EBU4375: SIGNALS AND SYSTEMS

LECTURE 8: PART 1

Queen Mary
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Response to an Impulse (DT Systems)

The impulse-response (IR) $h[n]$ of a discrete-time LTI system is defined to be the response following excitation by the signal $\delta[n]$ i.e.

$$h[n] = T\{\delta[n]\}$$

Response to a General Input (DT Systems)

From earlier we know

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

So for a **linear** system we can write

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$$

Time-invariance means

$$T\{\delta[n-k]\} = h[n-k]$$

$$\therefore\ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (5)$$

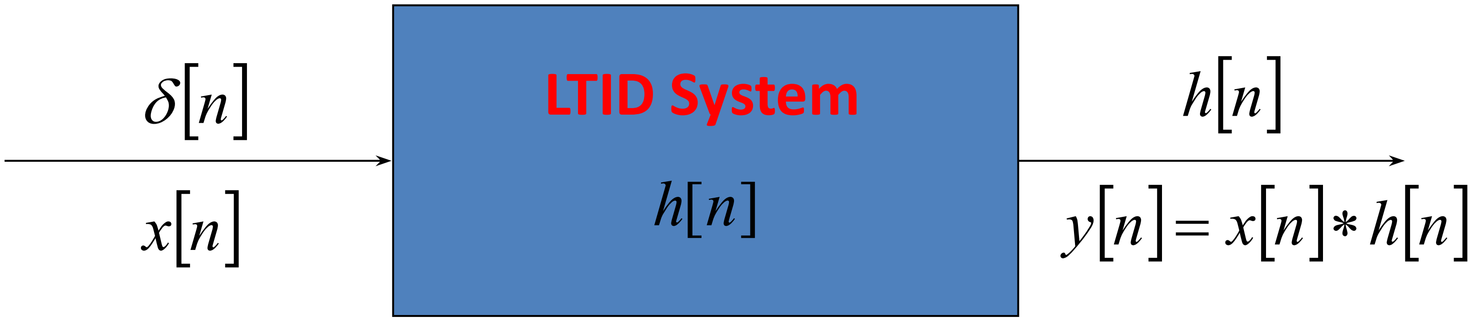
(5) Shows that a Discrete-time (DT) LTI system is completely characterised by its IR, $h[n]$.

Convolution Sum (DT Systems)

(5) Defines a convolution of two sequences, namely, $x[n]$ and $y[n]$; i.e.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (6)$$

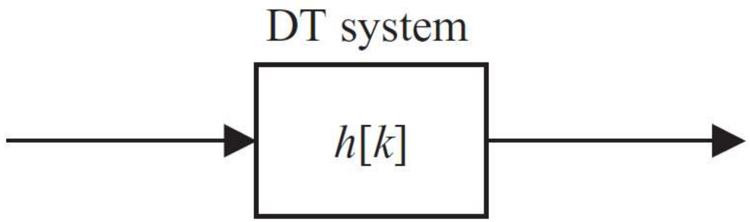
(6) is the convolution sum. In summary then



Convolution Sum (DT Systems)

$x[m]\delta[k-m]$

$x[m]h[k-m]$



$$x[k] = \sum_{m=-\infty}^{\infty} x[m]\delta[k-m]$$

$$y[k] = \sum_{m=-\infty}^{\infty} x[m]\,h[k-m] = x[m] * h[m]$$

Sequence Convolution Algebra (DT Systems)

Commutation: $$x[n]*h[n]=h[n]*x[n]$$

Association: $$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

Distribution: $$x[n]*\{h_1[n]+h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$$

Goto http://mathworld.wolfram.com/Convolution.html to get a dynamic appreciation of convolution.

Examples on LTID Systems

Definition *The impulse response $h[k]$ of an LTID system is the output of the system when a unit impulse $\delta[k]$ is applied at the input of the LTID system.*

$$\delta[k] \to h[k]$$

Note that an LTID system satisfies the linearity and the time-shifting properties. Therefore, if the input is a scaled and time-shifted impulse function $a\delta[k - k_0]$, the output of the DT system is also scaled by a factor of a and

$$a\delta[k - k_0] \to ah[k - k_0]$$

Example

Consider the LTID systems with the following input–output relationships:

$$y[k] = x[k - 1] + 2x[k - 3]$$

Calculate the impulse response. Also, determine the output response when the input is given by $x[k] = 2\delta[k] + 3\delta[k - 1]$.

