# EBU4375: SIGNALS AND SYSTEMS

## TOPIC 3-3: THE FOURIER TRANSFORM IN DISCRETE TIME

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# ACKNOWLEDGMENT

These slides are partially from lectures prepared by

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# AGENDA

1. Quick review
2. The Fourier transform of discrete-time signals
3. Some important properties
4. Discrete-time filters in the frequency domain

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# 1: QUICK REVIEW

- Please put the recording on hold and login to QM+
- Go to Topic 3
- Take 10 minutes to answer the questions in T3-Q4
- You can retry as many times as you wish.
- Take 5 minutes to understand your mistakes and discuss with your friends
- You are still unsure? Post your question on the MS Teams channel or QM+ forum.

# 1: QUICK REVIEW

- CT complex exponentials
- Always periodic
- Different frequencies produce different signals
- There exist infinite complex exponentials with period T, namely those of frequencies

$$\frac{2\pi}{T}, 2\frac{2\pi}{T}, 3\frac{2\pi}{T}, \dots$$

- DT complex exponentials
- Only periodic for $\Omega = \frac{2\pi k}{N}$; k, N integers
- Frequencies within an interval of size 2π produce different signals
- There only exist N complex exponentials with period N, namely those of frequencies

$$\frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, \dots, N\frac{2\pi}{N}$$

# 1: QUICK REVIEW

- Since a signal with frequency $\Omega_1$ is the same as a signal with frequency $\Omega_1+2\pi$, we will in general only consider an interval of frequencies of size $2\pi$, usually the interval $[-\pi; \pi]$. In this interval:
  - Low frequencies are close to $\Omega = 0$.
  - High frequencies are close to $\Omega = -\pi$ and $\Omega = \pi$.
- In the interval, $[\pi; 3\pi]$ low frequencies are around $\Omega = 2\pi$, and high frequencies around $\Omega = \pi$ and $= 3\pi$;
- In the interval $[3\pi; 5\pi]$ low frequencies are around $\Omega = 4\pi$, and high frequencies around $\Omega = 3\pi$ and $\Omega = 5\pi$, and so on…..

# Fourier series representation of DT periodic signals

- We have shown that a periodic signal $x_N[n]$ with period N can be expressed as a sum of N complex exponentials:

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}$$

- The frequencies of the complex exponentials are multiples of the fundamental frequency $\Omega_0 = 2\pi/N$ and they are distributed within an interval of size $2\pi$, for instance for the interval $[0,2\pi]$ :

$$0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \ldots, k\frac{2\pi}{N}, \ldots, (N-1)\frac{2\pi}{N}$$

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# 2-FOURIER TRANSFORM OF DT SIGNALS

Non-periodic discrete-time signals can also be expressed as a linear combination of complex exponentials with different frequencies. In other words, **DT signals also have a Fourier transform**. If we compare the Fourier transform of DT signals with the Fourier series of DT signals we note that:

- The Fourier **series** of DT signals consists of **N harmonic frequencies** in an interval of size $2\pi$.
- The Fourier **transform** of DT signals uses **every frequency** within an interval of size $2\pi$.

If we compare the Fourier transform of **DT signals** and the Fourier transform of **CT signals**, we note that:

- The Fourier transform of a CT signal uses **all the frequencies** within the **interval** $[-\infty; \infty]$.
- The Fourier transform of a DT uses **all the frequencies** within an **interval of size $2\pi$**.

# 2-FOURIER TRANSFORM OF DT SIGNALS

- Given a signal x[n], we denote by X(Ω) its Fourier transform: $x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$
- The equations for the Fourier Transform of Discrete time signals are:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

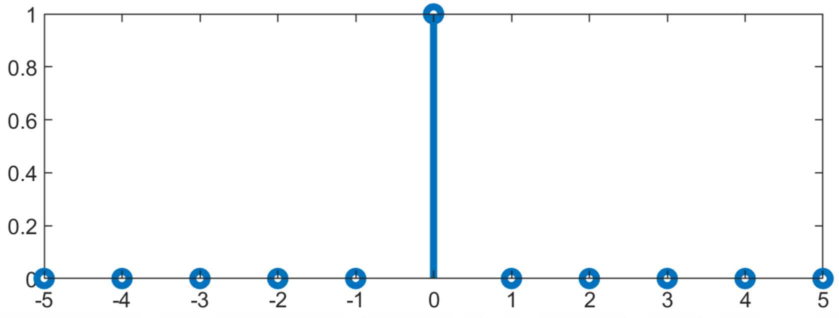
$$x[n] = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(\Omega)e^{j\Omega n}d\Omega$$

