Chapter 5 Introduction to Stochastic Processes

School of Sciences, BUPT

Introduction to Stochastic Processes

Probability provides models for analyzing random or unpredictable outcomes.
The main new ingredient in stochastic processes is the explicit role of time.

Contents

$F(x_1, x_2; t_1, t_2)$

$= P(X(t_1) \leq x_1, X(t_2) \leq x_2)$

$F(x;t) = P(X(t) \leq x)$

$E(X(t))$

$E(X(s)X(t))$

Introduction to Stochastic Processes

$X(t, \omega)$

A random variable X is a rule for assigning to every outcome ω of an experiment E a number $X(\omega)$.

Fix ω_0

A stochastic process $X(t)$ is a rule for assigning to every ω a function $X(t, \omega)$.

Thus a ***stochastic process*** is ***a family of time functions*** depending on the parameter ω. Each time function is called a sample path.

For $\omega_0 \in \Omega$,

$$\underset{\text{variable}}{\underset{\uparrow}{X(t}}, \underset{\text{Fixed}}{\underset{\uparrow}{\omega_0}}), \quad t \in T$$

Introduction to Stochastic Processes

Fix $t_0 \in T$, $X(t_0, \omega)$, $\omega \in \Omega$

Thus a ***stochastic process*** is ***a family of time functions*** depending on the parameter ω. Each time function is called a sample path.

On the other hand, a stochastic process is ***a family of random variables***. That is, for a given t, $X(t, \omega)$ is a function of sample point ω, i.e., a random variable.

Or, equivalently, a stochastic process is a family of function of t and ω. The domain of ω is the sample space Ω and the domain T of t is a subset of real numbers.

Introduction to Stochastic Processes

t is often interpreted as 'time'. But it doesn't always describe time.

Let T be the index set of the process. T is a subset of real numbers. T may be $[0, \infty)$, $[0, 1]$, $\{0, 1, 2, \cdots\}$ and so on. The complete description of a stochastic is $\{X(t, \omega), t \in T\}$, but we usually use the abbreviation $X(t)$ or X_t.

As a result, we refer to X_t as the state of the process at 'time' t.

The state space of a stochastic process is defined as the set of all possible values that the random variables X_t can assume.

$X = g(\Theta) = a\cos(\omega t_1 + \Theta)$

Example

Consider

$$X_t = acos(\omega t + \Theta),\ t \in (-\infty, +\infty),$$

Θ

where a, ω are constants and Θ has a uniform distribution in the interval $(0, 2\pi)$. Since for each fixed $t = t_1$, $X_{t_1} = acos(\omega t_1 + \Theta)$ is a random variable, $\{X_t\}$ is a stochastic process. If we select $\theta_i \in (0, 2\pi)$ arbitrarily, then

time

$$X_t = acos(\omega t + \theta_i),\ t \in (-\infty, +\infty),$$

fun.

is a sample path of the process. □

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Classical Types of Stochastic Processes

The main elements distinguishing stochastic processes are in the nature of the state space S, the index parameter T, and the dependence relations among the random variables X_t.

Index Parameter T	**State Space** S
discrete or continuous	discrete or continuous

When T is a countable set the stochastic process is said to be a discrete-time process, or a stochastic sequence. If T is an interval of the real line, then the stochastic process is said to be a continuous-time process.

For instance, $\{X_n, n = 0, 1, \cdots\}$ is a discrete-time stochastic process indexed by the nonnegative integers; while $\{X_t, t \geqslant 0\}$ is a continuous-time stochastic process indexed by the nonnegative real numbers.

Classical Types of Stochastic Processes

We now describe some of the classical types of stochastic processes characterized by different dependence relationships among X_t.

(a) ***Markov Processes*** Ch 7

Roughly speaking, a Markov process is a process with the property that, given the value of X_t, the values of $X_s, s > t$, do not depend on the values of $X_u, u < t$; that is, the probability of any particular future behavior of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behavior.

We should make it clear, however, that if our knowledge of the present state of the process is imprecise, then the probability of some future behavior will in general be altered by additional information relating to the past behavior of the system.

Introduction to Stochastic Processes

In formal terms a process is said to be Markovian if

$$
\begin{aligned}
&P(a < X_t \leqslant b \,|X_{t_1} = x_1, X_{t_2} = x_2, \cdots X_{t_n} = x_n) \\
=&P(a < X_t \leqslant b \,|X_{t_n} = x_n)
\end{aligned}
$$

whenever $t_1 < t_2 < \cdots < t_n < t$.