Examples on LTID Systems

Solution

(i) The impulse response of a system is the output of the system when the input sequence $x[k] = \delta[k]$. Therefore, the impulse response $h[k]$ of system (i) can be obtained by substituting $y[k]$ by $h[k]$ and $x[k]$ by $\delta[k]$ in Eq. (10.8). In other words, the impulse response for system (i) is given by

$$h[k] = \delta[k-1] + 2\delta[k-3].$$

To evaluate the output response resulting from the input sequence $x[k] = 2\delta[k] + 3\delta[k-1]$, we use the linearity and time-invariance properties of the system. The outputs resulting from the two terms $2\delta[k]$ and $3\delta[k-1]$ in the input sequence are as follows:

$$2\delta[k] \rightarrow 2h[k] = 2\delta[k-1] + 4\delta[k-3]$$

and

$$3\delta[k-1] \rightarrow 3h[k-1] = 3\delta[k-2] + 6\delta[k-4].$$

Examples on LTID Systems

Applying the superposition principle, the output $y[k]$ to input $x[k] = 2\delta[k] + 3\delta[k-1]$ is given by

$$2\delta[k] + 3\delta[k-1] \rightarrow 2h[k] + 3h[k-1]$$

or

$$\begin{aligned} y[k] &= (2\delta[k-1] + 4\delta[k-3]) + (3\delta[k-2] + 6\delta[k-4]) \\ &= 2\delta[k-1] + 3\delta[k-2] + 4\delta[k-3] + 6\delta[k-4]). \end{aligned}$$

Examples on LTID Systems

Example

The impulse response of an LTID system is given by $h[k] = 0.5^k u[k]$. Determine the output of the system for the input sequence $x[k] = \delta[k-1] + 3\delta[k-2] + 2\delta[k-6]$.

Solution

Because the system is LTID, it satisfies the linearity and time-shifting properties. The individual responses to the three terms $\delta[k-1]$, $3\delta[k-2]$, and $2\delta[k-6]$ in the input sequence $x[k]$ are given by

$$\delta[k-1] \rightarrow h[k-1] = 0.5^{k-1} u[k-1],$$
$$3\delta[k-2] \rightarrow 3h[k-2] = 3 \times 0.5^{k-2} u[k-2],$$

and

$$2\delta[k-6] \rightarrow 2h[k-6] = 2 \times 0.5^{k-6} u[k-6].$$

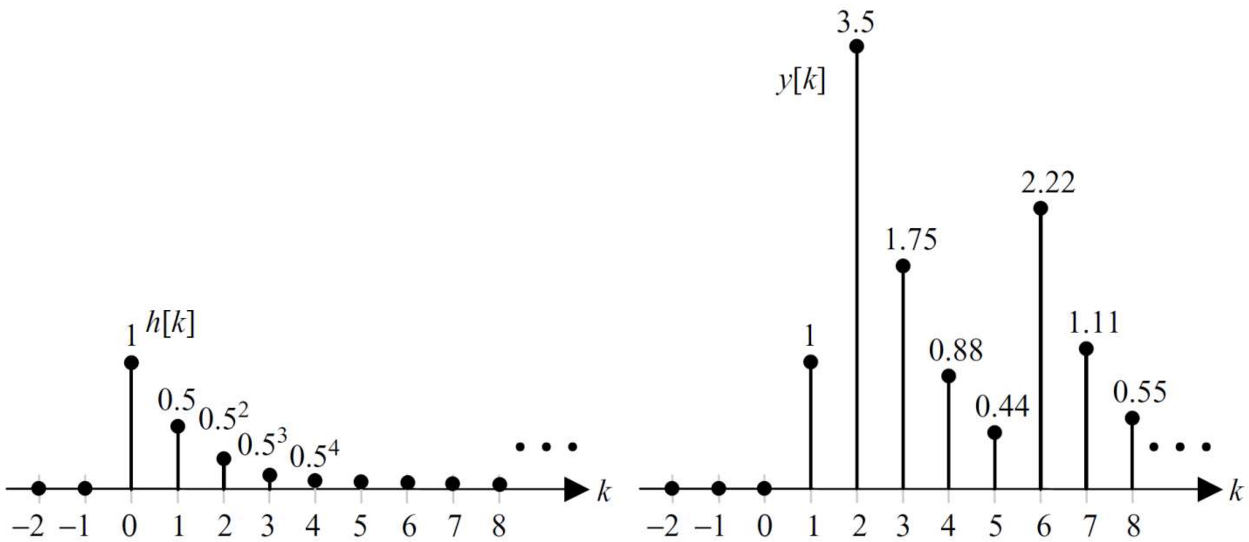
Examples on LTID Systems

Applying the principle of superposition, the overall response to the input sequence $x[k]$ is given by

$$y[k] = h[k-1] + 3h[k-2] + 2h[k-6].$$

Substituting the value of $h[k] = 0.5^k u[k]$ results in the output response:

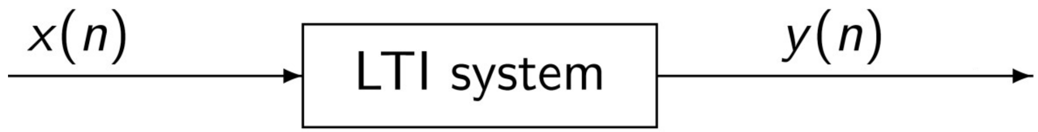
$$y[k] = 0.5^{k-1}u[k-1] + 3 \times 0.5^{k-2}u[k-2] + 2 \times 0.5^{k-6}u[k-6].$$

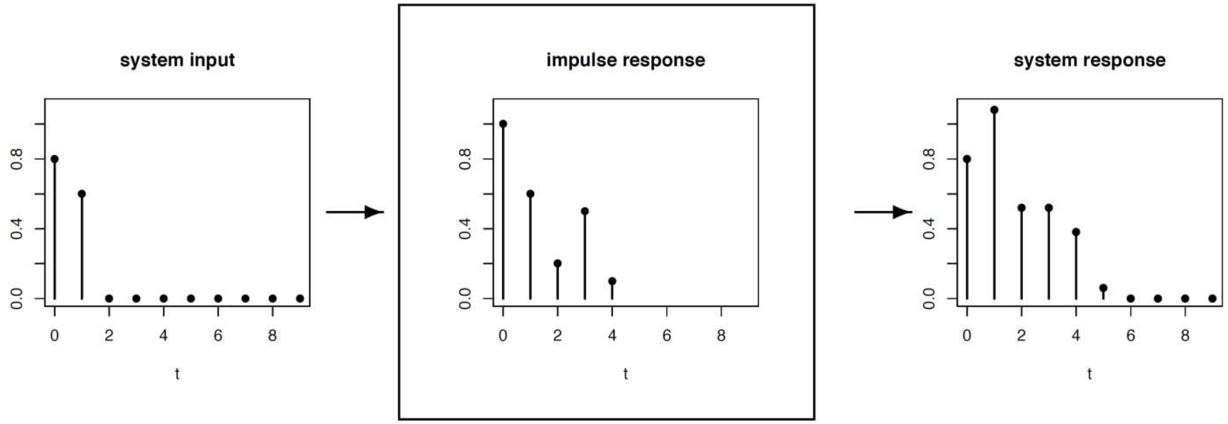


Convolution (DT Systems)

Convolution (DT Systems)

- we previously found that convolution with the impulse response gets us the system output:





- but *how* to calculate $y(n)$ from $x(n)$ and $h(n)$?
- that is: how do we calculate the convolution $y(n) = x(n) * h(n)$?

Convolution (DT Systems)

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

The sequences $h(k)$ and $x(k)$ are interchangeable.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = \ldots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + x(2)h(-2) +$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \ldots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1) +$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \ldots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) + \ldots$$

