Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

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3.1 Functions 函数

Definition 3.1.1 Let X and Y be sets. A function f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

We sometimes denote a function f from X to Y as $f\colon X \to Y$.

- An **ordered pair (有序对)** of elements, writeen (a, b).
- If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in$ Y. We call $X \times Y$ the **Cartesian product (笛卡尔积)** of X and Y.

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We sometimes denote a function f from X to Y as $f: X \to Y$.

- The set X is called the **domain (定义域)** of f.
- The set Y is called the **codomain (陪域)** of f.
- The set $\{y \mid (x, y) \in f\}$ is called the **range (值域)** of f.

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a subset of the codomain Y

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For each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.
This unique value y is denoted $f(x)$. In other words, $f(x) = y$ is another way to write $(x, y) \in f$.

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Example 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Example 3.1.4 The set $f = \{(1, a), (2, a), (3, b)\}$ is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$?

Example 3.1.5 The set $f = \{(1, a), (2, b), (3, c), (1, b)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Arrow Diagram 箭头图

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Yes.

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No.

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No.

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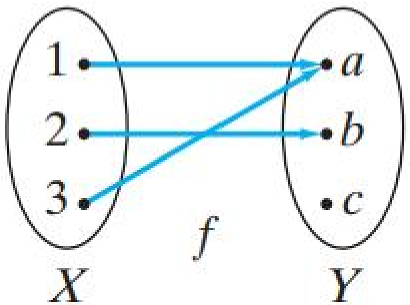

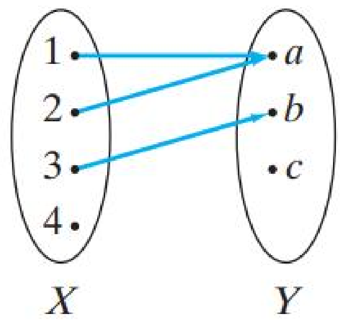

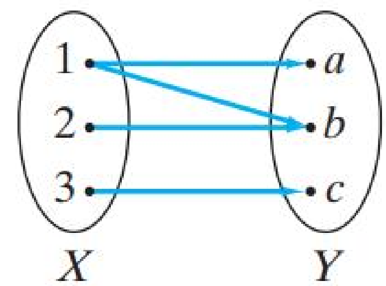

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Yes.

Example 3.1.4 The set $f = \{(1, a), (2, a), (3, b)\}$ is not a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$?

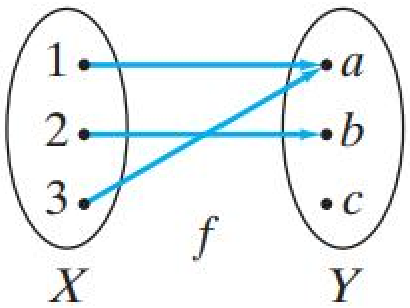
No.

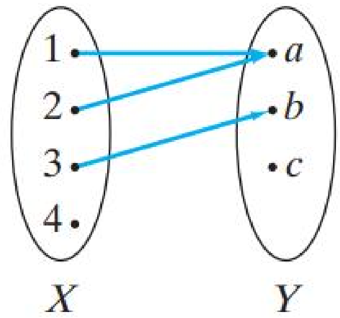
Example 3.1.5 The set $f = \{(1, a), (2, b), (3, c), (1, b)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

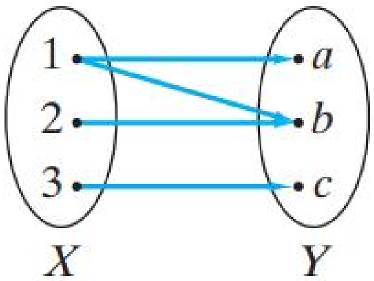
No.

Arrow Diagram 箭头图

A function: There is exactly one arrow from each element in X.







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No.

3.1 Functions 函数

	domain	range
$f(x) = e^{x}$		
$f(x) = \log(x)$		
$f(x) = \sin x$		
$f(x) = \sqrt{x}$		

3.1 Functions 函数

$$f(x) = e^x$$

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A set may contain any kind of element.
Examples:
{1, 2, Jason}
{1, 5, {3.5, 17}, Jason }

3.1 Functions 函数

$$f(S) = |S|$$

$$f(\text{string}) = \text{length}(\text{string})$$

$$f(\text{student}) = \text{student's ID}$$

$$f(p) = \text{is-prime}(p)$$

3.1 Functions 函数

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $\boldsymbol{x_1, x_2 \in X}$, if $\boldsymbol{f(x_1) = f(x_2)}$ then $\boldsymbol{x_1 = x_2}$**.

