Chapter 6 Counting Methods and Pigeonhole Principle
计数方法与鸽巢原理

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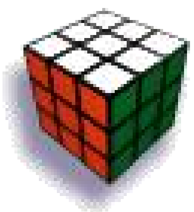
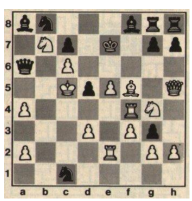
Counting in Games



- How many different configurations for a Rubik's cube?



- How many different chess positions after n moves?

- How many weighings to find the one counterfeit among 12 coins?

6.1 Basic Principles 基本原理

The menu for Kay’s Quick Lunch is shown as follows. It features two appetizers, three main courses, and four beverages.

APPETIZERS

Nachos	2.15
Salad	1.90

MAIN COURSES

Hamburger	3.25
Cheeseburger	3.65
Fish Filet	3.15

BEVERAGES

Tea	.70
Milk	.85
Cola	.75
Root Beer	.75

How many different dinners consist of one main course and one beverage?

How many different dinners consist of one appetizer, one main course and one beverage?

6.1 Basic Principles 基本原理

Multiplication Principle 乘法原理

If an activity can be constructed in t successive steps and step 1 can be done in n_1 ways, step 2 can then be done in n_2 ways, ..., and step t can be done in n_t ways, then the number of different possible activities is $n_1 n_2 \ldots n_t$.

We multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

6.1 Basic Principles 基本原理

Example 6.1.3

(a) How many strings of length 4 can be formed using the letters ABCDE if repetitions are not allowed?

(b) How many strings of part (a) begin with the letter B?

(c) How many strings of part (a) do not begin with the letter B?

6.1 Basic Principles 基本原理

Example 6.1.3

(a) How many strings of length 4 can be formed using the letters ABCDE if repetitions are not allowed? 120

(b) How many strings of part (a) begin with the letter B? 24

(c) How many strings of part (a) do not begin with the letter B? 96

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(1) How many elements in the power set of X ?

(2) How many ordered pairs (A, B) satisfy $A \subseteq B \subseteq X$?

(3) How many relations are there from X to Y ?

(4) How many functions are there from X to Y ?

(5) How many one-to-one functions are there from X to Y ?

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(1) How many elements in the power set of X ?

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set (幂集)** of X.

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(2) How many ordered pairs (A, B) satisfy $A \subseteq B \subseteq X$?

- An **ordered pair (有序对)** of elements, written (a, b).
- If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the **Cartesian product (笛卡尔积)** of X and Y.

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(3) How many relations are there from X to Y ?

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(4) How many functions are there from X to Y ?

Definition 3.1.1 Let X and Y be sets. A function f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(5) How many one-to-one functions are there from X to Y ?

Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

6.1 Basic Principles 基本原理

Exercise Let X be an n-element set and Y be an m-element set.

(1) How many elements in the power set of X ?

$\mathbf{2^n}$

(2) How many ordered pairs (A, B) satisfy $A \subseteq B \subseteq X$?

$\mathbf{3^n}$

(3) How many relations are there from X to Y ?

$\mathbf{2^{nm}}$

(4) How many functions are there from X to Y ?

$\boldsymbol{m^n}$

(5) How many one-to-one functions are there from X to Y ?

$\boldsymbol{m(m-1)(m-2)\ldots(m-n+1)}$

6.1 Basic Principles 基本原理

Example 6.1.9 How many eight-bit strings begin either 101 or 111?

6.1 Basic Principles 基本原理

Exercise 1: Counting Passwords

How many passwords satisfy the following requirements?

- between 6 & 8 characters long
- starts with a letter
- case sensitive
- other characters: digits or letters

6.1 Basic Principles 基本原理

Exercise 1: Counting Passwords

How many passwords satisfy the following requirements?

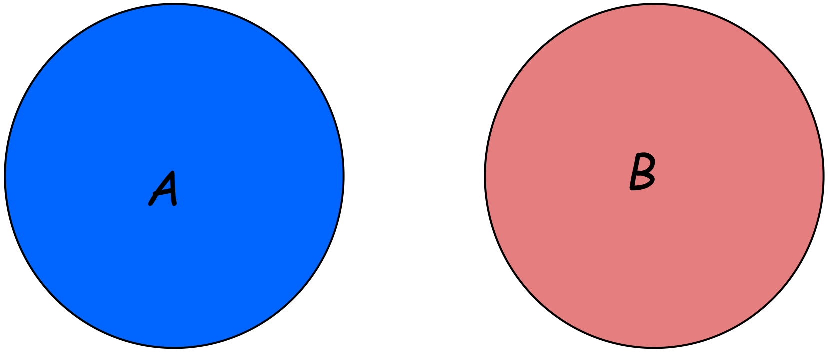
- between 6 & 8 characters long
- starts with a letter
- case sensitive
- other characters: digits or letters

