Chapter 8 Other Stochastic Processes

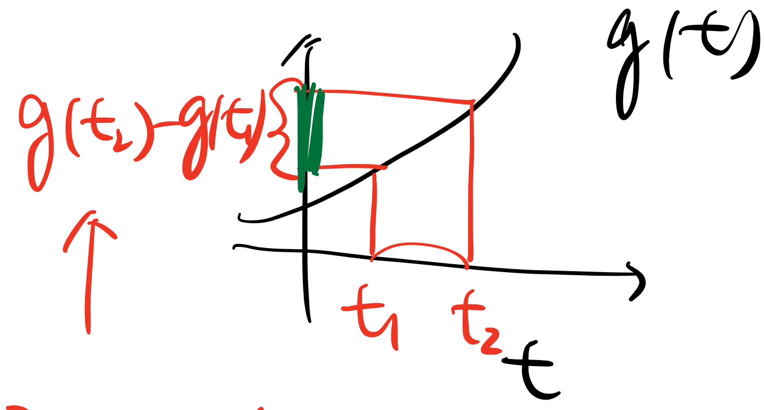
School of Sciences, BUPT

Other Stochastic Processes

Independent increment process is a special kind of Markov process. Poisson processes and Wiener processes are the two most important independent increment processes and they are the basis of Johnson noise theory.

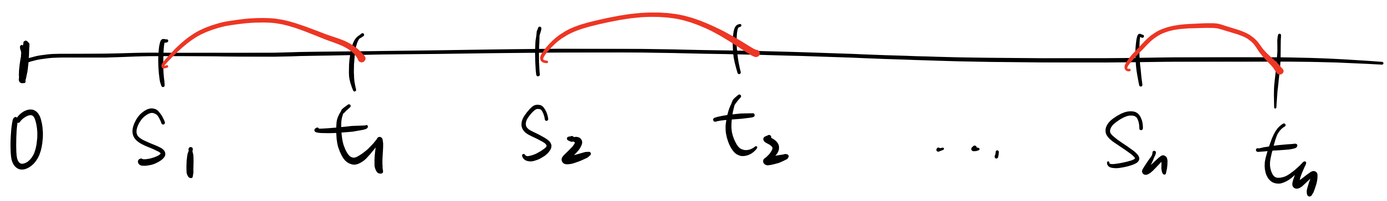
- Independent-Increment Processes
- Poisson Process $N(t)$
- Gaussian Processes
- Brownian Motion and Wiener Process $W(t)$

$$C_X(s,t) = \sigma^2_X(\min(s,t))$$



increment

X_t : $X_{t_2} - X_{t_1}$



$$0 \le s_1 < t_1 \le s_2 < t_2 \cdots \le s_n < t_n$$

$$\underbrace{X_{t_1} - X_{s_1}}, \underbrace{X_{t_2} - X_{s_2}}, \ldots, \underbrace{X_{t_n} - X_{s_n}}$$

independent

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Independent-Increment Processes

Definition

A stochastic process $\{X_t\}_{t\geqslant 0}$ is said to have independent increments if the set of the following random variables,

$$X_{s_1} - X_0,\ X_{t_1} - X_{s_1},\ X_{t_2} - X_{s_2},\ \cdots,\ X_{t_n} - X_{s_n}$$

are independent for all $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots \leqslant s_n < t_n < +\infty$, $n \geqslant 1$. Furthermore, if the distribution function of $X_{t+s} - X_s$ is independent of s, we say that $\{X_t\}_{t\geqslant 0}$ has stationary independent increments. The stochastic process $\{X_t\}_{t\geqslant 0}$ is called stationary independent-increment process or homogeneous independent-increment process.

Independent-Increment Processes

$X_t = X_t - X_0$ 是增量!

Theorem

Suppose that $X_0 = 0$. *For the independent-increment process* $\{X_t\}_{t \geqslant 0}$, *its finite-dimensional distribution function can be determined by the distribution function of* $X_t - X_s$, $0 \leqslant s < t$.

