2.28 Suppose that the cumulative distribution function of the random variable X is given by

$$F(x) = 1 - e^{-x^2}, \qquad x > 0.$$

Evaluate (a) $P(X > 2)$, (b) $P(1 < X < 3)$, (c) $E(X)$, and (d) $Var(X)$.

(a) $P(X>2) = 1 - P(X \le 2) = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4}$

(b) $P(1<X<3) = F(3^-) - F(1) = 1 - e^{-9} - 1 + e^{-1} = e^{-1} - e^{-9}$

(c) $f(x) = F'(x) = 2x \cdot e^{-x^2} \quad (x>0)$ $\qquad de^{-x^2} = (e^{-x^2})' dx = -2xe^{-x^2} dx$

$E(x) = \int_{-\infty}^{\infty} x f(x) = \int_0^{\infty} x \cdot 2x \cdot e^{-x^2} dx = \int_0^{\infty} x \cdot \frac{2x \cdot e^{-x^2}}{-2xe^{-x^2}} de^{-x^2} = \int_0^{\infty} -x \, de^{-x^2} = -xe^{-x^2} \Big|_0^{\infty} + \int_0^{+\infty} e^{-x^2} dx$

$= \frac{\sqrt{\pi}}{2}$

(d) $E(x^2) = \int_0^{+\infty} 2x^3 \cdot e^{-x^2} dx = \int_0^{+\infty} (-x^2) de^{-x^2} = -x^2 \cdot e^{-x^2} \Big|_0^{\infty} - \int_0^{+\infty} e^{-x^2} d(-x^2)$

$= \int_{-\infty}^{0} e^t dt = e^0 - e^{-\infty}$

$= 1$

$Var(x) = E(x^2) - E^2(x) = 1 - \frac{\pi}{4}$

2.30 The thickness X of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval [20, 40] microns. Find the mean, standard deviation and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick.

$X \sim U(20, 40) \qquad x \in [20, 40] \qquad f(x) = \begin{cases} \frac{1}{20} & 20 < x < 40 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_{20}^{40} x \cdot \frac{1}{20} dx = \frac{1}{20} \cdot \frac{1}{2} x^2 \Big|_{20}^{40} = 30$

$Var(x) = \int_{-\infty}^{\infty} (x - E(x))^2 f(x) dx = \int_{20}^{40} (x-30)^2 \cdot \frac{1}{20} dx$

$= \frac{1}{20} \int_{20}^{40} (x-30)^2 d(x-30)$

$= \frac{1}{20} \cdot \frac{1}{3} (x-30)^3 \Big|_{20}^{40}$

$= \frac{100}{3}$

$\sigma = \sqrt{Var(x)} = \frac{10\sqrt{3}}{3}$

$F(x) = \int_{-\infty}^{x} f(x) dx = \int_{20}^{x} \frac{1}{20} dx = \frac{1}{20} x \Big|_{20}^{x} = \frac{x}{20} - 1, \quad x \in [20, 40)$

$F(x) = \begin{cases} 0, & x < 20 \\ \frac{x}{20} - 1, & 20 \le x < 40 \\ 1, & x \ge 40 \end{cases}$

$P(x<35) = F(35^-) = \frac{3}{4}$

2.33 The number of automobiles sold weekly at a certain dealership is a random variable with expected value 16. Give an upper bound to the probability that

(a) next week's sales exceed 18,

(b) next week's sales exceed 25.

Let X = # automobiles sold weekly

(a) $P(X>18) = P(X \geq 19) \leq \frac{E(X)}{19} = \frac{16}{19}$

(b) $P(X>25) = P(X \geq 26) \leq \frac{E(X)}{26} = \frac{16}{26} = \frac{8}{13}$

2.34 Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 100/3$. Using Chebyshev's inequality, find an upper bound for the following probabilities.

(a) $P(|X-10| \geqslant 2)$.

(b) $P(|X-10| \geqslant 5)$.

(c) $P(|X-10| \geqslant 9)$.

(d) $P(|X-10| \geqslant 20)$.

(a) $P(|X-10| \geq 2) \leq \frac{\sigma^2}{2^2} = \frac{25}{3} \Rightarrow P(|X-10| \geq 2) \leq 1$

(b) $P(|X-10| \geq 5) \leq \frac{\sigma^2}{5^2} = \frac{4}{3} \Rightarrow P(|X-10| \geq 5) \leq 1$

(c) $P(|X-10| \geq 9) \leq \frac{\sigma^2}{9^2} = \frac{100}{243}$

(d) $P(|X-10| \geq 20) \leq \frac{\sigma^2}{20^2} = \frac{1}{12}$

2.35 Suppose that $P(X=a) = p$, $P(X=b) = 1-p$.

(a) Show that $\frac{X-b}{a-b}$ is a Bernoulli random variable.

(b) Find $Var(X)$.

(a) Let $Y = \frac{X-b}{a-b}$

$X = a,\ Y = 1$

$X = b,\ Y = 0$

$P(Y=1) = P(X=a) = p$

$P(Y=0) = P(X=b) = 1-p$

$P(Y=1) + P(Y=0) = 1$, so it's Bernoulli r.v.

