Chapter 6 Counting Methods and Pigeonhole Principle
计数方法与鸽巢原理

Lu Han

hl@bupt.edu.cn

6.1 Basic Principles 基本原理

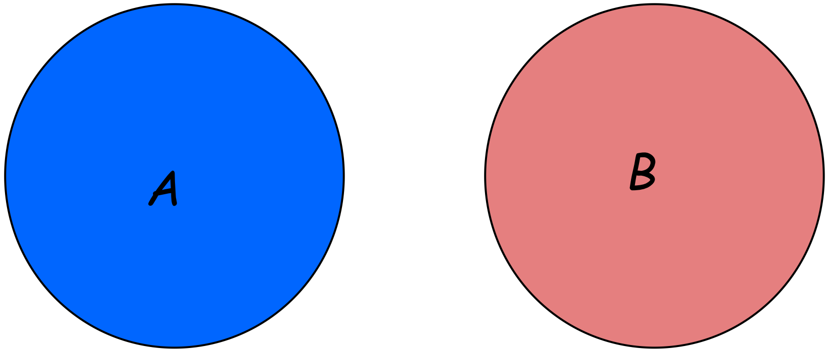
Multiplication Principle 乘法原理

If an activity can be constructed in t successive steps and step 1 can be done in n_1 ways, step 2 can then be done in n_2 ways, ..., and step t can be done in n_t ways, then the number of different possible activities is $n_1 n_2 \ldots n_t$.

We multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

6.1 Basic Principles 基本原理

Addition Principle 加法原理



If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

- Class has 43 women, 54 men, so total enrollment = 43 + 54 = 97
- 26 lower case letters, 26 upper case letters, and 10 digits, so total characters = 26+26+10 = 62

6.1 Basic Principles 基本原理

Addition Principle 加法原理

Suppose that $X_1, \ldots, X_t$ are sets and that the ith set X_i has n_i elements. If $\{X_1, \ldots, X_t\}$ is a pairwise disjoint family (i.e., if $i \neq j$, $X_i \cap X_j = \emptyset$), the number of possible elements that can be selected from X_1 or X_2 or ... or X_t is

$$n_1 + n_2 + \ldots + n_t.$$

(Equivalently, the union $X_1 \cup X_2 \cup \ldots \cup X_t$ contains $n_1 + n_2 + \ldots + n_t$ elements.)

We add the numbers of each subset when the elements being counted can be decomposed into pairwise disjoint subset.

6.1 Basic Principles 基本原理

Example 6.1.11 A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

(a) In how many ways can this be done?

(b) In how many ways can this be done if either Alice or Ben must be chairperson?

(c) In how many ways can this be done if Egbert must hold one of the offices?

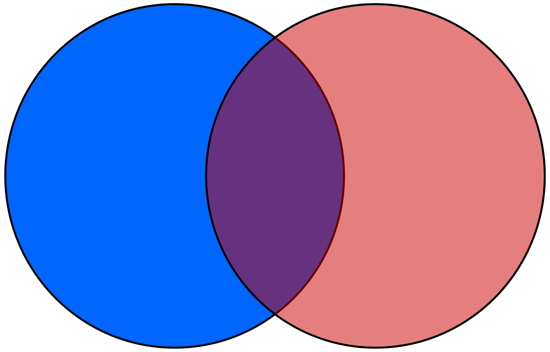
(d) In how many ways can this be done if both Dolph and Francisco must hold office?

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理

If X and Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$



6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理

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Proof

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理
If X and Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

Example 6.1.14 A committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which eitherAlice or Dolph or both are officers?

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理
If X and Y are finite sets, then

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Exercise Count the number of eight-bit strings that start 10 or end 011 or both.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.2 There are six permutation of three elements. If the elements are dnoted $A, B, \ldots, C$ the six permutations are ?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Theorem 6.2.3 There are $n!$ permutations of n elements.

Proof

- There are n choices for the first element.
- For each of these, there are $n-1$ remaining choices for the second element.
- For every combination of the first two elements, there are $\mathrm{n}-2$ ways to choose the third element, and so forth.
- Thus, there are a total of $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$ permutations of an n-element set.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.5 How many permutations of the letters ABCDEF contains the substring DEF?

