Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

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3.1 Functions 函数

Definition 3.1.1 Let X and Y be sets. A function f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

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A function can be defined by

- **Listing its members**

$$f = \{(a, 1), (b, 2), (c, 3)\}$$

- **A formula**

$$f = \{(x, x^2) \mid x \in \mathbf{Z}\}$$

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one-to-one (or injective) (单射)

onto (or surjective) (满射)

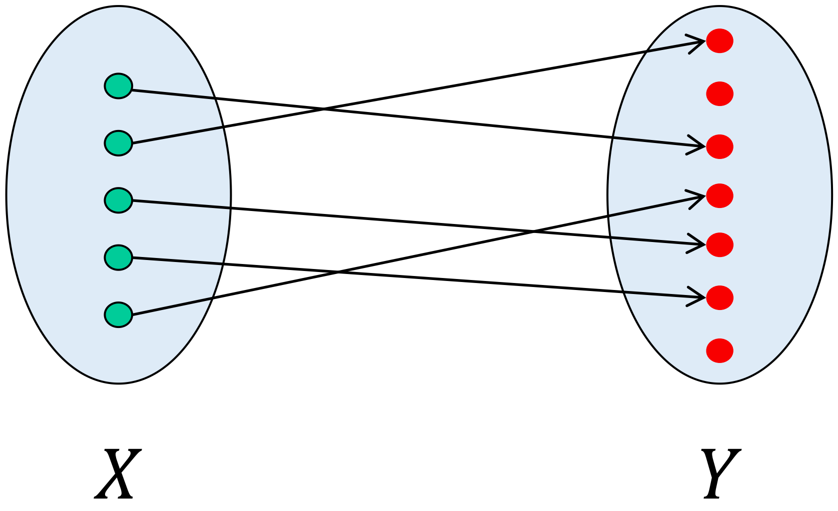
bijection (双射)

3.1 Functions 函数

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

injective

≤ 1 arrow in



$|X| \leq |Y|$

3.1 Functions 函数

Definition 3.1.29 A function f from X to Y is said to be **onto** Y **(or surjective) (满射的)** if **for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$**.

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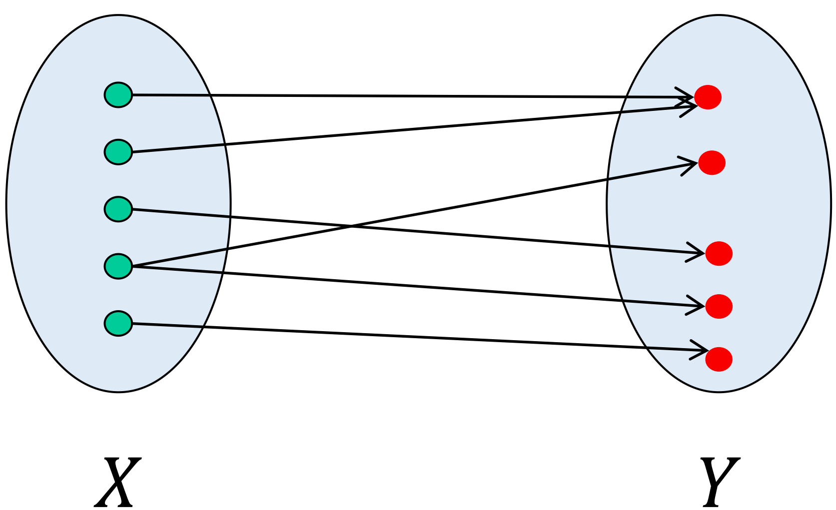
surjective?



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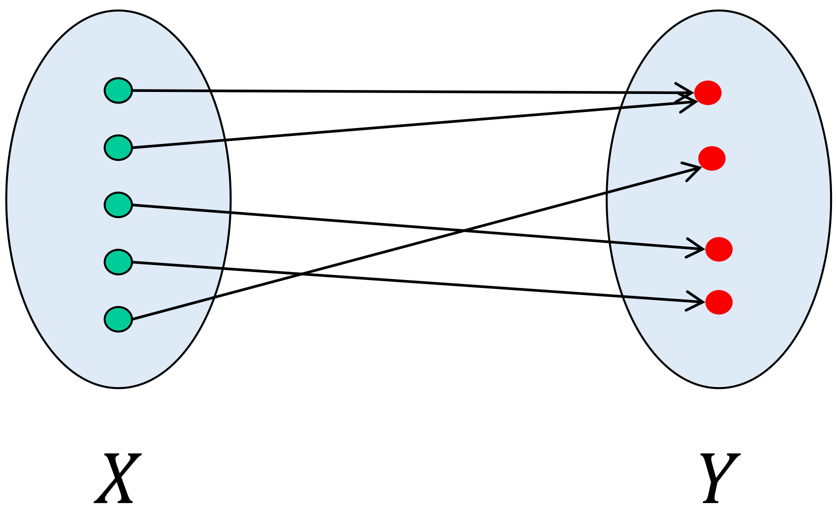