Theorem

Suppose that $\{X_t\}_{t \geqslant 0}$ *is an independent-increment process and* $X_0 = 0$. *Then the covariance function* $C_X(s,t) = Var_X[\min(s,t)]$.

finite-dim. distr.: $t_1, \dots, t_n$

$(X_{t_1} \dots X_{t_n})$

$\Uparrow$ $X_0 = 0$

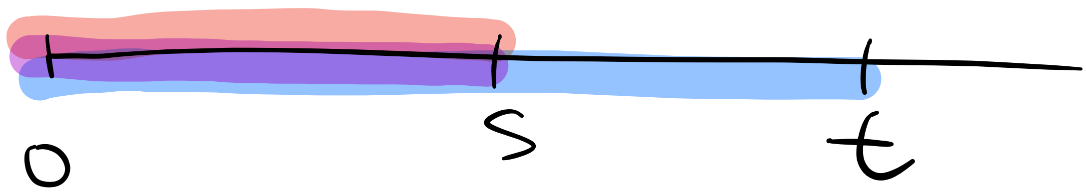
$$\{ X_t - X_s, \quad s, t \in T \}$$

$$\underbrace{\begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_n} \end{pmatrix}}_{n\times 1} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 \cdots & 0 \\ & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix}}_{n\times n} \underbrace{\begin{pmatrix} X_{t_1} - \cancel{X_0} \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_n} - X_{t_{n-1}} \end{pmatrix}}_{\substack{n\times 1 \\ \text{known}}}$$

$$X_{t_n} = X_{t_n} - X_{t_{n-1}} + X_{t_{n-1}} - X_{t_{n-2}}$$
$$+ \dots + -X_{t_1} + X_{t_1} - X_0 \quad \square$$

$$C_X(s,t) = \mathrm{Cov}(X_s, X_t)$$

If $s < t$:



$$\mathrm{Cov}(X_s - X_0,\ X_t - X_s + X_s - X_0)$$

$$= \mathrm{Cov}(X_s - X_0,\ X_t - X_s)$$

$$+ \mathrm{Cov}(X_s - X_0,\ X_s - X_0)$$

$$= 0 + \mathrm{Var}(X_s)$$

$$= \mathrm{Var}(\min(s,t))$$

□

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Poisson Processes

A Poisson process is a stochastic process which counts the number of events, which occur in a given time interval. The process is named after the French mathematician Siméon Denis Poisson. Examples to keep in mind are the moments at which telephone calls are received in a call centre, the times of customer/job arrivals at queues, or the moments at which Peking is hit by an earthquake. It is one of the most important models used among many phenomena.

Poisson Processes

Definition

The counting process $\{N_t\}_{t\geqslant 0}$ *is said to be a Poisson process having rate* λ, $\lambda > 0$, *if*
(i) $N_0 = 0$,
(ii) *the process has independent increments,*
(iii) *the number of events in any interval of length* t *is Poisson distributed with mean* λt. *That is, for all* $s, t \geqslant 0$,

$$P\{N_{t+s} - N_s = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \ n = 0, 1, \cdots$$

Poisson Processes

Note that it follows from condition (iii) that a Poisson process has stationary increments.

Proposition

Let N_t be a Poisson process with rate λ. Then
(i) $\mu_N(t) = \lambda t$,
(ii) $C_N(s,t) = Var_N[\min(s,t)] = \lambda \min(s,t)$,
(iii) $R_N(s,t) = \lambda \min(s,t) + \lambda^2 st$,
where

$$\min(s,t) = \frac{s+t-|s-t|}{2}.$$

Poisson Processes

Example

Let N_t be a Poisson process with parameter λ. Find the values of $E(N_t^2)$ and $E[(N_t - N_s)^2]$, for $t > s$.

Example

Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile.
(a) What is the probability that no defects appear in the first two miles of cable?
(b) Given that there are no defects in the first two miles of cable, what is the conditional probability of no defects between mile points two and three?

Poisson Processes

Example

Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour. Given that the store opens at 9:00 A.M., what is the probability that exactly one customer has arrived by 9:30 and a total of five have arrived by 11:30 A.M.?

Example

Patients arrive at the doctor's office according to a Poisson process with rate $\lambda = 1/10$ minute. The doctor will not see a patient until at least three patients are in the waiting room. What is the probability that nobody is admitted to see the doctor in the first hour?

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Gaussian Processes

The fundamental characterization of a Gaussian process is that all the finite dimensional distributions have a multivariate normal (or Gaussian) distribution. It is completely determined by its mean and covariance functions. This property facilitates model fitting as only the first- and second-order moments of the process require specification. Gaussian processes can be used as building blocks to construct more complex models that are appropriate for non-Gaussian data.

