No. Sun Haoran

Date .4.13

Week 7 Homework.

2.42 solution

离散可直接算.

(a). we know $X \sim Geom(p)$.

$p(x) = (1-p)^{x-1} \cdot p$

Let $Y = X^2$, we can get $F_Y(y) = P(Y \le y)$, $P_Y(y) = P(Y = y)$

$$\Rightarrow P_Y(y) = P(X^2 = y) = P(X = \sqrt{y}) = (1-p)^{\sqrt{y}-1} \cdot p$$

(b). Let $Y' = X+3$, we can get $F_{Y'}(y) = P(Y' \le y)$, $P_{Y'}(y) = P(Y' = y)$

$$\Rightarrow P_{Y'}(y) = P(X+3 = y) = (1-p)^{y-4} \cdot p$$

2.49

(a). Solution:

As we know $g(x) = 2F(x) + 4$, we can get $Y = 2F(X) + 4$.

$$\text{s.t. } F_Y(y) = P(Y \le y) = P(2F(X) + 4 \le y) = \int_{-\infty}^{F_X^{-1}\left(\frac{y-4}{2}\right)} f_X(x)\,dx$$

$$f_Y(y) = \left[F_X^{-1}\left(\frac{y-4}{2}\right)\right]' \cdot f_X\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right) \qquad \text{We know } f_X(x) = F_X'(x).$$

$$\text{And } \left(F_X^{-1}\left(\frac{y-4}{2}\right)\right)' = \frac{d}{dy}\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right).$$

$$\text{We know } \frac{y-4}{2} = F_X\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right)$$

$$\Rightarrow \frac{1}{2} = \left.\frac{dF_X}{du}\right|_{u = F_X^{-1}\left(\frac{y-4}{2}\right)} \cdot \frac{d F_X^{-1}\left(\frac{y-4}{2}\right)}{dy}$$

$$\frac{1}{2} = f_X\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right) \cdot \frac{d}{dy}\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right)$$

$$\text{So. } f_Y(y) = \frac{1}{2 f_X\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right)} \cdot f_X\left(F_X^{-1}\left(\frac{y-4}{2}\right)\right) = \frac{1}{2}.$$

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(b) Solution:

$Y \sim U(8,10)$

Then, we can write $f_Y(y) = \frac{1}{2}$ $(y \in (8,10))$

$F_Y(y) = \frac{y-8}{2}$

$$P(Y \le y) = F_Y(y) = P(X \le g^{-1}(y)) = \frac{y-8}{2}.$$

$$\frac{y-8}{2} = \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx$$

$$\left(\frac{y-8}{2}\right)' = (g^{-1}(y))' \cdot f_X(g^{-1}(y))' = F_X'(g^{-1}(y)).$$

$$\cancel{\frac{y-8}{2} = \frac{1}{\frac{dg}{dx}\big|_{x=g^{-1}(y)}} \cdot f_X(g^{-1}(y))}.$$

$\Rightarrow F_X(g^{-1}(y)) = \frac{y-8}{2}$ let $x = g^{-1}(y)$ $y = g(x)$

$$F_X(x) = \frac{g(x)-8}{2}$$

s.t. $g(x) = 2F_X(x) + 8$

2.50.

Proof: we know $X \sim Exp(\lambda)$.

$f(x) = \lambda e^{-\lambda x}$ $(x > 0)$

s.t. $F_Y(y) = P(Y \le y) = P(cX \le y) = P(X \le \frac{y}{c})$

$$= \int_{-\infty}^{\frac{y}{c}} \lambda e^{-\lambda x}\,dx = F\left(\frac{y}{c}\right)$$

$$\therefore f_Y(y) = F_Y'(y) = \left(\frac{y}{c}\right)' \cdot \lambda \cdot e^{-\lambda \frac{y}{c}} = \frac{\lambda}{c} \cdot e^{-y\frac{\lambda}{c}}$$

It is obvious to see $Y \sim Exp(\frac{\lambda}{c})$.

~~2.52~~ Solution.

According to the question, it gives the average time to queue theory. So we use the exponential distribution.

$X \sim Exp(\cancel{5})\ \frac{1}{5}$

p.d.f. $f(x) = 5 \cdot e^{-5x}$ c.d.f. $F(x) = 1 - e^{-5x}$ λ是单位时间内的次数.

s.t. $P(x \geq 4) = 1 - P(x \leq 4) = 1 - F(4) = \cancel{1 - e^{-0.08}}\ e^{-0.8}$.

2.53

(a). solution.

$X \sim N(\mu, 0.01\mu^2)$

The average is μ, s.t. $P(0 \leq x \leq 1.15\mu) = F(1.15\mu) - F(0)$

$= \Phi(1.5) - \Phi(-10)$

(b). solution:

$x \sim N(4, 0.16)$

p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,0.4} e^{-\frac{(x-4)^2}{0.32}}$

$F(x) = \Phi\left(\frac{x-4}{0.4}\right)$

$\Delta = \left|\Phi\left(\frac{|0-4|}{0.4}\right) - \Phi\left(\frac{|x-4|}{0.4}\right)\right| = 0.9$

$\Phi\left(\frac{|x-4|}{0.4}\right) = 0.9 \Rightarrow x = 4.516$ or 3.484 (discard).