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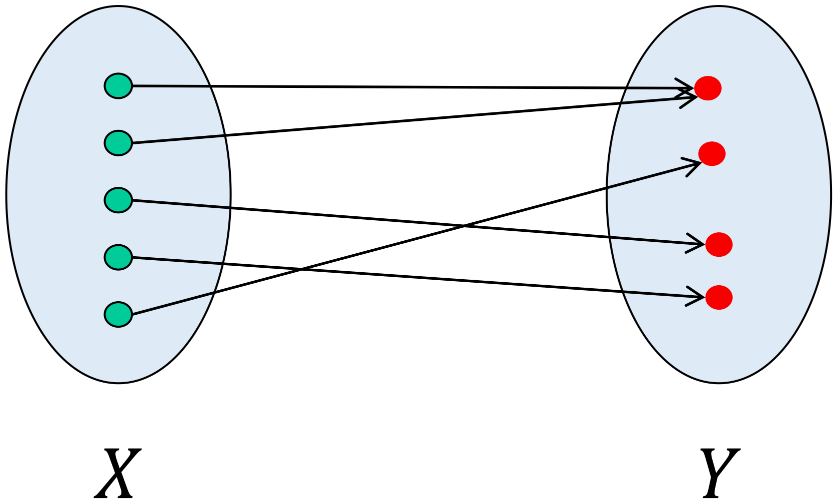
surjective **$\geq$ 1 arrow in**



$|X| \geq |Y|$

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Example 3.1.30 The set $f = \{(1, a), (2, c), (3, b)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Example 3.1.31 The set $f = \{(1, b), (3, a), (2, c)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$?

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Example 3.1.33 Prove that the function $f(x) = \frac{1}{x^2}$ from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.

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Definition 3.1.29 A function f from X to Y is said to be **onto** Y **(or surjective) (满射的)** if **for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$**.

Example 3.1.34 Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is not onto Y.

A function f from X to Y is not onto Y if for some $y \in Y$, for every $x \in X$, $f(x) \neq y$.

3.1 Functions 函数

Definition 3.1.35 A function that is **both one-to-one and onto** is called a **bijection (双射)**.

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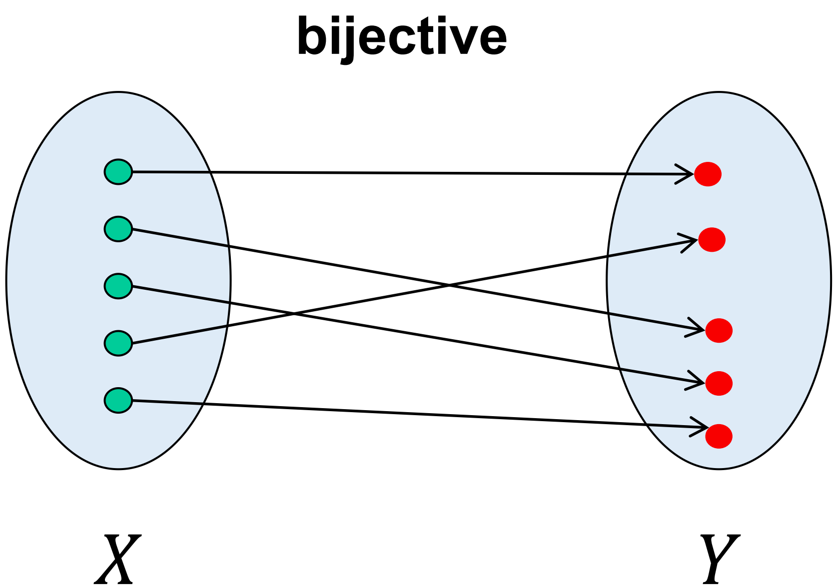

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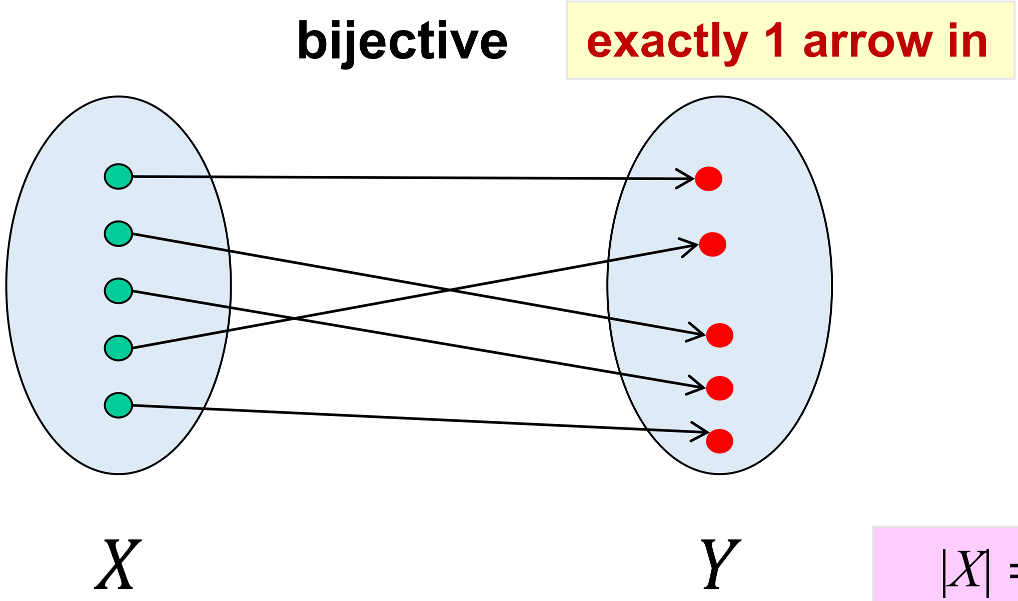

