Section 2.5 Continuous Random Variables

School of Sciences, BUPT

Contents

Uniform Distribution

Let a and b be two given real numbers such that $a < b$.

Definition

The distribution of the random variable X is called the ***uniform distribution*** *of the interval $[a, b]$ if the p.d.f. of X is*

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leqslant x \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

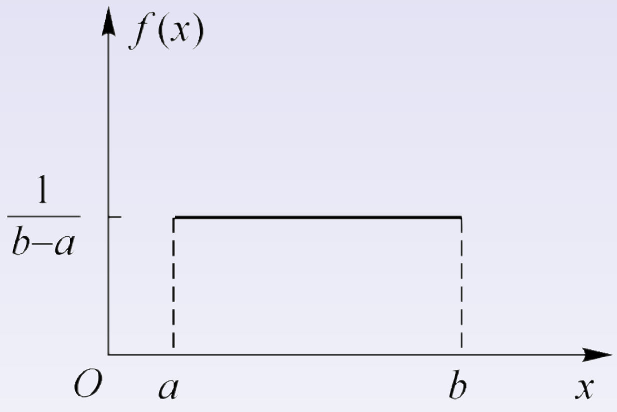
We write that by $X \sim U(a, b)$.

The corresponding d.f. of X is

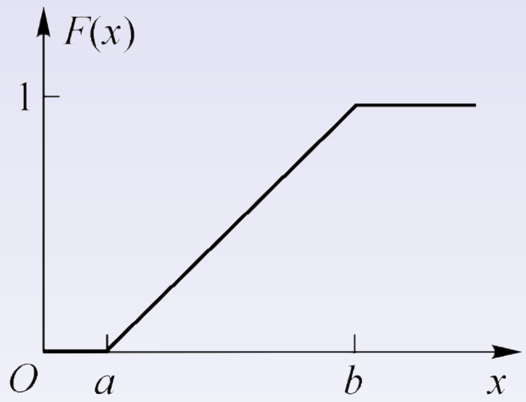
$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ \dfrac{x-a}{b-a} & \text{for } a \leqslant x < b, \\ 1 & \text{for } x \geqslant b. \end{cases}$$

Uniform Distribution

The constant a is the *location parameter* and the constant $b - a$ is the *scale parameter*.



(a) probability density function $f(x)$



(b) distribution function $F(x)$

Uniform Distribution

The case where $a = 0$ and $b = 1$ is called the ***standard uniform distribution***. The p.d.f. for the standard uniform distribution is

$$f(x) = \begin{cases} 1 & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

and the d.f. of the standard uniform distribution is

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } 0 \leqslant x < 1, \\ 1 & \text{for } x \geqslant 1. \end{cases}$$

Uniform Distribution

Proposition

Suppose that X is a random variable which has uniform distribution of the interval $[a, b]$. Then we have

$$E(X) = \frac{a+b}{2} \quad \text{and} \quad Var(X) = \frac{(b-a)^2}{12}. \tag{1}$$

Proof. Using the basic definition of expectation, we know

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{a+b}{2}.$$

We have

$$\begin{aligned} Var(X) =& E(X^2) - [E(X)]^2 \\ =& \int_a^b x^2 \cdot \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}. \end{aligned}$$

$\square$

Uniform Distribution

Example

The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval [0, 25]. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the expectation and variance of the distribution and find the distribution function $F(x)$.

Solution. Over the interval [0, 25] the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} \dfrac{1}{25-0} = 0.04 & \text{for } 0 \leqslant x \leqslant 25, \\ 0 & \text{otherwise.} \end{cases}$$

Using equation (1), we have

Uniform Distribution

Solution. Using equation (1), we have

$$E(X) = \frac{25+0}{2} = 12.5mA \text{ and } Var(X) = \frac{(25-0)^2}{12} = 52.08mA^2.$$

The distribution function is obtained by integrating the probability density function as shown below,

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Hence, choosing the three distinct regions $x < 0$, $0 \leqslant x < 25$ and $x \geqslant 25$ in turn gives:

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ \dfrac{x}{25} & \text{for } 0 \leqslant x < 25, \\ 1 & \text{for } x \geqslant 25. \end{cases}$$

$\square$

Uniform Distribution

Example

Suppose that $X \sim U(0,1)$. Let $Y = g(X) = aX + b$, $a > 0$.

(a) Find the p.d.f. $f_Y(y)$ of Y.

(b) Calculate the value of $E(Y)$ and $Var(Y)$.

Solution. (a) Obviously, the possible values taken by Y is between b and $a+b$. If $b \leqslant y \leqslant a+b$, then

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{a} = \frac{1}{a}.$$

Otherwise, we have $f_Y(y) = 0$. That means $Y \sim U(b, a+b)$.

(b) $E(Y) = E(aX+b) = aE(X) + b = \frac{a}{2} + b$.

$Var(Y) = Var(aX+b) = a^2 Var(X) = \frac{a^2}{12}$.