$$\ldots$$

Convolution (DT Systems) – Basic operations

Example

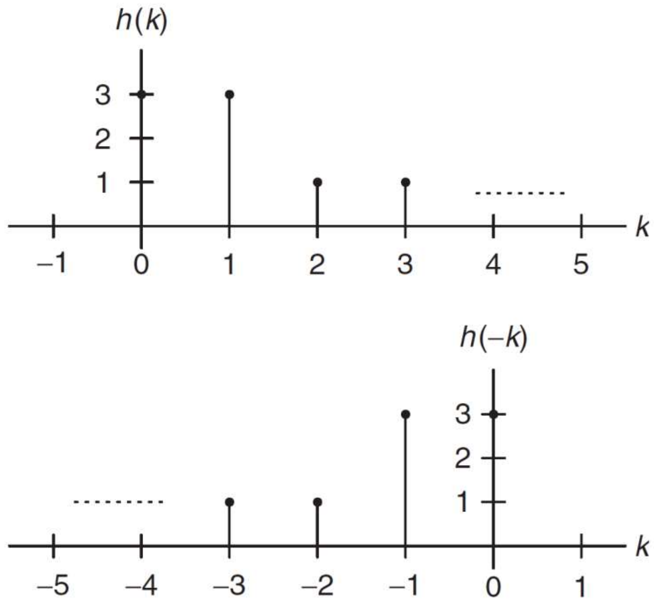
Given a sequence,

$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

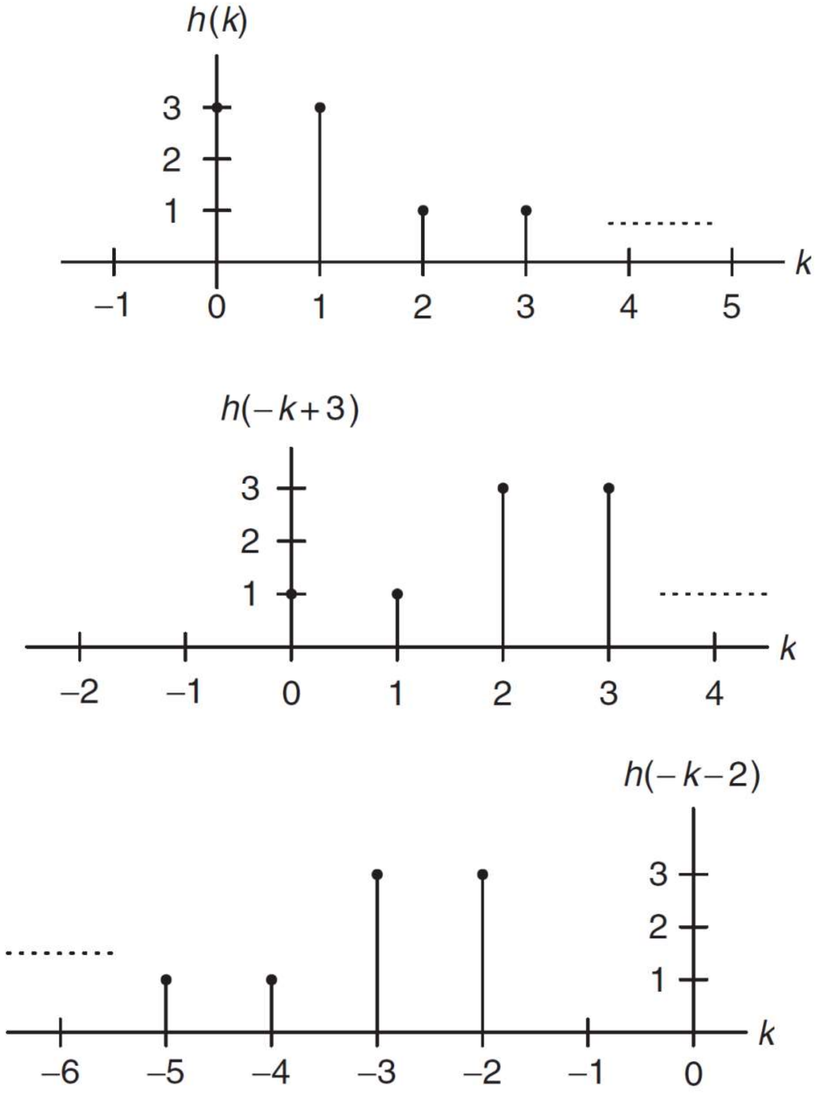
where k is the time index or sample number,

a. Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.

b. Sketch the shifted sequences $h(-k+3)$ and $h(-k-2)$.



Convolution (DT Systems) – Basic operations



Convolution (DT Systems) - Digital convolution using the graphical method

Digital convolution using the graphical method

Step 1. Obtain the reversed sequence $h(-k)$.

Step 2. Shift $h(-k)$ by $|n|$ samples to get $h(n-k)$. If $n \geq 0$, $h(-k)$ will be shifted to the right by n samples; but if $n < 0$, $h(-k)$ will be shifted to the left by $|n|$ samples.

