HW5

EX 2.2.19 $\quad Y = X^n \quad X \geq 0.$

$F_Y(y) = P(Y \leq y) = P(X^n \leq y) = \cancel{P(X \leq \sqrt[n]{y})}\ P(0 \leq X \leq \sqrt[n]{y})$

$= F_X(\sqrt[n]{y})$

$\therefore f_Y(y) = (F_Y(y))' = f_X(\sqrt[n]{y}) \cdot (\sqrt[n]{y})' = f_X(\sqrt[n]{y}) \cdot \frac{1}{n} y^{\frac{1}{n}-1}$

Prop. 2.4.4 $\quad X$ is a random variable which has Possion distribution with λ. Then we have $E(X) = Var(X) = \lambda$.

1° $\quad p(x) = \frac{\lambda^x}{x!} e^{-\lambda}$

$\therefore E(X) = \sum_{x=0}^{n} x p(x) = \sum_{x=0}^{+\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \cdot \sum_{x=1}^{+\infty} \frac{e^{-\lambda} \cdot \lambda^{x-1}}{(x-1)!} = \lambda \sum_{x=0}^{+\infty} e^{-\lambda} \cdot \frac{\lambda^x}{x!} = \lambda$

2° $\quad Var(X) = E(X^2) - (E(X))^2$

$= \sum_{x=0}^{+\infty} x^2 \frac{e^{-\lambda} \cdot \lambda^x}{x!} - \lambda^2$

$= \sum x(x-1) \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \lambda \sum_{x=0}^{+\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} - \lambda^2$

$= \lambda^2 + \lambda - \lambda^2 = \lambda. \quad \square$

Eg 2.5.3 $\quad Y \sim U(0,1) \quad X = h(Y)$ ~~Let $h(y)$ be inverse of~~

$\because F_X(x) = P(X \leq x) = P(h(Y) \leq X) = \cancel{P(Y \leq h^{-1}(X)) = h^{-1}(X)}$

$= \cancel{\int_0^{h^{-1}(X)} f(y)\,dy}$

And $F_X(x) = P(Y \leq F_X(x))$

$= P(F_X^{-1}(Y) \leq x)$

we obtain $X = F_X^{-1}(Y) = h(Y)$

$\therefore h(x) = F_X^{-1}(x)$

Ex 2.5.4 (a)

possible values by Y can between b and $a+b$

$\therefore f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{a}$

when $y < b$ or $y > a+b$, $\quad f_Y(y) = 0$

$\therefore Y \sim U(b, a+b)$

Ex:

2.23

X	-1	0	1
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$E(X) = 0 \qquad Var(X) = E(X^2) - E(x)^2 = \frac{2}{3} - 0 = \frac{2}{3}$

$SD(X) = \sqrt{Var(X)} = \frac{\sqrt{6}}{3}$

2.24

(a) Let event A: n th tomato is k th defective tomato

$P(A) = C_{n-1}^{k-1}\, p^{k-1} (1-p)^{n-k} \cdot p = C_{n-1}^{k-1}\, p^{k} (1-p)^{n-k}$

(b) Let event B be the acquired event.

$P(B) = C_{n-1}^{k-1}\, p^k r^k (1-pr)^{n-k}$

(c) $P(X=k) = C_n^k\, p^k (1-p)^{n-k}\, pr \,/\, (1-pr)^n \cdot pr = C_n^k \cdot \frac{p^k(1-p)^{n-k}}{(1-pr)^n}$

$E(X) = \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=0}^{n} k\, C_n^k \cdot \frac{p^k(1-p)^{n-k}\, pr}{(1-pr)^n \cdot pr}$

$= \sum_{k=0}^{n} k\, C_n^k \cdot \left(\frac{p}{1-pr}\right)^k \left(\frac{1-p}{1-pr}\right)^{n-k}$

(c) $P(X=k) = C_n^k\, p^k (1-r)^k (1-p)^{n-k} \cdot pr \,/\, (1-pr)^n \cdot pr$

$= C_n^k \cdot \frac{p^k (1-r)^k (1-p)^{n-k}}{(1-pr)^n}$

$E(X) = \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=0}^{n} k\, C_n^k \cdot \left(\frac{p(1-r)}{1-pr}\right)^k \cdot \left(\frac{1-p}{1-pr}\right)^{n-k}$

$= \sum_{k=1}^{n} k\, \frac{n}{k}\, C_{n-1}^{k-1} \cdot \frac{p(1-r)}{1-pr} \cdot \left(\frac{p(1-r)}{1-pr}\right)^{k-1} \cdot \left(\frac{1-p}{1-pr}\right)^{n-k}$

$= n \cdot \frac{p(1-r)}{1-pr} \cdot 1^{n-1}$

$= \frac{np(1-r)}{1-pr}$

2.25

(a) $E(X) = \int_0^1 x(ax+bx^2)\,dx = \frac{a}{3}x^3 + \frac{b}{4}x^4 \Big|_0^1 = \frac{a}{3} + \frac{b}{4} = 0.6$