Uniform Distribution

Example

Suppose that $X \sim U(-1, 1)$. Let $Y = g(X) = X^2$.
(a) Find the p.d.f. $f_Y(y)$ of Y.
(b) Calculate the value of $E(Y)$ and $Var(X)$.

Solution. (a) Since $X \sim U(-1, 1)$, the p.d.f. of X is

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{for } -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Because $y = g(x) = x^2$, the possible values taken by Y is between 0 and 1.

Uniform Distribution

Solution. If $0 < y < 1$, then

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] = \frac{1}{2\sqrt{y}}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2\sqrt{y}}.$$

If $y \leqslant 0$ or $y \geqslant 1$, then $f_Y(y) = 0$.

(b)

$$E(Y) = E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)dx = \int_{-1}^{1} x^2 \cdot \frac{1}{2}dx = \frac{1}{3}.$$

$$Var(Y) = E\left[(X^2)^2\right] - [E(X^2)]^2 = \int_{-1}^{1} x^4 \cdot \frac{1}{2}dx - \frac{1}{9} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

$\square$

Uniform Distribution

Example

Given a random variable X with distribution $F_X(x)$ which is strict increasing, prove that $Y = F_X(X)$ is uniformly distributed in the interval $(0, 1)$.

(*Hint: if* $Y = g(X) = F_X(X)$, *then* $F_Y(y) = y$ for $0 \leqslant y \leqslant 1$.)

HW

Example

Given a random variable Y with uniform distribution of the interval $(0, 1)$. Prove that the distribution of the random variable $X = F_X^{-1}(Y)$ is a specified function $F_X(x)$.

Solution. For the random variable $X = F_X^{-1}(Y)$ and $x \in \mathbb{R}$,

$$P(X \leqslant x) = P(F_X^{-1}(Y) \leqslant x) = P(Y \leqslant F_X(x)) = F_X(x).$$

□

$Y = F_X(X)$ $\qquad g(\cdot) = F_X(\cdot)$

$F_X^{-1}(Y) = X$

$y<0$ $\Rightarrow 0$

Prove: $Y \sim U(0,1)$.

If $0<y<1$ $\qquad y>1 \Rightarrow 1$

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$$

$$= P(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y))$$

$$F_Y(y) = \begin{cases} 0, & y<0 \\ y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases} = y$$

$$F(\heartsuit) = P(X \le \heartsuit)$$

Attandance. Due 9:00

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise}, \end{cases} \qquad Y = e^X.$$

Find $f_Y(y)$.

$$f_Y(y) = \begin{cases} \lambda y^{-\lambda - 1}, & y > 1 \\ 0, & \text{otherwise} \end{cases}$$

$\text{Exp}(\lambda)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \_\_\_\_ \end{cases}$$

$$F(x) = \begin{cases} \int_0^x \lambda e^{-\lambda t} \, dt = 1 - e^{-\lambda x} \\ 0, & x \le 0 \end{cases}$$

$$\boxed{\begin{array}{l} y = e^x, \ x > 0 \\ \text{Range: } (1, \infty) \end{array}}$$

d.f. method: If $y > 1$,

$$F_Y(y) = P(Y \le y) = P(e^X \le y)$$

$$= P(X \le \ln y) \qquad = \int_{-\infty}^{\ln y} f_X(x)\,dx$$

$$= F_X(\ln y) \qquad = \int_0^{\ln y} \lambda e^{-\lambda x}\,dx$$

$$= 1 - e^{-\lambda \ln y}$$

$$f_Y(y) = \lambda e^{-\lambda \ln y} \cdot \frac{1}{y}$$

$$= 1 - \left(e^{\ln y}\right)^{-\lambda}$$

$$= \lambda \cdot y^{-\lambda} \cdot y^{-1}$$

$$= 1 - y^{-\lambda}$$

$$= \lambda y^{-\lambda-1}$$

$$f_Y(y) = -(-\lambda)\, y^{-\lambda-1} = \lambda y^{-\lambda-1}, \quad y > 1.$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left[g^{-1}(y)\right]'$$

$$y = g(x) = e^x$$

$$\ln y = g^{-1}(y) = x$$

$$\left(g^{-1}(y)\right)' = (\ln y)' = \frac{1}{y}$$

$$= \lambda e^{-\lambda \ln y} \cdot \frac{1}{y}.$$

Contents

Exponential Distribution

Definition

Let X be a continuous random variable whose density function is of the form

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & for\ x > 0, \\ 0 & for\ x \leqslant 0, \end{cases}$$