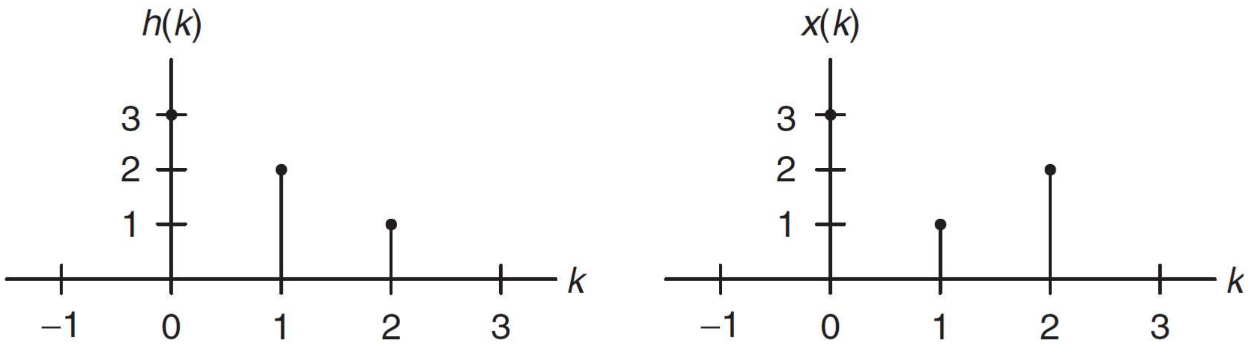
Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and $h(n-k)$ to get $y(n)$.

Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Convolution (DT Systems) - Digital convolution using the graphical method

Example

Using the following sequences



evaluate the digital convolution

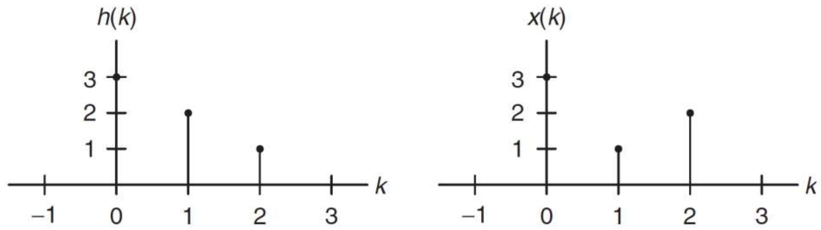
a. By the graphical method.

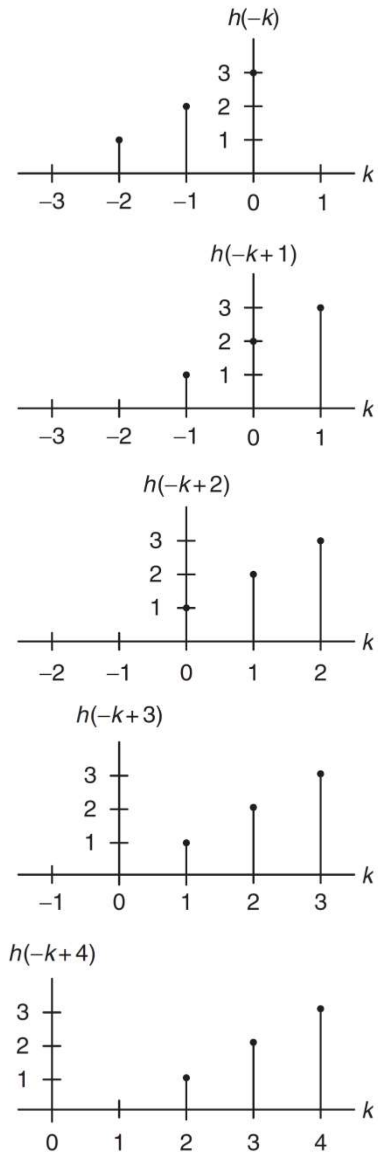
b. By applying the formula directly.

Convolution (DT Systems) - Digital convolution using the graphical method

Solution:

a. To obtain $y(0)$, we need the reversed sequence $h(-k)$; and to obtain $y(1)$, we need the reversed sequence $h(1-k)$, and so on.





sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.

