# 1.2 Propositions 命题

Logic is a system based on **propositions**.

A sentence that is either true or false, but not both, is called a **proposition**.

Truth Value of a proposition

**True: T**

**False: F**

**Conjunction 合取**
**Disjunction 析取**
**Negation 否定**

**Conditional Proposition 条件命题**

**Converse 逆**
**Biconditional Proposition 双条件命题**
**Contrapositive (Transposition) Proposition 逆否命题（转换命题）**

**Tautology 重言式/永真式**
**Contradiction 矛盾式**

# De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Statement: Tom is in the football team and the basketball team.

Negation: Tom is not in the football team or not in the basketball team.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Statement: The number 6 is divisible by 2 or 5.

Negation: The number 6 is not divisible by 2 and not divisible by 5.

# Contingency

$$((p \wedge r) \vee (q \wedge r)) \wedge (\neg(p \vee q) \vee r)$$

(A) It is a tautology.
(B) It is a contradiction.
(C) It is not sure.

- A compound proposition is called contingency if and only if it is neither a tautology nor a contradiction.
- It contains both T (True) and F (False) in last column of its truth table.

# Argument 论证

Definition1.4.1 An argument is a sequence of propositions written

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

hypotheses (假设)
or premises (前提)

conclusion (结论)

or

$$p_1, p_2, \ldots, p_n / \therefore q$$

$\therefore$ means therefore

# Argument 论证

Definition1.4.1 An argument is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

or

$$p_1, p_2, \ldots, p_n / \therefore q$$

Argument is said to be **valid (有效的)** if the conclusion follows from the hypotheses; that is, if $p_1$ and $p_2$ and … and $p_n$ are true, then $q$ must also be true. otherwise, the argument is invalid (or a fallacy).

# Rules of inference 推理规则

TABLE 1.4.1 ■ Rules of Inference for Propositions

| *Rule of Inference* | *Name* | *Rule of Inference* | *Name* |
|---|---|---|---|
| $p \to q$<br>$p$<br>$\therefore q$ | 假言推理或分离定律<br>Modus ponens | $p$<br>$q$<br>$\therefore p \wedge q$ | 合取<br>Conjunction |
| $p \to q$<br>$\neg q$<br>$\therefore \neg p$ | 拒取<br>Modus tollens | $p \to q$<br>$q \to r$<br>$\therefore p \to r$ | 假言三段论<br>Hypothetical syllogism |
| $p$<br>$\therefore p \vee q$ | 附加<br>Addition | $p \vee q$<br>$\neg p$<br>$\therefore q$ | 析取三段论<br>Disjunctive syllogism |
| $p \wedge q$<br>$\therefore p$ | 化简<br>Simplification | | |

# Rules of inference 推理规则

Exercise. It is known that

1. It is not sunny this afternoon, and it is colder than yesterday.
2. We will go swimming only if it is sunny this afternoon.
3. If we do not go swimming, we will play basketball.
4. If we play basketball, we will go home early.

Can you conclude “we will go home early”?

$p$ := It is sunny this afternoon
$q$ := It is colder than yesterday
$r$ := We will go swimming
$s$ := We will play basketball
$t$ := We will go home early

# Rules of inference 推理规则

Exercise. A student is trying to prove that propositions $p$, $q$, and $r$ are all true.

She proceeds as follows.

First, she proves three facts:

- $p$ implies $q$
- $q$ implies $r$
- $r$ implies $p$.

Then she concludes,

``Thus $p$, $q$, and $r$ are all true."

**What does its form of argument is like?**

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First, she proves three facts:

- $p$ implies $q$
- $q$ implies $r$
- $r$ implies $p$.

Then she concludes,

``Thus $p$, $q$, and $r$ are all true."

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ r \rightarrow p \\ \hline \therefore p \land q \land r \end{array}$$

**What does its form of argument is like?**

# Rules of inference 推理规则

$$\begin{array}{l} p \to q \\ q \to r \\ r \to p \\ \hline \therefore p \land q \land r \end{array}$$

| $p$ | $q$ | $r$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |

| $p \to q$ | $q \to r$ | $r \to p$ |
|---|---|---|
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| | | |
| | | |
| | | |

| $p \land q \land r$ |
|---|
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| |

To prove an argument is not valid, we just need to find a counterexample.

