EBU4375: SIGNALS AND SYSTEMS

TOPIC 5: LAPLACE TRANSFORMS

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DEFINITION OF LAPLACE TRANSFORM

The Laplace transform $X(s)$ of the time-domain signal $x(t)$ is defined by the integral

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

The integral differs from the Fourier transform in two main ways:

- The integration is from $t=0$ to $t=\infty$. (It is causal).
- The Laplace integral contains the factor e^{-st} (rather than $e^{-j\omega t}$ as in the Fourier transform).

THE LAPLACE VARIABLE S

The Laplace variable s is a complex number, where $s = \sigma + j\omega$.

Hence, $e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t}$

The term $e^{-j\omega t}$ is identical to the term in the Fourier integral.

Depending on the value and sign of σ , the factor represents a growing or decaying exponential.

The product $e^{-\sigma t} e^{-j\omega t}$, therefore represents an exponentially growing or decaying complex frequency component.

$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = e^{-\sigma t}(\cos \omega t - j \sin \omega t)$$

The Laplace variable $s = \sigma + j\omega$ is sometimes called the complex frequency. The s plane is called the complex frequency domain.

LAPLACE TRANSFORM PAIRS

As with Fourier transforms, the relationship between the time-domain model of a signal and its Laplace transform is unique.

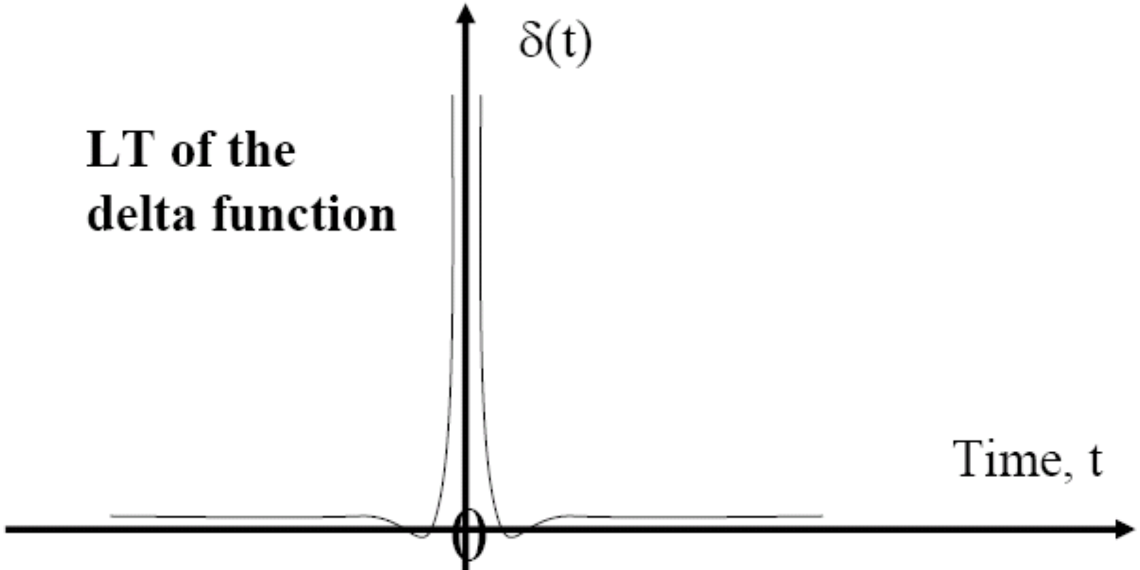
A signal and its Laplace transform form a transform pair, which is denoted by the double-headed arrow

Example 1: Evaluate the LT of the exponential decay: $x(t) = Ae^{-\alpha t} \overset{LT}{\longleftrightarrow} \dfrac{A}{s+\alpha}$

This provides a starting point from which the transforms of many other functions can be derived.

LAPLACE TRANSFORM PAIRS

Example 2:



$$X(s) = \int_{0}^{\infty} \delta(t).\ e^{-st}\, dt$$

$$X(s) = \int_{0}^{0} \delta(t).e^{-s0}\, dt = \int_{0}^{0} \delta(t).1\, dt = 1 \qquad \boxed{\boldsymbol{\delta(t) \longleftrightarrow 1}}$$

Note: strictly speaking, $\delta(t)$ is the Dirac Delta Function

APPLICATION OF THE LT TO SINUSOIDS

Example 3: $x(t) = A\cos(\omega t)$

Try the signal is a cosine $x(t) = A\cos(\omega t)$

$$X(s) = \int_0^\infty A\cos(\omega t).\ e^{-st}\,dt$$

$$X(s) = A\int_0^\infty [e^{j\omega t} + e^{-j\omega t}]/2\ .\ e^{-st}\,dt$$

$$X(s) = (A/2)\int_0^\infty [e^{j\omega t}\,e^{-st} + e^{-j\omega t}\,e^{-st}]\,dt$$

but these ----------- -----------

are standard results for the exponential, and allow solutions for sinusoids (via the LT) that are rigorous enough to satisfy everyone.

