1.5 Quantifiers 量词

Consider the following statement

n is an odd integer

Is this a proposition?

NO: Its truth value is based on the value of n.

An argument is a sequence of **propositions** written

$$
\begin{array}{l}
p_1 \\
p_2 \\
\vdots \\
p_n \\
\hline
\therefore\ q
\end{array}
$$

1.5 Quantifiers 量词

Definition 1.5.1 Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a propositional function **(命题函数)** or predicate **(谓词)** (with respect to D) if for each $x \in D$，$P(x)$ is a proposition.

We call D the domain of discourse **(论域)** of P.

The domain of discourse specifies the alllowable values for x.

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

$$\text{for every } x,\ P(x)$$

is said to be a **universally quantified statement (全称量词语句).**

It may be written

$$\forall x\ P(x).$$

The symbol ∀ means “for every”, and is called a **universal quantifier (全称量词).**

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

for every x, $P(x)$

is said to be a **universally quantified statement (全称量词语句).**

$$\forall x\, P(x)$$

The statement is true
The statement is false

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

for every x, $P(x)$

is said to be a **universally quantified statement (全称量词语句).**

$$\forall x \, P(x)$$

The statement is true if $P(x)$ is true for every x in D.
The statement is false if $P(x)$ is false for at least one x.

Universal Quantifiers 全称量词

Definition 1.5.4 Let P be a propositional function with domain of discourse D. The statement

for every x, $P(x)$

is said to be a **universally quantified statement (全称量词语句).**

for all x, $P(x)$
for any x,
$P(x)$

Alternative ways to write $\forall x\ P(x)$.

for all x in D, $P(x)$

Specify the domain of discourse.

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

there exists x, $P(x)$

is said to be an **existentially quantified statement (存在量词语句).**

$$\exists x\, P(x)$$

The statement is true
The statement is false

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

there exists x, $P(x)$

is said to be an **existentially quantified statement (存在量词语句).**

$$\exists x\, P(x)$$

The statement is true if $P(x)$ is true for at least one x in D.
The statement is false if $P(x)$ is false for every x in D.

Existential Quantifiers 存在量词

Definition 1.5.9 Let P be a propositional function with domain of discourse D. The statement

there exists x, $P(x)$

is said to be an **existentially quantified statement (存在量词语句).**

there exists x such that, $P(x)$
for some x, $P(x)$
for at least one x, $P(x)$

Alternative ways to write $\exists x\, P(x)$.

Generalized De Morgan’s Laws for Logic 广义德•摩根定律

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

De Morgan’s Laws for Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Generalized De Morgan's Laws for Logic 广义德•摩根定律

Suppose that the domain of discourse of the propositional function P is $\{-2, 0, 5\}$.

The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-2) \wedge P(0) \wedge P(5)$$

The propositional function $\exists x\, P(x)$ is equivalent to

$$P(-2) \vee P(0) \vee P(5)$$

Exercise

"All lions are fierce"

$P(x)$: x is a lion

$Q(x)$: x is fierce

(A) $\forall x\ (P(x) \wedge Q(x))$

(B) $\forall x\ (P(x) \rightarrow Q(x))$

Exercise

"All lions are fierce"

$P(x)$: x is a lion

$Q(x)$: x is fierce

(A) $\forall\ x\ (\ P(x) \land Q(x)\ )$

(B) $\forall\ x\ (P(x) \rightarrow Q(x)\ )$

$P(x)$	$Q(x)$	$P(x) \land Q(x)$
T	T	T
T	F	F
F	T	F
F	F	F

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise

"Some lions do not drink coffee"

$P(x)$: x is a lion

$Q(x)$: x drinks coffee

(A) $\exists\, x\ (\ P(x) \land \neg Q(x)\ )$

(B) $\exists\, x\ (P(x) \rightarrow \neg Q(x)\ )$

Exercise

"Some lions do not drink coffee"

$P(x)$: x is a lion

$R(x)$: x drinks coffee

(A) $\exists\, x\ (\ P(x) \land \neg Q(x)\ )$

(B) $\exists\, x\ (P(x) \rightarrow \neg Q(x)\ )$

$P(x)$	$R(x)$	$\neg R(x)$	$P(x)\ \land\ \neg R(x)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

$P(x)$	$R(x)$	$\neg R(x)$	$P(x) \rightarrow\ \neg R(x)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Exercise

$P(x)$: x is an accountant

$Q(x)$: x owns a Porsche

Write the following statements symbolically.