L ::= {a, b, …, z, A, B, …, Z}

D ::= {0, 1, …, 9}

6.1 Basic Principles 基本原理

Addition Principle 加法原理

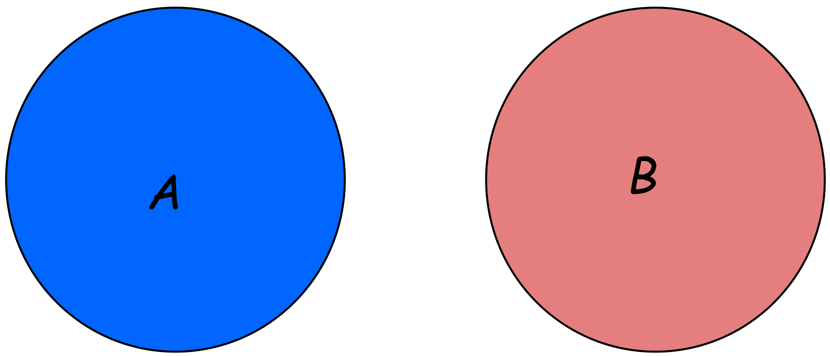


If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

6.1 Basic Principles 基本原理

Addition Principle 加法原理



If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

- Class has 43 women, 54 men, so total enrollment = 43 + 54 = 97
- 26 lower case letters, 26 upper case letters, and 10 digits, so total characters = 26+26+10 = 62

6.1 Basic Principles 基本原理

Addition Principle 加法原理

Suppose that $X_1, \ldots, X_t$ are sets and that the ith set X_i has n_i elements. If $\{X_1, \ldots, X_t\}$ is a pairwise disjoint family (i.e., if $i \neq j$, $X_i \cap X_j = \emptyset$), the number of possible elements that can be selected from X_1 or X_2 or ... or X_t is

$$n_1 + n_2 + \ldots + n_t.$$

(Equivalently, the union $X_1 \cup X_2 \cup \ldots \cup X_t$ contains $n_1 + n_2 + \ldots + n_t$ elements.)

We add the numbers of each subset when the elements being counted can be decomposed into pairwise disjoint subset.

APPETIZERS	
Nachos	2.15
Salad	1.90
MAIN COURSES	
Hamburger	3.25
Cheeseburger	3.65
Fish Filet	3.15
BEVERAGES	
Tea	.70
Milk	.85
Cola	.75
Root Beer	.75

6.1 Basic Principles 基本原理

Example 6.1.10 In how many ways can we select two books from different subjects among five distinct computer science books, three distinct mathematics books, and two distinct art books?

6.1 Basic Principles 基本原理

Exercise 2: At Least One Seven

How many # 4-digit numbers with at least one 7?

6.1 Basic Principles 基本原理

Exercise 2: At Least One Seven

How many # 4-digit numbers with at least one 7?

Method 1:

count by *1st occurrence* of 7:

7xxx + o7xx + oo7x + ooo7

$10^3 + 9\cdot10^2 + 9^2\cdot10 + 9^3 = 3439$

Method 2:

|4-digit numbers with at least one 7|

=|4-digit numbers| – |those with no 7s|

$= 10^4 - 9^4 = 3439$

6.1 Basic Principles 基本原理

Example 6.1.11 A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

(a) In how many ways can this be done?

(b) In how many ways can this be done if either Alice or Ben must be chairperson?

(c) In how many ways can this be done if Egbert must hold one of the offices?

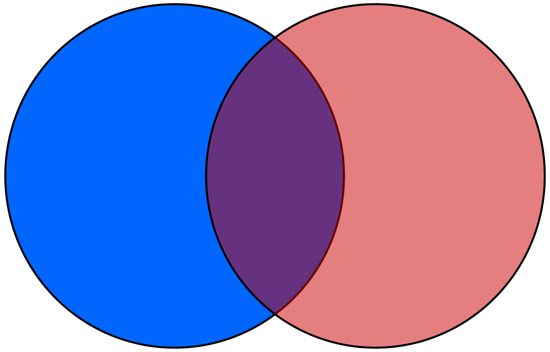
(d) In how many ways can this be done if both Dolph and Francisco must hold office?