$\because \int_0^1 f(x)\,dx = \int_0^1 (ax+bx^2)\,dx = \frac{a}{2}x^2 + \frac{b}{3}x^3 \Big|_0^1 = \frac{a}{2} + \frac{b}{3} = 1$

$\Rightarrow a = \frac{18}{5} \quad b = -\frac{12}{5}$

$\therefore f(x) = \begin{cases} \frac{18}{5}x - \frac{12}{5}x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$\therefore\ P(X<\frac{1}{2}) = \int_0^{\frac{1}{2}} \left(\frac{18}{5}x - \frac{12}{5}x^2\right) dx = \frac{9}{5}x^2 - \frac{4}{5}x^3 \Big|_0^{\frac{1}{2}} = \frac{9}{20} - \frac{1}{10} = \frac{7}{20}$

(b) $Var(X) = E(X^2) - E^2(X)$

$= \int_0^1 x^2\left(\frac{18}{5}x - \frac{12}{5}x^2\right) dx - \frac{9}{25} = 0.42 - 0.36 = 0.06$

2.26 ~~Let $p(X=0)=p$~~

$\because P(X=0) + P(X=1) = 1 \Rightarrow X \sim B(1,p)$

$\therefore E(X) = p \quad Var(X) = p(1-p)$

$\therefore p = 3p(1-p) \Rightarrow p = \frac{2}{3}$ or $p=0$

$\therefore$

X	0	1
p(x)	$\frac{1}{3}$	$\frac{2}{3}$

$\therefore P(X=0) = \frac{1}{3}$ or $P(X=0) = 1$

~~2.27 $F(x) = 1 - e^{-x^2}$ $x \geq 0$~~

~~(a) $P(X>2) =$~~

2.27

(a) we first consider we have won for the first five games, Let random variable X be the # extra game we play.

$\therefore P(X=k) = (1-p)\cdot p^k \quad k=0,1,2,3,\cdots$

$\therefore E(X) = \sum_{n=0}^{\infty} n\cdot p^n(1-p) = \lim_{n\to\infty}\left[p(1-p) + 2p^2(1-p) + \cdots + np^n(1-p)\right]$

$= p(1-p)\lim_{n\to\infty}(1+2p+3p^2+\cdots) = p(1-p)\cdot\left(\frac{p}{1-p}\right)' = \frac{p}{1-p}$

$\therefore$ the expect game number that we play is $5 + \frac{p}{1-p}$

(b) Define a random variable Y is about the game which lose before we start the 5th game.

$P(Y=y) = C_4^y \cdot (1-p)^y \cdot p^{4-y} \quad y = 0,1,2,3,4$

$Y \sim B(4, 1-p) \quad E(Y) = 4(1-p)$

$\therefore$ the expected number of num we lose will be $4(1-p)+1$

2.39

Let X be # questions that correct

question ~~tor~~ to be correct probability is $p = \frac{1}{5}$

$$P(X \geq 8) = C_{10}^{8} p^8 (1-p)^2 + C_{10}^{9} p^9 (1-p) + C_{10}^{10} p^{10}$$

$$= \frac{761}{5^{10}}$$

2.40

(a) Let X = # bits corrupted, $p = 0.1$ $X \sim B(12, 0.1)$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$ $Y \sim P(1.2)$ $\lambda = np = 1.2$

$= 0.9^{12} + C_{12}^{1} 0.1 \times 0.9^{11} + C_{12}^{2} 0.1^2 \times 0.9^{10}$ ~~$P(Y) = e^{-\lambda 2}$~~

$\approx e^{-1.2} + \frac{1.2 \times e^{-1.2}}{1} + \frac{1.2^2 e^{-1.2}}{2}$ $P(Y=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$= 2.92 \times e^{-1.2}$

(b) Let Y = # packets contain 3 or more corrupted bits.

$P(Y \geq 1) = \sum_{k=1}^{6} P(Y=k)$ $Y \sim B(6, 1-2.92e^{-1.2})$

$P(Y=k) = C_6^k (1-2.92e^{-1.2})^k \cdot (2.92e^{-1.2})^{6-k}$

$\therefore P(Y \geq 1) = 1 - P(Y=0) = 1 - (2.92e^{-1.2})^6$

2.44

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$ $X \sim B(50, 0.05)$ $\lambda = np = 2.5$

$\approx e^{-2.5} + 2.5e^{-2.5} + \frac{6.25}{2} e^{-2.5}$ $Y \sim P(2.5)$

$= 6.625e^{-2.5}$ $P_Y(y) = \frac{e^{-2.5} 2.5^y}{y!}$

2.46

(a) Let X be the random variable of this question

$X \sim B(50, \frac{1}{100})$, call $Y \sim P(\frac{1}{2})$ be the approximately Possion

distribution $P_Y(x) = \frac{e^{-\frac{1}{2}} (\frac{1}{2})^x}{x!}$

$P(a) = 1 - P(X=0) = 1 - e^{-\frac{1}{2}}$

(b) $P(X=1) = \frac{1}{2} e^{-\frac{1}{2}}$

(c) $P(X \geq 2) = 1 - \frac{3}{2} e^{-\frac{1}{2}}$