6.2 Permutations and Combinations 排列与组合

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Example 6.2.6 How many permutations of the letters ABCDEF contain the letters DEF togerther in any oder?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.7 In how many ways can six persons be seated around a circular table? If a seating is obtained from another seating by having everyone move n seats clockwise, the seatings are considered identical.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Definition 6.2.8 An ***r*-permutation (*r*排列)** of n (distinct) elements x_1, ..., x_n is an ordering of an r-element subset of $\{x_1, \ldots, x_n\}$. The **number of *r*-permutations** of set of n distinct elements is denoted $\boldsymbol{P(n, r)}$.

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Example 6.2.9 Examples of 2-permutations of a, b, c are ab, ba, and ca.

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$$\boxed{P(n, n) = n!}$$

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.10 The number of r-permutations of a set of n distinct objects is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n!}{(n-r)!} \qquad r \leq n.$$

Proof

- There are n choices for the first element.
- For each of these, there are $n-1$ remaining choices for the second element.
- There are $n-r+1$ remaining choices for the last element.
- Thus, there are a total of $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ to choose r element.

6.2 Permutations and Combinations 排列与组合

Example 6.2.13 In how many ways can seven distinct Martians and five distinct Jovians wait in line if no two Jovians stand together?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.14 Given a set $X = \{x_1, \ldots, x_n\}$ containing n (distinct) elements,
(a) An ***r*-combination (*r*组合)** of X is an unordered selection of r-elements of X (i.e., an r-element subset of X).
(b) The number of r-combinations of a set of n distinct elements is denoted $C(n, r)$ or $\binom{n}{r}$.

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6.2 Permutations and Combinations 排列与组合

Example 6.2.15 A group of five students, Mary, Boris, Rosa, Ahmad, and Nguyen, has decided to talk with the Mathematics Department chairperson about having the Mathematics Department offer more courses in discrete mathematics. The chairperson has said that she will speak with three of the students. In how many ways can these five students choose three of their group to talk with the chairperson?

6.2 Permutations and Combinations 排列与组合

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We can construct r-permutations of an n-element set X in two successive steps:

- Select an r-combination of X (an unordered subset of r items).
- Order the r-combination.

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.16 The number of r-combinations of a set of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \leq n.$$

Proof

- There are n choices for the first element.
- For each of these, there are $n-1$ remaining choices for the second element.
- There are $n-r+1$ remaining choices for the last element.
- Thus, there are a total of $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ to choose r element.

Any ordering of the first k elements give the same subset!

6.2 Permutations and Combinations 排列与组合

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$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \leq n.$$

Example 6.2.17 In how many ways can we select a committee of three from a group of 10 distinct persons?

6.2 Permutations and Combinations 排列与组合

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Example 6.2.18 In how many ways can we select a committee of two women and three men from a group of five distinct women and six distinct men?

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.16 The number of r-combinations of a set of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \leq n.$$

Example 6.2.19 How many eight-bit strings contain exactly four 1's?

6.2 Permutations and Combinations 排列与组合

Example 6.2.23 What is wrong with the following argument, which purports to show that there are $C(8,5)2^3$ bit strings of length 8 containing at least five 0's?

$$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

Poker Hands

There are 52 cards in a deck.
Each card has a suit and a value.

4 suits (♠ ♥ ♦ ♣)

13 values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

Five-Card Draw is a card game in which each player is initially dealt a hand, a subset of 5 cards.

How many different hands?

Poker Hands

There are 52 cards in a deck.
Each card has a suit and a value.

4 suits $(\spadesuit \quad \heartsuit \quad \diamondsuit \quad \clubsuit)$

13 values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

Five-Card Draw is a card game in which each player is initially dealt a hand, a subset of 5 cards.

How many different hands?

$$\binom{52}{5} = 2598960$$

Poker Hands

Example 1: Four of a Kind

A Four-of-a-Kind is a set of four cards with the same value.

$$\{ \quad 8\spadesuit, \quad 8\diamondsuit, \quad Q\heartsuit, \quad 8\heartsuit, \quad 8\clubsuit \quad \}$$
$$\{ \quad A\clubsuit, \quad 2\clubsuit, \quad 2\heartsuit, \quad 2\diamondsuit, \quad 2\spadesuit \quad \}$$

How many different hands contain a Four-of-a-Kind?

Poker Hands

Example 1: Four of a Kind

A Four-of-a-Kind is a set of four cards with the same value.