$|X| = |Y|$

3.1 Functions 函数

Exercise

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x) = \sin(x)$	$\mathbf{R}$	$\mathbf{R}$			
$f(x) = 2^x$	$\mathbf{R}$	$\mathbf{R}^+$			
$f(x) = x^2$	$\mathbf{R}$	$\mathbf{R}^{\boldsymbol{nonneg}}$			
Reverse String	Bit Strings of length n	Bit Strings of length n			

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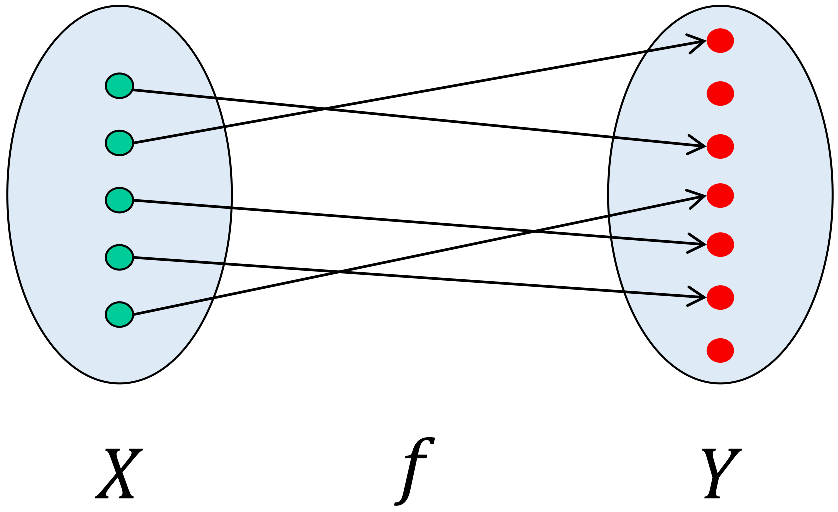
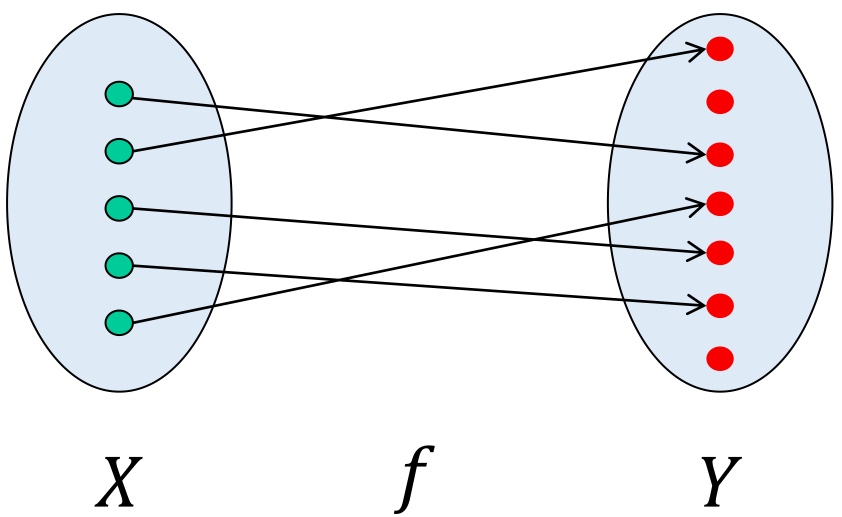
Arrow Diagram 箭头图



Arrow Diagram 箭头图

arrow out of X and into Y

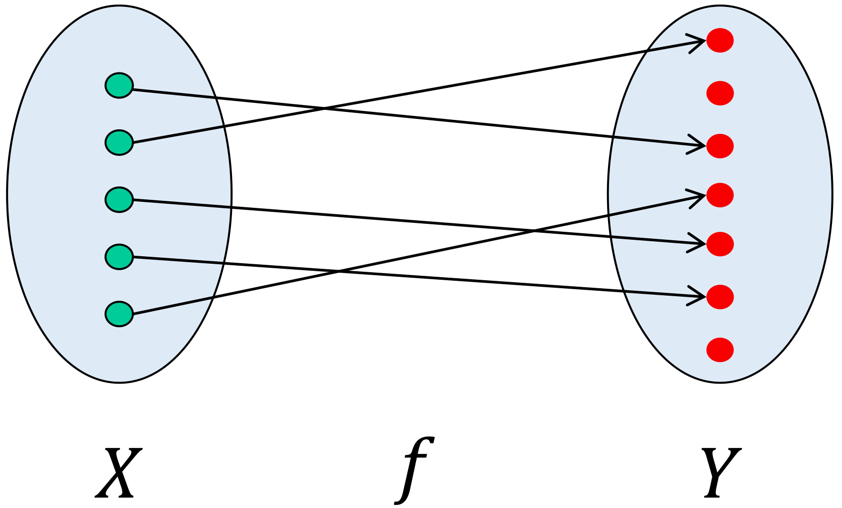


Arrow Diagram 箭头图

arrow out of X and into Y

A function: exactly one arrow goes out of X.