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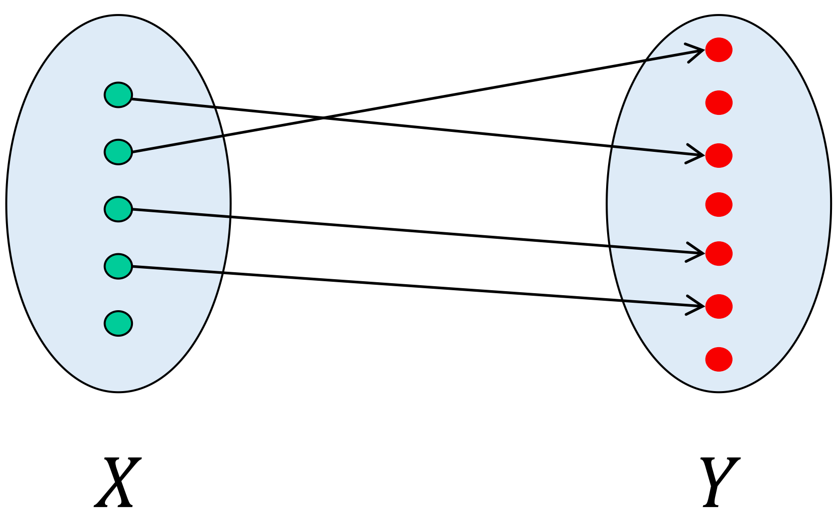
injective?



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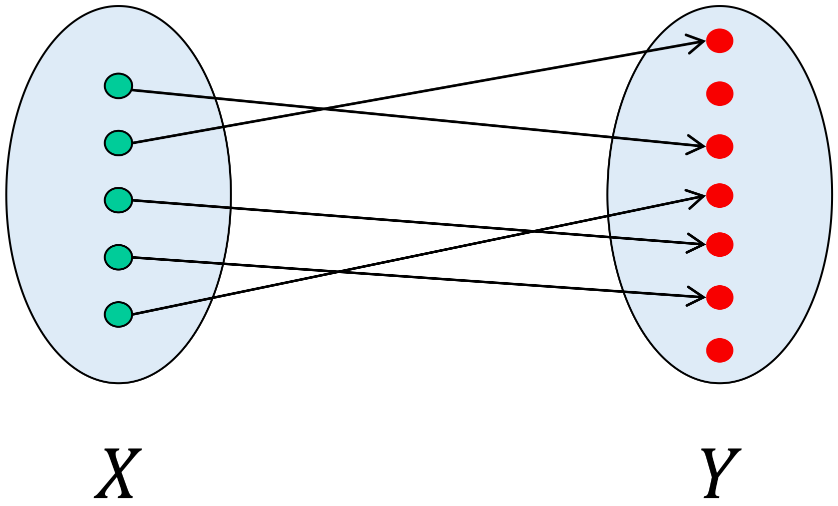

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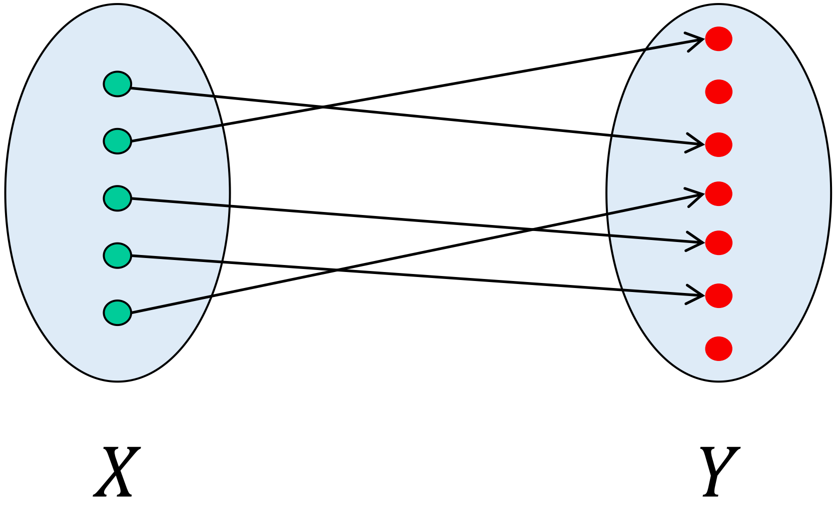
injective ≤ 1 arrow in



$|X| \leq |Y|$

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Example 3.1.23 The set $f = \{(1, b), (3, a), (2, c)\}$ is a one-to-one function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$?