Markov chains, as statistical models of real-world processes, have many applications in a wide range of topics such as physics, chemistry, medicine, music, game theory and sports. It will be introduced in chapter 7.

(b) ***Stationary Processes*** Ch 6

A stochastic process X_t for $t \in T$ is said to be *strictly stationary* if the joint distribution functions of the families of random variables

$$(X_{t_1+h}, X_{t_2+h}, \cdots, X_{t_n+h}) \quad \text{and} \quad (X_{t_1}, X_{t_2}, \cdots, X_{t_n})$$

are the same for all $h > 0$ and arbitrary selections $t_1, t_2, \cdots, t_n$ from T. This condition asserts that in essence the process is in probabilistic equilibrium and that the particular times at which we examine the process are of no relevance. In particular, the distribution of X_t is the same for each t.

Introduction to Stochastic Processes

A stochastic process X_t for $t \in T$ is said to be *wide sense stationary* or *covariance stationary* if it possesses finite second moments and if

$$Cov(X_t, X_{t+h}) = E(X_t X_{t+h}) - E(X_t)E(X_{t+h})$$

depends only on h for all $t \in T$. A strict stationary process that has finite second moments is wide stationary. Stationary

processes are appropriate for describing many phenomena that occur in communication theory, astronomy, biology, and sometimes economics and are discussed in more detail in Chapter 6.

Ch8

(c) ***Process with Stationary Independent Increments***

We say that X_t is *a process with independent increments* if for any $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_n$,

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent. If the index set is discrete, that is, $T = (0, 1, \cdots)$, then a process with independent increments reduces to a sequence of independent random variables $Z_0 = X_0, Z_i = X_i - X_{i-1} (i = 1, 2, 3, \cdots)$, in the sense that knowing the individual distributions of $Z_0, Z_1, \cdots$. enables one to determine (as should be fairly clear to the reader) the joint distribution of any finite set of the X_i.

In fact,

$$X_i = Z_0 + Z_1 + \cdots + Z_i, \quad i = 0, 1, 2, \cdots$$

If the distribution of the increments $X_{t_1+h} - X_{t_1}$ depends only on the length h of the interval and not on the time t_1, then the process is said to have *stationary increments*. For a process with stationary increments the distribution of $X_{t_1+h} - X_{t_1}$ is the same as the distribution of $X_{t_2+h} - X_{t_2}$, no matter what the values of t_1, t_2 and h.

We will put our attention to two independent increments processes, Poisson process and Wiener process in Chapter 8, which are useful in modeling a large number of random phenomena.

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$$\sim \left\{ F(x_1, \ldots, x_n; t_1, \ldots, t_n), \ \forall n \geq 1, \ \forall t_i \in T, \ i = 1, \ldots, n \right\}$$

The Distribution Family and the Moment Functions

Similar to random sequences, we can define the family of distribution functions of a stochastic process which has an uncountable infinity of random variables, one for each t. For a specific t, X_t is a random variable with distribution

$$F(x;t) = P\{X_t \leqslant x\}.$$

The distribution function of this random variable, in general, depend on t. The function $F(x;t)$ is called the first-order distribution function of the process X_t. Its derivative with respect to x:

$t \in T$

$$f(x;t) = \frac{\partial F(x;t)}{\partial x}$$

is the first-order density of X_t.

The Distribution Family and the Moment Functions

The second-order distribution of the process X_t is the joint distribution

$$F(x_1, x_2; t_1, t_2) = P\big(X_{t_1} \leq x_1, X_{t_2} \leq x_2\big).$$

The corresponding density equals

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2, ; t_1, t_2)}{\partial x_1 \partial x_2}.$$

The Distribution Family and the Moment Functions

Similar to random sequence, if we want to determine the whole statistical property of a stochastic process X_t, then we need the information of the ***family of distribution functions*** of X_t,

$$\{F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n), \forall n \geqslant 1,\ \forall t_i \in T, i = 1, 2, \ldots, n\}.$$

That is, we should master the probability distribution of this vector for all times t_1 through t_n and for all positive integers n.

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The Moments of the Stochastic Processes

For the determination of the statistical properties of a stochastic process, knowledge of the family of distribution functions of the stochastic process is required for every x_i, t_i and n. However, for many applications, only certain averages are used, in particular, the expected value of X_t and of X_t^2.