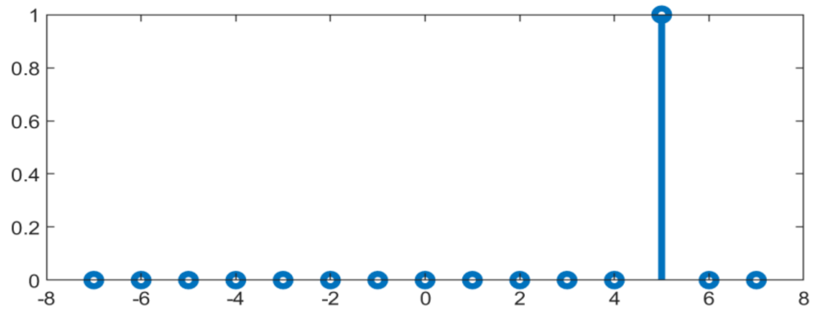
# 2- TEST YOUR SKILLS!

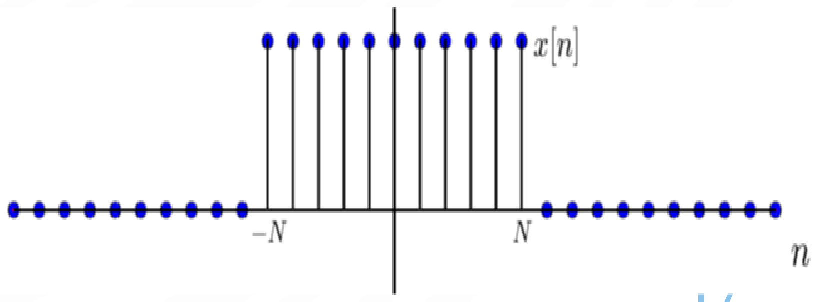
Take 10 minutes to calculate the Fourier transform of the signals shown in the adjacent plots.

Log in to QM+ and take 10 minutes to answer the questions in T3-Q5. You can try as many times as you wish. Please note that this is not graded.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$







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# 2- LET'S SOLVE THESE TOGETHER!

- Obtain the Fourier transform of the signal $x[n] = \delta[n]$ and sketch it.

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- Obtain the Fourier transform of the signal $x[n] = \delta[n-N_0]$ and sketch it.

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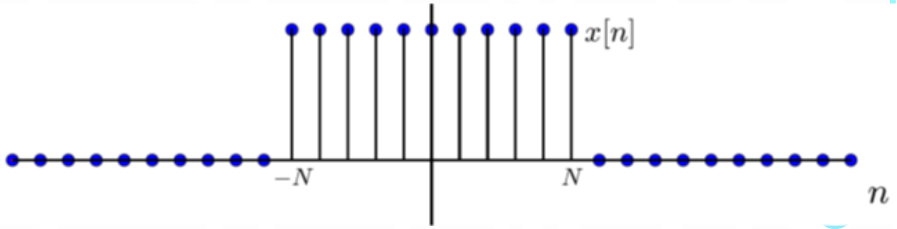


- For the following signal x[n], get the Fourier Transform and sketch it.

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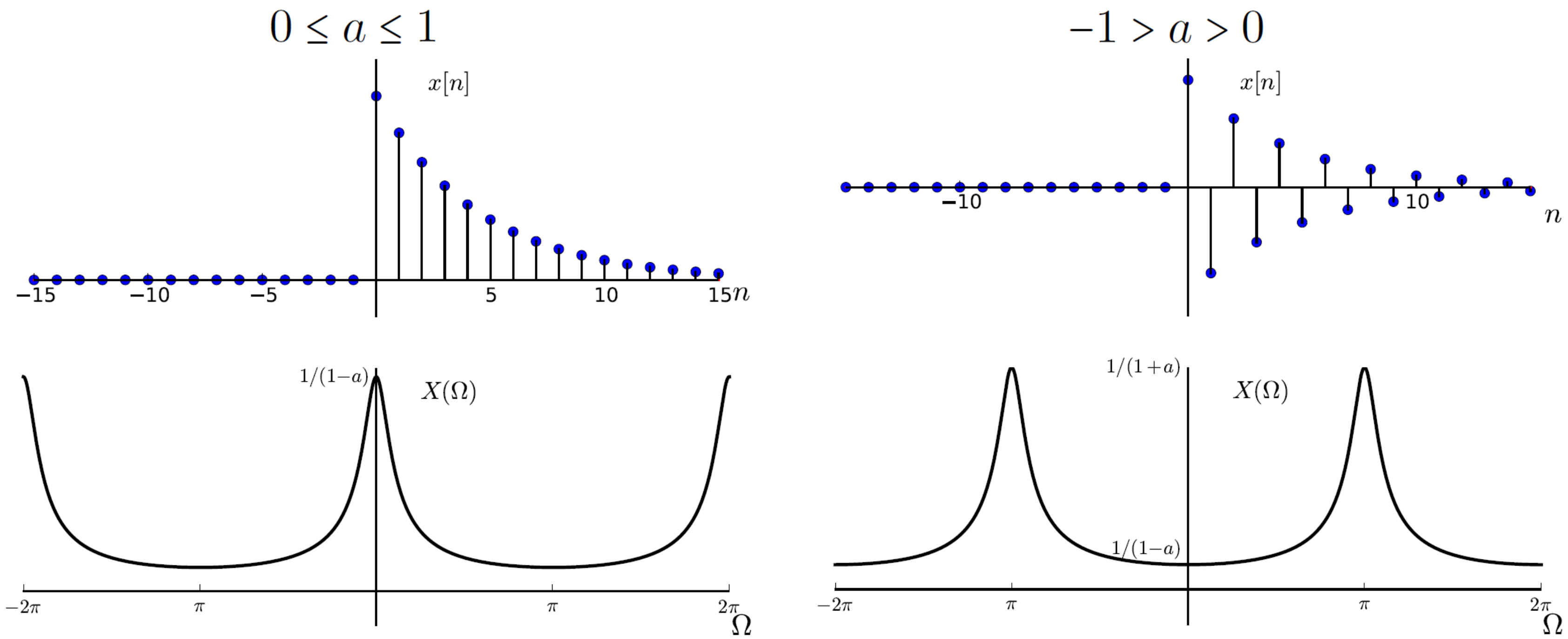
# 2- LET'S SOLVE THESE TOGETHER!