Gaussian Processes

Definition

A real-valued stochastic process $\{X_t\}_{t \in T}$ *is said to be a Gaussian process if all the finite-dimensional distributions have a multivariate normal distribution. That is, for any choice of distinct values* $t_1, \cdots, t_k \in T$*, the random vector* $\boldsymbol{X} = (X_{t_1}, \cdots, X_{t_k})'$ *has a multivariate normal distribution with mean vector* $\mu = E(\boldsymbol{X})$ *and covariance matrix* $\Sigma = Cov(\boldsymbol{X}, \boldsymbol{X})$*, which will be denoted by*

$$\boldsymbol{X} \sim N(\mu, \Sigma).$$

Gaussian Processes

Provided the covariance matrix Σ is nonsingular, the random vector $\boldsymbol{X}$ has a Gaussian probability density function given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-n/2}(det\Sigma)^{-1/2} \times \exp\left(-\frac{1}{2}(\boldsymbol{x}-\mu)'\Sigma^{-1}(\boldsymbol{x}-\mu)\right).$$

The mean and covariance functions of a Gaussian process are defined by

$$\mu_X(t) = E(X_t) \quad \text{and} \quad C_X(s,t) = Cov(X_s, X_t)$$

respectively.

Gaussian Processes

While Gaussian processes depend only on these two quantities, modeling can be difficult without introducing further simplifications on the form of the mean and covariance functions. The assumption of stationarity frequently provides the proper level of simplification without sacrificing much generalization. Moreover, after applying elementary transformations to the data, the assumption of stationarity of the transformed data is often quite plausible.

Gaussian Processes

A Gaussian time series $\{X_t\}$ is said to be stationary if

1. $\mu_X(t) = E(X_t) = \mu_X$ is independent of t, and

2. $C_X(t, t+h) = Cov(X_t, X_{t+h}) = C_X(h)$ is independent of t for all h.

Gaussian Processes

Proposition

For a wide-stationary Gaussian process $\{X_t\}$, *we have*
(i) $X_t \sim N(\mu_X, C_X(0))$ *for all* t, *and*
(ii) $(X_t, X_{t+h})'$ *has a bivariate normal distribution with covariance matrix*

$$\begin{pmatrix} C_X(0) & C_X(h) \\ C_X(h) & C_X(0) \end{pmatrix}$$

for all t *and* h.

Proposition

(i) *Suppose that* $\{X_t\}_{t \in T}$ *is a Gaussian process. Then it is strict-sense stationary if and only if it is wide-sense stationary.*
(ii) $\{X_t\}_{t \in T}$ *is a Gaussian process if and only if any linear combination of any finite number of random variables has a normal distribution, i.e., for any* $t_1, t_2, \cdots, t_n \in T$ *and* $c_1, c_2, \cdots, c_n \in \mathbb{R}$, *the random variable* $c_1 X_{t_1} + c_2 X_{t_2} + \cdots + c_n X_{t_n}$ *has a (univariate) normal distribution.*
*(iii) *Suppose that* $\{X_t\}_{t \in T}$ *is a Gaussian process and* S_t *is a deterministic function. Then for all* t, $X_t + S_t$ *has a normal distribution.*

Gaussian Processes

Proposition

*(iv) *Suppose that a Gaussian process $\{X_t\}_{t\in T}$ is mean-square differentiable. Then its derivative $\{X_t'\}_{t\in T}$ is also a Gaussian process.*

*(v) *Suppose that a Gaussian process $\{X_t\}_{t\in T}$ is mean-square integrable, then for all a, b and $t \in T$,*

$$Y_t = \int_a^t X_s ds, \quad \textit{and} \quad Z_t = \int_a^b X_s h(s,t) ds$$

are both Gaussian processes.

Gaussian Processes

Example

Suppose that $\{X_t\}_{t \in T}$ is a Gaussian process with the mean function $E(X_t) = 0$ and autocorrelation function $R_X(\tau) = \frac{1}{4}e^{-2|\tau|}$. For all t, find $P\{0.5 \leqslant X_t \leqslant 1\}$.

Example

Suppose that the stochastic process $X_t = A + Bt + Ct^2, t \in (-\infty, +\infty)$, where the random variables A, B and C are independent, and each has a normal distribution with mean 0 and variance σ^2. Show that X_t is a Gaussian process.