Example 3.1.24 The set $f = \{(1, a), (2, b), (3, a)\}$ is a one-to-one function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

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Example 3.1.27 Prove that the function $f(n) = 2n + 1$ from the set of positive integers to the set of positive integers is one-to-one.

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Example 3.1.28 Prove that the function $f(n) = 2^n - n^2$ from the set of positive integers to the set of integers is not one-to-one.

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Definition 3.1.29 A function f from X to Y is said to be **onto** Y **(or surjective) (满射的)** if **for every $\boldsymbol{y} \in \boldsymbol{Y}$, there exists $\boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$**.

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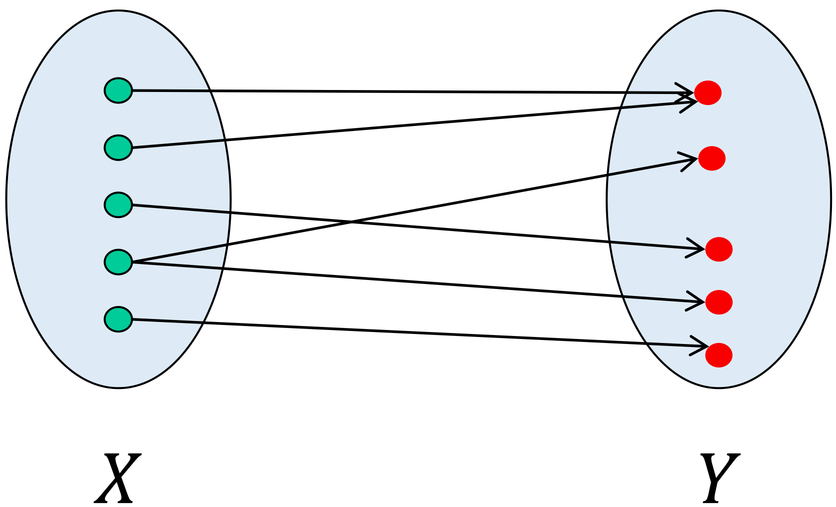
surjective?



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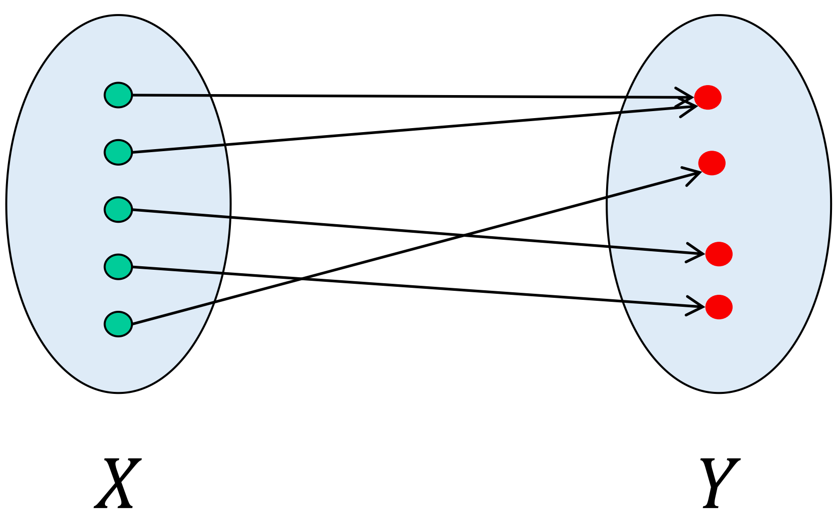