The Moments of the Stochastic Processes

Mean, Autocorrelation and Autocovariance

If X_t is a real stochastic process, then its mean function is

$\mu(t)$ 自

$$\mu_X(t) = E(X_t), \; t \in T.$$

$d=1$

The variance function is

$$\sigma_X^2(t) = Var(X_t).$$

The autocorrelation function is

$R(s,t)$

$$R_X(t_1, t_2) = R_{XX}(t_1, t_2) = E(X_{t_1} X_{t_2}).$$

$d=2$

Correlation functions have special properties. First,

$$R_X(t_1, t_2) = E(X_{t_1} X_{t_2}) = E(X_{t_2} X_{t_1}) = R_X(t_2, t_1).$$

X	$g(X)$
$E(X) = \int x f(x)\,dx$	$E(g(X)) = \int g(x) f(x)\,dx$

$d=1$ $\quad X_t, \quad X(t)$

$d=2$ $\quad (X_s, X_t), \quad (X(s), X(t))$

(X, Y)

$$\cancel{E(h(X,Y)) = \iint h(x,y) f(x,y)\,dx\,dy}$$

eg 1. $X_t = Y \underbrace{\cos\omega t}_{a} + Z \underbrace{\sin\omega t}_{b}$

$$E(aX) = a \cdot E(X)$$

Y, Z r.v.

eg2. $X_t = a \cdot \cos(\omega t + \Theta)$

a, ω const. $t \in (-\infty, +\infty)$

$\Theta \sim U(0, 2\pi)$.

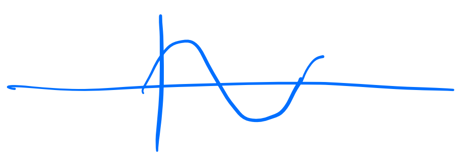
$$\mu_X(t) = E(X_t)$$

$$= E\left(a \cos(\omega t + \Theta) \right)$$

$$E\left(g(\Theta)\right) = \int g(\theta) \cdot f(\theta)\, d\theta$$

$$= \int_0^{2\pi} a \cos(\omega t + \theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= 0$$



$$(X_s, X_t)$$

$$R_X(s,t) = E(X_s \cdot X_t)$$

Eg1. $E(X_s \cdot X_t)$

$$= E\left((Y \cdot \cos ws + Z \sin ws)(Y \cos wt + Z \sin wt) \right)$$

$$\begin{cases} E(Y) = E(Z) = 0, \\ Var(Y) = Var(Z) = \sigma^2 \\ Y \perp Z \end{cases}$$

$$EY^2 = Var(Y) + E^2(Y) = \sigma^2$$

$$= E\big(Y^2 \cos \omega s \cos \omega t + YZ \cos \omega s \cdot \sin \omega t$$

$$+ YZ \sin \omega s \cos \omega t + Z^2 \sin \omega s \sin \omega t \big)$$

$$= \sigma^2 \cos[\omega(t-s)]$$

Eg 2. $X_t = a \cos(\omega t + \Theta)$

$R_X(s,t)$

$$E(h(\Theta))$$

$$= E(X_s \cdot X_t)$$

$$= E\big(a \cos(\omega s + \Theta) \cdot a \cos(\omega t + \Theta) \big)$$

$$= a^2 \cdot \int_0^{2\pi} \cos(\omega s + \theta) \cos(\omega t + \theta) \cdot \frac{1}{2\pi} \cdot d\theta .$$

$$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \quad ①$$

$$\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \quad ②$$

$$\frac{① + ②}{2} = \cos\alpha \cdot \cos\beta$$

$$\cos(\omega s + \theta) \cdot \cos(\omega t + \theta)$$

$$= \frac{1}{2}\left[\cos(\omega s + \omega t + 2\theta) + \boxed{\cos(\omega s - \omega t)} \right]$$

$$R_x(s,t) = \frac{a^2}{2\pi} \cdot \frac{1}{2} \int_0^{2\pi} [\sim] \, d\theta$$

$$= \frac{a^2}{2} \cdot \cos(\omega(t-s))$$

$\square$.

