1.3 Conditional Propositions and Logical Equivalence

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"If p then q"

In symbols: $p \to q$.

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In symbols: $p \rightarrow q$.

p : hypothesis (or antecedent) 假设（前件）

q : conclusion (or consequent) 结论（后件）

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In symbols: $p \to q$.

p	q	$p \to q$

?

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In symbols: $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F

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In symbols: $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional proposition that is true because the hypothesis is false is said to be **true by default** （默认为真） or **vacuously true**（空虚真）.

1.3 Conditional Propositions and Logical Equivalence

Your parents say: "If your got at least 85 in the this course, then I will buy you a gift."

When is the above sentence false?

- It is false when you get an 85 but your parents do not buy you a gift.
- In particular, it is not false if your score is below 85.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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- In particular, it is not false if your score is below 85.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\wedge$ ::= AND $\quad \vee$ ::= OR $\quad \neg$ ::= NOT

?

1.3 Conditional Propositions and Logical Equivalence

Some statements may be rephrased as conditional propositions.

Example 1.3.6

(a) Mary will be a good student if she studies hard.

(b) John takes calculus only if he has sophomore, junior, or senior standing.

(c) When you sing, my ears hurt.

(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.

(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

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(b) John takes calculus only if he has sophomore, junior, or senior standing.

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(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.

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(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

Logic Operators

$\wedge$ ::= AND $\quad \vee$ ::= OR $\quad \neg$ ::= NOT $\quad \rightarrow$ ::= IMPLIES

Operator Precedence 操作符的优先级

In the absence of parentheses,
we first evaluate $\neg$,
then $\wedge$,
then $\vee$,
and then $\rightarrow$.

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In the absence of parentheses,
we first evaluate $\neg$,
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Example: $p \vee q \rightarrow \neg r$

Logic Operators

$\wedge \ ::= \ \text{AND} \qquad \vee \ ::= \ \text{OR} \qquad \neg \ ::= \ \text{NOT} \qquad \rightarrow \ ::= \ \text{IMPLIES}$

Example 1.3.5

Assume that p is true, q is false, and r is true, which proposition is false?

(a) $\boldsymbol{p \wedge q \longrightarrow r}$

(b) $\boldsymbol{p \wedge (q \longrightarrow r)}$

(c) $\boldsymbol{p \rightarrow (q \longrightarrow r)}$

(d) $\boldsymbol{p \vee q \longrightarrow \neg r}$

Definition 1.3.10 Suppose that the propositions P and Q are made up of the propositions $p_1, p_2, p_3, \ldots, p_n$. We said that P and Q are **logically equivalent (逻辑等价)**, and write

$$P \equiv Q,$$

provided that given any truth value of $p_1, p_2, p_3, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

$$p \to q \equiv ?$$

- If you don't give me all your money, then you will be killed.
- Either you give me all your money or you will be killed (or both).

p	q		

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

$$\vdots$$

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$\boxed{(p \wedge \neg q) \vee (\neg p \wedge q)}$$

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$$

Idea 1: Look at the true rows

Exclusive-Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

$$\boxed{\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)}$$

Idea 2: Look at the false rows

$$p \to q \equiv ?$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 1:

Idea 2:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
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F	T	F	T
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T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 1:

$$(p \wedge q \wedge \neg r)$$
$$\vee (p \wedge \neg q \wedge r)$$
$$\vee (\neg p \wedge q \wedge r)$$
$$\vee (\neg p \wedge q \wedge \neg r)$$
$$\vee (\neg p \wedge \neg q \wedge r)$$

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 2:

Writing Logical Formula for a Truth Table

p	q	r	Output
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Idea 2:

$$\neg(p \wedge q \wedge r)$$

$$\wedge \neg(p \wedge \neg q \wedge \neg r)$$

$$\wedge \neg(\neg p \wedge \neg q \wedge \neg r)$$

Example 1.3.13

Which proposition is logically equivalent to the negation of $p \longrightarrow q$?

Example 1.3.13

Show that the negation of $p \longrightarrow q$ is logically equivalent to $p \wedge \neg q$.

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Statement: The number 6 is divisible by 2 or 5.

Negation: The number 6 is not divisible by 2 and not divisible by 5.

De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Truth Table

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

Are these two propositions logically equivalent?

Converse 逆

The converse of $p \longrightarrow q$ is $q \longrightarrow p$.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Are these two propositions logically equivalent?

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

Converse 逆

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words.

Solution: Let p: Jerry receives a scholarship, and q: Jerry goes to college. The given proposition can be written symbolically as $p \longrightarrow q$. The coverse of the proposition is

If Jerry goes to college, then he receives a scholarship.

The converse can be written as $q \longrightarrow p$.

Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

The original proposition is true and its converse is false.

Biconditional Proposition 双条件命题

Definition 1.3.8

If p and q are propositions, the proposition

$$p \text{ if and only if } q,$$

is called a biconditional proposition and is denoted

$$p \leftrightarrow q.$$

$\neg$

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T	F	F
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p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
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$$\leftrightarrow \ ::= \mathrm{IFF}$$

Contrapositive (Transposition) Proposition 逆否命题（转换命题）

Definition 1.3.16

The contrapositive (or transposition) of the conditional proposition $p \to q$ is the proposition $\neg q \longrightarrow \neg p$.

Contrapositive (Transposition) Proposition
逆否命题（转换命题）

Definition 1.3.16

The contrapositive (or transposition) of the conditional proposition $p \to q$ is the proposition $\neg q \longrightarrow \neg p$.

The conditional proposition $p \to q$ and its contrapositive $\neg q \longrightarrow \neg p$ are logically equivalent.

Proof by Truth Table

If p, then q.

If $\neg q$, then $\neg p$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
F	F	T
T	F	F
F	T	T
T	T	T

Propositional Equivalent

$\neg(\neg A) \equiv A$

$A \vee A \equiv A$

$A \wedge A \equiv A$

$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

$(A \vee B) \vee C \equiv A \vee (B \vee C)$

$A \vee B \equiv B \vee A$

$A \wedge B \equiv B \wedge A$

Propositional Equivalent

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

$\neg (A \vee B) \equiv \neg A \wedge \neg B$

$\neg (A \wedge B) \equiv \neg A \vee \neg B$

$A \wedge (A \vee B) \equiv A$

$A \vee (A \wedge B) \equiv A$

$A \wedge T \equiv$?

$A \vee F \equiv$?

Propositional Equivalent

$A \wedge F \equiv$?

$A \vee T \equiv$?

$A \vee (\neg A) \equiv$?

$A \wedge (\neg A) \equiv$?

$A \rightarrow B \equiv \neg A \vee B$

$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

$A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$? **Use Truth table to prove it?**

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$?

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \equiv \neg A$$

Propositional Equivalent

$$A \rightarrow B \equiv \neg B \rightarrow \neg A$$

$$A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$$

$$(A \rightarrow B) \land (A \rightarrow \neg B) \equiv \neg A$$?