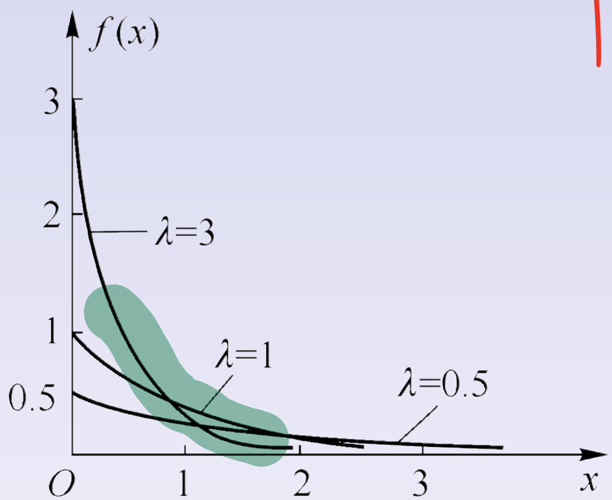
where $\lambda > 0$ is the scale parameter. We say that X follows a ***exponential distribution*** *with parameter λ. We write that $X \sim Exp(\lambda)$. The case where $\lambda = 1$ is called the standard exponential distribution.*

The corresponding d.f. of X is

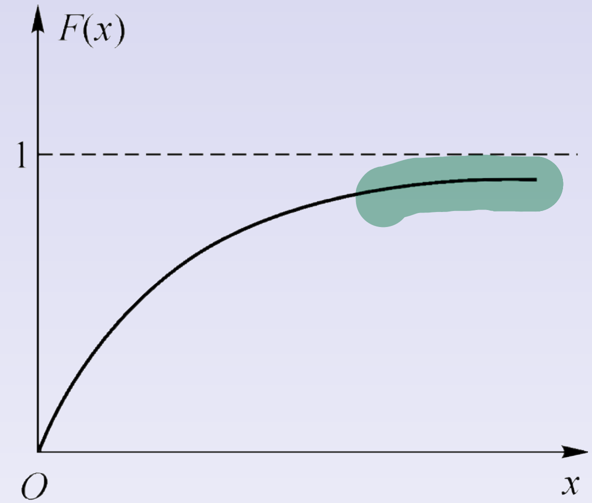
$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Exponential Distribution

$S(x) = P(X > x) = e^{-\lambda x}$, $\lambda > 0$



(a) probability density function $f(x)$



(b) distribution function $F(x)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & for\ x > 0, \\ 0 & for\ x \leqslant 0, \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0. \end{cases}$$

$= \lambda\!\!\!/ \; e^{-\lambda x}$

$\sim e^{-x^2}$

Exponential Distribution

$\int_0^\infty e^{-3x} dx = \frac{1}{3}$

We are sure about $f(x)$ is a p.d.f. since

$$\int_{-\infty}^{\infty} f(x)dx = \boxed{\int_0^{+\infty} \lambda e^{-\lambda x} dx = 1.}$$

The exponential distribution is usually used to model the time until something happens in the process.

Proposition

Suppose that X is a random variable which has exponential distribution with parameter λ. Then we have $EX^2 - EX^2$

$$E(X) = \frac{1}{\lambda} \quad \text{and} \quad Var(X) = \frac{1}{\lambda^2}. \tag{2}$$

$EX = \int_{-\infty}^{+\infty} x f(x) dx.$

$x e^{-\lambda x}, \quad x^2 e^{-\lambda x}$

Exponential Distribution

Solution. By using integration by parts,

$$
\begin{aligned}
E(X) &= \int_0^{+\infty} \lambda x e^{-\lambda x} dx \\
&= \lambda \Big(\frac{-xe^{-\lambda x}}{\lambda} \Big|_0^{+\infty} + \frac{1}{\lambda} \int_0^{+\infty} e^{-\lambda x} dx \Big) \\
&= \lambda \Big(0 + \frac{1}{\lambda} \frac{-e^{-\lambda x}}{\lambda} \Big|_0^{+\infty} \Big) = \frac{1}{\lambda}.
\end{aligned}
$$

From the first and second moments we can compute the variance as

$$
\begin{aligned}
Var(X) =& E(X^2) - [E(X)]^2 \\
=& \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx - (1/\lambda)^2 \\
=& \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.
\end{aligned}
$$

$\square$

Exponential Distribution: memoryless property

无记忆性.

~~If $X \sim Exp(\lambda)$, then~~

Def : $P(X > t + s | X > t) = P(X > s)$ for $s, t \geqslant 0$.

$$F(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Pf.

$$\text{L.H.S.} = \frac{P(X>s+t,\ X>t)}{P(X>t)}$$

$$= \frac{P(X>s+t)}{P(X>t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(X>s) \quad \square$$

$X \sim Geom(p)$.

$$p(k) = P(X=k) = (1-p)^{k-1} \cdot p.$$

$$k = 1, 2, \ldots$$

$$\frac{e^{\underset{\sim}{a+b}}}{e^{\underset{\sim}{a}}} = e^{\underset{\sim}{b}}$$

$$P(X > m+n \mid X > n) = P(X > m)$$

$$P(X > n) = (1-p)^n$$

$$= \sum_{k=n+1}^{\infty} (1-p)^{k-1} p$$

$$= \frac{(1-p)^{n+1-1} \cdot p}{1-(1-p)} = (1-p)^n$$

Exponential Distribution

Example

The lifetime (in years) of a radio has an exponential distribution with parameter $\lambda = 1/10$. If we buy a five-year-old radio, what is the probability that it will work for less than 10 additional years?