APPLICATION OF THE LT TO SINUSOIDS

$A.e^{-\alpha t} \longleftrightarrow A/(\alpha + s)$, and $A.e^{+\alpha t} \longleftrightarrow A/(s - \alpha)$

apply to:

$$X(s) = (A/2) \int_0^\infty [e^{j\omega t} e^{-st} + e^{-j\omega t} e^{-st}]\, dt$$

$$X(s) = (A/2)\, [1/(s - j\omega) + 1/(s + j\omega)]$$

$$X(s) = (A/2) \frac{[s + j\omega + s - j\omega]}{[(s - j\omega).(s + j\omega)]} = (A/2) \frac{2s}{[s^2 - j\omega s - j^2\omega^2 + j\omega s]}$$

$$X(s) = (\cancel{2}As/\cancel{2}).[1/(s^2 + \omega^2)] = As/(s^2 + \omega^2)$$

USING STANDARD LT RESULTS FOR COMPOSITE SIGNALS

The Laplace Transform is a Linear transform

If we have e.g.

$X(t) = (2/3)u(t) - e^{-2t} + (1/3)e^{-3t}$

we can take the LT directly:

$X(s) = (2/3) \int u(t).e^{-st}\,dt \;-\; \int e^{-2t}.e^{-st}\,dt \;+\; (1/3) \int e^{-3t}\,e^{-st}\,dt$

but also note these are standard results:

$X(s) = (2/3)1/s \;-\; 1/(s + 2) + (1/3).(1 / (s + 3))$

and this is the quicker way to the answer.

PROPERTIES OF THE LAPLACE TRANSFORM

- Linearity
- Right shift in time
- Time Scaling
- S-plane shift
- Multiplication by a Power of t
- Differentiation in the Time Domain
- Integration

PROPERTIES: LINEARITY

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

Example :

$$L\{\sinh(t)\} =$$

$$y\{\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\} =$$

$$\frac{1}{2}L\{e^{t}\} - \frac{1}{2}L\{e^{-t}\} =$$

$$\frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+1}) =$$

$$\frac{1}{2}(\frac{(s+1)-(s-1)}{s^2-1}) = \frac{1}{s^2-1}$$

Proof :

$$L\{c_1 f_1(t) + c_2 f_2(t)\} =$$

$$\int_0^\infty [c_1 f_1(t) + c_2 f_2(t)] e^{-st} dt =$$

$$c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t) e^{-st} dt =$$

$$c_1 F_1(s) + c_2 F_2(s)$$

PROPERTIES: SCALING IN TIME

$$L\{f(at)\} = \frac{1}{a} F(\frac{s}{a})$$

Example :

$$L\{\sin(\omega t)\}$$

$$\frac{1}{\omega}(\frac{1}{(s/\omega)^2}+1) =$$

$$\frac{1}{\omega}(\frac{\omega^2}{s^2+\omega^2}) =$$

$$\frac{\omega}{s^2+\omega^2}$$

Proof :

$$L\{f(at)\} =$$

$$\int_0^\infty f(at)e^{-st}dt =$$

let $u = at, t = \frac{u}{a}, dt = \frac{1}{a}du$

$$\frac{1}{a}\int_0^{\frac{\infty}{a}} f(u)e^{-(\frac{s}{a})u}du =$$

$$\frac{1}{a}F(\frac{s}{a})$$

PROPERTIES: TIME SHIFT

$$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$$

Example :

$$L\{e^{-a(t-10)}u(t-10)\} =$$

$$\frac{e^{-10s}}{s+a}$$

Proof :

