6.3 Consider the random process

$$X_t = A\sin(2\pi t), \quad t \in \mathbb{R},$$

where A is some real-valued random variable. Is the process X_t WSS? Why or why not?

Solution: $E(X_t) = E(A\sin(2\pi t)) = E(A)\sin(2\pi t)$

$E(X_t)$ is a function of t. it is not constant. This already violates the first condition for WSS process.

It is not WSS.

6.5 Suppose the process $X_t = t^2 + A\sin t + B\cos t$, where A and B are random variables, $E(A) = E(B) = 0$, $Var(A) = Var(B) = 10$, and $E(AB) = 0$. Discuss the stationarity of the processes X_t and $Y_t = X_t - \mu_X(t)$.

Solution: $E(X_t) = E(t^2 + A\sin t + B\cos t) = E(t^2) + \sin t E(A) + \cos t E(B) = t^2$

$E(X_t)$ is a function of t, it is not WSS

$E(X_m X_n) = E[(m^2 + A\sin m + B\cos n)(n^2 + A\sin n + B\cos n)] = m^2n^2 + 10\sin m\sin n + 10\cos m\cos n$

$R_X(m,n)$ depends on m, n. it is not WSS.

$Y_t = X_t - \mu_X(t) = A\sin t + B\cos t$

$E(Y_t) = E(A\sin t + B\cos t) = 0$ is a constant

$R_Y(m,n) = E(Y_m Y_n) = E[(A\sin m + B\cos m)(A\sin n + B\cos n)] = 10\sin m\sin n + 10\cos m\cos n = 10\cos(m-n) = 10\cos(\tau)$

$R_Y(m,n)$ only depends on $m-n$.

So. it is WSS process.

6.11 Suppose that the process $\{X_t\}_{t\in R}$ is a stationary process with the autocorrelation function $R_X(t) = 4\cos\omega t$, where ω is constant. Find the average power of X_t.

Solution: $Q = R_X(0) = 4$

6.12 A stochastic stationary process has an autocorrelation function of

$$R_X(n) = 5\sin(n/80) + 4.$$

(a) Find the variance of this stochastic process.

Solution: $R_X(n) = R_X(t_1, t_1 + n)$

$Var(X_t) = R_X(t,t) = R_X(0) = 4$

6.13 A stationary stochastic process has a power spectral density of $S_X(\omega) = \frac{500}{\omega^2+9}$. Find the autocorrelation function of X_t.

Solution: $R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)\, e^{i\omega\tau}\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{500}{\omega^2+9}\, e^{i\omega\tau}\, d\omega$

$= \frac{500}{6} e^{-3|r|} = \frac{250}{3} e^{-3|r|}$

Example (HW)

Suppose that $\{X_n\}_{n \geqslant 1}$ are IID random sequences, each of which has a standard normal distribution, and $\{Y_n\}_{n \geqslant 1}$ are also IID random sequences, while each of which has a uniform distribution in the interval $(-\sqrt{3}, \sqrt{3})$. Suppose that $\{X_n\}_{n \geqslant 1}$ and $\{Y_n\}_{n \geqslant 1}$ are independent. Let

$$Z_n = \begin{cases} X_n & \text{if n is odd,} \\ Y_n & \text{if n is even.} \end{cases}$$

Prove $\{Z_n\}_{n \geqslant 1}$ is WSS, but not SSS.

Prove: n is odd: $E(Z_n) = E(X_n) = 0$

n is even: $E(Z_n) = E(Y_n) = 0$

$\therefore E(Z_n) = 0$, is a constant

$R_Z(m, n) = E(Z_m Z_n)$

when $m \neq n$

m, n both odd: $R_Z(m,n) = E(X_m)E(X_n) = 0$

m, n both even: $R_Z(m,n) = E(Y_m)E(Y_n) = 0$

m is odd, n is even. $R_Z(m,n) = E(X_m)E(Y_n) = 0$

m is even, n is odd: $R_Z(m,n) = E(X_n)E(Y_m) = 0$

$\therefore R_Z(r) = 0$

when $m = n$

$R_Z(m,n) = E(Z_m^2) = Var(X_m) = 1$

$R_Z(r) = \begin{cases} 1, & r = 0 \\ 0, & r \neq 0 \end{cases}$, only depends on $m-n$

$\therefore$ It is WSS

when n, $n+k$ both odd, $[Z_n, Z_{n+k}]$ consists of two independent standard normal distributions

when n, $n+k$ both even, $[Z_n, Z_{n+k}]$ consists of two independent uniform distribution.

Therefore, the joint distribution changes with time and doesn't satisfy requirments for SSS