(b) $E(Y) = p \quad E(Y^2) = p \quad Var(Y) = E(Y^2) - E^2(Y) = p - p^2$

$= p(1-p)$

$Y = \frac{X-b}{a-b} \Rightarrow X = (a-b)Y + b$

$Var(X) = Var((a-b)Y + b)$

$= (a-b)^2 \, Var(Y)$

$= (a-b)^2 \, p(1-p)$

2.39 An examination consists of 10 multi-choice questions, in each of which a candidate has to deduce which one of five suggested answers is correct. A completely unprepared student may be assumed to guess each answer completely randomly. What is the probability that this student gets 8 or more questions correct? Draw the appropriate moral!

Let X = # question(s) correct

question correct probability $p = \frac{1}{5}$, $\lambda = np = 10 \times \frac{1}{5} = 2$

$P(X \geq 8) = C_{10}^{8} (\frac{1}{5})^8 (1-\frac{1}{5})^2 + C_{10}^{9} (\frac{1}{5})^9 (1-\frac{1}{5}) + (\frac{1}{5})^{10}$

$= 45 \times \frac{1}{390625} \times \frac{16}{25} + 10 \times \frac{1}{1953125} \times \frac{4}{5} + \frac{1}{9765625}$

$= \frac{720 + 40 + 1}{9765625}$

$= \frac{761}{9765625} = \frac{761}{5^{10}}$

2.43 Based upon past experience, 1% of the telephone bills mailed to households are incorrect. If a sample of 20 bills is selected, using the binomial distribution and the Poisson approximation to the binomial distribution, find the probability that at least one bill is incorrect. Briefly compare and explain your results.

Let $X = \#$ bill(s) incorrect , $\lambda = np = 20 \times 0.01 = 0.2$

binomial distribution

$P_b(X \geq 1) = 1 - P(X=0) = 1 - 0.99^{20}$

≈ 0.1821

Poisson approximation

$P_p(X \geq 1) = 1 - P(X=0) = 1 - (0.99)^{20}$

$\approx 1 - e^{-\frac{1}{5}}$

≈ 0.1813

$P_b(X \geq 1) \approx P_p(X \geq 1)$

2.51 Assume X is an exponentially distributed random variable whose expected value is $1/\lambda$. Compute the laws of the random variables e^X and $\min\{X, 3\}$.

$X \sim Exp(\lambda)$ $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$ $F(x) = \begin{cases} 1 - e^{-\lambda x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$

suppose $Y = e^X$

$F_Y(y) = P(Y \leq y)$

$= P(e^X \leq y)$

$= P(X \leq \ln y)$

$= F_X(\ln y)$

$= \begin{cases} 1 - y^{-\lambda} & y > 1 \\ 0 & 0 < y \leq 1 \end{cases}$

$f_Y(y) = F_Y'(y) >$

$f_Y(y) = \begin{cases} \lambda y^{-\lambda - 1} & y > 1 \\ 0 & 0 < y \leq 1 \end{cases}$

$Z = \min\{X, 3\} \leq 3$

1° $Z = 3$ $F_Z(z) = P(Z \leq z)$

$= P(3 \leq z)$

$= 1$

2° $X < 3$, $Z = X < 3$ $F_Z(z) = P(Z \leq z)$

$= P(X \leq z)$

$= F_X(z)$

$= \begin{cases} 1 - e^{-\lambda z} & , 0 < z < 3 \\ 0 & , z \leq 0 \end{cases}$

$\Rightarrow F_Z(z) = \begin{cases} 0 & , z \leq 0 \\ 1 - e^{-\lambda z} & , 0 < z < 3 \\ 1 & , z \geq 3 \end{cases}$

$f_Z(z) = F_Z'(z) >$

$f_Z(z) = \begin{cases} \lambda e^{-\lambda z} & , 0 < z < 3 \\ 0 & , \text{otherwise} \end{cases}$

2.52 At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

$T \sim Exp(\lambda) \qquad E(X) = \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$

$P(T > t+4 \mid T > t) = P(T > 4) = 1 - P(T \le 4) = 1 - F(4) = 1 - (1 - e^{-\frac{1}{5} \times 4}) = e^{-0.8}$

2.56 Evaluate the integral $\int_0^{+\infty} e^{-4x^2} dx$.

we know from standard normal distribution: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} dx = 1$

$\Rightarrow \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

and $e^{-\frac{x^2}{2}}$ is even

$\Rightarrow \int_0^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{2}$

so $\int_0^{+\infty} e^{-4x^2} dx = \frac{1}{2\sqrt{2}} \int_0^{+\infty} e^{-\frac{(2\sqrt{2}x)^2}{2}} d(2\sqrt{2}x)$

$\overset{t=2\sqrt{2}x}{=} \frac{1}{2\sqrt{2}} \int_0^{+\infty} e^{-\frac{t^2}{2}} dt$

$= \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2\pi}}{2}$

$= \frac{\sqrt{\pi}}{4}$