$$L\{f(t-t_0)u(t-t_0)\} =$$

$$\int_0^\infty f(t-t_0)u(t-t_0)e^{-st}\,dt =$$

$$\int_{t_0}^\infty f(t-t_0)e^{-st}\,dt =$$

let $u = t-t_0,\, t = u+t_0$

$$\int_0^{\infty-t_0} f(u)e^{-s(u+t_0)}\,du =$$

$$e^{-st_0}\int_0^\infty f(u)e^{-su}\,du = e^{-st_0}F(s)$$

PROPERTIES: S-PLANE (FREQUENCY) SHIFT

$$L\{e^{-at} f(t)\} = F(s+a)$$

Example :

$$L\{e^{-at}\sin(\omega t)\} =$$

$$\frac{\omega}{(s+a)^2+\omega^2}$$

Proof :

$$L\{e^{-at} f(t)\} =$$

$$\int_0^\infty e^{-at} f(t) e^{-st} dt =$$

$$\int_0^\infty f(t) e^{-(s+a)t} dt =$$

$$F(s+a)$$

PROPERTIES: MULTIPLICATION BY T^N

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example :

$$L\{t^n u(t)\} =$$

$$(-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) =$$

$$\frac{n!}{s^{n+1}}$$

Proof :

$$L\{t^n f(t)\} = \int_0^\infty t^n f(t) e^{-st} dt =$$

$$\int_0^\infty f(t) t^n e^{-st} dt =$$

$$(-1)^n \int_0^\infty f(t) \frac{\partial^n}{\partial s^n} e^{-st} dt =$$

$$(-1)^n \frac{\partial^n}{\partial s^n} \int_0^\infty f(t) e^{-st} dt = (-1)^n \frac{\partial^n}{\partial s^n} F(s)$$

THE "D" OPERATOR

1. Differentiation shorthand

$$Df(t) = \frac{df(t)}{dt}$$

$$D^2 f(t) = \frac{d^2}{dt^2} f(t)$$

2. Integration shorthand

$$\text{if} \quad g(t) = \int_{-\infty}^{t} f(t)dt$$

$$\text{then} \quad Dg(t) = f(t)$$

$$\text{if} \quad g(t) = \int_{a}^{t} f(t)dt$$

$$\text{then} \quad g(t) = D_a^{-1} f(t)$$

PROPERTIES: DIFFERENTIATION

$$L\{Df(t)\} = sF(s) - f(0^+)$$

Example :

$$L\{D\cos(t)\} =$$

$$\frac{s^2}{s^2+1} - f(0^+) =$$

$$\frac{s^2}{s^2+1} - 1 =$$

$$\frac{s^2-(s^2+1)}{s^2+1}$$

$$\frac{-1}{s^2+1} = L\{-\sin(t)\}$$

Proof :

$$L\{Df(t)\} = \int_0^\infty \frac{d}{dt} f(t)e^{-st}dt$$

$$u = e^{-st}, du = -se^{-st}$$

let $dv = \frac{d}{dt} f(t)dt, v = f(t)$

$$[e^{-st} f(t)]_0^\infty + s\int_0^\infty f(t)e^{-st}dt =$$

$$-f(0^+) + sF(s)$$

PROPERTIES: INTEGRATION

$$L\{D_0^{-1} f(t)\} = \frac{F(s)}{s}$$

Example :

$$L\{D_0^{-1} \cos(t)\} =$$

$$(\frac{1}{s})(\frac{s}{s^2+1}) = \frac{1}{s^2+1}$$

$$L\{\sin(t)\}$$

DIFFERENCE IN $f(0^+), f(0^-) \,\&\, f(0)$

- The values are only different if f(t) is not continuous @ t=0
- Example of discontinuous function: u(t)

$$f(0^-) = \lim_{t \to 0^-} u(t) = 0$$

$$f(0^+) = \lim_{t \to 0^+} u(t) = 1$$

$$f(0) = u(0) = 1$$

THE INVERSE LAPLACE TRANSFORM

Like the Fourier transform the Laplace transform has an inverse which is formally defined by an integral

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

The integral in above equation is evaluated along the path $s = c + j\omega$ in the complex plane from $c - j\infty$ to $c + j\infty$.

The evaluation of this integral requires a knowledge of contour integration and complex variable theory and is seldom if ever used in routine Laplace transform work.

A frequently-used method is the partial-fraction technique.

THE INVERSE LAPLACE TRANSFORM

The method of partial fractions relies on the fact that finding the Laplace transform or its inverse is a linear operation.

1. Split up a transform into a sum of simpler transforms

2. Find the inverse of the overall transform by finding the inverse of each simpler function separately.

3. Add them together

Extensive tables of Laplace transform pairs have been prepared to remove the need to work out a transform or its inverse from first principles.