Solution. Let X be the total lifetime of the radio. We have that $X \sim Exp(\lambda = 1/10)$. We seek

$$
\begin{aligned}
P(X \leqslant 15|X > 5) &= 1 - P(X > 15|X > 5)\\
&= 1 - P(X > 10) = P(X \leqslant 10)\\
&= \int_0^{10} \frac{1}{10} e^{-x/10} dx = -e^{-x/10}\Big|_0^{10}\\
&= 1 - e^{-1} \approx 0.6321.
\end{aligned}
$$

$= P(Y<10)$

$= 1-e^{-\lambda x}$ with $\lambda = \frac{1}{10}$ and $x = 10$

$= 1-e^{-1}$

$\square$

Exponential Distribution

Example

Jobs are sent to a printer at an average of 3 jobs per hour. (a) What is the expected time between jobs? (b) What is the probability that the next job is sent within 5 minutes?

Solution. Job arrivals represent rare events, thus the time T between them is exponential with rate 3 jobs/hour, i.e., $\lambda = 3$.
(a) Thus $E(T) = 1/\lambda = 1/3$ hours or 20 minutes.
(b) Using the same units (hours) we have 5 min.$= 1/12$ hours.
Thus we compute

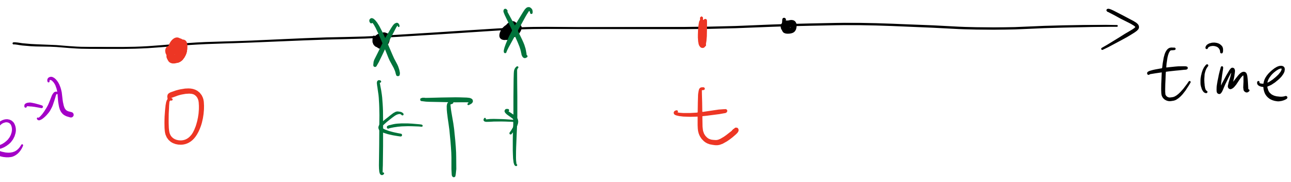
$$P(T < 1/12) = 1 - e^{-3\cdot\frac{1}{12}} = 1 - e^{-\frac{1}{4}} = 0.2212.$$

$P(X<t) = 1 - e^{-\lambda t}$

$\square$

λ: rate (#arrivals per unit time)

$\frac{\lambda^k}{k!} e^{-\lambda}$



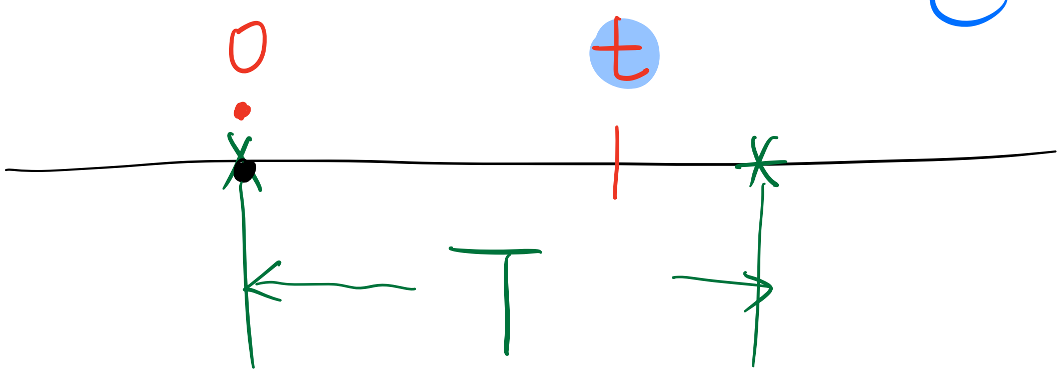
$N(t) \sim P(\lambda t) \Rightarrow P(N(t)=0) = e^{-\lambda t}$

Poisson Process.