(1) All accountants own Porsches.

(2) Some accountant owns a Porsche.

(3) All owners of Porsches are accountants.

(4) Someone who owns a Porsche is an accountant.

Rules of Inference for Quantified Statements 量词推理规则

TABLE 1.5.1 Rules of Inference for Quantified Statements†

Rule of Inference	*Name*
$\forall x\, P(x)$ / $\therefore P(d)$ if $d \in D$	Universal instantiation
$P(d)$ for every $d \in D$ / $\therefore \forall x\, P(x)$	Universal generalization
$\exists x\, P(x)$ / $\therefore P(d)$ for some $d \in D$	Existential instantiation
$P(d)$ for some $d \in D$ / $\therefore \exists x\, P(x)$	Existential generalization

† The domain of discourse is D.

Universal Instantiation 全称例化

$$\frac{\forall x\, P(x)}{\therefore\ P(d) \text{ if } d \in D}$$

Universal Generalization 全称一般例化

$$\frac{P(d) \text{ for every } d \in D}{\therefore \ \forall x\, P(x)}$$

Existential Instantiation 存在例化

$$\frac{\exists x\, P(x)}{\therefore\ P(d) \text{ for some } d \in D}$$

Existential Generalization 存在一般例化

$$\frac{P(d) \text{ for some } d \in D}{\therefore\ \exists x\, P(x)}$$

Rules of Inference for Quantified Statements 量词推理规则

Practice Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

Every BUPT student is a genius. Zhang is a BUPT student. Therefore, Zhang is a genius.

Rules of Inference for Quantified Statements 量词推理规则

Example 1.5.23 Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

For every real number x, if x is an integer, then x is a rational number. The number $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

Arguments with Quantified Statements

Universal instantiation:

$$\forall x, P(x)$$
$$\therefore P(a)$$

Universal modus ponens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$P(a)$$
$$\therefore\ Q(a)$$

Universal modus tollens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$\neg Q(a)$$
$$\therefore\ \neg P(a)$$

Practice

$$\begin{array}{l} \forall x\big(p(x)\vee q(x)\big) \\ \forall x\Big(\big(\neg p(x)\wedge q(x)\big)\rightarrow r(x)\Big) \\ \hline \therefore \forall x\big(\neg r(x)\rightarrow p(x)\big) \end{array}$$

$$\begin{array}{l} \forall x\left(p(x) \vee q(x)\right) \\ \forall x\left(\left(\neg p(x) \wedge q(x)\right) \rightarrow r(x)\right) \\ \hline \therefore \forall x\left(\neg r(x) \rightarrow p(x)\right) \end{array}$$

$(1)\forall x\left(p(x) \vee q(x)\right)$ premise

$(2)p(c) \vee q(c)$ step1+rule of universal specification

$(3)\forall x\left(\left(\neg p(x) \wedge q(x)\right) \rightarrow r(x)\right)$ premise

$(4)\left(\neg p(c) \wedge q(c)\right) \rightarrow r(c)$ step3+rule of univ. specif.