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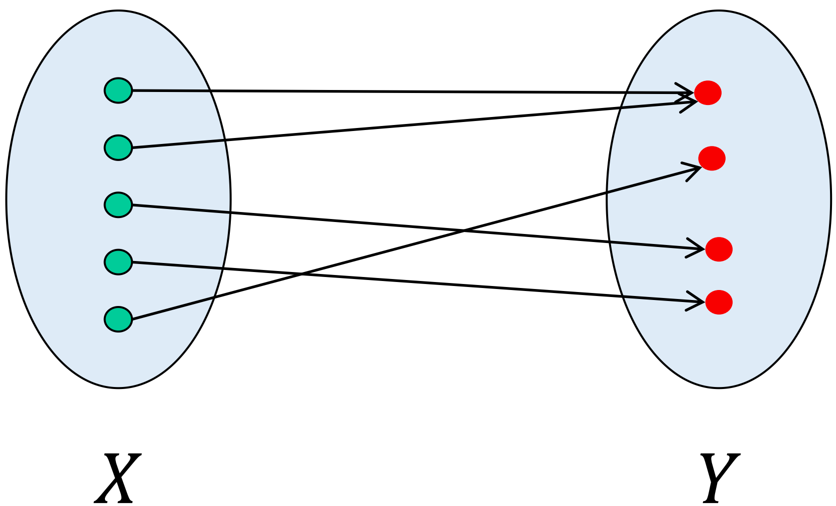
surjective **$\geq$ 1 arrow in**



$|X| \geq |Y|$

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Example 3.1.30 The set $f = \{(1, a), (2, c), (3, b)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$?

Example 3.1.31 The set $f = \{(1, b), (3, a), (2, c)\}$ is onto Y from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$?

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Example 3.1.33 Prove that the function $f(x) = \frac{1}{x^2}$ from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.

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Example 3.1.34 Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is not onto Y.

A function f from X to Y is not onto Y if for some $y \in Y$, for every $x \in X$, $f(x) \neq y$.

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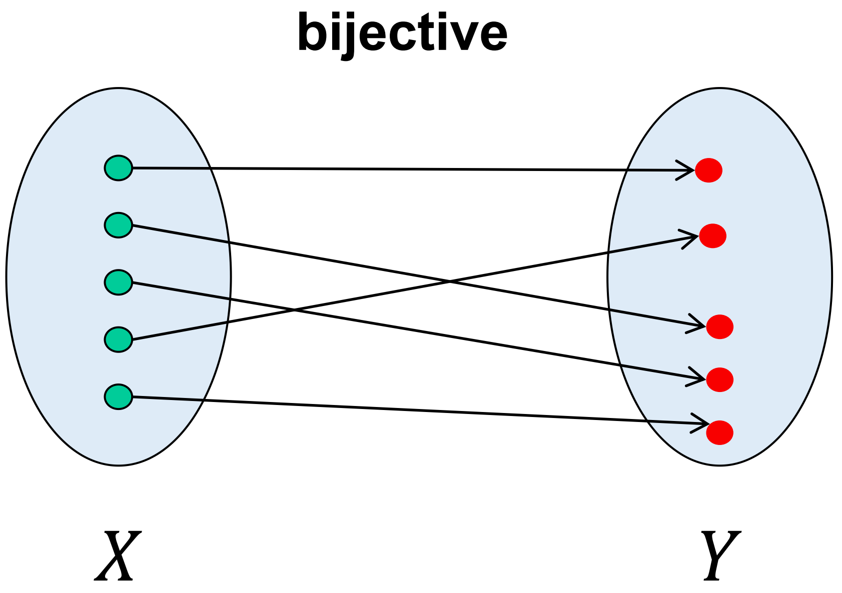

