A Half-Blood Half-Pipe, A Perfect Performance

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TABLE OF CONTENTS

Abstract

Our basic model has two parts: to find a half-pipe shape that can maximize vertical air, and to adapt the shape to maximize the possible total angle of rotation. In an extended model, we analyze the snowboarder's effect on vertical air and on rotation. Finally, we discuss the feasibility and the tradeoffs of building a practical course.

The major assumption is that resistance includes the friction of snow plus air drag, with the former proportional to the normal force. We find air drag negligible.
We first obtain and solve a differential equation for energy lost to friction and drag based on force analysis and energy conservation. We calculate vertical air by analyzing projectile motion. We then calculate the angular momentum before the flight and discuss factors influencing it. In an extended model, we take the snowboarder's influence into account.

We compare analytical and numerical results with reality, using default parameters; we validate that our method is correct and robust. We analyze the effects on vertical air of width, height, and gradient angle of the half-pipe. We find that a wider, steeper course with proper depth and the path of a skilled snowboarder are best for vertical air. Using a genetic algorithm, we globally optimize the course shape to provide either the greatest vertical air or maximal potential rotation; there is a tradeoff. Implementing a hybrid scoring system as the objective function, we optimize the course shape to a "half-blood" shape that would provide the eclectically best snowboard performance.

Background

A half-pipe is the venue for extreme sports such as snowboarding and skateboarding. It usually consists of two concave ramps (including a transition and a vert), topped by copings and decks, facing each other across a transition as shown in Figure 1. Half-pipe snowboarding has been a part of the Winter Olympics since 2002; the riders take two runs, performing tricks such as straight airs, grabs, spins, flips, and inverted rotations.

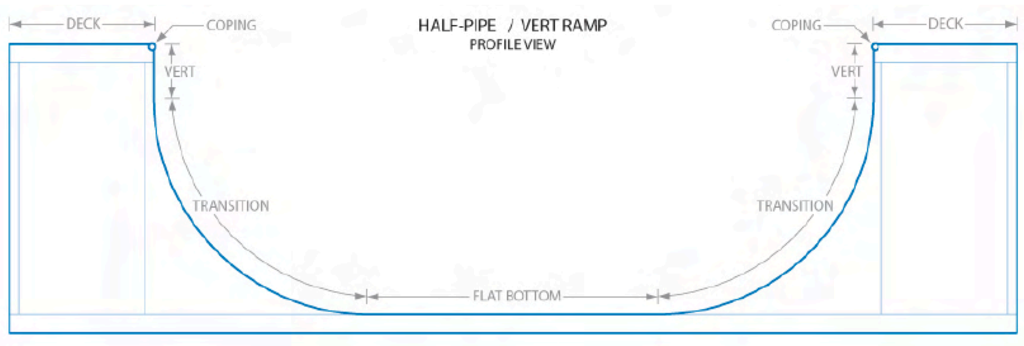

Figure 1. End-on schematic view of a half-pipe. (Source: Wikimedia Commons; created by Dennis Dowling.)

We find no analysis of the "best" shape for a half-pipe. However, usually it is 100–150mlong, 17–19.5mwide, and 5.4–6.5mfrom floor to crown, with slope angle 16–18.5° [Postins n.d.]. In addition, the F´ed´eration Internationalede Ski (FIS) recommends that the width, height, transition, and the bottom flat be 15 m, 3.5 m, 5 m, and 5 m, respectively [2003, 36].

Half-pipe snowboarding is currently judged using subjective measures. Still, there is strong community perception that airtime and degree of rotation play a major role in competition success [Harding et al. 2008a; Harding et al. 2008b].

According to Harding et al. [2008b] and Harding and James[2010], who have attempted to introduce objective analysis into the scoring, air time and total rotation are the two most critical evaluation criteria.

Terminology and Definitions

Cycle: The start of a cycle is when the snowboarder reaches the edge of the half-pipe after a flight, and the end of a cycle is the next start.