The Moments of the Stochastic Processes

The autocovariance function is

$$\begin{aligned} C_X(t_1, t_2) &:= Cov(X_{t_1}, X_{t_2}) \\ &= E\left[\left(X_{t_1} - \mu_X(t_1)\right)\left(X_{t_2} - \mu_X(t_2)\right)\right]. \end{aligned}$$

Notice that the covariance function is also symmetric, i.e., $C_X(t_1, t_2) = C_X(t_2, t_1)$. By easy calculations,

$$\sigma^2_{X_t} = R_X(t,t) - \mu^2_X(t),$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

$E(XY) = Cov(X, Y) + \mu_X \cdot \mu_Y$ ☆

$$\mathrm{Var}(X) = E(X^2) - E^2(X)$$

$$\sigma_X^2(t) = \mathrm{Var}(X_t)$$

$$= E(X_t^2) - E^2(X_t)$$

$$= E(X_t \cdot X_t) - \mu_X^2(t)$$

$$= R_X(t,t) - \mu_X^2(t)$$

$$C_X(s,t) = \mathrm{Cov}(X_s, X_t)$$

$$= R_X(s,t) - \mu_X(s)\,\mu_X(t)$$

Indep. Increment. 先求 $C_X(s,t)$ [好求]

再求 $R_X(s,t) = C_X(s,t) + \mu_X(s)\mu_X(t)$

The Moments of the Stochastic Processes

Example

An extreme example of a stochastic process is a deterministic signal $X_t = f(t)$. In this case,

$$\mu_X(t) = E(f(t)) = f(t), \;\; R_X(t_1, t_2) = E\big(f(t_1)f(t_2)\big) = f(t_1)f(t_2).$$

Example (HW)

Consider the stochastic process

$$X_t = Y cos\omega t + Z sin\omega t, \quad t \geqslant 0,$$

where Y and Z are independent, real-valued random variables and each has a normal distribution with mean 0 and variance σ^2, ω is a constant. Find the mean function $\mu_X(t)$ and autocorrelation function $R_X(s, t)$.

The Moments of the Stochastic Processes

Solution. First

$$\mu_X(t) = E(X_t) = E(Y cos\omega t + Z sin\omega t)$$

$$= E(Y)cos\omega t + E(Z)sin\omega t = 0.$$

Then, since Y and Z are independent, for the autocorrelation,

$$\begin{aligned} R_X(s,t) &= E(X_s X_t) \\ &= E[(Y cos\omega s + Z sin\omega s)(Y cos\omega t + Z sin\omega t)] \\ &= cos\omega s \cdot cos\omega t \cdot E(Y^2) + sin\omega s \cdot sin\omega t \cdot E(Z^2) \\ &= \sigma^2 cos\omega(t-s). \end{aligned}$$

□

The Moments of the Stochastic Processes

Example (HW)

Consider the stochastic process

$$X_t = a\sin(\omega t + \Theta), \quad t \in (-\infty, +\infty),$$

where a and ω are constants and Θ has a uniform distribution in the interval $(0, 2\pi)$. This kind of X_t is called a sine wave with random phase. Find the mean function $\mu_X(t)$, the autocorrelation function $R_X(s,t)$ and the variance function $\sigma_X^2(t)$.

The Moments of the Stochastic Processes

Solution. Since the probability density function of Θ is

$$f(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{for } 0 < \theta < 2\pi, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\mu_X(t) = E[a\sin(\omega t + \Theta)] = \frac{a}{2\pi}\int_0^{2\pi} \sin(\omega t + \theta)d\theta = 0,$$

$$\begin{aligned} R_X(t_1, t_2) &= E[a\sin(\omega t_1 + \Theta) \cdot a\sin(\omega t_2 + \Theta)] \\ &= \frac{a^2}{2\pi}\int_0^{2\pi} \sin(\omega t_1 + \theta)\sin(\omega t_2 + \theta)d\theta \\ &= \frac{a^2}{2} cos(\omega(t_2 - t_1)). \end{aligned}$$

The Moments of the Stochastic Processes

Let $t_2 = t_1 = t$. we get

$$\sigma_X^2(t) = R_X(t,t) - \mu_X^2(t) = \frac{a^2}{2}.$$

$\square$

The Moments of the Stochastic Processes

Example

Suppose that X_t is a process with

$$\mu_X(t) = 3, \qquad R_X(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}.$$

Determine the mean, the variance, and the covariance of the random variables $Z = X_5$ and $W = X_8$.

Solution. Clearly, $E(Z) = \mu_X(5) = 3$ and $E(W) = \mu_X(8) = 3$. Furthermore,

$$E(Z^2) = R(5,5) = 13, \quad E(W^2) = R(8,8) = 13.$$

$$R_X(5,8) = E(ZW) = R(5,8) = 9 + 4e^{-0.6} = 11.195.$$

Thus Z and W have the same variance $\sigma^2 = 13 - 9 = 4$ and their covariance equals
$C_X(5,8) = R_X(5,8) - E(Z)E(W) = 2.195.$ $\square$

The Moments of the Stochastic Processes

Example

Suppose that the stochastic process X_t is

$$X_t = Y + Zt,$$

where Y and Z are independent and each has a normal distribution $N(1, 1)$.

