# EBU4375: SIGNALS AND SYSTEMS

## TOPIC 5: LAPLACE TRANSFORMS

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# ACKNOWLEDGMENT

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# DEFINITION OF LAPLACE TRANSFORM

The Laplace transform $X(s)$ of the time-domain signal $x(t)$ is defined by the integral

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

The integral differs from the Fourier transform in two main ways:

- The integration is from $t = 0$ to $t = \infty$. (It is causal).
- The Laplace integral contains the factor $e^{-st}$ (rather than $e^{-j\omega t}$ as in the Fourier transform).

# THE LAPLACE VARIABLE S

The Laplace variable s is a complex number, where $s = \sigma + j\omega$ .

Hence, $e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t}$

The term $e^{-j\omega t}$ is identical to the term in the Fourier integral.

Depending on the value and sign of $\sigma$ , the factor represents a growing or decaying exponential.

The product $e^{-\sigma t} e^{-j\omega t}$ , therefore represents an exponentially growing or decaying complex frequency component.

$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = e^{-\sigma t}(\cos \omega t - j \sin \omega t)$$

The Laplace variable $s = \sigma + j\omega$ is sometimes called the complex frequency. The s plane is called the complex frequency domain.

# LAPLACE TRANSFORM PAIRS

As with Fourier transforms, the relationship between the time-domain model of a signal and its Laplace transform is unique.

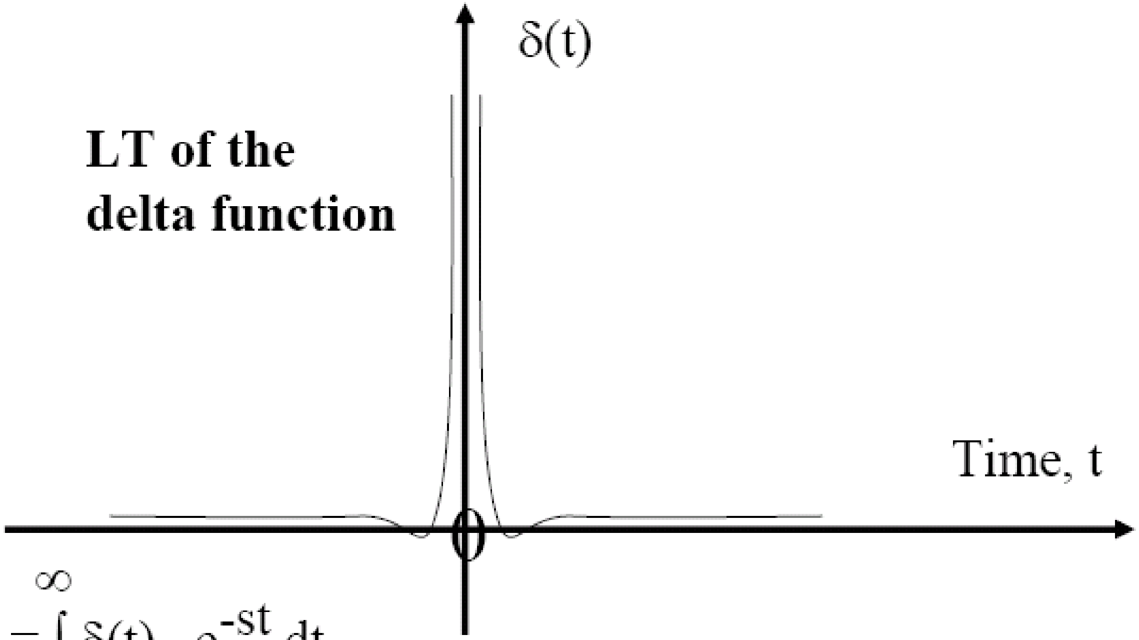
A signal and its Laplace transform form a transform pair, which is denoted by the double-headed arrow

**Example 1**: Evaluate the LT of the exponential decay: $x(t) = Ae^{-\alpha t} \xleftrightarrow{LT} \dfrac{A}{s+\alpha}$

This provides a starting point from which the transforms of many other functions can be derived.