T: inter-event time

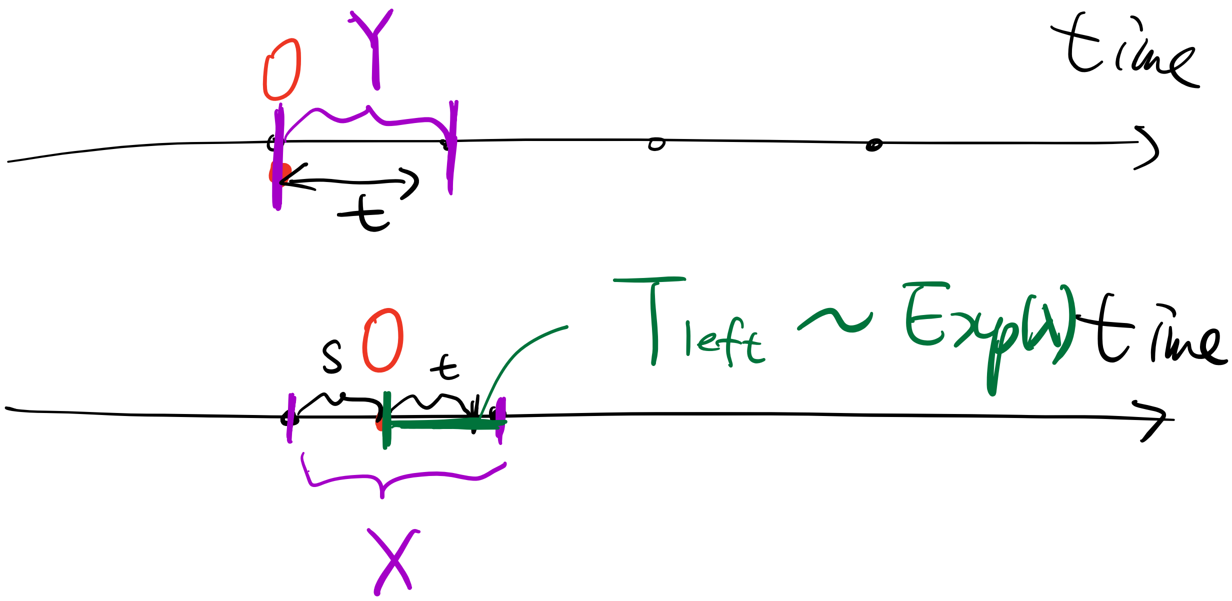
$T \sim Exp(\lambda)$ $\quad E(T) = \frac{1}{\lambda}$

Proof. $P(T>t) = P(N(t)=0) = e^{-\lambda t}$



$\therefore F_T(t) = 1 - e^{-\lambda t}, \quad t \geq 0.$

$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & \underline{\qquad\qquad} \end{cases}$



$$P(X > t+s \mid X > s) = P(X > t)$$

Exponential Distribution

Example

There is an equipment. Let $N(t)$ be the failure time of this equipment at any time length t. Assume that $N(t)$ has the Poisson distribution $P(\lambda t)$. Find the distribution of the interval time T between two failure time.

Solution. Since $N(t) \sim P(\lambda t)$,

$$P(N(t) = x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}, \qquad x = 0, 1, 2, \cdots$$

If $t > 0$, $\{T > t\} = \{N(t) = 0\}$, then

$$P(T > t) = P(N(t) = 0) = e^{-\lambda t}.$$

So

$$P(T \leqslant t) = \begin{cases} 1 - \exp(-\lambda t) & \text{for } t > 0, \\ 0 & \text{for } t \leqslant 0. \end{cases}$$

That means $T \sim Exp(\lambda)$. $\square$

Exponential Distribution

In fact, the exponential distribution is the probability distribution that describes the time between events in a Poisson process, i.e., a process in which events occur continuously and independently at a constant average rate.

Contents

1. Uniform Distribution
2. Exponential Distribution
3. Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$\boxed{\frac{X-\mu}{\sigma} \sim N(0,1)}$$

