3.17 Three fair coins are tossed. Let X denote the number of heads on the first two coins, and Y denote the number of tails on the last two coins.

(a) Find the joint distribution of X and Y.

(b) Find the condition distribution of Y given that $X = 1$.

Solution: (a)

$y \backslash x$	0	1	2
0	0	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$
1	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$
2	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$	0

(b) $P(X=1) = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$

$$P(Y=0 \mid X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = \frac{1}{4}$$

$$P(Y=1 \mid X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{1}{2}$$

$$P(Y=2 \mid X=1) = \frac{P(Y=2, X=1)}{P(X=1)} = \frac{1}{4}$$

3.18 Suppose that two standard, fair dice are rolled and the sequence of scores (X_1, X_2) is recorded. Let $U = \min\{X_1, X_2\}$ and $V = \max\{X_1, X_2\}$ denote the minimum and maximum scores, respectively.

(a) Find the conditional density of U given $V = v$ for each $v \in \{1, 2, 3, 4, 5, 6\}$.

(b) Find the conditional density of V given $U = u$ for each $u \in \{1, 2, 3, 4, 5, 6\}$.

Solution: (a) when $v=1$

$$P(U=u \mid V=1) = \begin{cases} 1, & u=1 \\ 0, & \text{otherwise} \end{cases}$$

when $v=2$

$$P(U=u \mid V=2) = \begin{cases} \frac{1}{2}, & u=1,2 \\ 0, & \text{otherwise} \end{cases}$$

when $v=3$

$$P(U=u \mid V=3) = \begin{cases} \frac{1}{3}, & u=1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

when $v=4$

$$P(U=u \mid V=4) = \begin{cases} \frac{1}{4}, & u=1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$$

when $v=5$

$$P(U=u \mid V=5) = \begin{cases} \frac{1}{5}, & u=1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$$

when $v=6$

$$P(U=u \mid V=6) = \begin{cases} \frac{1}{6}, & u=1,2,3,4,5,6 \\ 0, & \text{otherwise} \end{cases}$$

(b) when $u=1$

$$P(V=v \mid U=1) = \begin{cases} \frac{1}{6}, & v=1,2,3,4,5,6 \\ 0, & \text{otherwise} \end{cases}$$

when $u=2$

$$P(V=v \mid U=2) = \begin{cases} \frac{1}{5}, & v=2,3,4,5,6 \\ 0, & \text{otherwise} \end{cases}$$

when $u=3$

$$P(V=v \mid U=3) = \begin{cases} \frac{1}{4}, & v=3,4,5,6 \\ 0, & \text{otherwise} \end{cases}$$

when $u=4$

$$P(V=v \mid U=4) = \begin{cases} \frac{1}{3}, & v=4,5,6 \\ 0, & \text{otherwise} \end{cases}$$

when $u=5$

$$P(V=v \mid U=5) = \begin{cases} \frac{1}{2}, & v=5,6 \\ 0, & \text{otherwise} \end{cases}$$

when $u=6$

$$P(V=v \mid U=6) = \begin{cases} 1, & v=6 \\ 0, & \text{otherwise} \end{cases}$$

3.20 Suppose that N has the Poisson distribution with parameter 1, and given $N = n$, Y has the binomial distribution with parameters n and p.

(a) Find the joint probability density function of (N, Y).

(b) Find the probability density function of Y.

(c) Find the conditional probability density function of N given $Y = k$.

Solution: (a) $P(N=n, Y=k) = P(N=n) \cdot P(Y=k \mid N=n)$

$$P(N=n) = \frac{1}{n!} e^{-1}$$

$$P(Y=k \mid N=n) = C_n^k p^k (1-p)^{n-k}$$

$$P(N=n, Y=k) = \frac{e^{-1}}{n!} C_n^k p^k (1-p)^{n-k}$$

(b) $P(Y=k) = \sum_{n=k}^{\infty} P(N=n, Y=k) = \sum_{n=k}^{\infty} \frac{e^{-1}}{n!} C_n^k p^k (1-p)^{n-k}$

(c) $P(N=n \mid Y=k) = \dfrac{P(N=n, Y=k)}{P(Y=k)} = \dfrac{\frac{e^{-1}}{n!} C_n^k p^k (1-p)^{n-k}}{\sum_{n=k}^{\infty} \frac{e^{-1}}{n!} C_n^k p^k (1-p)^{n-k}} = \dfrac{\frac{1^n}{n!} C_n^k (1-p)^{n-k}}{\sum_{n=k}^{\infty} \frac{1^n}{n!} C_n^k (1-p)^{n-k}}$

3.21 Suppose that the joint density of X and Y is

$$f(x, y) = \frac{12}{5} x(2 - x - y) \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1$$

and zero otherwise.

(a) Find the conditional probability density function $f_{X|Y}(x|y)$ for $0 < y < 1$.

(b) Determine the probability $P(X > 1/2 | Y = 1/3)$.

