true, find *one* value of x in the domain of discourse for which the proposition $P(x)$ is true. *One* value suffices.
- To prove that the universally quantified statement $\forall x P(x)$ is false, find *one* value of x (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false.
- To prove that the existentially quantified statement $\exists x P(x)$ is false, show that for *every* x in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for a *particular* value x does *not* prove that $\exists x P(x)$ is false.

1.6 Nested Quantifiers 嵌套量词

Consider writing the statement

"The sum of any two positive real numbers is positive"

symbolically.

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$P(x, y):\ (x > 0) \wedge (y > 0) \longrightarrow (x + y > 0)$

Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\, P(x, y)$

Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers**.

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Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\ P(x, y)$
Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers**.

Any other Nested Quantifiers?

1.6 Nested Quantifiers 嵌套量词

Example 1.6.1 Restate $\forall m \exists n\ (m < n)$ in words.

1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
“Everybody loves somebody.”
symbolically, letting $L(x, y)$ be the statement “x loves y”.

1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
“Everybody loves somebody.”
symbolically, letting $L(x, y)$ be the statement “x loves y”.

(A) $\forall x \exists y\, L(x, y)$

(B) $\exists x \forall y\, L(x, y)$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.13

$$\neg(\forall x \exists y P(x, y)) \equiv$$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.13

$$\neg(\forall x \exists y P(x,y)) \equiv \exists x \forall y \neg P(x,y)$$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.14 Write the negation of $\exists x \forall y(xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

Problem-Solving Tips

$$\forall x \forall y$$

- To prove that $\forall x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols x and y to stand for *arbitrary* elements in X and Y.
- To prove that $\forall x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ false.

Problem-Solving Tips

$$\forall x \exists y$$

- To prove that $\forall x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.
- To prove that $\forall x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always false.

Problem-Solving Tips

$$\exists x \forall y$$

- To prove that $\exists x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always true.
- To prove that $\exists x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.

Problem-Solving Tips

$$\exists x \exists y$$

- To prove that $\exists x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ true.
- To prove that $\exists x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols x and y to stand for *arbitrary* elements in X and Y.

Problem-Solving Tips

- To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \to q$ is equivalent to $p \wedge \neg q$.