$(5)\neg r(c) \rightarrow \neg\left(\neg p(c) \wedge q(c)\right)$ step4+Contrapositive

$(6)\neg r(c) \rightarrow \left(p(c) \vee \neg q(c)\right)$ DeMorgan's law+Double Negation

$(7)\neg r(c)$ premise assumed

$(8)p(c) \vee \neg q(c)$ Step 7+6+Modus ponens

$(9)\left(p(c) \vee q(c)\right) \wedge \left(p(c) \vee \neg q(c)\right)$ Step2+8+Rule Conjunction

$(10)p(c) \vee \left(q(c) \wedge \neg q(c)\right)$ Step 9+ Distrutive law

$(11)p(c)$ Step 10+$q(c) \wedge \neg q(c) \Leftrightarrow F + P(c) \vee F = P(c)$

$(12)\therefore \forall x\left(\neg r(x) \rightarrow p(x)\right)$ Step 7+11+rule univ generalization

Problem-Solving Tips

- To prove that the universally quantified statement $\forall x P(x)$ is true, show that for *every* x in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for a *particular* value x does *not* prove that $\forall x P(x)$ is true.
- To prove that the existentially quantified statement $\exists x P(x)$ is true, find *one* value of x in the domain of discourse for which the proposition $P(x)$ is true. *One* value suffices.
- To prove that the universally quantified statement $\forall x P(x)$ is false, find *one* value of x (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false.
- To prove that the existentially quantified statement $\exists x P(x)$ is false, show that for *every* x in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for a *particular* value x does *not* prove that $\exists x P(x)$ is false.

1.6 Nested Quantifiers 嵌套量词

Consider writing the statement

“The sum of any two positive real numbers is positive”

symbolically.

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- Two numbers are involved,
- need two universal quantifiers

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Consider writing the statement

"The sum of any two positive real numbers is positive"

symbolically.

- Two numbers are involved,
- need two universal quantifiers

$P(x, y):\ (x > 0) \wedge (y > 0) \longrightarrow (x + y > 0)$

Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\ P(x, y)$

Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers** **（嵌套量词）.**

1.6 Nested Quantifiers 嵌套量词

Consider writing the statement

"The sum of any two positive real numbers is positive"

symbolically.

- Two numbers are involved,
- need two universal quantifiers

$P(x, y): (x > 0) \wedge (y > 0) \longrightarrow (x + y > 0)$

Statement can be written as $\forall \boldsymbol{x} \forall \boldsymbol{y}\ P(x, y)$

Multiple quantifiers such as $\forall \boldsymbol{x} \forall \boldsymbol{y}$ said to be **nested quantifiers.**

Any other Nested Quantifiers?

1.6 Nested Quantifiers 嵌套量词

Example 1.6.1 Restate $\forall m \exists n\ (m < n)$ in words.

1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
“Everybody loves somebody.”
symbolically, letting $L(x, y)$ be the statement “x loves y”.

1.6 Nested Quantifiers 嵌套量词

Example 1.6.2 Write the following statement
"Everybody loves somebody."
symbolically, letting $L(x, y)$ be the statement "x loves y".

(A) $\forall x \exists y\, L(x, y)$

(B) $\exists x \forall y\, L(x, y)$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.13

$$\neg(\forall x \exists y P(x,y)) \equiv$$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.13

$$\neg(\forall x \exists y P(x, y)) \equiv \exists x \forall y \neg P(x, y)$$

1.6 Nested Quantifiers 嵌套量词

Example 1.6.14 Write the negation of $\exists x \forall y (xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

Problem-Solving Tips

$$\forall x \forall y\, P(x, y)$$

- To prove that $\forall x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols x and y to stand for *arbitrary* elements in X and Y.
- To prove that $\forall x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ false.

Problem-Solving Tips

$$\forall x \exists y\, P(x, y)$$

- To prove that $\forall x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.
- To prove that $\forall x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always false.

Problem-Solving Tips

$$\exists x \forall y\, P(x, y)$$

- To prove that $\exists x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always true.
- To prove that $\exists x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.

Problem-Solving Tips

$$\exists x \exists y\, P(x, y)$$

- To prove that $\exists x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ true.
- To prove that $\exists x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols x and y to stand for *arbitrary* elements in X and Y.

Problem-Solving Tips

- To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \rightarrow q$ is equivalent to $p \wedge \neg q$.