- **injective: ≤ 1 arrow into Y**
- **surjective: ≥ 1 arrow into Y**
- **bijective: exactly one arrow into Y**



3.1 Functions 函数

Inverse Function (反函数)

Suppose that f is one-to-one, onto function from X to Y. It can be shown that $\{(y, x) \mid (x, y) \in f\}$ is a one-to-one, onto function from Y to X. This new function, denote $\boldsymbol{f^{-1}}$, is called f **inverse (逆)**.

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Example 3.1.38 For function $f = \{(1, a), (2, c), (3, b\ )\}$, we have

$$f^{-1}=?$$

3.1 Functions 函数

Definition 3.1.41 Let g be a function X to Y and let f be a function from Y to Z. The **composition of f with g (f 与 g 的复合函数)**, denoted $f \circ g$, is the function

$$(f \circ g)(x) = f(g(x))$$

from X to Z.

3.1 Functions 函数

Example 3.1.42 Given $X = \{1,\ 2,\ 3\}$, $Y = \{a,\ b,\ c\}$, $Z = \{x,\ y,\ z\}$.
The function $g = \{(1,\ a),\ (2,\ a),\ (3,\ c)\}$ from X to Y.
The function $f = \{(a,\ y),\ (b,\ x),\ (c,\ z)\}$ from Y to Z.

The composition function from X to Z is $f \circ g =$?

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Example 3.1.43 Draw the arrow diagram of the function $f \circ g$.

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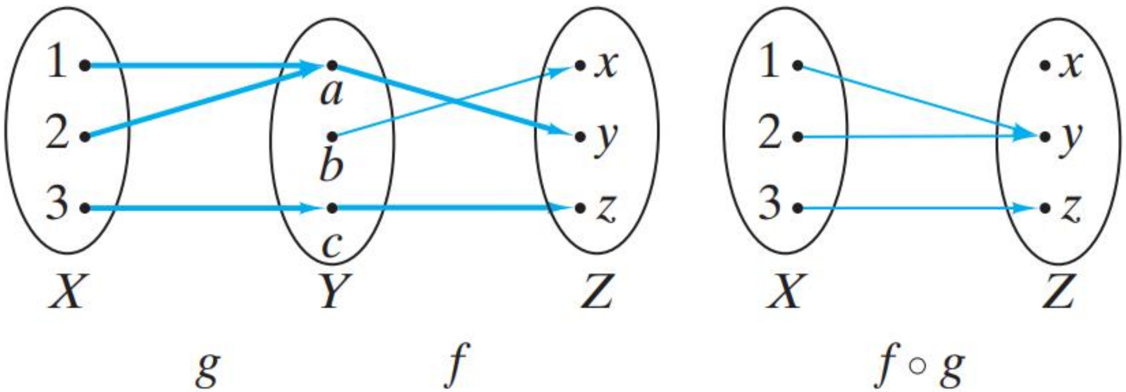
Example 3.1.43 Draw the arrow diagram of the function $f \circ g$.



3.1 Functions 函数

Example 3.1.44 If $f(x) = \log_3 x$ and $g(x) = x^4$,

then $(f \circ g)(x) = ?$

and $(g \circ f)(x) = ?$

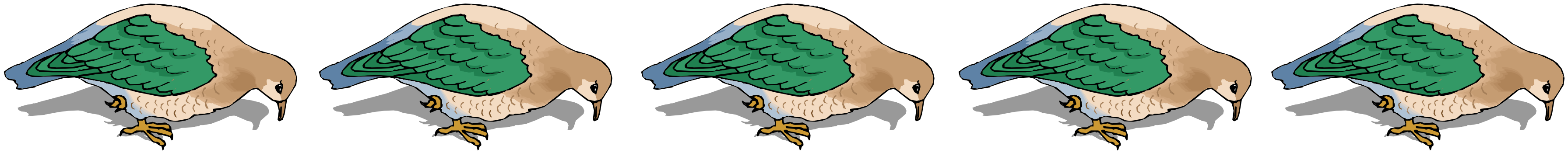
3.1 Functions 函数

Exercise: $g \circ f$

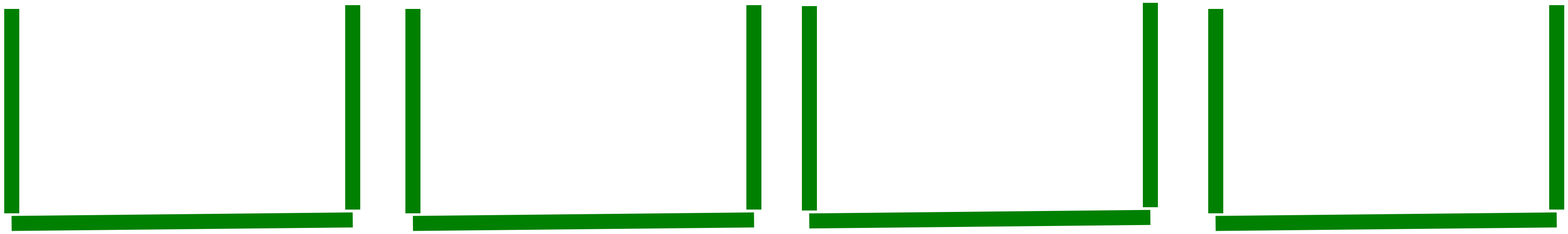
$f: X \to Y$	$g: Y \to Z$	Injective?	Surjective?	Bijective?
f is injective	g is injective	1	2	3
f is surjective	g is surjective	4	5	6
f is injective	g is surjective	7	8	9
f is surjective	g is injective	10	11	12
f is bijective	g is bijective	13	14	15