A SHORT TABLE OF LAPLACE TRANSFORM PAIRS

A short table of some of the more commonly used transform pairs is listed:

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$

EXAMPLE OF PARTIAL FRACTIONS (1)

If $X(s) = (s + 1) / (s (s + 2))$

find the original signal, $x(t)$.

$RHS = (s + 1) / (s (s + 2)) = A/s + B/(s + 2)$

where we need to find A and B.

$(s + 1) / (s (s + 2)) = A/s + B/(s + 2)$

now multiply by $s(s + 2)$ gives:

$(s + 1) = A(s + 2) + Bs$

Then:

EXAMPLE OF PARTIAL FRACTIONS (2)

$(s + 1) = A(s + 2) + Bs$

let $s = 0$

$1 = 2A$ therefore $A = 1/2$

let $s = -2$

$-1 = -2B$ therefore $B = 1/2$

therefore:

$x(t) = (1/2).u(t) + (1/2).e^{-2t}$

TUTORIAL QUESTIONS:

- Find the Inverse Laplace transform of

(1) $$F(s) = \frac{2}{s+k}$$

(2) $$F(s) = \frac{2}{s^2 + 3s + 2}$$

(3) $$F(s) = \frac{3s+5}{s^2 + 7}$$

THE TRANSFER FUNCTION FOR CTS

We know that

$$y(t) = x(t) * h(t)$$

output signal

input signal

Unit impulse response

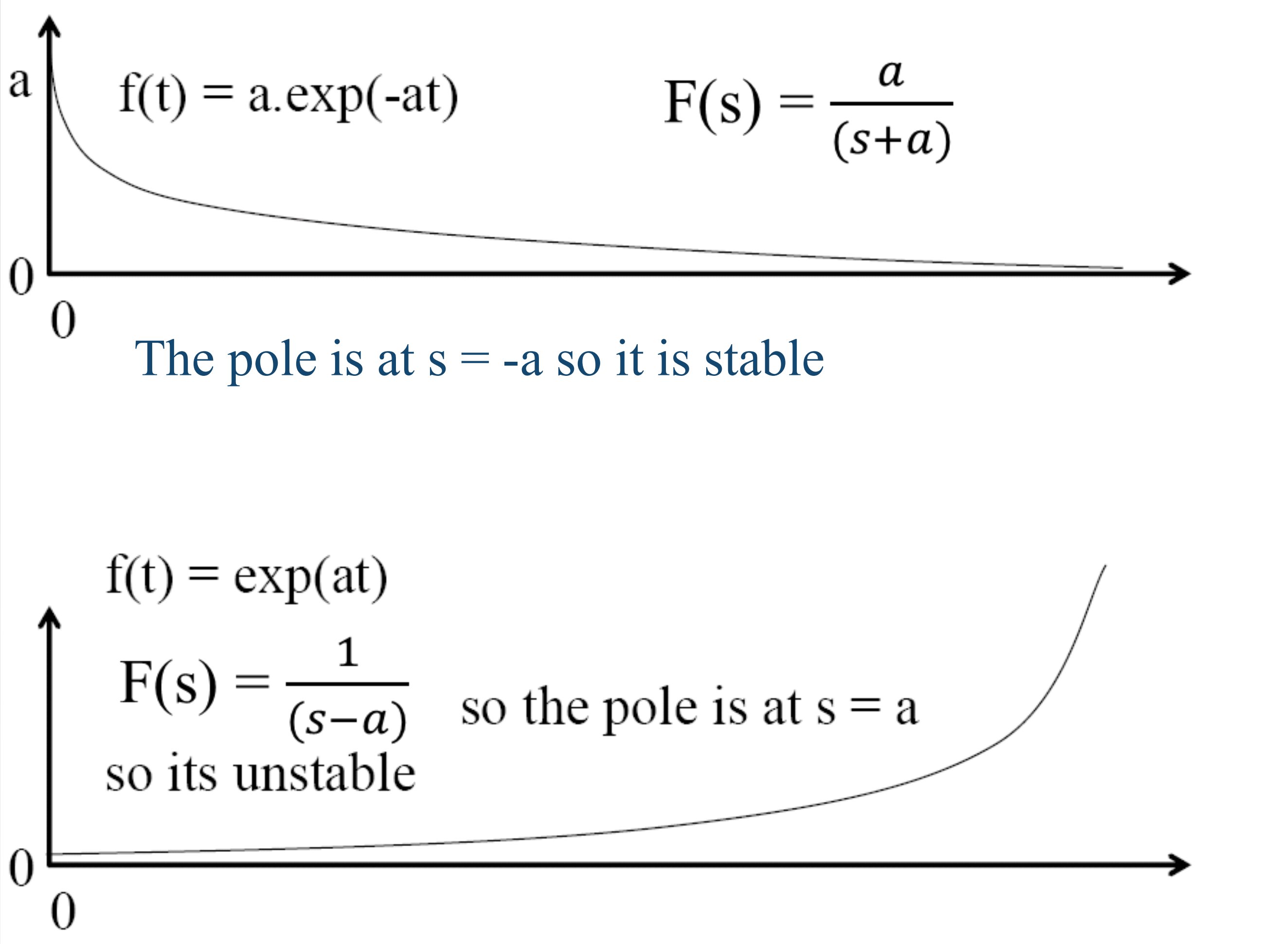
Convolution in the 'continuous time domain' can be replaced by the multiplication in s-transform domain.