Flight: the part of the movement when the snowboarder is airborne.

Flight distance (S_f): displacement along the z direction during the flight.

Flight time (t_f): duration of the flight.

Cycle distance (S_c): displacement along z direction during a cycle.

Assumptions

• The cross section of the half-pipe is a convex curve that is smooth (has second-order derivative) everywhere except the endpoints.

• The snowboarder crouches during the performance until standing up to gain speed right at the edge of the half-pipe before the flight.

• We neglect the rotational kinetic energy of the snowboarder before considering the twist performance.

• The friction of the snow is proportional to the normal force of the snow exerted on the snowboarder but has nothing to do with velocity (that is, the angle between the direction of the snowboard and the snowboarder's velocity is constant).

• Air drag is proportional to the square of speed.

• The snowboarder's body is perpendicular to the tangential surface of the half-pipe during movement on the half-pipe.

• The force exerted on the board can be considered as acting at its center.

• We neglect the influence of natural factors such as uneven sunshine (which may result from an east or west orientation), altitude, etc.

Basic Model

Model Overview

A cycle can be divided into two parts: movement on the half-pipe, and the airborne performance.

For the first, we focus on the conversion and conservation of energy. The loss of mechanical energy E_{lost} due to the resistance of snow and air is the key. We derive a differential equation for it. We cannot neglect the snowboarder's increasing the mechanical energy by stretching the body(standing up) and doing work against the centrifugal force.

To derive an expression for vertical air, we apply Newton's Second Law. If we neglect air drag during the flight (we later show that it is indeed negligible), we can calculate vertical air, duration of the flight, flight distance, gravitational potential decrease, etc.

Next, we discuss the airborne rotation of the snowboarder. Since the shape of the half-pipe directly influences the initial angular momentum of the snowboarder, and the angular momentum cannot change during the flight, the relationship between the half-pipe shape and the initial angular momentum is the key to our discussion. After deriving an expression for the initial angular momentum, we can find the optimal shape of the half-pipe.

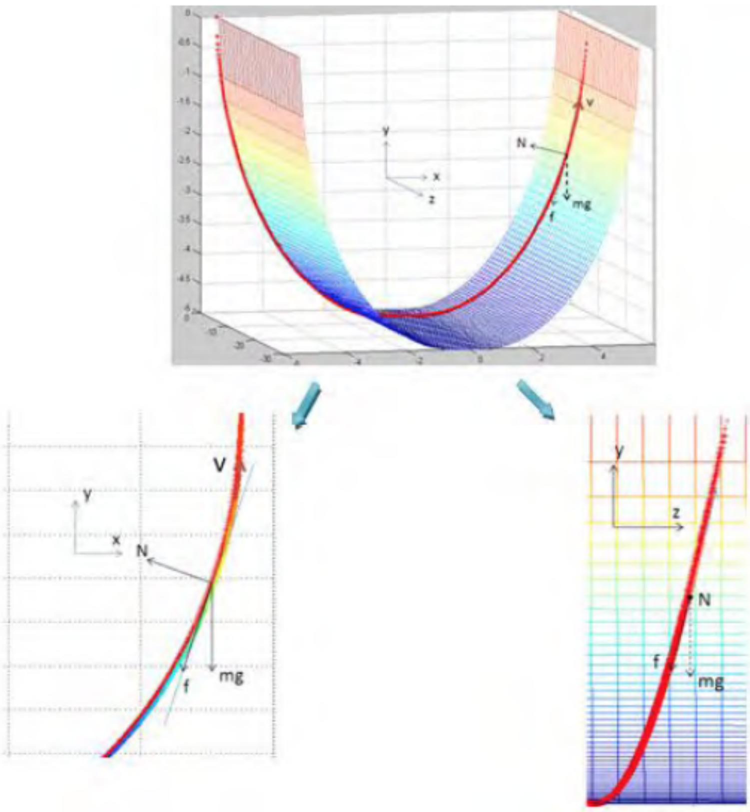

Figure 2. Force analysis.