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

If more pigeons

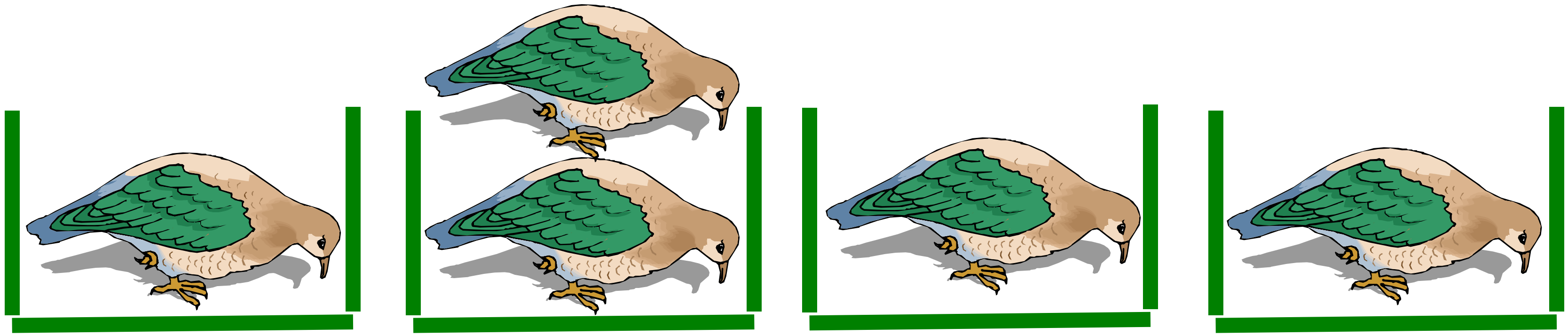


than pigeonholes,



Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

then some hole must have at least two pigeons!



Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Pigeonhole Principle (First Form)
If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

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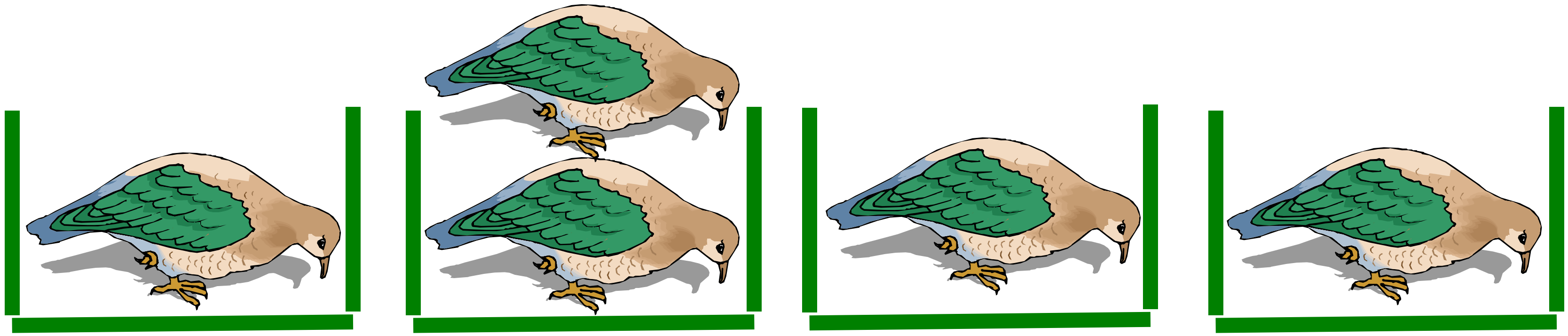

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If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then ...

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A function from a larger set to a smaller set cannot be injective.
(There must be at least two elements in the domain that have the same image in the codomain.)

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Example
In a group of 366 people, there must be two people having the same birthday.

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If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

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If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$.

Extension select 5 numbers from 1-8, there exists two number that their sum is exactly 9.

3.1 Functions 函数

Definition 3.1.17 The floor of x, denote $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The ceiling of x, denote $\lceil x \rceil$, is the least integer greater than or equal to x.