2.54 Solution $X \sim N(1, 4)$

(a). $P(x \leq 3) = F(3) = \Phi\left(\frac{3-1}{2}\right) = \Phi(1) = 0.84134$

(b). $P(x > 1.5) = 1 - F(1.5) = 1 - \Phi\left(\frac{1}{4}\right) = 0.40129$

(c). $P(x = 1) = 0$

(d). $P(2 < X < 5) = F(5) - F(2) = \Phi\left(\frac{5-1}{2}\right) - \Phi\left(\frac{2-1}{2}\right) = 0.26841$

(e). $P(x \geq 0) = 1 - P(x \leq 0) = 1 - \Phi\left(\frac{0-1}{2}\right) = 1 - (1 - \Phi(0.5)) = 0.70884$

(f). $P(-1 < x < 0.5) = F(0.5) - F(-1) = \Phi\left(\frac{0.5-1}{2}\right) - \Phi\left(\frac{-1-1}{2}\right) = \Phi(0.25) - \Phi(-1)$

$= (1 - \Phi(0.25)) - (1 - \Phi(1)) = 0.24863$

(g). $P(-2 \leq X \leq 2) = F(2) - F(-2) = \Phi\left(\frac{2-1}{2}\right) - \Phi\left(\frac{-2-1}{2}\right) = \Phi(0.5) - (1 - \Phi(1.5))$

$= 0.64827$

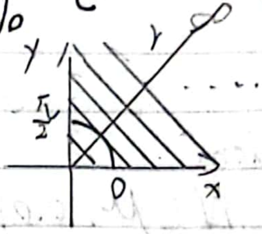
(h). $P(1 \le -2X+3 \le 8) = P(-\frac{5}{2} \le X \le 1) = \Phi(0) - \Phi(\frac{-2.5-1}{2})$

$= 0.5 - 1 + \Phi(1.75) = 0.45994$

2.56. Solution.

$$I = \int_0^{+\infty} e^{-4x^2} dx \Rightarrow I^2 = \int_0^{+\infty} e^{-4x^2} dx \int_0^{+\infty} e^{-4y^2} dy$$

$$= \int_0^{+\infty}\int_0^{+\infty} e^{-4x^2-4y^2} dx\,dy = \int_0^{+\infty}\int_0^{+\infty} e^{-4(x^2+y^2)} dx\,dy$$



$x = r\cos\theta$
$y = r\sin\theta$.

$$\Rightarrow \int_0^{+\infty}\int_0^{\frac{\pi}{2}} e^{-4r^2} r\,dr\,d\theta = +\frac{\pi}{16}$$

s.t. $I = \frac{\sqrt{\pi}}{4}$.

2.57. Solution: $X \sim N(0, 2)$

(a). $P(1 \le X \le 2) = F(2) - F(1) = \Phi(\frac{2}{\sqrt{2}}) - \Phi(\frac{1}{\sqrt{2}}) = \Phi(1.414) - \Phi(0.$

$= 0.15968$

(b) $= P(1 \le X \le 2 \mid X \ge 1) = \frac{0.15968}{1-\Phi(\frac{\sqrt{2}}{2})} = 0.6683$

2.58 (a). solution:

$X \sim N(5, 2) \Rightarrow Y \sim N(14, 8)$

P.f: $f_Y(y) = \frac{1}{\sqrt{2\pi}\,2\sqrt{2}} e^{-\frac{(y-14)^2}{8\cdot 2}} = \frac{1}{4\sqrt{\pi}} e^{-\frac{(y-14)^2}{16}}$

(b). $E(Y) = 14$, $Var(Y) = 8$

2.59 solution:

$P(X < 116) = \Phi(\frac{116-\mu}{\sigma}) = 0.2$. $P(X < 328) = \Phi(\frac{328-\mu}{\sigma}) = 0.9$

$$\begin{cases} -\frac{116-\mu}{\sigma} = 0.84 \\ \frac{328-\mu}{\sigma} = 1.28 \end{cases} \Rightarrow \begin{cases} \mu = 200 \\ \sigma = 100 \end{cases}$$

s.t. $E(X) = 200$, $Var(X) = 10000$

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2.60 solution

(a). $X \sim U(a,b)$.

If it has a median s.t. $F(m) = \frac{1}{2}$; then $b-a$ must be 2.
And the median is $\frac{a+b}{2}$

(b). $Y \sim N(\mu, \sigma)$. when $x = \mu$ and $F(m) = \frac{1}{2}$, it means it is a standard normal distribution., $F(m) = F(\mu) = \frac{1}{2}$
And the median is μ.

(c). $Z \sim Exp(\lambda)$

~~It is impossible to exist a median that $F(m) = \frac{1}{2}$.~~

$F(m) = \frac{1}{2} = \cancel{t}\, 1 - e^{-\lambda m}$

$$\Rightarrow \quad m = \frac{\ln 2}{\lambda}$$