Solution: (a) $f_Y(y) = \int_0^1 \frac{12}{5} x(2-x-y)\,dx = \frac{8}{5} - \frac{6}{5} y$, $0<y<1$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{12}{5} x(2-x-y)}{\frac{2}{5}(4-3y)} = \frac{6x(2-x-y)}{4-3y}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{6x(2-x-y)}{4-3y}, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(X > \frac{1}{2} \mid Y = \frac{1}{3}) = \int_{\frac{1}{2}}^{1} 2x(2-x-\frac{1}{3})\,dx = \frac{2}{3}$

3.23 Suppose that (X, Y) has probability density function

$$f(x, y) = 15x^2 y \quad \text{for } 0 \leqslant x \leqslant y \leqslant 1.$$

(a) Find the conditional density of X given $Y = y$.

(b) Find the conditional density of Y given $X = x$.

(c) Are X and Y independent?

Solution: (a) $f_Y(y) = \int_0^y 15x^2y\,dx = 5x^3y\Big|_0^y = 5y^4,\ 0 \le y \le 1$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{3x^2}{y^3}, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $f_X(x) = \int_x^1 15x^2y\,dy = \frac{15}{2}x^2(1-x^2),\ 0 \le x \le 1$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{2y}{1-x^2}, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(c) $f_X(x) \cdot f_Y(y) = 5y^4 \times \frac{15}{2}x^2(1-x^2) \neq f(x,y)$

$\therefore X$ and Y are not independent.

3.25 Suppose that X is uniformly distributed in the interval $(0,1)$, and that given $X = x$, Y is uniformly distributed in the interval $(0,x)$.

(a) Find the joint density of (X,Y).

(b) Find the probability density function of Y.

(c) Find the conditional probability density function of X given $Y = y$ for $y \in (0,1)$.

Solution: (a) $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$ $\qquad f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0<y<x \\ 0, & \text{otherwise} \end{cases}$

$$f(x,y) = f(x) \cdot f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0<y<x<1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $f(y) = \int_y^1 \frac{1}{x}\,dx = -\ln y,\ 0<y<1$

(c) $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} -\frac{1}{x\ln y}, & 0<y<x<1 \\ 0, & \text{otherwise} \end{cases}$$

3.26 Suppose that (X,Y) has probability density function

$$f(x,y) = \frac{1}{12\pi}e^{-\left(\frac{x^2}{8}+\frac{y^2}{18}\right)}, \quad (x,y) \in \mathbb{R}^2.$$

(a) Find the conditional density function of X given $Y = y$.

(b) Find the conditional density function of Y given $X = x$.

(c) Are X and Y independent?

Solution: (a) $f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{12\pi} e^{-\left(\frac{x^2}{8}+\frac{y^2}{18}\right)}\,dx = \frac{\sqrt{2}}{6\sqrt{\pi}} e^{-\frac{y^2}{18}}, \quad y \in \mathbb{R}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}} \quad , (x,y) \in R^2$$

(b) $f_X(x) = \int_{-\infty}^{\infty} \frac{1}{12\pi} e^{-(\frac{x^2}{8}+\frac{y^2}{18})} dy = \frac{\sqrt{2}}{4\sqrt{\pi}} e^{-\frac{x^2}{8}} \quad , x \in R$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{3\sqrt{\pi}} e^{-\frac{y^2}{18}} \quad , (x,y) \in R^2$

(c) $f_X(x) \cdot f_Y(y) = \frac{1}{12\pi} e^{-(\frac{x^2}{8}+\frac{y^2}{18})} = f(x,y)$

They are independent.

3.27 Suppose that a box has 12 balls labeled $1, 2, \cdots, 12$. Two independent repetitions are made of the experiment of selecting a ball at random from the box. Let X denote the numbers on the balls selected. Compute the probability function of X.

Solution:

X	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
P	$\frac{1}{144}$	$\frac{2}{144}$	$\frac{3}{144}$	$\frac{4}{144}$	$\frac{5}{144}$	$\frac{6}{144}$	$\frac{7}{144}$	$\frac{8}{144}$	$\frac{9}{144}$	$\frac{10}{144}$	$\frac{11}{144}$	$\frac{12}{144}$	$\frac{11}{144}$	$\frac{10}{144}$	$\frac{9}{144}$	$\frac{8}{144}$	$\frac{7}{144}$	$\frac{6}{144}$	$\frac{5}{144}$	$\frac{4}{144}$	$\frac{3}{144}$	$\frac{2}{144}$	$\frac{1}{144}$

3.28 Let X be geometrically distributed random variable having parameter p. Let $Y = X$ if $X > M$ and let $Y = M$ if $X \geqslant M$; that is, $Y = \min\{X, M\}$. Compute the probability function of Y.

Solution: $P(X=k) = (1-p)^{k-1} \cdot p$

when $Y=k$, $k<M$

$P(Y=k) = P(X=k) = (1-p)^{k-1} \cdot p$

when $Y=M$

$P(Y=M) = P(X \geq M) = \sum_{k=M}^{\infty} (1-p)^{k-1} p = (1-p)^{M-1}$

3.36 Suppose that X and Y are independent Poisson random variables such that $Var(X) + Var(Y) = 5$. Evaluate $P(X + Y < 2)$.