# Valid Arguments?

$$\begin{array}{c} p \to q \\ q \\ \hline \therefore\ p \end{array}$$

# Valid Arguments?

$$\begin{array}{c} p \to q \\ q \\ \hline \therefore\ p \end{array}$$

If you are a fish, then you drink water.
You drink water.

You are a fish.

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$$\begin{array}{l} p \to q \\ q \\ \hline \therefore\ p \end{array}$$

If you are a fish, then you drink water.
You drink water.

You are a fish.

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

# Valid Arguments?

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore\ p \end{array}$$

If you are a fish, then you drink water.
You drink water.

You are a fish.

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

If you are a fish, then you drink water.
You are not a fish.

You do not drink water.

# Exercises

$$\frac{p}{\therefore\ p \vee q}$$

**Addition**

$$\frac{p}{\therefore\ p \wedge q}$$

$$\frac{p \wedge q}{\therefore\ p}$$

**Simplification**

$$\frac{p \vee q}{\therefore\ p}$$

# Exercises

(A)
$$\begin{array}{l} \neg p \rightarrow q \\ \neg q \\ \hline \therefore\ p \end{array}$$

(B)
$$\begin{array}{l} \neg p \rightarrow \neg q \\ \hline \therefore\ p \rightarrow q \end{array}$$

(C)
$$\begin{array}{l} \neg p \rightarrow \neg q \\ \hline \therefore\ q \rightarrow p \end{array}$$

# Honest man and Liar

Honest man always tell the truth.

Liar always lie.

A says: “B is an honest man.”

B says: “A and I are of opposite type.”

What are the identities of A and B?

# Honest man and Liar

Honest man always tell the truth.

Liar always lie.

A says: “B is a liar.”

B says: “A is an honest man.”

What are the identities of A and B?

# A game

Imagine there are 3 coins on the table. Gold, silver and copper.

If you say a truthful sentence, you will get one coin. If you say a false sentence, you get nothing.

Which sentence can guarantee gaining the gold coin?

# A game

Imagine there are 3 coins on the table. Gold, silver and copper.

If you say a truthful sentence, you will get one coin. If you say a false sentence, you get nothing.

Which sentence can guarantee gaining the gold coin?

(A) You will give me the gold coin.
(B) You will give me all the coins.
(C) You will not give me any of the coins.
(D) You will give me either silver or copper coin.
(E) You will give me neither silver nor copper coin.

# Problem-Solving Tips

The validity of a very short argument or proof might be verified using a truth table. In practice, arguments and proofs use rules of inference.

# Whether each argument is valid?

**15.**
$$\begin{array}{l} (p \vee q) \rightarrow (r \vee s) \\ p \\ \neg r \\ \hline \therefore s \end{array}$$

**16.**
$$\begin{array}{l} (r \vee s) \rightarrow p \\ s \rightarrow q \\ p \vee s \\ \hline \therefore r \end{array}$$

**17.**
$$\begin{array}{l} (p \rightarrow r) \rightarrow q \\ (q \rightarrow s) \rightarrow p \\ r \wedge s \\ p \vee q \\ \hline \therefore p \wedge q \end{array}$$

**18.**
$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \neg r \\ s \rightarrow r \\ \hline \therefore \neg s \end{array}$$

**19.**
$$\begin{array}{l} p \rightarrow (q \vee r) \\ q \rightarrow (p \vee s) \\ p \vee \neg q \\ \neg s \\ \hline \therefore p \vee r \end{array}$$

# 1.5 Quantifiers 量词

Consider the following statement

$n$ is an odd integer

Is this a proposition?

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Consider the following statement

$n$ is an odd integer

Is this a proposition?

NO: Its truth value is based on the value of $n$.

# 1.5 Quantifiers 量词

Consider the following statement

$n$ is an odd integer

Is this a proposition?

NO: Its truth value is based on the value of $n$.

An argument is a sequence of **propositions** written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

# 1.5 Quantifiers 量词

**Definition 1.5.1** Let $P(x)$ be a statement involving the variable $x$ and let $D$ be a set. We call $P$ a propositional function (命题函数) or predicate (谓词) (with respect to $D$) if for each $x \in D$，$P(x)$ is a proposition.

We call $D$ the domain of discourse (论域) of $P$.

The domain of discourse specifies the alllowable values for $x$.