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理

If X and Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$



6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理

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Proof

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理
If X and Y are finite sets, then

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Example 6.1.14 A committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which eitherAlice or Dolph or both are officers?

6.1 Basic Principles 基本原理

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets 容斥原理
If X and Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

Exercise Count the number of eight-bit strings that start 10 or end 011 or both.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.2 There are six permutation of three elements. If the elements are dnoted $A, B, \ldots, C$ the six permutations are ?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Theorem 6.2.3 There are $n!$ permutations of n elements.

Proof

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- For every combination of the first two elements, there are $\mathrm{n} - 2$ ways to choose the third element, and so forth.
- Thus, there are a total of $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$ permutations of an n-element set.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.5 How many permutations of the letters ABCDEF contains the substring DEF?

6.2 Permutations and Combinations 排列与组合

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Example 6.2.6 How many permutations of the letters ABCDEF contain the letters DEF togerther in any oder?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Example 6.2.7 In how many ways can six persons be seated around a circular table? If a seating is obtained from another seating by having everyone move n seats clockwise, the seatings are considered identical.

6.2 Permutations and Combinations 排列与组合

Definition 6.2.1 A **permutation (排列)** of n distinct elements x_1, ..., x_n is an ordering of the n elements x_1, ..., x_n.

Definition 6.2.8 An ***r*-permutation (*r*排列)** of n (distinct) elements x_1, ..., x_n is an ordering of an r-element subset of $\{x_1, \ldots, x_n\}$. The **number of *r*-permutations** of set of n distinct elements is denoted $\boldsymbol{P(n, r)}$.

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$$\boxed{P(n, n) = n!}$$

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.10 The number of r-permutations of a set of n distinct objects is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n!}{(n-r)!} \qquad r \leq n.$$

Proof

- There are n choices for the first element.
- For each of these, there are $n-1$ remaining choices for the second element.
- There are $n-r+1$ remaining choices for the last element.
- Thus, there are a total of $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ to choose r element.

6.2 Permutations and Combinations 排列与组合

Example 6.2.13 In how many ways can seven distinct Martians and five distinct Jovians wait in line if no two Jovians stand together?

6.2 Permutations and Combinations 排列与组合

Definition 6.2.14 Given a set $X = \{x_1, \ldots, x_n\}$ containing n (distinct) elements,
(a) An ***r*-combination (*r*组合)** of X is an unordered selection of r-elements of X (i.e., an r-element subset of X).
(b) The number of r-combinations of a set of n distinct elements is denoted $C(n, r)$ or $\binom{n}{r}$.

6.2 Permutations and Combinations 排列与组合

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Example 6.2.9 Examples of 2-permutations of a, b, c are ab, ba, and ca.

6.2 Permutations and Combinations 排列与组合

Example 6.2.15 A group of five students, Mary, Boris, Rosa, Ahmad, and Nguyen, has decided to talk with the Mathematics Department chairperson about having the Mathematics Department offer more courses in discrete mathematics. The chairperson has said that she will speak with three of the students. In how many ways can these five students choose three of their group to talk with the chairperson?

6.2 Permutations and Combinations 排列与组合

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We can construct r-permutations of an n-element set X in two successive steps:

- Select an r-combination of X (an unordered subset of r items).
- Order the r-combination.

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.16 The number of r-combinations of a set of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \leq n.$$

Proof

- There are n choices for the first element.
- For each of these, there are $n-1$ remaining choices for the second element.
- There are $n-r+1$ remaining choices for the last element.
- Thus, there are a total of $n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$ to choose r element.

Any ordering of the first k elements give the same subset!

6.2 Permutations and Combinations 排列与组合

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$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \leq n.$$

Example 6.2.17 In how many ways can we select a committee of three from a group of 10 distinct persons?

6.2 Permutations and Combinations 排列与组合

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Example 6.2.18 In how many ways can we select a committee of two women and three men from a group of five distinct women and six distinct men?

6.2 Permutations and Combinations 排列与组合

Theorem 6.2.16 The number of r-combinations of a set of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \qquad r \le n.$$

Example 6.2.19 How many eight-bit strings contain exactly four 1's?

6.2 Permutations and Combinations 排列与组合

Example 6.2.23 What is wrong with the following argument, which purports to show that there are $C(8,5)2^3$ bit strings of length 8 containing at least five 0's?

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