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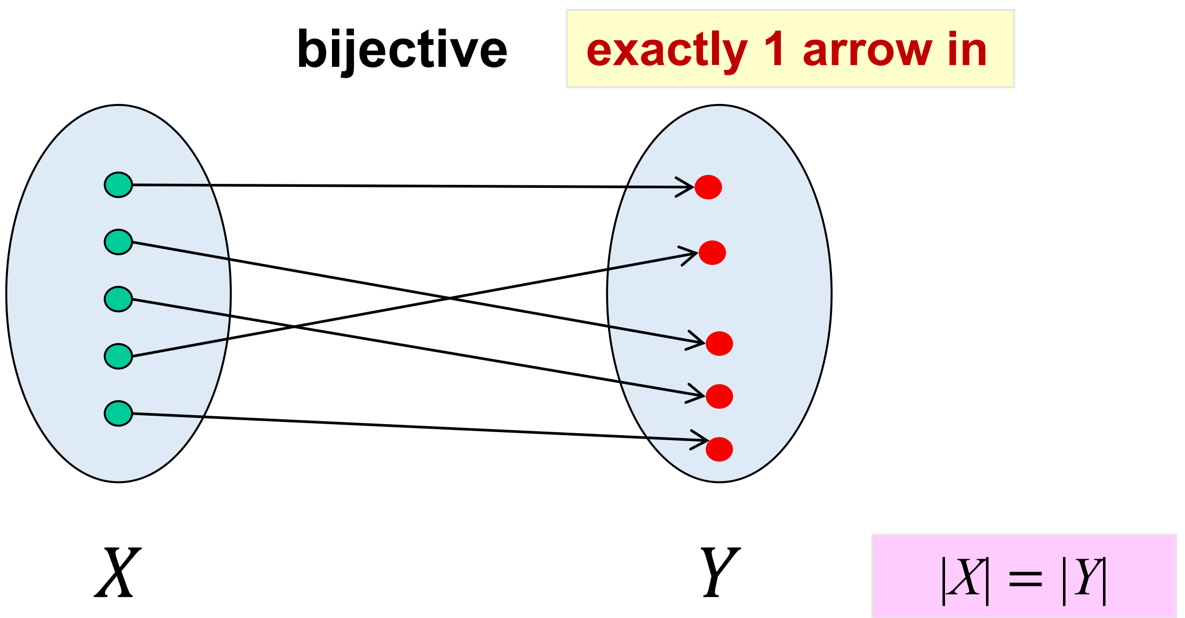

3.1 Functions 函数

Exercise

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x) = \sin(x)$	$\mathbf{R}$	$\mathbf{R}$			
$f(x) = 2^x$	$\mathbf{R}$	$\mathbf{R}^+$			
$f(x) = x^2$	$\mathbf{R}$	$\mathbf{R}^{nonneg}$			
Reverse String	Bit Strings of length n	Bit Strings of length n			

3.1 Functions 函数

Inverse Function (反函数)

Suppose that f is one-to-one, onto function from X to Y. It can be shown that $\{(y, x) \mid (x, y) \in f\}$ is a one-to-one, onto function from Y to X. This new function, denote $\boldsymbol{f^{-1}}$, is called f **inverse (逆)**.

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Example 3.1.38 For function $f = \{(1, a), (2, c), (3, b\ )\}$, we have

$$f^{-1}=?$$

3.1 Functions 函数

Definition 3.1.41 Let g be a function X to Y and let f be a function from Y to Z. The **composition of f with g (f 与 g 的复合函数)**, denoted $f \circ g$, is the function

$$(f \circ g)(x) = f(g(x))$$

from X to Z.

Example 3.1.42 Given $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$, $Z = \{x, y, z\}$.
The function $g = \{(1, a), (2, a), (3, c)\}$ from X to Y.
The function $f = \{(a, y), (b, x), (c, z)\}$ from Y to Z.

The composition function from X to Z is $f \circ g =$?

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The function $g = \{(1, a), (2, a), (3, c)\}$ from X to Y.
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The composition function from X to Z is $f \circ g = \{(1, y), (2, y), (3, z)\}$.

Example 3.1.43 Draw the arrow diagram of the function $f \circ g$.

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Given $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$, $Z = \{x, y, z\}$,
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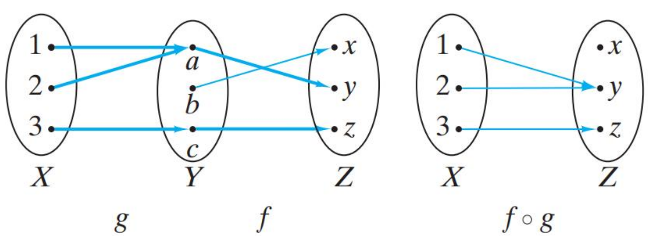

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from X to Z.

Example 3.1.44 If $f(x) = \log_3 x$ and $g(x) = x^4$,
then $(f \circ g)(x) = ?$
and $(g \circ f)(x) = ?$

