HW 15

Thm 7.3.1 $\{X_n\}$ homogeneous Markov chain, Then

$$P_{ij}(n+m) = \sum_k P_{ik}(n) P_{kj}(m) \quad \text{for all } m,n \geq 0 \quad \text{all } i,j \quad (\text{C-K equ.})$$

Proof: $P_{ij}(n+m) = P(X_{n+m}=j \mid X_0 = i)$

$= \sum_{k\in S} P(X_{n+m}=j, X_n = k \mid X_0 = i)$

$= \sum_{k\in S} P(X_{n}=k \mid X_0 = i) \cdot P(X_{n+m}=j \mid X_n = k, X_0 = i)$

0

$= \sum_{k\in S} P_{ik}(n) \cdot P_{kj}(m) \quad i,j \in S$

$\Rightarrow P(n) = P^n = P(0)\cdot P^n$

for k th dimentional $0 \leq n_1 < n_2 < \cdots < n_k$ for all $i_1, \cdots, i_k \in S$

$P(X_{n_1}=i_1, X_{n_2}=i_2, \cdots, X_{n_k}=i_k)$

$= P(X_{n_1}=i_1)\cdot P(X_{n_2}=i_2 \mid X_{n_1}=i_1) \cdots P(X_{n_k}=i_k \mid X_{n_{k-1}}=i_{k-1})$

$= P_{i_1}(n_1) \cdot P_{i_1 i_2}(n_2-n_1) \cdots P_{i_{k-1} i_k}(n_k - n_{k-1})$

Eg. 7.3.4

Sol: To find the prob. we first define an appropriate Markov chain

Lets define X_n to be the number of red balls in the urn after the nth selection and subsequent replacement.

$\{X_n, n=0,1,2\cdots\}$ is a markov chain with states 0,1,2

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

$P(\text{fifth selection is red})$

$= \sum_{i=0}^{2} P(\text{fifth} = \text{red} \mid X_4 = i)\cdot P(X_4 = i \mid X_0 = 2)\cdot P(X_0 = 2)$

$= 0 \cdot P_{20}(4) + 0.5 \times P_{21}(4) + 1 \cdot P_{22}(4)$

$P(4) = P^4 = (\quad)$

$\therefore P_{21}(4) = 0.4352 \quad P_{22}(4) = 0.4872$

$\therefore P(\text{fifth} = \text{red}) = 0.7048$

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Eg 7.3.5 $P = \begin{pmatrix} 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $E(X_2)$

$p(0) = (p_0(0), p_1(0), p_2(0), p_3(0), p_4(0))$

$p(2) = (p_0(2), p_1(2) \cdots\cdots)$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0.24 & 0 & 0.16 & 0 \\ 0.36 & 0 & 0.48 & 0 & 0.16 \\ 0 & 0.36 & 0 & 0.24 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore p(2) = p(0)P^2$

$= (p_0(0) + 0.6p_1(0) + 0.36p_2(0),\ 0.24p_1(0) + 0.36p_3(0),\ 0.48p_2(0),$
$0.16p_1(0) + 0.24p_3(0),\ 0.16p_2(0) + 0.4p_3(0) + p_4(0))$

$E(X_2) = \sum_{i=0}^{4} i\,P(X_2 = i) = \sum_{i=0}^{4} i\,p_i(2)$

$= 0.72p_1(0) + 1.6p_2(0) + 1.22p_3(0) + 4p_4(0)$

Ex 7.5

(a) A gets B's penny is $0.6^2 + 0.4^2 = 0.52$

B gets A's $\cdots$ $0.6 \times 0.4 \times 2 = 0.48$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.48 & 0 & 0.52 & 0 & 0 \\ 0 & 0.48 & 0 & 0.52 & 0 \\ 0 & 0 & 0.48 & 0 & 0.52 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $$P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.48 & 0.2496 & 0 & 0.2704 & 0 \\ 0.2304 & 0 & 0.4992 & 0 & 0.2704 \\ 0 & 0.2304 & 0 & 0.2496 & 0.52 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$p_{24}(2) = 0.2704$

(c) $p_{24}(3) = p_{21} \times p_{14}(2) + p_{23} \times p_{34}(2) = 0.2704$

(d) $p_{24}(2) + p_{20}(2) = 0.2704 + 0.2304 = 0.5008$

(e) $4 - 1 \times (0 + 0 + 2 \times 0.4992 + 0 + 4 \times 0.2704) = 1.92$

(f) $p_2(n) = \begin{cases} p_2(n-2) \cdot p_{22}^2 & n = 2, 4, 6 \cdots \\ 1 & n = 0 \\ 0 & n = 1, 3, 5, \cdots \end{cases}$

for $n \equiv 2 \pmod 2$, $p_2(n) = 0.4992^{\frac{n}{2}}$

$\therefore \sum_{k=0}^{\infty} p_4(k) = \sum_{k=0}^{\infty} p_2(2k) \cdot p_{24}(2) = 0.2704 \cdot \sum_{k=0}^{\infty} 0.4992^k \approx 0.54$

(g)

The prob. that A broke up:

$\sum_{n=0}^{\infty} P_{20}(n) = \sum_{n=0}^{\infty} P_2(2n) \cdot P_{20}(2) = 0.46$