Solution: $Var(X) = \lambda_1$

$Var(Y) = \lambda_2$

$\lambda_1 + \lambda_2 = 5$

$X+Y \sim P(\lambda_1+\lambda_2) \Rightarrow X+Y \sim P(5)$

$P(X+Y<2) = P(X+Y=0) + P(X+Y=1) = e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!} = 6e^{-5}$

Example 3.3.3 A store has a certain goods. Let X be the number of customers entering this store in a specified period of time. Suppose that $X \sim P(\lambda)$ and the probability of the event that each customer purchases the certain goods is p. If customs are independent, then find the probability function of the number of customers who purchase the certain goods.

Solution: Y is the number of customers who purchase the certain goods.

$P(X=m) = \frac{\lambda^m}{m!} e^{-\lambda}$, $m=0,1,2\cdots$

$P(Y=k|X=m) = C_m^k p^k (1-p)^{m-k}$, $k=0,1,2,\cdots,m$

$P(Y=k) = \sum_{m=k}^{+\infty} P(X=m) P(Y=k|X=m) = \frac{\lambda^k}{k!} e^{-\lambda p}$, $k=0,1,2,\cdots$

Example 3.3.4 Given

$$f(x,y) = \begin{cases} k & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Solution: $\iint_{R^2} f(x,y)\,dx\,dy = \int_0^1 \int_0^y k\,dx\,dy = \frac{k}{2} = 1$

$k=2$

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x)$, $0<x<1$

$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_0^y 2\,dx = 2y$, $0<y<1$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y}, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{1-x}, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

Example 3.3.5 Given

$$f(x,y) = \begin{cases} \frac{3}{16}(4-2x-y) & \text{for } x>0,\ y>0,\ 2x+y<4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine $f_{Y|X}(y|x)$.
(b) Determine the value of $P(Y \geqslant 2 \mid X \leqslant 1/2)$.
(c) Determine the value of $P(Y \geqslant 2 \mid X = 1/2)$.

Solution: (a) $0<x<2$, $f_X(x) = \int_0^{4-2x} \frac{3}{16}(4-2x-y)\,dy = \frac{3}{8}(2-x)^2$

$x \leq 0$ or $x \geq 1$, $f_X(x) = 0$.

$0<x<2$, $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{4-2x-y}{2(2-x)^2}, & 0<y<4-2x \\ 0, & \text{otherwise.} \end{cases}$

(b) $P(Y\geq 2|X\leq \frac{1}{2}) = \dfrac{\int_0^{\frac{1}{2}} dx \int_2^{4-2x} \frac{3}{16}(4-2x-y)\,dy}{\int_0^{\frac{1}{2}} \frac{3}{8}(2-x)^2\,dx} = \frac{7}{64}$

(c) $P(Y\geq 2|X=\frac{1}{2}) = \int_2^{4-1} \frac{2(3-y)}{9}\,dy = \frac{1}{9}$

Example 3.4.3 Let X and Y be independent geometric variables with common probability function

$$p(k) = p(1-p)^{k-1}, \qquad k = 1, 2, \cdots$$

Determine the probability function of $X + Y$.

Solution: $X+Y=n \Rightarrow X=k, Y=n-k$

$$P(X+Y=n) = \sum_{k=1}^{n-1} P(X=k, Y=n-k)$$

$$= \sum_{k=1}^{n-1} P(X=k) P(Y=n-k)$$

$$= \sum_{k=1}^{n-1} p(1-p)^{k-1} p(1-p)^{n-k-1}$$

$$= (n-1)p^2(1-p)^{n-2} \qquad n = 2, 3, \cdots$$

Example 3.4.4 Assume X and Y are independent, and $X \sim B(n_1, p)$, $Y \sim B(n_2, p)$. Prove $X + Y \sim B(n_1 + n_2, p)$.

Solution: $$P(X+Y=k) = \sum_{k_1} P(X=k_1) P(Y=k-k_1)$$

$$= \sum_{k_1=0}^{k} C_{n_1}^{k_1} p^{k_1} (1-p)^{n_1-k_1} \cdot C_{n_2}^{k-k_1} p^{k-k_1} (1-p)^{n_2-k+k_1}$$

$$= p^k (1-p)^{n_1+n_2-k} \sum_{k_1=0}^{k} C_{n_1}^{k_1} \cdot C_{n_2}^{k-k_1}$$

$$= C_{n_1+n_2}^{k} p^k (1-p)^{n_1+n_2-k}, \quad k = 0, 1, 2 \cdots n_1+n_2$$

Example 3.4.5 Assume X and Y are independent, and $X \sim P(\lambda)$, $Y \sim P(\mu)$. Prove $X + Y \sim P(\lambda + \mu)$.

Solution: $$P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)$$

$$= \sum_{k=0}^{n} P(X=k) P(Y=n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{n-k}}{(n-k)!}$$

$$= e^{-\lambda+\mu} \sum_{k=0}^{n} \frac{\lambda^k \mu^{n-k}}{k!(n-k)!}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} (\lambda+\mu)^n, \quad n = 1, 2, \cdots$$