The Model

Vertical Air

Step 1. Force Analysis: The top part of Figure 2 shows the definition of the coordinate variables: x is the free variable, while y and z are functions of x. The relationship between y and x depends on the shape of the half-pipe, while the relationship between z and x depends on the path chosen by the snowboarder. Three forces act on the snowboarder: gravity (mg), normal force (N), and resistance (f).

Resistance can be represented as

$$f = \alpha N + \beta\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \alpha N + \beta\left(1 + y'^2 + z'^2\right)\dot{x}^2.$$

For the normal force, only the part of centripetal acceleration that is parallel.

Table 1.
Model parameters.

Parameter	Meaning
x, y, z	Coordinate variables
$\dot{x}$, $\dot{y}$, $\dot{z}$	Velocities
y', y''	$\partial y/\partial x$, $\partial^2 y/\partial x^2$
z'	$\partial z/\partial x$
s	Length of the path
E_0	Initial mechanical energy at the beginning of a cycle
E_{leave}	Kinetic energy right before the flight
E_{reach}	Kinetic energy at the end of the flight
E_{lost}	Mechanical energy lost due to friction of the snow and air drag
N	Normal force of the snow exerted on the snowboarder
W_{human}	Work done by the snowboarder at the edge of the half-pipe when (s)he stands up
W_G	Decrease in gravitational potential during the flight
f	Friction of the snow plus air drag
m	Mass of the snowboarder
α	Friction coefficient between the snow and snowboard
β	Drag coefficient of air
θ	Angle between z-axis and the horizontal plane
Δh	Rise of the mass point of the snowboarder when (s)he stands up from a crouching position
ρ	Radius of curvature at a point on the cross section of the half-pipe
x_t, y_t, z_t, y'_t, z'_t, $\dot{y}_t$, $\dot{z}_t$, y''_t	Values right before the flight
H_f	Vertical air

to the direction of N needs to be considered:

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{y''},$$
$$N = \frac{\dot{x}^2+\dot{y}^2}{\rho} m + \frac{mg\cos\theta}{\sqrt{1+y'^2}} = \left(y''\dot{x}^2 + g\cos\theta\right)\frac{m}{\sqrt{1+y'^2}}.$$

Path length unit can be represented as:

$$ds = \sqrt{1+\dot{y}^2+\dot{z}^2}\,dx.$$

Step 2. Energy Conservation: According to the Energy Conservation Principle, we have:

$$\dot{x}^2 = \frac{2}{m\left(1+y'^2+z'^2\right)}\left[E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta)\right].$$

Step 3. E_{lost}:

$$\begin{aligned} E_{\text{lost}} &= \int_{-x_0}^{x} f\cdot ds \\ &= \int_{-x_0}^{x} \left[\alpha N\sqrt{1+y'^2+z'^2} + \beta\left(1+y'^2+z'^2\right)^{3/2}\cdot\dot{\tau}^2\right] d\tau \\ &= \int_{-x_0}^{x} \Bigg\{\left(\alpha\frac{my''}{\sqrt{\left(1+y'^2\right)\left(1+y'^2+z'^2\right)}} + \beta\right)\times \\ &\qquad \frac{2}{m}\left[E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta)\right] \\ &\qquad\qquad + \alpha mg\cos\theta\frac{\sqrt{1+y'^2+z'^2}}{\sqrt{1+y'^2}}\Bigg\}\, d\tau. \end{aligned}$$

The integral has a variable upper limit. Differentiating both sides and solving the resulting first-order linear ordinary differential equation, we get an expression for E_{lost}, which we would like to minimize. However, since the relationship between y and x is unknown, as is that between z and x, the expression is a functional. The expression and the calculation are too complicated, so we use a numerical method to solve the problem.