- Draw the signal $x[n] = a^n u[n]$ and its Fourier transform $X(\Omega)$ for $0 \leq a \leq 1$ and $-1 < a < 0$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n = \frac{1}{1 - ae^{-j\Omega}}$$



# 2- REMEMBER

- The Fourier transform describes a signal in the frequency domain, i.e. describes its frequency components.
- Because of the nature of DT complex exponentials, we only need an interval of frequencies of size 2π.
- If we want to consider all the frequencies, we only need to replicate the Fourier transform in the original interval. Consequently, the **Fourier transform of DT signals can be seen as a periodic function.**
- Finally, **low frequency** components are located around the frequencies
  - $\Omega = 0; \pm 2\pi; \pm 4\pi; \dots ; \pm k2\pi; \dots,$
- whereas **high frequencies** are around
  - $\Omega = ; \pm 3\pi; \dots; \pm(2k+1)\pi; \dots.$

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# 3- SOME IMPORTANT PROPERTIES (I)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$
$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$
$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Periodicity: $X(\Omega + 2\pi) = X(\Omega)$

-

- Linearity: $Ax_1[n] + Bx_2[n] \overset{FT}{\Longleftrightarrow} AX_1(\Omega) + BX_2(\Omega)$

# 3- SOME IMPORTANT PROPERTIES (II)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$

$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$

$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Time shift:

$$x[n-n_0] \overset{FT}{\Longleftrightarrow} e^{-j\Omega n_0} X(\Omega)$$

- Frequency shift:

$$e^{j\Omega_0 n} x[n] \overset{FT}{\Longleftrightarrow} X(\Omega-\Omega_0)$$

# 3- SOME IMPORTANT PROPERTIES (III)

- Reflexion: $x[-n] \overset{FT}{\Longleftrightarrow} X(-\Omega_0)$

- Real signals $x[n]\text{real} \quad \Longrightarrow \quad X(\Omega_0) = X^*(-\Omega_0)$

$$\Longrightarrow \quad |X(\Omega_0)| = |X(-\Omega_0)|$$

$$\Longrightarrow \quad \angle X(\Omega_0) = -\angle X(-\Omega_0)$$

# 3- SOME IMPORTANT PROPERTIES (IV)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$
$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$
$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Convolution:

$$x_1[n] * x_2[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)X_2(\Omega)$$

- Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\Omega)|^2 d\Omega$$

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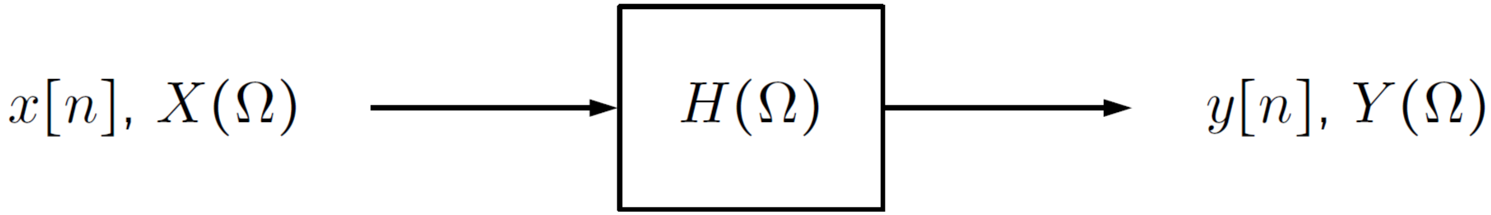
# Discrete-Time LTI systems and Fourier transform