## 1.5 Quantifiers 量词

**Example 1.5.3** Whether the following are propositional functions？
(命题函数)

(a) $n^2 + 2n$ is an odd integer (domain of discourse $= \mathbf{Z}^+$).

(b) $x^2 + x - 6 = 0$ (domain of discourse = $\mathbf{R}$).

(c) The baseball player hit over $.300$ in 2015 (domain of discourse = set of baseball players).

(d) The film is rated over $20\%$ by Rotten Tamatoes (domain of discourse = set of films rated by Rotten Tomatoes).

# Practice

Let $D = \{2, 3, 4, 5, 6, 7, 8\}$

For each of the following propositional function (or predicates), list those elements of $D$ that make the statement true:

1. $P(x)$ is "$x^2 + 3$ is divisible by 5".

2. $Q(x)$ is "$x > 1$ and $2x \leq 10$".

3. $R(x)$ is "$x$ is even or prime".

# Universal Quantifiers 全称量词

**Definition 1.5.4** Let $P$ be a propositional function with domain of discourse $D$. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句)**.

It may be written

$$\forall x\ P(x).$$

The symbol ∀ means “for every”, and is called a **universal quantifier (全称量词)**.

The statement is true if
The statement is false if
?

## Universal Quantifiers 全称量词

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The statement is true if $P(x)$ is true for every $x$ in $D$.
The statement is false if $P(x)$ is false for at least one $x$.

## Universal Quantifiers 全称量词

**Definition 1.5.4** Let $P$ be a propositional function with domain of discourse $D$. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句)**.

for all $x$, $P(x)$
for any $x$,
$P(x)$

Alternative ways to write $\forall x\ P(x)$.

for all $x$ in $D$, $P(x)$

Specify the domain of discourse.

# Universal Quantifiers 全称量词

**Example 1.5.8** Whether the universally quantified statement for every real number $x$, if $x > 1$, then $x + 1 > 1$ is true for every real number $x$?

The statement is true if $P(x)$ is true for every $x$ in $D$.
The statement is false if $P(x)$ is false for at least one $x$.

# Existential Quantifiers 存在量词

**Definition 1.5.9** Let $P$ be a propositional function with domain of discourse $D$. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an **existentially quantified statement (存在量词语句)**.

It may be written

$$\exists x\, P(x).$$

The symbol ∃ means “there exists”, and is called a **existential quantifier (存在量词)**.

The statement is true if
The statement is false if
?

# Existential Quantifiers 存在量词

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$$\text{there exists } x,\ P(x)$$

is said to be an **existentially quantified statement (存在量词语句)**.

It may be written

$$\exists x\, P(x).$$

The symbol $\exists$ means "there exists", and is called a **existential quantifier (存在量词)**.

The statement is true if $P(x)$ is true for at least one $x$ in $D$.
The statement is false if $P(x)$ is false for every $x$ in $D$.

## Existential Quantifiers 存在量词

**Definition 1.5.9** Let $P$ be a propositional function with domain of discourse $D$. The statement

there exists $x$, $P(x)$

is said to be an **existentially quantified statement (存在量词语句)**.

there exists $x$ such that, $P(x)$
for some $x$, $P(x)$
for at least one $x$, $P(x)$

Alternative ways to write $\exists x\ P(x)$.

## Existential Quantifiers 存在量词

**Example 1.5.11** Verify that the existentially quantified statement

$$\exists x \in \boldsymbol{R}\left(\frac{1}{x^2+1} > 1\right)$$

is false.

# Existential Quantifiers 存在量词

**Example 1.5.13** Whether the following existentially quantified statement is true or false?

For some $n$, if $n$ is prime, then $n+1$, $n+2$, $n+3$, $n+4$ are not prime. The domain of discourse is $\mathbf{Z}^{+}$.

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

De Morgan's Laws for Logic

?

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

**De Morgan's Laws for Logic**

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\, P(x)\ ) \equiv \ ?$$

**De Morgan's Laws for Logic**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\ P(x)\ ) \equiv\ ?$$

Everyone likes football.

What is the negation of this statement?

**De Morgan's Laws for Logic**

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\ P(x)\ ) \equiv \exists x\ \neg P(x)$$

**De Morgan's Laws for Logic**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\,\forall x\, P(x)\,) \equiv \exists x\, \neg P(x)$$

$$\neg(\,\exists x\, P(x)\,) \equiv \text{?}$$

## De Morgan's Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\, P(x)\ ) \equiv \exists x\, \neg P(x)$$

$$\neg(\ \exists x\, P(x)\ ) \equiv\ ?$$

There is a plant that can fly.