(a) Find the first-order distribution of $X_{2.5}$. 1

(b) Find the first-order distribution of X_t.

(c) Find the second-order distributions of X_t.

2 $R_X(s,t) = ?$

$X_t = Y + Zt$

$\underline{t=2.5}$ $\qquad X_{2.5} \sim N(3.5,\ 7.25)$

$$Var(X_t) = Var(Y) + t^2 \cdot Var(Z)$$
$$= 1 + 6.25 \cdot 1 = 7.25$$

$$f^{t}(x;\ 2.5) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{7.25}} e^{-\frac{(x-3.5)^2}{2\times 7.25}}$$

$E(YZ) = \rho\sigma_1\sigma_2 + \mu_1\mu_2$

$\square$

$\rho = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y}$

$$R_X(s,t) = E(X_s \cdot X_t)$$

$E(XY) - \mu_x \cdot \mu_y = Cov(X,Y) = \rho\sigma_x \cdot \sigma_y$

$$= E((Y+Zs)(Y+Zt))$$
$$= E(Y^2 + YZt + YZs + Z^2st)$$
$$= 2 + t + s + 2st.$$

$\square$

The Moments of the Stochastic Processes

Solution. (a) Since Y and Z are independent and each has a normal distribution with mean 1 and variance 1, the linear combination of Y and Z

$$X_{2.5} = Y + 2.5Z$$

has a normal distribution with mean 3.5 and variance 7.25.
(b) For a given $t > 0$, X_t is a linear combination of Y and Z. So X_t has a normal distribution. Since

$$\mu_X(t) = E(X_t) = 1 + t, \quad \sigma_X^2(t) = Var(X_t) = 1 + t^2,$$

X_t has a normal distribution with mean $1 + t$ and variance $1 + t^2$.

The Moments of the Stochastic Processes

(c) For two specific s, t and $s \neq t$,

$$\begin{pmatrix} X_s \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}.$$

Since Y and Z are independent and each has a normal distribution, (Y, Z) has a bivariate normal distribution. Thus, (X_s, X_t) has a bivariate normal distribution. By an easy calculation, $E(X_s) = 1 + s$, $E(X_t) = 1 + t$,

$$Var(X_s) = 1 + s^2, \ Var(X_t) = 1 + t^2,$$

$$Cov(X_s, X_t) = E\big((Y + Zs)(Y + Zt)\big) = 1 + st.$$

Thus

$$(X_s, X_t) \sim N\left(1 + s, 1 + t, 1 + s^2, 1 + t^2, \frac{1 + st}{\sqrt{(1 + s^2)(1 + t^2)}}\right).$$

$\square$

Contents

The Distribution Family and the Moment Functions

Sometimes we need consider two stochastic processes at the same time. For example, when considering a wide-sense stationary processes through linear time-invariant systems, it is conventional to consider both input and output processes.

Definition

Suppose that X_t and Y_t are two stochastic processes defined on the same sample space Ω and they have the same parameter set T. If for all $t \in T$, (X_t, Y_t) is a random variable, then we say that $\{(X_t, Y_t), t \in T\}$ is a two-dimensional process.

The Distribution Family and the Moment Functions

Definition

Suppose that $\{(X_t, Y_t), t \in T\}$ *is a two-dimensional process. For all positive integers* n *and* m, *and for all* $t_1, t_2, \ldots, t_n;$ $t'_1, t'_2, \ldots, t'_m \in T$, *the* $(n+m)$*th-order probability distribution function of the process* (X_t, Y_t) *is defined by*

$$
\begin{aligned}
& F(x_1, \ldots, x_n; y_1, \ldots, y_m; t_1, \ldots, t_n; t'_1, \ldots, t'_m) \\
= {} & P\{X(t_1) \leqslant x_1, \ldots, X(t_n) \leqslant x_n; Y(t'_1) \leqslant y_1, \ldots, Y(t'_m) \leqslant y_m\}.
\end{aligned}
$$

The Distribution Family and the Moment Functions

The processes $\{X_t\}$ and $\{Y_t\}$ are independent if the group $\{X(t_1), \cdots, X(t_n)\}$ is independent of the group $\{Y(t'_1), \cdots, Y(t'_m)\}$ for any $n, m \in \mathbb{N}$, $t_1, t_2, \ldots, t_n$, $t'_1, t'_2, \ldots, t'_m \in T$. That is

$$\begin{aligned} & F(x_1, \ldots, x_n; y_1, \ldots, y_m; t_1, \ldots, t_n; t'_1, \ldots, t'_m) \\ = \; & F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) \cdot F_Y(y_1, \ldots, y_m; t'_1, \ldots, t'_m). \end{aligned}$$

The Distribution Family and the Moment Functions

Definition

Suppose that $\{X_t\}$ and $\{Y_t\}$ are two real-valued processes, we say that

$$Z_t = X_t + iY_t$$

is a complex process. A complex process is a family of complex functions.