$\sum_{n=0}^{\infty} P_2(2n) \cdot P_{24}(2) \cdot 2(n+1) + \sum_{n=0}^{\infty} P_2(2n) P_{20}(2) 2(n+1)$

$= 2 + 1.0016 \sum_{n=0}^{\infty} n \cdot 0.4992^n = 3.9931$

7.9

(a) $P_{20} = 0$ (b) $= P(X_2=0 | X_1=2) = 0$

$P^2 = \begin{pmatrix} 0.17 & 0.53 & 0.3 \\ 0.22 & 0.72 & 0.06 \\ 0.06 & 0.45 & 0.49 \end{pmatrix}$

(c) $P_{20}(2) = 0.06$

(d) $\vec{P}(2) = \vec{P}(0) \cdot P^2 = (0.15 \quad \frac{1.7}{3} \quad \frac{0.85}{3})$

$\therefore P(X_2=0) = 0.157$

(e) $\vec{P}(1) = \vec{P}(0) \cdot P = (\frac{0.5}{3} \quad 0.5 \quad \frac{1}{3})$ $E(X_1) = 0.5 + \frac{2}{3} = \frac{7}{6}$

$E(X_1X_2) = \sum_{i=0}^{2}\sum_{j=0}^{2} ij\, P_i(1) P_{ij} = 0.8 \times 1 \times 1 \times 0.5 + 0.3 \times 2 \times 1 \times \frac{1}{3} + 0.7 \times 2 \times 2 \times \frac{1}{3} = \frac{4.6}{3}$

$E(X_2) = \frac{1.7}{3} + 2 \times \frac{0.85}{3} = \frac{3.4}{3}$

$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{1.9}{9}$

7.10

(a) $P_{12}(1) = 0.4$

$P^2 = \begin{pmatrix} 0.09 & 0.63 & 0.28 \\ 0.16 & 0.4 & 0.44 \\ 0.32 & 0.39 & 0.29 \end{pmatrix}$

(b) $P_{12}(2) = 0.44$

(c) $P_{12}(2) \cdot \cancel{P_{21}(1) \cdot P(X_0=2) = 0.44 \times 0.1 \times 0.6 = 0.0264} = 0.44$

(d) $P(X_2=1) = \cancel{P(0) \cdot P}\; P_0(0) \cdot P_{01}(2) + P_1(0) \cdot P_{11}(2) + P_2(0) \cdot P_{21}(2)$

$= 0.1 \times 0.63 + 0.3 \times 0.4 + 0.6 \times 0.39 = 0.417$

(e) $= P(X_1=2 | X_0=3) \cdot P(X_2=1 | X_1=2, X_0=3) = P_{32}(1) \cdot P_{21}(1) = 0$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & P = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.7 \\ 0.8 & 0 & 0.2 \end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
P^2 = \begin{pmatrix} 0.3 & 0.25 & 0.45 \\ 0.61 & 0.09 & 0.3 \\ 0.4 & 0.4 & 1.2 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

7.12

(a) $= P(X_0=1)\cdot P_{13}(2)\cdot P_{33}(1)$

$= 0.3\times 0.45\times 0.2 = 0.027$

(b) $= P(X_2=1)\cdot P_{12}(2)\cdot P_{23}(1)$

$= \left(p_1(0)\cdot P_{11}(2) + p_2(0)\cdot P_{21}(2) + p_3(0)\cdot P_{31}(2)\right) P_{12}(2)\cdot P_{23}(1)$

$= 0.475\times 0.25\times 0.7 = 0.083125$

Eg 8.2.3 $\quad P\{N_{t+s}-N_s=n\} = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

$P(N_{\frac{1}{2}}=1, N_{\frac{5}{2}}=5)$ Using Independence $N_{\frac{5}{2}}-N_{\frac{1}{2}}$ and $N_{\frac{1}{2}}$

$= P(N_{\frac{1}{2}}=1, N_{\frac{5}{2}}-N_{\frac{1}{2}}=4)$

$= \dfrac{e^{-4\cdot(\frac{1}{2})}\cdot 4\cdot(\frac{1}{2})}{1!}\cdot\dfrac{e^{-4\cdot 2}(4\cdot 2)^4}{4!} = 0.015$

Ex 8.6

8.3. $= P(N_1=0)\cdot P(N_2=2\mid N_1=0)\cdot P(N_3=3\mid N_2=2)$

$= e^{-\lambda}\cdot e^{-\lambda}\cdot\dfrac{\lambda^2}{2!}\cdot e^{-\lambda}\cdot\lambda = \dfrac{\lambda^3}{2}\cdot e^{-3\lambda}$

8.5

(a) $E(N_2) = 2\lambda = 4$

(b) $E(N_1^2) = Var(N_1) + E^2(N_1) = \lambda + \lambda^2 = 6$

(c) $E(N_1N_2) = C_N(1,2) + E(N_1)E(N_2) = \lambda(1\wedge 2) + 2\lambda^2 = 10$

8.19 $\quad C_W(4,5) = 4 = \sigma^2\cdot 4 \Rightarrow \sigma^2 = 1 \quad W_t \sim N(0,\sigma^2 t)$

$W_1 \sim N(0,1)$

$\mu_W(t) = 0,\ C_W(s,t) = R_W(s,t) = \sigma^2\min(s,t)$