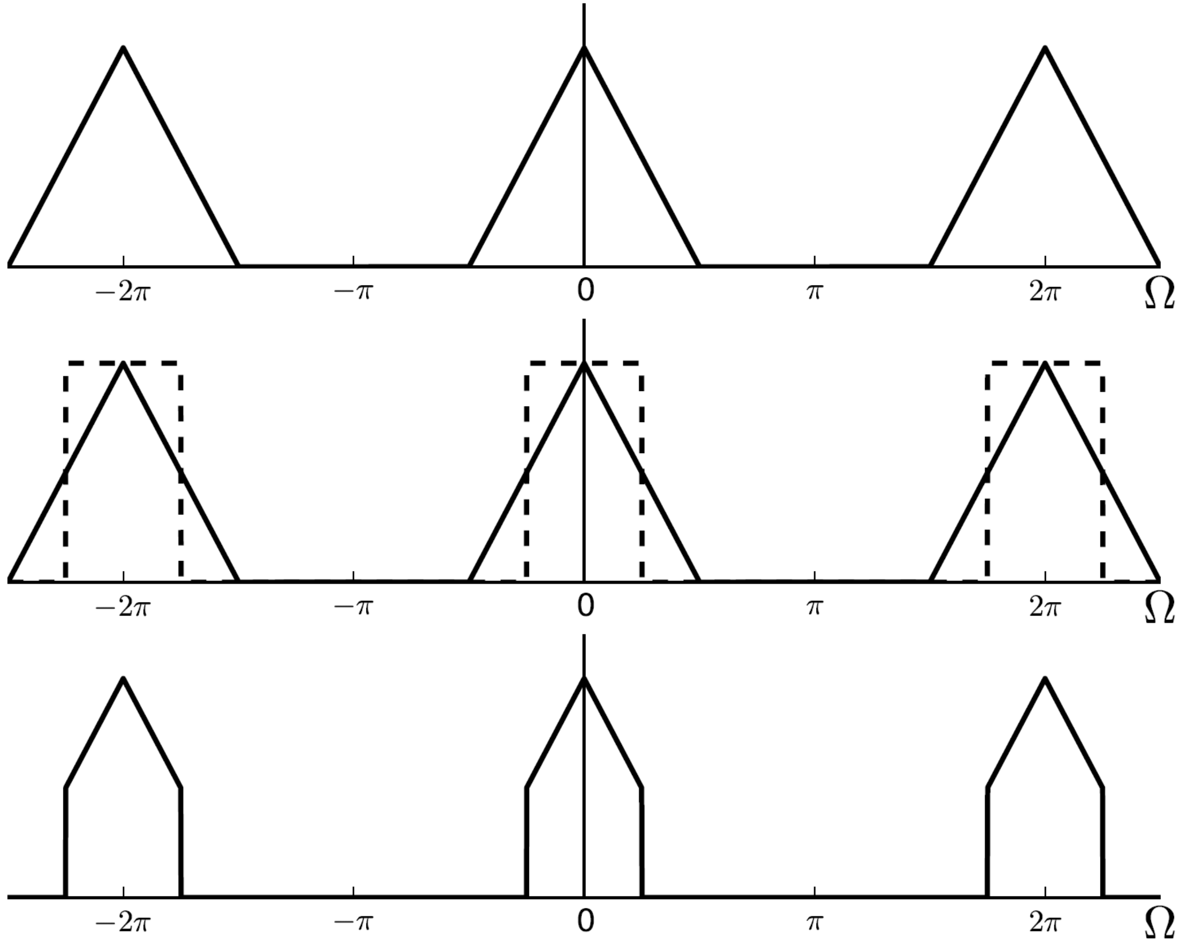

$$y[n] = x[n] \star h[n] \quad \overset{FT}{\Longleftrightarrow} \quad Y(\Omega) = X(\Omega)H(\Omega)$$

$$x[n] \quad \overset{FT}{\Longleftrightarrow} \quad X(\Omega)$$

$$h[n] \quad \overset{FT}{\Longleftrightarrow} \quad H(\Omega)$$

$$y[n] \quad \overset{FT}{\Longleftrightarrow} \quad Y(\Omega)$$

# Discrete-Time LTI systems and Fourier transform



# Discrete-Time Filters