Gaussian Processes

Example

Suppose that the process $\{X_t\}_{t\in T}$ is defined by

$$X_t = U\cos\omega_0 t + V\sin\omega_0 t, \quad t \geqslant 0,$$

where ω_0 is a constant, U and V are two independent r.v.'s with

$$E(U) = E(V) = 0, \quad E(U^2) = E(V^2) = \sigma^2.$$

Show that $\{X_t\}_{t\in T}$ is a Gaussian process and determine the first-order and second-order probability density functions.

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Brownian Motion and Wiener Processes

Definition

A real-valued process $\{W_t\}_{t \geqslant 0}$ *is said to be a Wiener process (also called Brownian motion) with parameter* σ^2*, if*
(i) $W_0 = 0$,
(ii) *the increments*

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_m} - W_{t_{m-1}}$$

are independent for all $0 = t_0 < t_1 < \cdots < t_m$,
(iii) $W_t - W_s$ *has a normal distribution with mean 0 and variance* $\sigma^2(t-s)$ *for all* s, t, $0 \leqslant s < t$.

Brownian Motion and Wiener Processes

From (ii) and (iii), we know the process $\{W_t\}_{t \geqslant 0}$ has stationary, independent increments. A Wiener process with initial value $W_0 = x$ is gotten by adding x to a standard Wiener process. From (i) and (iii) we know

$$W_t \sim N(0, \sigma^2 t).$$

Proposition

Suppose that W_t is a Wiener process. Then its mean and auto-correlation function are given by

$$\mu_W(t) = 0, \quad C_W(s,t) = R_W(s,t) = \sigma^2 \min(s,t), \quad s,t \geqslant 0.$$

Brownian Motion and Wiener Processes

By using the property (iii) in Definition, we have

Proposition

The Wiener process W_t is a Gaussian process.

Proof. For all $0 \leqslant t_1 < t_2 < \cdots < t_n$, $a_1, a_2 \ldots, a_n \in \mathbb{R}$, and for all integers $n > 0$, we have the linear combination

$$\sum_{k=1}^{n} a_k W_{t_k} = \sum_{k=1}^{n} b_k (W_{t_k} - W_{t_{k-1}}), \quad t_0 = 0,$$

where $b_k = \sum_{i=k}^{n} a_i, k = 1, 2, \ldots, n$. Since W_t has independent increments, we know $\sum_{k=1}^{n} a_k W_{t_k}$ has the normal distribution. Thus we obtain that W_t is a Gaussian process. $\square$

Example

Suppose that $\{W_t\}_{t \geqslant 0}$ is a Wiener process. Find the autocorrelation functions of the stochastic processes $Y_t = W_{t+l} - W_t,\ t \geqslant 0$, ($l > 0$ is a constant).

Proposition

Suppose that W_t is a Wiener process. Then the following processes are all Gaussian processes.

(i) $U_t = -W_t$ *for* $t \geqslant 0$.

(ii) $Y_t = \begin{cases} tW_{1/t} & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$

(iii) $X_t = \dfrac{1}{c} W_{c^2 t}$ for $c > 0$.

Summary

- Suppose that $\{X_t\}_{t\geqslant 0}$ is an independent-increment process and $X_0 = 0$. Then the covariance function

$$C_X(s,t) = Var_X[\min(s,t)].$$

- Let N_t be a Poisson process with rate λ. Then
  (i) $\mu_N(t) = \lambda t$,
  (ii) $C_N(s,t) = Var_N[\min(s,t)] = \lambda \min(s,t)$,
  (iii) $R_N(s,t) = \lambda \min(s,t) + \lambda^2 st$,
  where

$$\min(s,t) = \frac{s+t-|s-t|}{2}.$$

Summary

- A Gaussian process is completely determined by its mean and covariance functions.
- $\{X_t\}_{t \in T}$ is a Gaussian process if and only if any linear combination of any finite number of random variables has a normal distribution, i.e., for distinct $t_1, t_2, \cdots, t_n \in T$ and $c_1, c_2, \cdots, c_n \in \mathbb{R}$, the random variable $c_1 X_{t_1} + c_2 X_{t_2} + \cdots + c_n X_{t_n}$ has a (univariate) normal distribution.
- Suppose that W_t is a Gaussian process. Then its mean and autocorrelation function are given by

$$\mu_W(t) = 0, \quad C_W(s,t) = R_W(s,t) = \sigma^2 \min(s,t), \quad s, t \geqslant 0.$$

The end

Thank you !