Convolution (DT Systems) - Digital convolution using the graphical method

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
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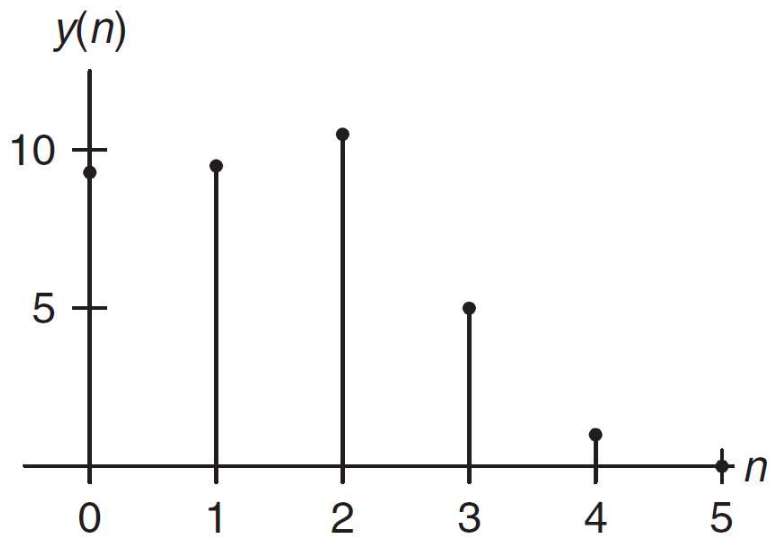


Convolution (DT Systems) - Digital convolution using the graphical method

Applying

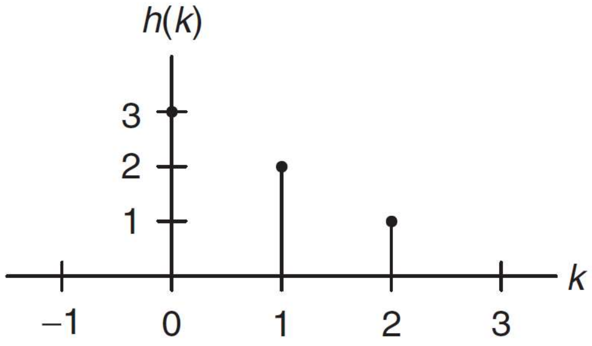
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

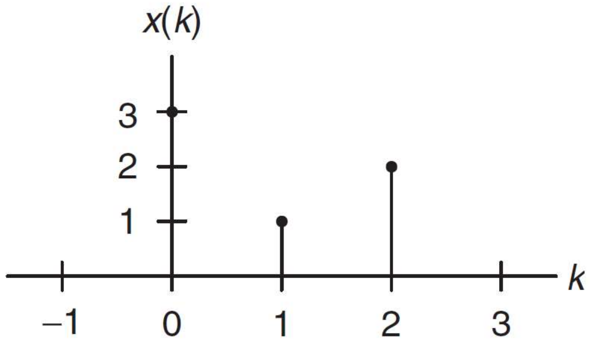
we get

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0$$

Convolution (DT Systems) - Digital convolution using the table method





Convolution sum using the table method

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution (DT Systems) - Digital convolution using the table method

Example

Given the following two rectangular sequences,

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

a. Convolve them using the table method.

k:	-2	-1	0	1	2	3	4	5	...	
$x(k)$:			**1**	**1**	**1**				...	
$h(-k)$:	1	1	0							$y(0) = 0$ (no overlap)
$h(1-k)$		1	1	0						$y(1) = 1 \times 1 = 1$
$h(2-k)$			1	1	0					$y(2) = 1 \times 1 + 1 \times 1 = 2$
$h(3-k)$				1	1	0				$y(3) = 1 \times 1 + 1 \times 1 = 2$
$h(4-k)$					1	1	0			$y(4) = 1 \times 1 = 1$
$h(n-k)$						1	1	0		$y(n) = 0,\ n \geq 5$ (no overlap) Stop

Convolution (DT Systems) – Example using the graphical method and the table method

- $\{x(n)\} = \{\dots, 0, \underline{4}, 5, 6, 0, \dots\}$,
  $\{h(n)\} = \{\dots, 0, \underline{1}, 2, 3, 4, 0, \dots\}$ (position zero underlined)
- calculate $\{y(n)\} = \{x(n) * h(n)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$
- need only calculate for $0 \leq n \leq (2+3)$
- $\{y(n)\} = \{\dots, 0, \underline{4}, 13, 28, 43, 38, 24, 0, \dots\}$

Note:

- if both x and h have finite duration, then convolution sum is non-zero only from $(n_{\text{begin } h} + n_{\text{begin } x})$ to $(n_{\text{end } h} + n_{\text{end } x})$

Convolution (DT Systems) - Theorem

Theorem

Convolution is the time domain equivalent to multiplication in the frequency domain: if

$$Y(\omega) = X(\omega) \times H(\omega)$$

then

$$y(n) = x(n) * h(n)$$