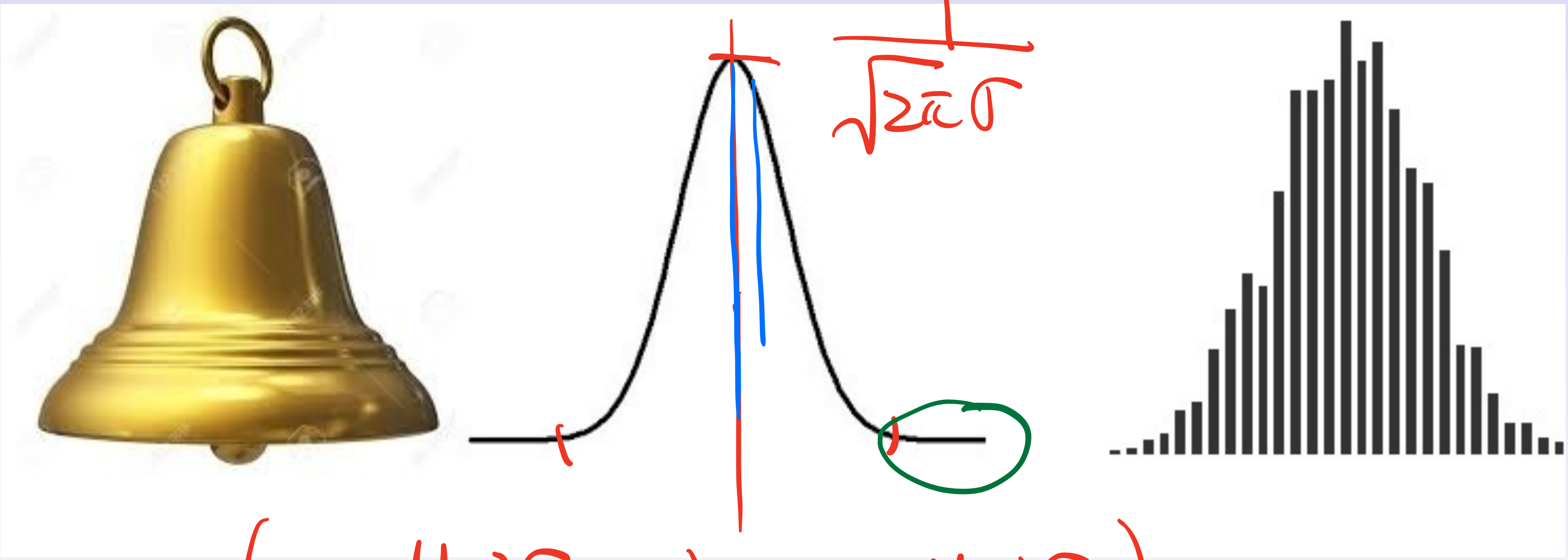
$\Phi(x)$

$\Phi(1) = 0.8413$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Normal Distribution

Bell-shaped curve. , 钟型曲线



$P(\mu-3\sigma < X < \mu+3\sigma) \approx 0.9973$

$P(|x-\mu| < 2\sigma) > 0.95$

Normal Distribution

$P(|X-\mu| < \sigma) = 0.68$

Definition

Let X be a continuous random variable that can take any real value. If its density function is given by

$\neq \varphi$!

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad for -\infty < x < \infty, \tag{3}$$

then we say that X has a ***normal (or Gaussian) distribution*** *with parameters μ and σ^2, where $\mu \in \mathbb{R}$ and $\sigma > 0$. We write that $X \sim N(\mu, \sigma^2)$.*

Normal Distribution

Now, let's verify

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad for - \infty < x < \infty,$$

is a valid probability density function by showing that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

If we let $y = (x - \mu)/\sigma$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}y^2} \sigma dy.$$

$$(\uparrow dx = \sigma dy)$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\boxed{\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \boxed{dx}$$

$$t = \frac{x-\mu}{\sigma}$$

$$dt = \frac{1}{\sigma} dx$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \boxed{dt}$$

$$I = \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt}.$$

$$\boxed{I^2 = 2\pi.}$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx \cdot \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy.$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}} dx\, dy.$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} \underbrace{r\, dr}\, d\theta$$

$$1 = \int_0^{\infty} e^{-t} dt = \int_0^{\infty} e^{-\frac{r^2}{2}} d\frac{r^2}{2} \qquad \square$$

Normal Distribution

Next, let's prove $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi}$. Let

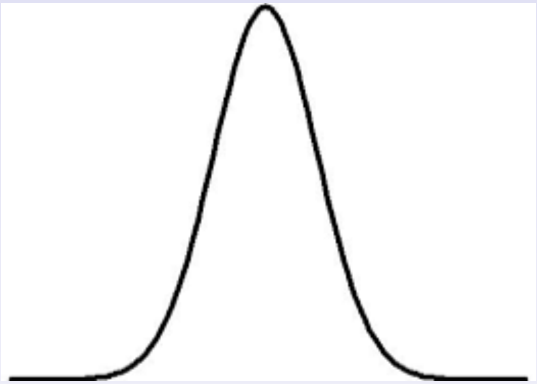
$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy.$$

It follows that

$$\begin{aligned}
I^2 =& I \cdot I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2+z^2)} dydz \\
=& \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\frac{1}{2}r^2} rdrd\theta \quad (y = r\cos\theta, z = r\sin\theta) \\
=& 2\pi.
\end{aligned}$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

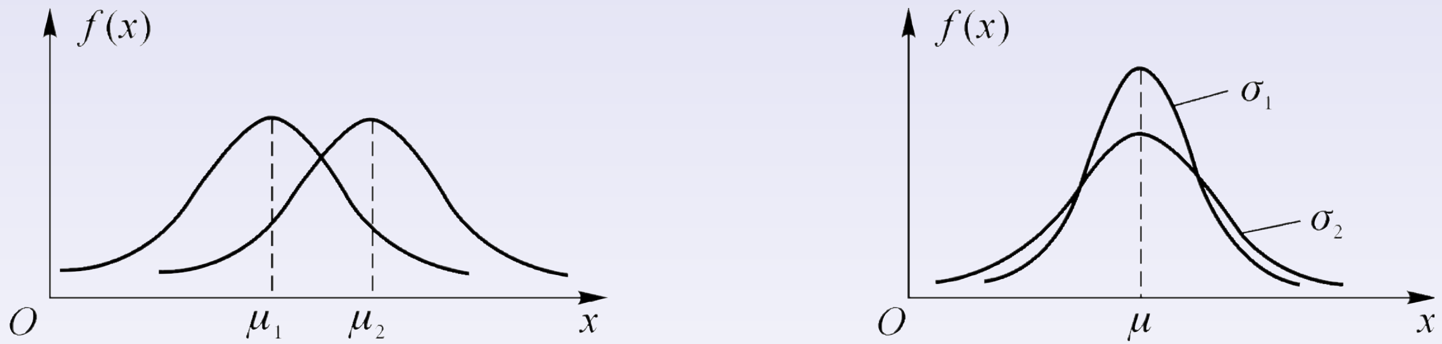


1. $f(\mu - x) = f(\mu + x)$ for all $x \in R$.
2. $f_{\max} = f(\mu) = \frac{1}{\sqrt{2\pi}\sigma}$.