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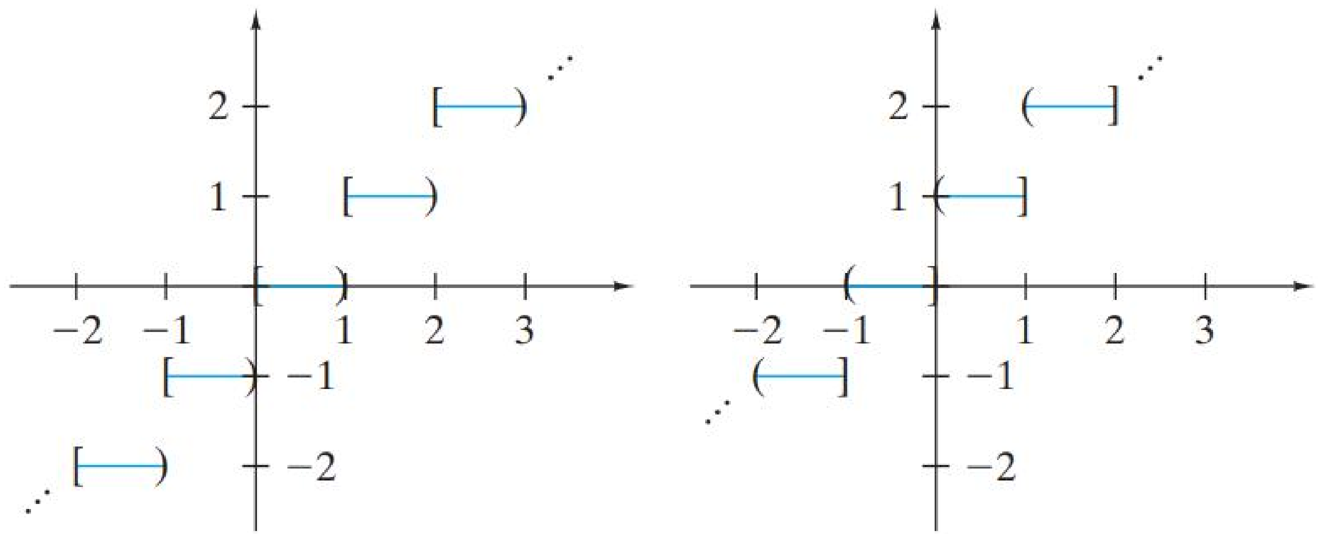

Figure 3.1.7 The graphs of the floor (left graph) and ceiling (right graph) functions.

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $\boldsymbol{x \bmod y}$ to be the remainder when x is divided by y.

- **mod** is called the **modulus operator(模算子)**

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Example 3.1.12 We have

$11 \bmod 7 = 4$, $-11 \bmod 7 = ?$

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Quotient-Remainder Theorem 商和余数定理

If d and n are integers, $d > 0$, there exist integers q (quotient) and r (remainder) satisfying $n = dq + \boldsymbol{r}$ $\boldsymbol{(0 \le r < d)}$

Furthermore, q and r are unique; that is, if

$n = dq_1 + r_1$ $(0 \le r_1 < d)$ and $n = dq_2 + r_2$ $(0 \le r_2 < d)$,

then $q_1 = q_2$ and $r_1 = r_2$.

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $\boldsymbol{x} \textbf{ mod } \boldsymbol{y}$ to be the remainder when x is divided by y.

- **mod** is called the **modulus operator(模算子)**

Example 3.1.12 We have
$11 \bmod 7 = 4, -11 \bmod 7 = 3.$

Example 3.1.14 What day of the week will it be 365 days from Wednesday?

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $\boldsymbol{x \bmod y}$ to be the remainder when x is divided by y.

- **mod** is called the **modulus operator(模算子)**

Exercise Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by $f(x) = 4x \bmod 5$. Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

3.1 Functions 函数

Definition 3.1.47 A function from $X \times X$ to X is called a **binary operator on X**.
(X上的二元操作符)

Definition 3.1.50 A function from X to X is called a **unary operator on X**.
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3.1 Functions 函数

Definition 3.1.47 A function from $X \times X$ to X is called a **binary operator on X**. **(X上的二元操作符)**

Example Let $X = \{\mathbf{1}, \mathbf{2}, \ldots\}$. If we define $f(x, y) = x + y$, where $x, y \in X$, then f is a binary operator on X.

3.1 Functions 函数

Definition 3.1.50 A function from X to X is called a **unary operator on X**.
(X上的一元操作符)

Example Let U be a universal set. If we define $f(X) = \overline{X}$, where $X \in \mathcal{P}(U)$, then f is a unary operator on $\mathcal{P}(U)$.

f: $\mathcal{P}(U) \rightarrow \mathcal{P}(U)$

3.2 Sequences and Strings 序列和串

Definition 3.2.1 A **sequence (序列)** s is a function whose domain D is a subset of integers.

The notation s_n is typically used instead of the more general function notation $s(n)$. The term n is called the **index(下标)** of the sequence.

If D is a finite set, we call s a finite sequence **(有限序列)**;
otherwise, s is an infinite sequence **(无限序列)**.

3.2 Sequences and Strings 序列和串

A sequence s is denoted s or $\{s_n\}$ if n is the index of the sequence.

- Notation s_n denotes the single element of the sequence s at index n.

If s is a sequence $\{s_n\}$, where $n = 1, 2, 3, \ldots$

- s_1 denotes the first element,
- s_2 the second element,
- s_n the nth element…

3.2 Sequences and Strings 序列和串

A sequence s is denoted s or $\{s_n\}$ if n is the index of the sequence.

- Notation s_n denotes the single element of the sequence s at index n.

- We will frequently use $\mathbf{Z}^+$ or $\mathbf{Z}^{nonneg}$ as the domain of a sequence.

If the domain of a sequence s is $\{k, k+1, k+2, \ldots\}$ and the index of s is n, we can denote the sequence s as $\{s_n\}_{n=k}^{\infty}$.