# LAPLACE TRANSFORM PAIRS

**Example 2:**




$$X(s) = \int_0^{\infty} \delta(t).\ e^{-st}\, dt$$

$$X(s) = \int_0^{0} \delta(t).e^{-s0}\, dt = \int_0^{0} \delta(t).1\, dt = 1 \qquad \boxed{\boldsymbol{\delta(t) \longleftrightarrow 1}}$$

Note: strictly speaking, δ(t) is the Dirac Delta Function

# APPLICATION OF THE LT TO SINUSOIDS

**Example 3:** $x(t) = A\cos(\omega t)$

$$X(s) = \int_0^{\infty} A\cos(\omega t).\ e^{-st}\, dt$$

$$X(s) = A\int_0^{\infty} [e^{j\omega t} + e^{-j\omega t}]/2\ .\ e^{-st}\, dt$$

$$X(s) = (A/2)\int_0^{\infty} [e^{j\omega t}\, e^{-st} + e^{-j\omega t}\, e^{-st}]\, dt$$

but these ----------- -----------

are standard results for the exponential, and allow solutions for sinusoids (via the LT) that are rigorous enough to satisfy everyone.

# APPLICATION OF THE LT TO SINUSOIDS

**Example 4:** $x(t) = A\sin(\omega t)$

A. $X(s) = \frac{A\omega}{s^2+\omega^2}$

B. $X(s) = \frac{As}{s^2+\omega^2}$

C. $X(s) = \frac{As}{s^2-\omega^2}$

D. $X(s) = \frac{A\omega}{s^2-\omega^2}$

# PROPERTIES OF THE LAPLACE TRANSFORM

- Linearity
- Right shift in time
- Time Scaling
- S-plane shift
- ~~Multiplication by a Power of t~~
- ~~Differentiation in the Time Domain~~
- ~~Integration~~

# PROPERTIES: LINEARITY

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

Proof :

$$L\{c_1 f_1(t) + c_2 f_2(t)\} =$$

$$\int_0^\infty [c_1 f_1(t) + c_2 f_2(t)] e^{-st} dt =$$

$$c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t) e^{-st} dt =$$

$$c_1 F_1(s) + c_2 F_2(s)$$

Example : $x(t) = \sinh(t)$

$$L\{\sinh(t)\} =$$

$$y\{\frac{1}{2} e^{t} - \frac{1}{2} e^{-t}\} =$$

$$\frac{1}{2} L\{e^{t}\} - \frac{1}{2} L\{e^{-t}\} =$$

$$\frac{1}{2} \left(\frac{1}{s-1} - \frac{1}{s+1}\right) =$$

$$\frac{1}{2} \left(\frac{(s+1) - (s-1)}{s^2 - 1}\right) = \frac{1}{s^2 - 1}$$

# PROPERTIES: SCALING IN TIME

$$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$$

Proof :

$$L\{f(at)\} =$$

$$\int_0^\infty f(at)e^{-st}dt =$$

$$u = at, t = \frac{u}{a}, dt = \frac{1}{a}du$$

$$\frac{1}{a}\int_0^{\frac{\infty}{a}} f(u)e^{-(\frac{s}{a})u}du =$$

$$\frac{1}{a}F(\frac{s}{a})$$

Example : $x(t) = \sin(\omega t)$

$$L\{\sin(\omega t)\}$$

$$\frac{1}{\omega}(\frac{1}{(s/\omega)^2}+1) =$$

$$\frac{1}{\omega}(\frac{\omega^2}{s^2+\omega^2}) =$$

$$\frac{\omega}{s^2+\omega^2}$$

# PROPERTIES: TIME SHIFT

$$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$$

Proof :

$$L\{f(t-t_0)u(t-t_0)\} =$$

$$\int_0^\infty f(t-t_0)u(t-t_0)e^{-st}\,dt =$$

$$\int_{t_0}^\infty f(t-t_0)e^{-st}\,dt =$$

let $u = t - t_0, t = u + t_0$

$$\int_0^{\infty - t_0} f(u)e^{-s(u+t_0)}\,du =$$

$$e^{-st_0}\int_0^\infty f(u)e^{-su}\,du = e^{-st_0}F(s)$$

Example : $x(t) = e^{-a(t-10)}u(t-10)$

$$L\{e^{-a(t-10)}u(t-10)\} =$$

$$\frac{e^{-10s}}{s+a}$$

# PROPERTIES: S-PLANE (FREQUENCY) SHIFT

$$L\{e^{-at} f(t)\} = F(s+a)$$

Proof :