What is the negation of this statement?

**De Morgan's Laws for Logic**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

**De Morgan's Laws for Logic**

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

## Generalized De Morgan’s Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function $P$ is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function $P$ is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-2) \wedge P(0) \wedge P(5)$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function $P$ is $\{-2, 0, 5\}$.

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The propositional function $\exists x\, P(x)$ is equivalent to

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

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The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-2) \wedge P(0) \wedge P(5)$$

The propositional function $\exists x\, P(x)$ is equivalent to

$$P(-2) \vee P(0) \vee P(5)$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

**De Morgan's Laws for Logic**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

# Generalized De Morgan's Laws for Logic 广义德•摩根定律

$$\neg(\ \forall x\ P(x)\ ) \equiv \exists x\ \neg P(x)$$

$$\neg(\ \exists x\ P(x)\ ) \equiv \forall x\ \neg P(x)$$

## Truth Table

# Exercise

"All lions are fierce"

$P(x)$: $x$ is a lion

$Q(x)$: $x$ is fierce

(A) $\forall x\ (\ P(x) \wedge Q(x)\ )$

(B) $\forall x\ (P(x) \rightarrow Q(x)\ )$

# Exercise

“All lions are fierce”

$P(x)$: $x$ is a lion

$Q(x)$: $x$ is fierce

(A) $\forall x\ (P(x) \land Q(x))$

(B) $\forall x\ (P(x) \rightarrow Q(x))$

| $P(x)$ | $Q(x)$ | $P(x) \land Q(x)$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

| $P(x)$ | $Q(x)$ | $P(x) \rightarrow Q(x)$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

# Exercise

"Some lions do not drink coffee"

$P(x)$: $x$ is a lion

$R(x)$: $x$ drinks coffee

(A) $\exists\ x\ (\ P(x) \land \neg Q(x)\ )$

(B) $\exists\ x\ (P(x) \rightarrow \neg Q(x)\ )$

# Exercise

"Some lions do not drink coffee"

$P(x)$: $x$ is a lion

$R(x)$: $x$ drinks coffee

(A) $\exists\, x\ (\ P(x) \wedge \neg Q(x)\ )$

(B) $\exists\, x\ (P(x) \rightarrow \neg Q(x)\ )$

| $P(x)$ | $R(x)$ | $\neg R(x)$ | $P(x)\ \wedge\ \neg R(x)$ |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |

| $P(x)$ | $R(x)$ | $\neg R(x)$ | $P(x) \rightarrow\ \neg R(x)$ |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |

# Rules of Inference for Quantified Statements 量词推理规则

**TABLE 1.5.1** Rules of Inference for Quantified Statements†

| *Rule of Inference* | *Name* |
|---|---|
| $\forall x\, P(x)$ <br> ——— <br> $\therefore P(d)$ if $d \in D$ | Universal instantiation |
| $P(d)$ for every $d \in D$ <br> ——— <br> $\therefore \forall x\, P(x)$ | Universal generalization |
| $\exists x\, P(x)$ <br> ——— <br> $\therefore P(d)$ for some $d \in D$ | Existential instantiation |
| $P(d)$ for some $d \in D$ <br> ——— <br> $\therefore \exists x\, P(x)$ | Existential generalization |

† The domain of discourse is $D$.

# Universal Instantiation 全称例化

$$\frac{\forall x\, P(x)}{\therefore\ P(d) \text{ if } d \in D}$$

# Universal Generalization 全称一般例化

$$\frac{P(d) \text{ for every } d \in D}{\therefore\ \forall x\, P(x)}$$

# Existential Instantiation 存在例化

$$\frac{\exists x\, P(x)}{\therefore\ P(d) \text{ for some } d \in D}$$

# Existential Generalization 存在一般例化

$$\frac{P(d) \text{ for some } d \in D}{\therefore \ \exists x\, P(x)}$$

## Rules of Inference for Quantified Statements 量词推理规则

**Practice** Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

Every BUPT student is a genius. Zhang is a BUPT student. Therefore, Zhang is a genius.