A sequence s whose domain is $\mathbf{Z}^{nonneg}$:

A sequence s whose domain is $\{i, i+1, \ldots, j\}$:

A sequence s whose domain is $\{-1, 0, 1, 2, 3\}$:

3.2 Sequences and Strings 序列和串

Example 3.2.5 Define a sequence b by the rule b_n is the nth letter in the word digital. If the domain of b is $\{1, 2, \ldots, 7\}$,

then $b_1 = ?$, $b_2 = ?$, $b_4 = ?$ and $b_7 = ?$

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then $b_1 = ?$, $b_2 = ?$, $b_4 = ?$ and $b_7 = ?$

Example 3.2.6 If x is the sequence defined by

$$x^2 = \frac{1}{2^n} \qquad -1 \leq n \leq 4,$$

the elements of x are ?

3.2 Sequences and Strings 序列和串

Example 3.2.7 Define a sequence s as

$$s_n = 2^n + 4 \times 3^n \quad n \geq 0.$$

(a) Find s_0.

(b) Find s_1.

(c) Find a formula for s_i.

(d) Find a formula for s_{n-1}.

(e) Find a formula for s_{n-2}.

(f) Prove that $\{s_n\}$ satisties

$$s_n = 5s_{n-1} - 6s_{n-2} \text{ for all } n \geq 2.$$

Important Types of Sequences

- **Increasing Sequences (递增序列)**

A sequence s is **increasing** if for all i and j in the domain of s,

if $i < j$, then $s_i < s_j$.

- **Decreasing Sequences (递减序列)**

- **Nonincreasing Sequences (非增序列)**

- **Nondecreasing Sequences (非减序列)**

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- **Increasing Sequences (递增序列)**

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if $i < j$, then $s_i < s_j$.

- **Decreasing Sequences (递减序列)**

A sequence s is **decreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i > s_j$.

- **Nonincreasing Sequences (非增序列)**

A sequence s is **nonincreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \geq s_j$.

- **Nondecreasing Sequences (非减序列)**

A sequence s is **nondecreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \leq s_j$.

Important Types of Sequences

- **Increasing Sequences (递增序列)**
- **Decreasing Sequences (递减序列)**
- **Nonincreasing Sequences (非增序列)**
- **Nondecreasing Sequences (非减序列)**

Examples

1. The sequence $100, 90, 90, 74, 74, 74, 30$.
2. The sequence $\{s_n\}$ defined by the rule $s_n = 2n - 1$, for all $n \geq 1$.

Subsequences 子序列

Definition 3.2.12 Let s be **a sequence. A subsequence of** s is a sequence obtained from s by choosing certain terms of s in the same order in which they appear in s.

Example

Let $s = \{s_n = n \mid n = 1, 2, 3, \ldots\}$.

Let $t = \{t_n = 2n \mid n = 1, 2, 3, \ldots\}$.

t is a subsequence of s

Sigma and Pi Notation

Definition 3.2.17 If $\{a_i\}_{i=m}^{n}$ is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n,$$

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \ldots a_n.$$

The formalism $\sum_{i=m}^{n} a_i$ is called the **sum (or sigma) notation (求和符号)** and $\prod_{i=m}^{n} a_i$ is called the **product (or pi) notation (乘积符号)**.

Sigma and Pi Notation

Exercise 1 For the sequence w defined by

$$w_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1.$$

(1) Find $\sum_{i=1}^{3} w_i$.

(2) Find $\sum_{i=1}^{10} w_i$.

(3) Find a formula for the sequence c defined by $c_n = \sum_{i=1}^{n} w_i$.

(4) Find a formular for the sequence d defined by $d_n = \prod_{i=1}^{n} w_i$.

(5) Is w increasing?

(6) Is w decreasing?

(7) Is w nonincreasing?

(8) Is w nondecreasing?

Sigma and Pi Notation

Exercise 2 For the sequence a defined by

$$a_n = \frac{n-1}{n^2(n-2)^2}, n \geq 3.$$

and the sequence z defined by $z_n = \sum_{i=3}^{n} a_i$

(1) Find a_3.

(2) Find a_4.

(3) Find z_3.

(4) Find z_4.

(5) Find z_{100}.

(6) Is z increasing, decreasing, nondecreasing and nonincreasing?

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(3) Find z_3.

(4) Find z_4.

(5) Find z_{100}.

Hint: Show that $a_n = \frac{1}{4}\left[\frac{1}{(n-2)^2} - \frac{1}{n^2}\right]$

and use this form in the sum. Write out $a_3 + a_4 + a_5 + a_6$to see what is going on.

(6) Is z increasing, decreasing, nondecreasing and nonincreasing?

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

- Finite sequences are also called strings.
- The string with no elements is called **null string (空串)** and is denoted λ.
- Let X^* denote the set of all strings over X.
- Let X^+ denote the set of all nonnull strings over X.

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Example 3.2.5 Let $X = \{a, b\}$. Some elements in X^* are λ, a, b, $abab$ and $b^2a^{50}ba$.

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

● The **length (长度)** of a string α is the number of elements in α. The length of α is denoted $|\alpha|$.

Example 3.2.26 If $\alpha = aabab$ and $\beta = a^3b^4a^{32}$, then $|\alpha| = 5$ and $|\beta| = 39$.

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y.

If $X = Y$, we call R a (binary) relation on X.

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- **The relationship between Function, Sequence and Relation**

Definition 3.2.1 A **sequence (序列)** s is a function whose domain D is a subset of integers.

