HW 12

Eg 3.4.7 $X \sim Exp(\lambda_1)$ $Y \sim Exp(\lambda_2)$ Find $X+Y$

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\therefore f_{X+Y}(z) = \int_0^z \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2 (z-x)} \, dx$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} \cdot \int_0^z e^{(-\lambda_1+\lambda_2)x} \, dx$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 z} - e^{-\lambda_2 z} \right)$$

Eg 3.4.10 $X_1 \sim Exp(\alpha)$ $X_2 \sim Exp(\beta)$ $X_1 \perp X_2$

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad F_Y(y) = \begin{cases} 1 - e^{-\beta y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

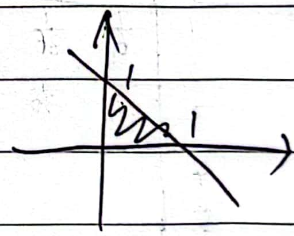
$$F_Z(z) = F(z,z) = F_X(x) F_Y(y) = \begin{cases} (1 - e^{-\alpha z})(1 - e^{-\beta z}) & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$F_W(w) = 1 - [1 - F_X(w)][1 - F_Y(w)] = \begin{cases} 1 - e^{-(\alpha+\beta) w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

$$\therefore f_W(w) = \begin{cases} (\alpha+\beta) e^{-(\alpha+\beta) w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

$$\therefore W \sim Exp(\alpha+\beta)$$

Ex 3.31



$$f_U(u) = \int_{-\infty}^{\infty} f(x, u-x) \, dx$$

$x \geq 0$, $u - x \geq 0$ $u \leq 1$ $\Rightarrow 0 \leq x \leq u$

$$f_U(u) = \int_0^u 6x \, dx = 3u^2$$

3.37 $X, Y, Z \sim N(0,1)$

$$P(3X+2Y < 6Z-7) = P(3X+2Y-6Z < -7)$$

$$= P(-3X-2Y+6Z > 7)$$

$$= 1 - P(-3X-2Y+6Z \leq 7)$$

Let $W = -3X-2Y+6Z$, $W \sim N(0, 49)$

$$= 1 - P(W \leq 7)$$

$$= 1 - P\left(\frac{W}{7} \leq 1\right)$$

$$= 1 - \Phi(1) \approx 0.1587$$

3.39

(a) $\int_y^1 \frac{1}{x}\,dx = -\ln y \quad (0<y<1)$

(b) $\int_0^x \frac{1}{x}\,dy = 1 \quad (y<x<1)$

(c) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$

(d) $E(Y) = \int_0^1 y\cdot(-\ln y)\,dy$

3.41 $X \sim U(0,b)$ $\quad f(x) = \begin{cases} \frac{1}{b}, & 0\le x\le b \\ 0, & \text{else} \end{cases}$

$Cov(X,X^2) = E(X\cdot X^2) - E(X)E(X^2)$

$E(X^2) = Var(X) + E^2(X) = \frac{b^2}{12} + \frac{b^2}{4} = \frac{b^2}{3}$

$E(X^3) = \int_0^b x^3\cdot\frac{1}{b}\,dx = \frac{1}{4}b^3$

$\therefore Cov(X,X^2) = \frac{b^3}{4} - \frac{b}{2}\cdot\frac{b^2}{3} = \frac{b^3}{12}$

$\rho(X,X^2) = \frac{Cov(X,X^2)}{\sqrt{Var(X)\cdot Var(X^2)}}$ $\quad Var(X) = \frac{b^2}{12}$

$Var(X^2) = E(X^4) - E(X^2)^2$

$E(X^4) = \int_0^b x^4 \frac{1}{b}\,dx = \frac{1}{5}b^4$, $\quad Var(X^2) = \frac{4}{45}b^4$

$\therefore \rho(X,X^2) = \frac{\sqrt{15}}{4}$

3.42 $U=\min\{X,Y\}$ $\quad V=\max\{X,Y\}$

(a)

U \ X	1	2	3	4	5	6
1	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2		$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3			$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4				$\frac{1}{12}$	$\frac{1}{36}$	$\frac{1}{36}$
5					$\frac{1}{18}$	$\frac{1}{36}$
6						$\frac{1}{36}$

(b)