$$Y(s) = X(s) \cdot H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$ is defined as transfer function.

$$H(s) \leftrightarrow h(t)$$

CONSIDER CERTAIN SIGNALS:



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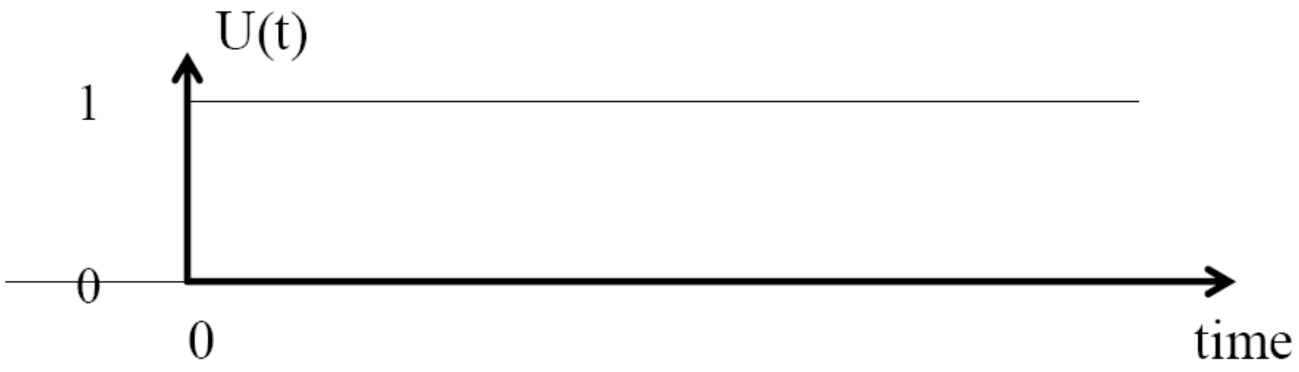
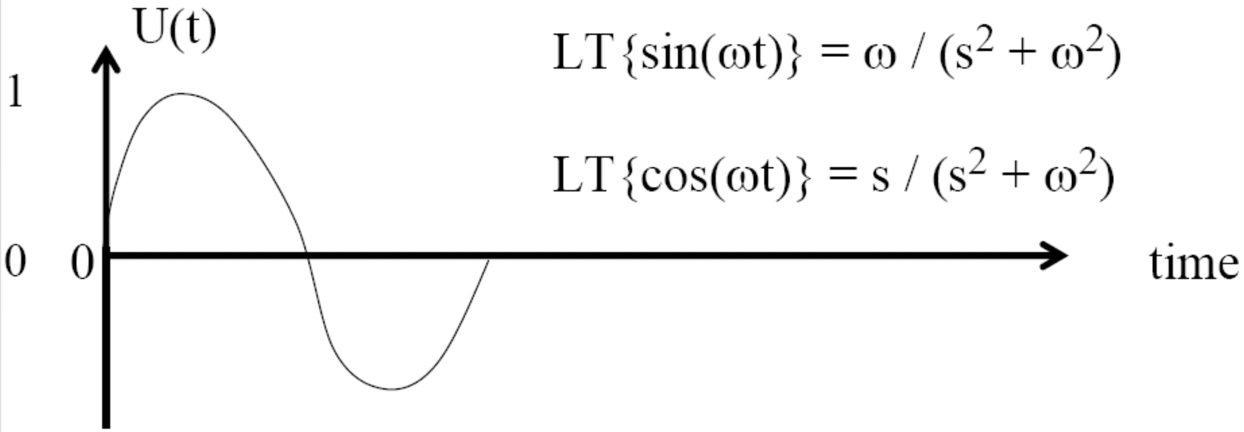


If x(t) = u(t), as above, then X(s) = 1/s

so the single pole is at s = 0

this is right on the border between stable and unstable and this is what you would expect since u(t) neither grows or decays with increasing time.