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Special case

sequence $\longleftarrow$ **function**

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The relationship between Function, Sequence and Relation

A function f from X to Y is a relation from X to Y having the properties:
(a) The domain of f is equal to X.
(b) For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$.

$$\text{sequence} \xleftarrow{\text{Special case}} \text{function} \xleftarrow{\text{Special case}} \text{relation}$$

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A relation can be defined by

- simply specifying which ordered pairs belong to the relation

$$R = \{(\text{Bill}, \text{CompSci}), (\text{Mary}, \text{Math}), (\text{Bill}, \text{Art}), (\text{Beth}, \text{History}), (\text{Beth}, \text{CompSci}), (\text{Dave}, \text{Math})\}$$

TABLE 3.3.1 ■ Relation of Students to Courses

Student	*Course*
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

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- defining a relation by giving a rule for membership in the relation

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R = ?$

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = ?$

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A relation on a set

draw its **digraph (有向图)**

Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

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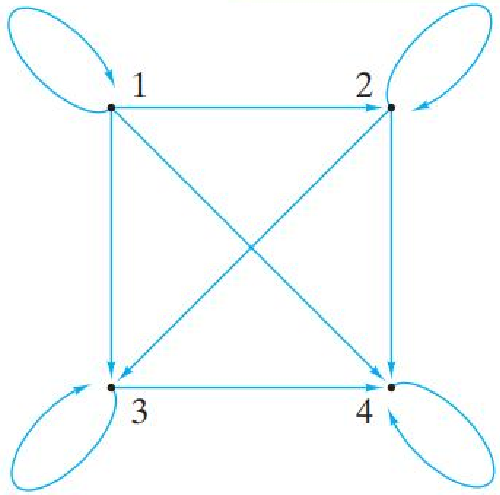

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- draw its **digraph (有向图)**
- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

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How many different relations can we define on a set X with n elements?

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How many different relations can we define on a set X with n elements?

A relation on a set X is a subset of $X \times X$.
How many elements are in $X \times X$?
There are n^2 elements in $X \times X$, so how many subsets (= relations on X) does $X \times X$ have?
Therefore, 2^{n^2} subsets can be formed out of $X \times X$.

Answer: We can define 2^{n^2} different relations on X.

3.3 Relations 关系

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$

$R = \{(1, 1), (2, 2), (3, 3)\}$

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Exercise Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

$R = \{(1, 1), (2, 2), (3, 3)\}$ No.

3.3 Relations 关系

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

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for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

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Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

symmetric? antisymmetric?

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not symmetric = antisymmetric?

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not symmetric = antisymmetric?

$R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$.

Both symmetric and antisymmetric!

3.3 Relations 关系

Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$

$R = \{(1, 3), (3, 2), (2, 1)\}$

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$

3.3 Relations 关系

Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ Yes.

$R = \{(1, 3), (3, 2), (2, 1)\}$ No.

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$ No.

The relation $R = \emptyset$

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

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Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Negation

Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

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Definition 3.3.17 A relation R on a set X is **transitive (传递的)**
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Negation

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not reflexive: if there exists $x \in X$, such that $(x, x) \notin R$.

Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

not symmetric: if there exists x and y, such that $(x, y) \in R$ and $(y, x) \notin R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

not antisymmetric: if there exists x and y, $x \neq y$, such that $(x, y) \in R$ and $(y, x) \in R$.

Definition 3.3.17 A relation R on a set X is **transitive (传递的)**

if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

not transitive: if there exists $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \notin R$.

3.3 Relations 关系

Exercise Give examples of relations on $\{1, 2, 3, 4\}$ having the properties specified as follows.

(a) Reflexive, symmetric, and not transitive

(b) Reflexive, not symmetric, and not transitive

(c) Reflexive, antisymmetric, and not transitive

(d) Not reflexive, symmetric, not antisymmetric, and transitive

(e) Not reflexive, not symmetric, and transitive

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

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If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

We say that

- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \npreceq y$ or $y \npreceq x$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

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- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

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Example ?

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \le y$, $x, y \in X$.
Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

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we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

3.3 Relations 关系

Definition 3.3.23 Let R be a relation from X to Y. The **inverse of R (R的逆)**, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

$R^{-1} =$

3.3 Relations 关系

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 =$

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Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

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is $R_2 \circ R_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

(2) $R \cap S$

(3) $R \circ S$

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

$$R = \{(1, 2)\}, S = \{(2, 3)\}, R \cup S = \{(1, 2), (2, 3)\}.$$

(2) $R \cap S$

If $\{x, y\}, \{y, z\} \in R \cap S$, then $\{x, z\} \in R \cap S$.

(3) $R \circ S$

$$R = \{(5, 2), (6, 3)\}, S = \{(1,5), (2, 6)\}, R \circ S = \{(1, 2), (2, 3)\}.$$

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

3.4 Equivalence Relations 等价关系

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- A **collection of sets (集族)**：A set $\mathcal{S}$ whose elements are sets.
Example: $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$.

- A **partition (划分)**：A collection $\mathcal{S}$ of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of $\mathcal{S}$.
Example: $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is a a partition of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

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Example 3.4.2 Consider the partition $\mathcal{S} = \{\{1, 3, 5\}, \{2. 6\}, \{4\}\}$ of $X = \{1, 2, 3, 4, 5, 6\}$. The relation R on X is given by Theorem 3.4.1. Then

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Exercise Which of the following relation is an equivalence relation?

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$.

The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$.

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