U \ V	1	2	3	4	5	6	~~Ege~~ Edge
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{11}{36}$
2		$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{4}$
3			$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{7}{36}$
4				$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{5}{36}$
5					$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$
6						$\frac{1}{36}$	$\frac{1}{36}$

$Cov(U,V) = E(U\cdot V) - E(U)\cdot E(V)$

3.43

(a) $Cov(3X+2Y, X+5Y+10) = Cov(3X, X+5Y+10) + Cov(2Y, X+5Y+10)$

$= Cov(3X, X+10) + Cov(3X, 5Y) + Cov(2Y, X+10) + Cov(2Y, 5Y)$

$= 3Var(X) + 15Cov(X,Y) + 2Cov(Y, X+10) + 10Cov(Y,Y)$

$\underline{X \perp Y}$ $\quad 3 + 0 + 0 + 10 = 13$

(b) $P(X+4Y \geq 2)$ $\qquad Z = X+4Y \sim N(0,17)$

$= P(Z \geq 2) = 1 - P(Z<2) = 1 - P\left(\frac{Z}{\sqrt{17}} < \frac{2}{\sqrt{17}}\right) = 1 - \Phi\left(\frac{2}{\sqrt{17}}\right) \approx 0.31$

(c) $P((X-Y)^2 > 9)$ $\qquad W = X-Y \sim N(0,2)$

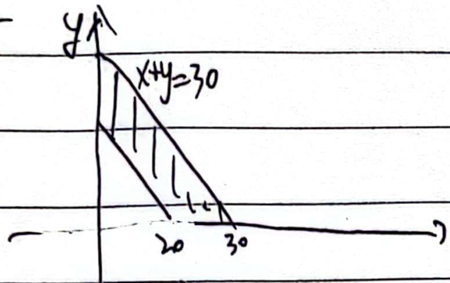
$= P(W>3) + P(W<-3)$

$= 1 - P(W \leq 3) + P(W<-3) = 1 - \Phi\left(\frac{3}{\sqrt{2}}\right) + \left(1 - \Phi\left(\frac{3}{\sqrt{2}}\right)\right)$

$= 2 - 2 \cdot \Phi(2.12) = 0.034$

3.45

(a)



$k = \frac{3}{81250}$

$\Uparrow$

$\iint f(x,y)\,dx\,dy = 1$

$\int_0^{20}\int_{20-x}^{30-x} kxy\,dy\,dx + \int_{20}^{30}\int_0^{30-x} kxy\,dx\,dy = 1$

$\leftarrow$ when $0 \leq y \leq 30-x$

(b) $f_X(x) = \cancel{\int_{20}^{30} kxy}\ \int_{-\infty}^{+\infty} f(x,y)\,dy =$

$$= \begin{cases} \int_{20-x}^{30-x} kxy\,dy = k(250x - 10x^2) & 0 \leq x \leq 20 \\ \int_0^{30-y} kxy\,dy = k\left(450x - 30x^2 + \frac{1}{2}x^3\right) & 20 \leq x \leq 30 \end{cases}$$

$\because f_X(25) > 0,\ f_Y(25) > 0,\quad f(25,25) = 0$

$\therefore f_X(x) \cdot f_Y(y) \neq f(x,y)$ $\Rightarrow$ X, Y are not independent.

(c) $P(X+Y \leq 25) = \int_0^{20}\int_{20-x}^{25-x} kxy\,dy\,dx + \int_{20}^{25}\int_0^{25-x} kxy\,dy\,dx$

$= 355$

(d) $E(X+Y) = E(X) + E(Y)$

$= 2\left(\int_0^{20} x \cdot k(250x - 10x^2)\,dx + \int_{20}^{30} x \cdot k\left(450x - 30x^2 + \frac{1}{2}x^3\right)dx\right)$

$= 25.969$

(e) $E(XY) = \int_0^{20}\int_{20-x}^{30-x} kx^2y^2\,dy\,dx + \int_{20}^{30}\int_0^{30-x} kx^2y^2\,dy\,dx$

$= \frac{k}{3} \cdot \frac{33250000}{3}$

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = -32.19$

$E(X^2) = E(Y^2) = 204.61$

$\sigma_X^2 = \sigma_Y^2 = 204 - (12.98)^2 = 36.01$ $\quad \rho = -0.894$

(f) $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = 7.66$