Step 4. W_{human}: When the snowboarder stands up, (s) he does work overcoming the centrifugal force. At high speed, the centrifugal force is huge, so W_{human} is considerable and cannot be neglected. The work done by the snowboarder is

$$W_{\text{human}} = \frac{\dot{t}_t^2}{\rho} m \cdot \Delta h = \frac{y_t'^2 y_t''}{(1+y_t')^{3/2}} m \cdot \Delta h.$$

Step 5. Vertical Air: At the edge of the half-pipe before the flight, we have $\dot{x} = 0$. From the Energy Conservation Principle, we get

$$\tfrac{1}{2} m \left(\dot{y}_t^2 + \dot{z}_t^2\right) = E_0 - E_{\text{lost}}(x_t) = mg \cdot z_t \sin\theta + \frac{y_t'^2 y_t''}{(1+y_t')^{3/2}} m \cdot \Delta h.$$

Since $\dfrac{\dot{y}_t}{\dot{z}_t} = \dfrac{y_t'}{z_t'}$, we have $\dot{z}_t = \dfrac{z_t'}{y_t'} \dot{y}_t$ and

$$\dot{y}_t^2 = \frac{2}{m\left[1 + \left(\frac{z_t'}{y_t'}\right)^2\right]} \left(E_0 - E_{\text{lost}}(x_t) + mg \cdot z_t \sin\theta + \frac{y_t'^2 y_t''}{(1+y_t')^{3/2}} m \cdot \Delta h\right).$$

If we neglect air drag during the flight, Figure 3 shows that vertical air is

$$H_f = \frac{1}{\cos\theta} \frac{\dot{y}_t^2}{2g\cos\theta} = \frac{E_0 - E_{\text{lost}}(x_t) + mg \cdot z_t \sin\theta + \frac{y_t'^2 y_t''}{(1+y_t')^{3/2}} m \cdot \Delta h}{mg \cdot \cos^2\theta \left[1 + \left(\frac{z_t'}{y_t'}\right)^2\right]}.$$

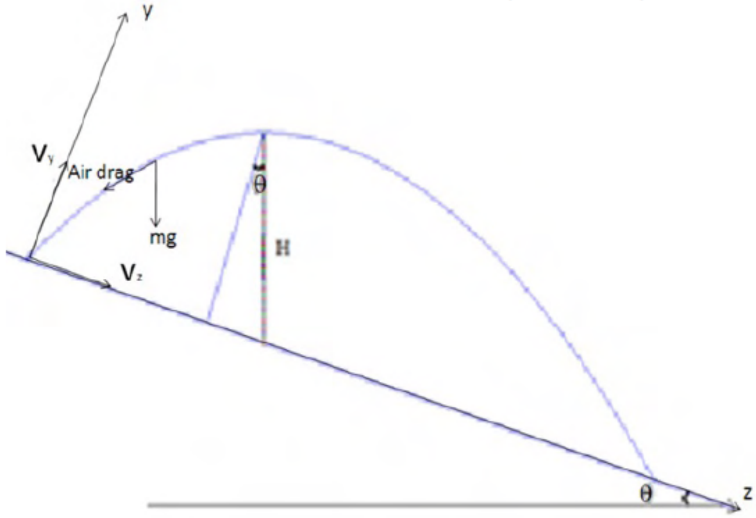

Figure 3. The path and force analysis of the flight.

Strengths and Weaknesses

• The model describes the motion in detail, with coordination among the many physical quantities.

• The numerical computations are precise.

• The results generated by numerical computation agree with empirical data, lending support to the model.

• The model takes the subjective influence of snowboarders into account.

• We establish an objective function to compare different course shapes.

• We optimize the course locally (to learn the individual impact of the parameters) as well as globally (to shed light on the design of a half-pipe), and obtain numerical solutions.

• The model does not provide an analytic solution for the optimal course.

• The model does not take into account detailed mechanical characteristics and on-snow performance of snowboards.

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