The complex process $Z_t = X_t + iY_t$ is specified in terms of the joint statistics of the real processes X_t and Y_t.

A vector process (n-dimensional process) is a family of n stochastic processes.

The Study of Two Stochastic Processes

Cross-correlation and Cross-covariance

When we observe two stochastic processes, we will concern the cross-corre1ation and cross-covariance of two processes X_t and Y_t.
The cross-correlation function is

$$R_{XY}(t_1, t_2) = E(X_{t_1}\overline{Y}_{t_2}) = \overline{R_{YX}(t_2, t_1)}.$$

Similarly, $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\overline{\mu_Y(t_2)}$ is their cross-covariance function.

The Study of Two Stochastic Processes

Since we mainly concern real-valued processes,

$$\overline{Y}_{t_2} = Y_{t_2}, \quad \overline{\mu_Y(t)} = \mu_Y(t), \quad \text{and} \quad R_{XY}(t_1, t_2) = R_{YX}(t_2, t_1).$$

Two processes $\{X_t\}$ and $\{Y_t\}$ are called *uncorrelated* if, for all t_1 and t_2, we have

$$C_{XY}(t_1, t_2) = 0 \quad \text{or} \quad R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2).$$

They are called *orthogonal* if for all t_1 and t_2,

$$R_{XY}(t_1, t_2) = 0.$$

The Study of Two Stochastic Processes

Theorem

If two stochastic processes are independent, then they are uncorrelated.

Proof. We prove the continuous case here.

$$\begin{aligned}
C_{XY}(s,t) &= E\left[\big(X_s - \mu_X(s)\big)\big(Y_t - \mu_Y(t)\big)\right] \\
&= \iint (x-\mu_X(s))(y-\mu_Y(t)) f_{XY}(x,y;s,t)\,dx\,dy \\
&= \int (x-\mu_X(s)) f_X(x,s)\,dx \cdot \int (y-\mu_Y(t)) f_Y(y,t)\,dy \\
&= E\big(X_s - \mu_X(s)\big) E\big(Y_t - \mu_Y(t)\big) = 0.
\end{aligned}$$

$\square$

The Study of Two Stochastic Processes

Example

The processes $\{X_t\}$ and $\{Y_t\}$ are given by

$$
\begin{aligned}
X_t &= U\cos t + V\sin t,\ \ t \in (-\infty, +\infty),\\
Y_t &= U\sin t + V\cos t,\ \ t \in (-\infty, +\infty),
\end{aligned}
$$

where U and V are two independent random variables, with $E(U) = E(V) = 0$, $E(U^2) = E(V^2) = \sigma^2$. Find the cross-correlation function $R_{XY}(s,t)$.

Solution. The cross-correlation function $R_{XY}(s,t)$ is

$$
\begin{aligned}
R_{XY}(s,t) &= E[(U\cos s + V\sin s)(U\sin t + V\cos t)]\\
&= E(U^2)\cos s\sin t + E(UV)(\cos s\cos t + \sin s\sin t) + E(V^2)\\
&= \sigma^2\sin(s+t).
\end{aligned}
$$

$\square$

The Study of Two Stochastic Processes

Example

Suppose that X_t is a signaling process and Y_t is a noise process. Let $W_t = X_t + Y_t$. Find the mean and autocorrelation of W_t.

Solution. The mean function of W_t is

$$\mu_W(t) = \mu_X(t) + \mu_Y(t),$$

and the autocorrelation function is

$$\begin{aligned} R_W(s,t) &= E[(X_s + Y_s)(X_t + Y_t)] \\ &= R_X(s,t) + R_{XY}(s,t) + R_{YX}(s,t) + R_Y(s,t). \end{aligned}$$

In particular, if the mean functions of X_t and Y_t are 0, then they are uncorrelated, and hence

$$R_W(s,t) = R_X(s,t) + R_Y(s,t).$$

$\square$

The end

Thank you for your patience !