$$L\{e^{-at} f(t)\} =$$

$$\int_0^\infty e^{-at} f(t) e^{-st} dt =$$

$$\int_0^\infty f(t) e^{-(s+a)t} dt =$$

$$F(s+a)$$

Example : $x(t) = e^{-at}\sin(\omega t)$

$$L\{e^{-at}\sin(\omega t)\} =$$

$$\frac{\omega}{(s+a)^2+\omega^2}$$

# THE INVERSE LAPLACE TRANSFORM

- Like the Fourier transform the Laplace transform has an inverse which is formally defined by an integral

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

The integral in above equation is evaluated along the path $s = c + j\omega$ in the complex plane from $c - j\infty$ to $c + j\infty$.

The evaluation of this integral requires a knowledge of contour integration and complex variable theory and is seldom if ever used in routine Laplace transform work.

A frequently-used method is the partial-fraction technique.

# THE INVERSE LAPLACE TRANSFORM

The method of partial fractions relies on the fact that finding the Laplace transform or its inverse is a linear operation.

1. Split up a transform into a sum of simpler transforms

2. Find the inverse of the overall transform by finding the inverse of each simpler function separately.

3. Add them together

Extensive tables of Laplace transform pairs have been prepared to remove the need to work out a transform or its inverse from first principles.

# A SHORT TABLE OF LAPLACE TRANSFORM PAIRS

A short table of some of the more commonly used transform pairs is listed:

| Time function $x(t)$ | Laplace transform $X(s)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $1/s$ |
| $t$ | $1/s^2$ |
| $t^n$ | $n!/s^{n+1}$ |
| $e^{-\alpha t}$ | $1/(s+\alpha)$ |
| $te^{-\alpha t}$ | $1/(s+\alpha)^2$ |
| $te^{-\alpha t}/n!$ | $1/(s+\alpha)^{n+1}$ |
| $\sin \omega t$ | $\omega/(s^2+\omega^2)$ |
| $\cos \omega t$ | $s/(s^2+\omega^2)$ |
| $e^{-\alpha t}\sin \omega t$ | $\omega/[(s+\alpha)^2+\omega^2]$ |
| $e^{-\alpha t}\cos \omega t$ | $(s+\alpha)/[(s+\alpha)^2+\omega^2]$ |

## Find the Laplace transform of:

(1) $f(t) = [\cos(3t)]^2$

(2) $f(t) = 1 - 2e^{-2t}$

(3) $f(t) = \sin(2t)\cos(2t)$

(4) $f(t) = (e^{-2t} - 1)^2$

# EXAMPLE OF PARTIAL FRACTIONS (1)

$$X(s) = \frac{s+1}{s(s+2)}$$

$$X(s) = \frac{A}{s} + \frac{B}{(s+2)} = \frac{As+2A+Bs}{s(s+2)}$$

$$(A+B)s + 2A = s + 1$$

$$2A = 1 \Rightarrow A = 1/2$$

$$(A+B) = 1 \Rightarrow B = 1 - A = 1 - \frac{1}{2} = 1/2$$

$$X(s) = \frac{1}{2s} + \frac{1}{2(s+2)}$$

$$\boxed{x(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-t}}$$

# TUTORIAL QUESTION:

- Find the Inverse Laplace transform of $$F(s) = \frac{2}{s^2 + 3s + 2}$$

A. $f(t) = 2(e^{t} + e^{-2t})$

B. $f(t) = 2(e^{-t} - e^{-2t})$

C. $f(t) = 2(e^{-t} - e^{2t})$

D. $f(t) = 2(e^{t} + e^{2t})$

# THE TRANSFER FUNCTION FOR CONTINUOUS TIME SIGNAL

We know that

$$y(t) = x(t) * h(t)$$

output signal

input signal

Unit impulse response

Convolution in the ‘continuous time domain’ can be replaced by the multiplication in s-transform domain.

$$Y(s) = X(s) \cdot H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$ is defined as transfer function.

$$H(s) \leftrightarrow h(t)$$