Normal Distribution

$$X_1 \sim N(\mu_1, \sigma_1^2), \quad f_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}},$$

$$X_2 \sim N(\mu_2, \sigma_2^2), \quad f_2(x) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-(x-\mu_2)^2}{2\sigma_2^2}},$$



(a) $\mu_1 \neq \mu_2, \sigma_1 = \sigma_2 = \sigma$ (b) $\mu_1 = \mu_2 = \mu, \sigma_1 \neq \sigma_2$

3. μ : positional parameter
4. σ^2 : shape parameter $\quad (f_{\max} = f(\mu) = \frac{1}{\sqrt{2\pi}\sigma}.)$

Normal Distribution

Proposition

Suppose that $X \sim N(\mu, \sigma^2)$*. Then we have*

$$E(X) = \mu \quad \text{and} \quad Var(X) = \sigma^2. \tag{4}$$

Proof.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{+\infty} x f(x) dx = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&\overset{\frac{x-\mu}{\sigma}=z}{=} \int_{-\infty}^{+\infty} (\mu + \sigma z) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2/2} \sigma dz \\
&= \mu + \sigma \int_{-\infty}^{+\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \mu + \sigma \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, d\frac{z^2}{2} = \mu,
\end{aligned}$$

Normal Distribution

Proof.

$$\begin{aligned}
Var(X) &= \int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx \\
&= \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= -\sigma^2 \cdot (x-\mu) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big|_{-\infty}^{+\infty} \\
&\qquad + \sigma^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \sigma^2.
\end{aligned}$$

$\square$

Standard Normal Distribution

If $\mu = 0, \sigma^2 = 1$, then the distribution is called ***standard normal distribution***.

We denote the r.v. by Z. The p.d.f. of Z is

$$\phi(z) \quad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for } -\infty < z < \infty, \tag{5}$$

Proposition

If $X \sim N(\mu, \sigma^2)$, then the d.f. $F(x)$ of X is given by $\Phi\left(\frac{x-\mu}{\sigma}\right)$, i.e.,

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Proof: $F(x) = P(X \le x)$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad \square$$

Standard Normal Distribution

Proposition

If $X \sim N(\mu, \sigma^2)$, then the d.f. $F(x)$ of X is given by $\Phi\Big(\dfrac{x-\mu}{\sigma}\Big)$, i.e.,

$$F(x) = \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

Proof. Let $s = \frac{t-\mu}{\sigma}$.

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{s^2}{2}}\,ds = \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

$\square$

Thus

$$P(a \leqslant X \leqslant b) = F(b) - F(a) = \Phi\Big(\frac{b-\mu}{\sigma}\Big) - \Phi\Big(\frac{a-\mu}{\sigma}\Big),$$

$$P(X \leqslant a) = \Phi(\frac{a-\mu}{\sigma}), \qquad P(X \geqslant b) = 1 - \Phi(\frac{b-\mu}{\sigma}).$$

Standard Normal Distribution

Proposition

If $\boxed{X \sim N(\mu, \sigma^2)}$, *then*

$$Z = g(X) = \boxed{\frac{X - \mu}{\sigma} \sim N(0,1)}$$

has a standard normal distribution. Thus

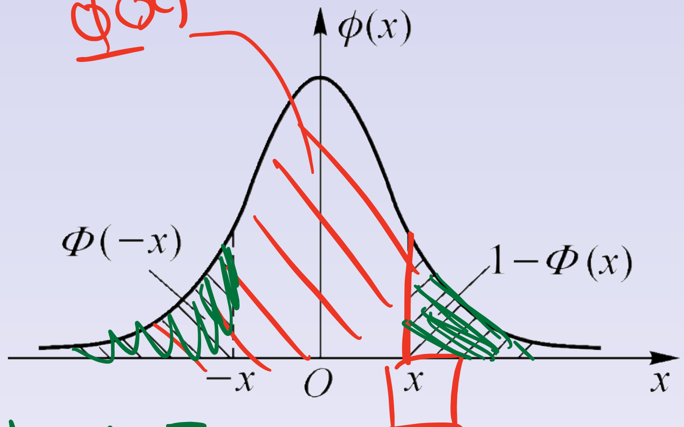
$$\begin{aligned} P(a \leqslant X \leqslant b) =& P\left(\frac{a-\mu}{\sigma} \leqslant Z \leqslant \frac{b-\mu}{\sigma}\right) \\ =& \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right), \end{aligned}$$