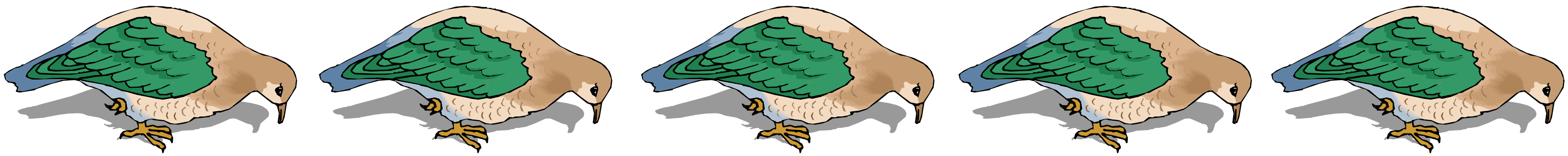
3.1 Functions 函数

Exercise: $g \circ f$

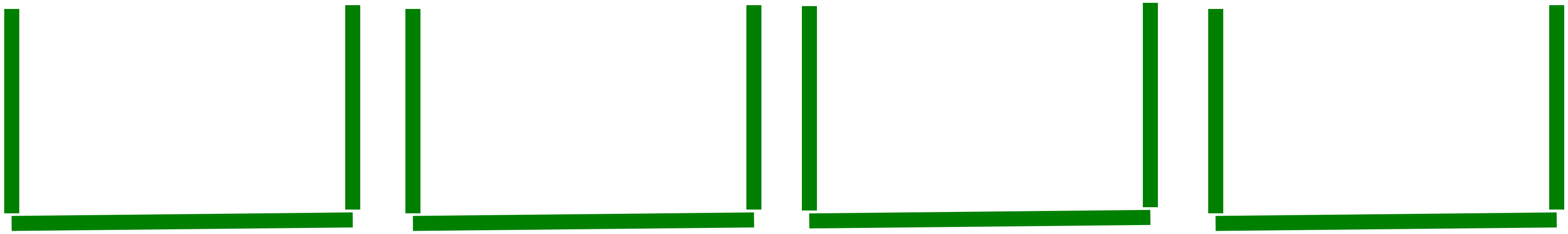
$f: X \to Y$	$g: Y \to Z$	Injective?	Surjective?	Bijective?
f is injective	g is injective			
f is surjective	g is surjective			
f is injective	g is surjective			
f is surjective	g is injective			
f is bijective	g is bijective			

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

If more pigeons

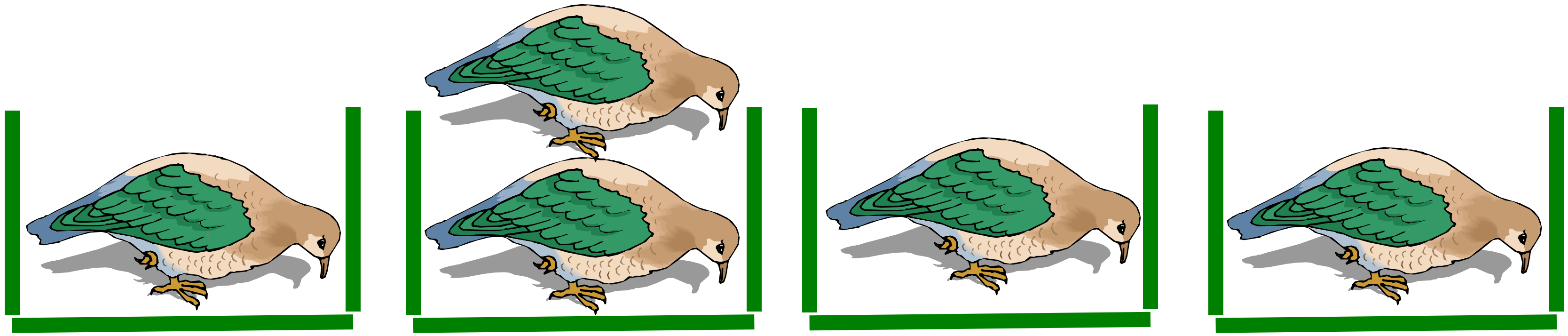


than pigeonholes,



Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

then some hole must have at least two pigeons!