A hand with a Four-of-a-Kind is completely described by a sequence specifying:
(1) The value of the four cards.
(2) The value of the extra card.
(3) The suit of the extra card.

$$
\begin{array}{lcllllll}
(8, Q, \heartsuit) & \leftrightarrow & \{ & 8\spadesuit, & 8\diamondsuit, & 8\heartsuit, & 8\clubsuit, & Q\heartsuit \ \} \\
(2, A, \clubsuit) & \leftrightarrow & \{ & 2\clubsuit, & 2\heartsuit, & 2\diamondsuit, & 2\spadesuit, & A\clubsuit \ \}
\end{array}
$$

Poker Hands

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A hand with a Four-of-a-Kind is completely described by a sequence specifying:
(1) The value of the four cards.
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(3) The suit of the extra card.

$$(8, Q, \heartsuit) \leftrightarrow \{\ 8\spadesuit,\ 8\diamondsuit,\ 8\heartsuit,\ 8\clubsuit,\ Q\heartsuit\ \}$$
$$(2, A, \clubsuit) \leftrightarrow \{\ 2\clubsuit,\ 2\heartsuit,\ 2\diamondsuit,\ 2\spadesuit,\ A\clubsuit\ \}$$

There are 13 choices for (1), 12 choices for (2), and 4 choices for (3).
By generalized product rule, there are 13x12x4 = 624 hands.

Only 1 hand in about 4165 has a Four-of-a-Kind!

Poker Hands

Example 2: Full House

A Full House is a hand with three cards of one value and two cards of another value.

$$\{\ 2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\ \}$$
$$\{\ 5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\ \}$$

How many different hands contain a Full House?

Poker Hands

Example 2: Full House

A Full House is a hand with three cards of one value and two cards of another value.

There is a bijection between Full Houses and sequences specifying:
(1) The value of the triple, which can be chosen in 13 ways.
(2) The suits of the triple, which can be selected in $C(4,3)$ ways.
(3) The value of the pair, which can be chosen in 12 ways.
(4) The suits of the pair, which can be selected in $C(4,2)$ ways.

$$(2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) \leftrightarrow \{\ 2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\ \}$$
$$(5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) \leftrightarrow \{\ 5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\ \}$$

By generalized product rule, there are $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$

Only 1 hand in about 634 has a Full House!

Poker Hands

Example 3: Two Pairs

A Two Pairs is a set of two cards of one value, two cards of another value, and one card of a third value.

$$\{\ 3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\ \}$$
$$\{\ 9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\ \}$$

How many different hands contain a Two Pairs?

Poker Hands

Example 3: Two Pairs

A Two Pairs is a set of two cards of one value, two cards of another value, and one card of a third value.

(1) The value of the first pair, which can be chosen in 13 ways.
(2) The suits of the first pair, which can be selected (4 2) ways.
(3) The value of the second pair, which can be chosen in 12 ways.
(4) The suits of the second pair, which can be selected in (4 2) ways
(5) The value of the extra card, which can be chosen in 11 ways.
(6) The suit of the extra card, which can be selected in 4 ways.

Number of Two pairs = $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$

Poker Hands

Example 3: Two Pairs

A Two Pairs is a set of two cards of one value, two cards of another value, and one card of a third value.

Double Count!

$$(3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \searrow$$

$$\{\ 3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\ \}$$

$$(Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) \nearrow$$

Number of Two pairs = $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$

So the answer is $\frac{1}{2} \cdot 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4 = 123552$

Poker Hands

Example 4: Every Suit

How many hands contain at least one card from every suit?

$$\{ \quad 7\diamondsuit, \quad K\clubsuit, \quad 3\diamondsuit, \quad A\heartsuit, \quad 2\spadesuit \quad \}$$

Poker Hands

Example 4: Every Suit

How many hands contain at least one card from every suit?

$$\{\ 7\diamondsuit,\ K\clubsuit,\ 3\diamondsuit,\ A\heartsuit,\ 2\spadesuit\ \}$$

(1) The value of each suit, which can be selected in 13x13x13x13 ways.
(2) The suit of the extra card, which can be selected in 4 ways.
(3) The value of the extra card, which can be selected in 12 ways.

$$(7, K, A, 2, \diamondsuit, 3) \leftrightarrow \{\ 7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\ \}$$

Poker Hands

Example 4: Every Suit

How many hands contain at least one card from every suit?