CONSIDER CERTAIN SIGNALS:



$$LT\{\sin(\omega t)\} = \omega / (s^2 + \omega^2)$$

$$LT\{\cos(\omega t)\} = s / (s^2 + \omega^2)$$

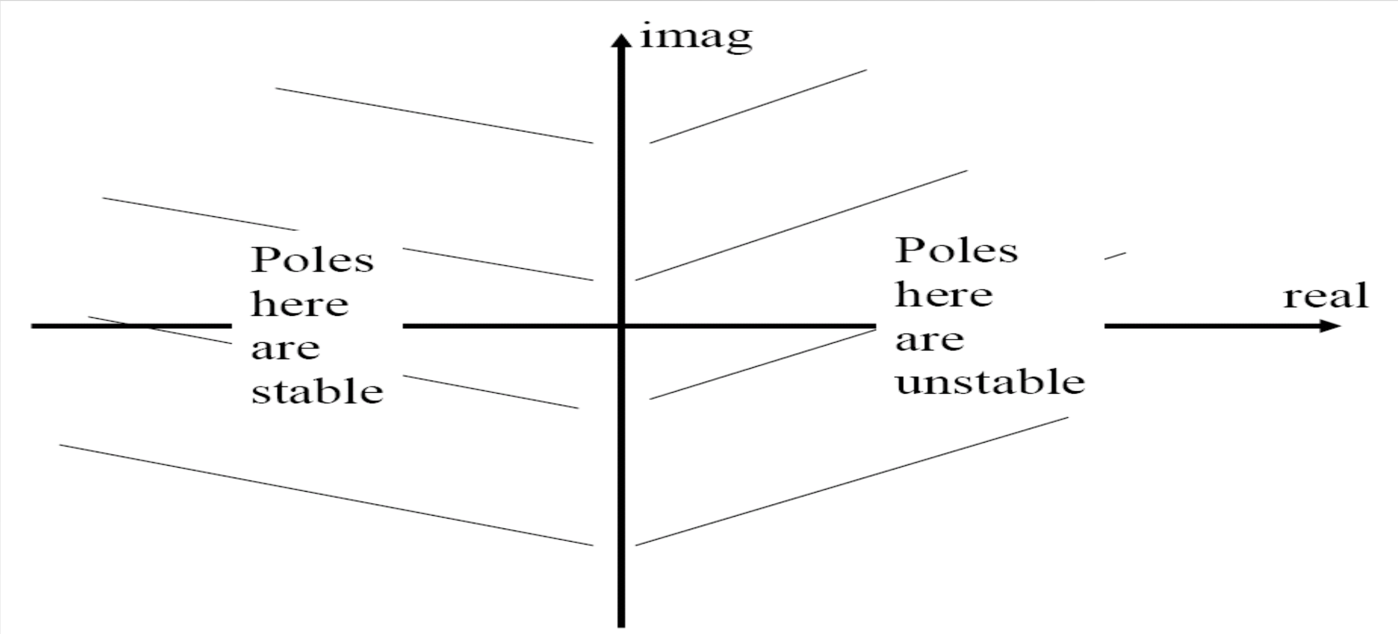
So for either type of sinusoid the pole is at:

$(s^2 + \omega^2) = 0$, which can again be solved by using the quadratic formula:

$s = [\ -0 +/- \sqrt{(0 - 4\omega^2)}\ ] / 2 = +/- j\omega$ this is again

right on the border between stability and instability.

POLE ZERO STABILITY RULE FOR LT



The Poles of the Transfer Function are the values of s that make H(s) infinite.
The Zeroes are the values of s that make H(s) zero.

EXAMPLES ON EVALUATING STABILITY VIA TRANSFER FUNCTION

1) $H(s) = (s^2 + 1) / (s + 1)^2$

2) $H(s) = (s^2 - s + 1) / (s^2 + s + 1)$

3) $H(s) = (s^2 + 2) / (s + 1)(s^2 + 0.2s + 1)$