# Rules of Inference for Quantified Statements 量词推理规则

**Example 1.5.23** Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

For every real number $x$, if $x$ is an integer, then $x$ is a rational number. The number $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

# Arguments with Quantified Statements

**Universal instantiation:**

$$\forall x, P(x)$$
$$\therefore P(a)$$

**Universal modus ponens:**

$$\forall x, P(x) \rightarrow Q(x)$$
$$P(a)$$
$$\therefore\ Q(a)$$

**Universal modus tollens:**

$$\forall x, P(x) \rightarrow Q(x)$$
$$\neg Q(a)$$
$$\therefore\ \neg P(a)$$

# Practice

$$
\begin{array}{l}
\forall x\big(p(x) \vee q(x)\big) \\
\forall x\Big(\big(\neg p(x) \wedge q(x)\big) \rightarrow r(x)\Big) \\
\hline
\therefore \forall x\big(\neg r(x) \rightarrow p(x)\big)
\end{array}
$$

# Problem-Solving Tips

- To prove that the universally quantified statement $\forall x P(x)$ is true, show that for *every* $x$ in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for a *particular* value $x$ does *not* prove that $\forall x P(x)$ is true.
- To prove that the existentially quantified statement $\exists x P(x)$ is true, find *one* value of $x$ in the domain of discourse for which the proposition $P(x)$ is true. *One* value suffices.
- To prove that the universally quantified statement $\forall x P(x)$ is false, find *one* value of $x$ (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false.
- To prove that the existentially quantified statement $\exists x P(x)$ is false, show that for *every* $x$ in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for a *particular* value $x$ does *not* prove that $\exists x P(x)$ is false.

# 1.6 Nested Quantifiers 嵌套量词

Consider writing the statement

"The sum of any two positive real numbers is positive"

symbolically.

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- Two numbers are involved,
- need two universal quantifiers

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Consider writing the statement

"The sum of any two positive real numbers is positive"

symbolically.

- Two numbers are involved,
- need two universal quantifiers

$P(x, y): (x > 0) \wedge (y > 0) \longrightarrow (x + y > 0)$

Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\, P(x, y)$

Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers**.

## 1.6 Nested Quantifiers 嵌套量词

Consider writing the statement

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symbolically.

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Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\ P(x, y)$

Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers**.

**Any other Nested Quantifiers?**

# 1.6 Nested Quantifiers 嵌套量词

Example 1.6.1 Restate $\forall m \exists n\ (m < n)$ in words.

# 1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
“Everybody loves somebody.”
symbolically, letting $L(x, y)$ be the statement “$x$ loves $y$”.

# 1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
"Everybody loves somebody."
symbolically, letting $L(x, y)$ be the statement "$x$ loves $y$".

(A) $\forall x \exists y\, L(x, y)$

(B) $\exists x \forall y\, L(x, y)$

# 1.6 Nested Quantifiers 嵌套量词

Example 1.6.13

$$\neg(\forall x \exists y P(x, y)) \equiv$$

# 1.6 Nested Quantifiers 嵌套量词

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$$\neg(\forall x \exists y P(x,y)) \equiv \exists x \forall y \neg P(x,y)$$

# 1.6 Nested Quantifiers 嵌套量词

Example 1.6.14 Write the negation of $\exists x \forall y (xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

## Problem-Solving Tips

$$\forall x \forall y$$

- To prove that $\forall x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols $x$ and $y$ to stand for *arbitrary* elements in $X$ and $Y$.
- To prove that $\forall x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for $x$ and one for $y$) that make $P(x, y)$ false.

## Problem-Solving Tips

$$\forall x \exists y$$

- To prove that $\forall x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let $x$ stand for an arbitrary element in $X$ and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.
- To prove that $\forall x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for $x$, let $y$ stand for an arbitrary element of $Y$ and show that $P(x, y)$ is always false.

## Problem-Solving Tips

$$\exists x \forall y$$

- To prove that $\exists x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for $x$, let $y$ stand for an arbitrary element of $Y$ and show that $P(x, y)$ is always true.
- To prove that $\exists x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let $x$ stand for an arbitrary element in $X$ and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.

# Problem-Solving Tips

$$\exists x \exists y$$

- To prove that $\exists x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for $x$ and one for $y$) that make $P(x, y)$ true.
- To prove that $\exists x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols $x$ and $y$ to stand for *arbitrary* elements in $X$ and $Y$.

# Problem-Solving Tips

- To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \to q$ is equivalent to $p \wedge \neg q$.