$$\{\ 7\diamondsuit,\ K\clubsuit,\ 3\diamondsuit,\ A\heartsuit,\ 2\spadesuit\ \}$$

(1) The value of each suit, which can be selected in 13x13x13x13 ways.
(2) The suit of the extra card, which can be selected in 4 ways.
(3) The value of the extra card, which can be selected in 12 ways.

$$(7, K, A, 2, \diamondsuit, 3) \leftrightarrow \{\ 7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\ \}$$

Double count!

$$(7, K, A, 2, \diamondsuit, 3) \searrow$$
$$\{\ 7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\ \}$$
$$(3, K, A, 2, \diamondsuit, 7) \nearrow$$

So the answer is 13^4x4x12/2

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.1 How many strings can be formed using the following letters?

M I S S I S S I P P I

The answer is not 11!

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.1 How many strings can be formed using the following letters?

M I S S I S S I P P I

The answer is not 11!

By the Multiplication Principle, the number of ways of ordering the letters is

$$C(11,2)C(9,4)C(5,4)=\frac{11!}{2!\,9!}\frac{9!}{4!\,5!}\frac{5!}{4!\,1!}=\frac{11!}{2!\,4!\,4!\,1!}=34650.$$

The solution to Example 6.3.1 assumes a nice form. The number 11 that appears in the numerator is the total number of letters. The value in the denominator give the numbers of duplicates of each letter.

6.3 Generalized Permutations and Combinations 广义的排列与组合

Theorem 6.3.2 Suppose that a sequence S of n items has n_1 identical objects of type 1, n_2 identical objects of type 2, ..., and n_t identical objects of type t. Then the number of orderings of S is

$$\frac{n!}{n_1!\,n_2!\ldots n_t!}.$$

Proof

- Assign positions to the n_1 items of type 1 in $C(n, n_1)$ ways.
- Assign positions to the n_2 items of type 2 in $C(n-n_1, n_2)$ ways, and so on.
- By the Multiplication Principle, the number of orderings is

$$C(n,n_1)C(n-n_1,n_2)C(n-n_1-n_2,n_3)\ldots C(n-n_1-\ldots-n_{t-1},n_t)$$
$$=\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\ldots-n_{t-1})!}{n_T!0!}$$
$$=\frac{n!}{n_1!n_2!\ldots n_t!}.$$

6.3 Generalized Permutations and Combinations 广义的排列与组合

Theorem 6.3.2 Suppose that a sequence S of n items has n_1 identical objects of type 1, n_2 identical objects of type 2, ..., and n_t identical objects of type t. Then the number of orderings of S is

$$\frac{n!}{n_1!\,n_2!\ldots n_t!}.$$

Example 6.3.3 In how many ways can eight distinct books be divided among three students if Bill gets four books and Shizuo and Marian each get two books?

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.4 Consider three books: a computer science book, a physics book, and a history book. Suppose that the library has at least six copies of each of these books. In how many ways can we select six books?

Method 1

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.4 Consider three books: a computer science book, a physics book, and a history book. Suppose that the library has at least six copies of each of these books. In how many ways can we select six books?

Method 2

$$XX|XX|XX$$

$$XX|XXX|X$$

$$XX|XXXX|$$

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.4 Consider three books: a computer science book, a physics book, and a history book. Suppose that the library has at least six copies of each of these books. In how many ways can we select six books?

Method 3

$$\_X\_X\_X\_X\_X\_X\_$$

What is wrong with $\frac{C(7,1)C(7,1)}{2}$?

6.3 Generalized Permutations and Combinations 广义的排列与组合

Theorem 6.3.5 If X is a set containing t elements, the number of unordered, k-element selections from X, repetitions allowed, is

$$C(k+t-1, t-1) = C(k+t-1, k).$$

6.3 Generalized Permutations and Combinations 广义的排列与组合

Example 6.3.6 Suppose that there are piles of red, blue, and green balls and that each pile contains at least eight balls.

(a) In how many ways can we select eight balls?

(b) In how many ways can we select eight balls if we must have at least one ball of each color?

6.3 Generalized Permutations and Combinations 广义的排列与组合

The following table summarizes the various formulas:

	No Repetitions	*Repetitions Allowed*
Ordered Selections	$n!$	$n!/(n_1!\cdots n_t!)$
Unordered Selections	$C(n, r)$	$C(k+t-1, t-1)$