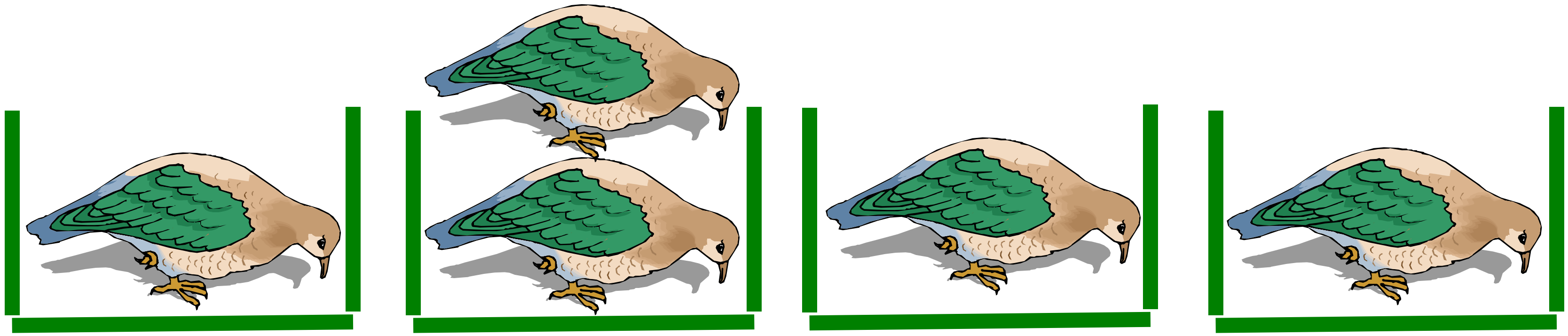


Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Pigeonhole Principle (First Form)

If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

then some hole must have at least two pigeons!



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Pigeonhole Principle (Second Form)
If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then ...

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If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

Pigeonhole Principle (Second Form)
If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.

A function from a larger set to a smaller set cannot be injective.
(There must be at least two elements in the domain that have the same image in the codomain.)

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Example
In a group of 366 people, there must be two people having the same birthday.

3.1 Functions 函数

Definition 3.1.11 If x is an integer and y is a positive integer, we define $\boldsymbol{x} \textbf{ mod } \boldsymbol{y}$ to be the remainder when x is divided by y.

Example 3.1.12 We have
$6 \bmod 2 = 0,\ 199673 \bmod 2 = 1.$

Example 3.1.14 What day of the week will it be 365 days from Wednesday?

3.1 Functions 函数

Definition 3.1.17 The floor of x, denote $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The ceiling of x, denote $\lceil x \rceil$, is the least integer greater than or equal to x.

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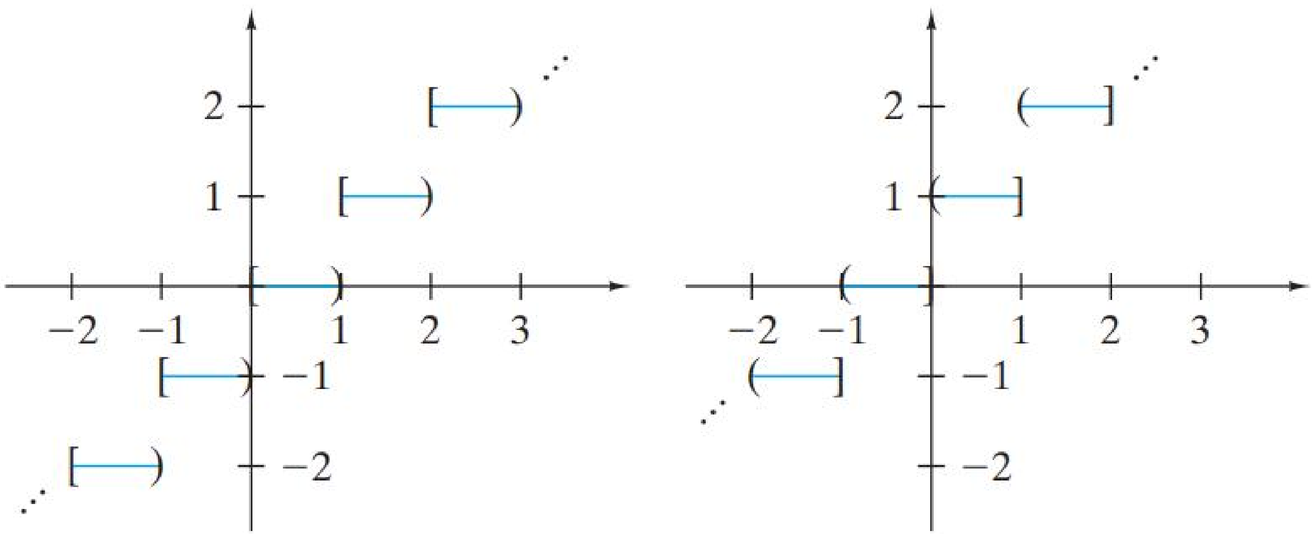

Figure 3.1.7 The graphs of the floor (left graph) and ceiling (right graph) functions.