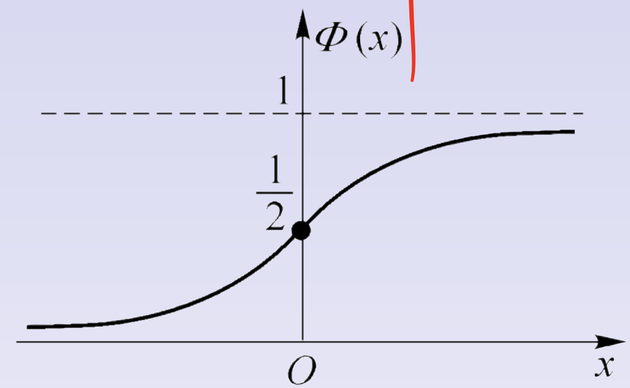
$$P(X \leqslant a) = \Phi\left(\frac{a-\mu}{\sigma}\right), \qquad P(X \geqslant b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right).$$

Standard Normal Distribution

	0.00	0.01
0.0	0.5000	
0.1		

$\Phi(x)$





$\Phi(-x) = 1-\Phi(x)$.

Proposition

(i) $\Phi(x) + \Phi(-x) = 1$, (ii) $\Phi(0) = 1/2$.

Table of Normal Probabilities in Appendix gives $\Phi(z) = P(Z \leqslant z)$, the area under the standard normal density curve to the left of z, for $z = 0, 0.01, \cdots, 3.98, 3.99$.

3σ - principle !

Standard Normal Distribution

Example

Let us determine the following standard normal probabilities: (a)$P(Z \leqslant 1.25)$, (b) $P(Z > 1.25)$, (c) $P(Z \leqslant -1.25)$, (d) $P(-0.38 \leqslant Z \leqslant 1.25)$.

Solution. (a) $P(Z \leqslant 1.25) = \Phi(1.25) = 0.89435$.

(b) $P(Z > 1.25) = 1 - P(Z \leqslant 1.25) = 1 - \Phi(1.25) = 0.10565$.

(c) $P(Z \leqslant -1.25) = \Phi(-1.25) = 1 - \Phi(1.25) = 0.10565$.

(d)

$$\begin{aligned} P(-0.38 \leqslant Z \leqslant 1.25) &= \Phi(1.25) - \Phi(-0.38) \\ &= \Phi(1.25) - [1 - \Phi(0.38)] \\ &= 0.89435 - 0.35197 = 0.54238. \end{aligned}$$

Normal Distribution

Example

The time that it takes a driver to react to the brake lights on a decelerating vehicle is critical in helping to avoid rear-end collisions. Suppose that reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled with a normal distribution having mean value 1.25sec and standard deviation of 0.46sec. What is the probability that reaction time is between 1.00sec and 1.75sec?

Solution. If we let X denote reaction time, then

$$\begin{aligned}
&P(1 \leqslant X \leqslant 1.75) \\
&= P\Big(\frac{1-1.25}{0.46} \leqslant Z \leqslant \frac{1.75-1.25}{0.46}\Big) \\
&= P(-0.54 \leqslant Z \leqslant 1.09) = \Phi(1.09) - (1 - \Phi(0.54)) \\
&= 0.86214 - (1 - 0.70540) = 0.56754.
\end{aligned}$$

$\square$

Normal Distribution

Similarly, if we view 2sec as a critically long reaction time, the probability that actual reaction time will exceed this value is

$$
\begin{aligned}
P(X > 2) &= P\Big(Z > \frac{2 - 1.25}{0.46}\Big) \\
&= P(Z > 1.63) \\
&= 1 - \Phi(1.63) \\
&= 0.05155.
\end{aligned}
$$

Normal Distribution

Observe the Table of Normal Probabilities in Appendix, we find the largest value of z is 3.99.

$z = 4.5,\ 8.9,\ ?$

Example

The breakdown voltage of a randomly chosen diode of a particular type is known to be normally distributed. What is the probability that a diode's breakdown voltage is within 1 standard deviation(SD) of its mean value?

Normal Distribution

Solution. This question can be answered without knowing either μ or σ, as long as the distribution is known to be normal; the answer is the same for any normal distribution:

$$\begin{aligned}
& P(X \text{ is within 1 standard deviation of its mean}) \\
= {} & P(|X-\mu| \leqslant \sigma) = P(\mu-\sigma \leqslant X \leqslant \mu+\sigma) \\
= {} & P\Big(\frac{\mu-\sigma-\mu}{\sigma} \leqslant Z \leqslant \frac{\mu+\sigma-\mu}{\sigma}\Big) \\
= {} & P(-1 \leqslant Z \leqslant 1) \\
= {} & \Phi(1)-\Phi(-1) = 0.6826. \quad \square
\end{aligned}$$

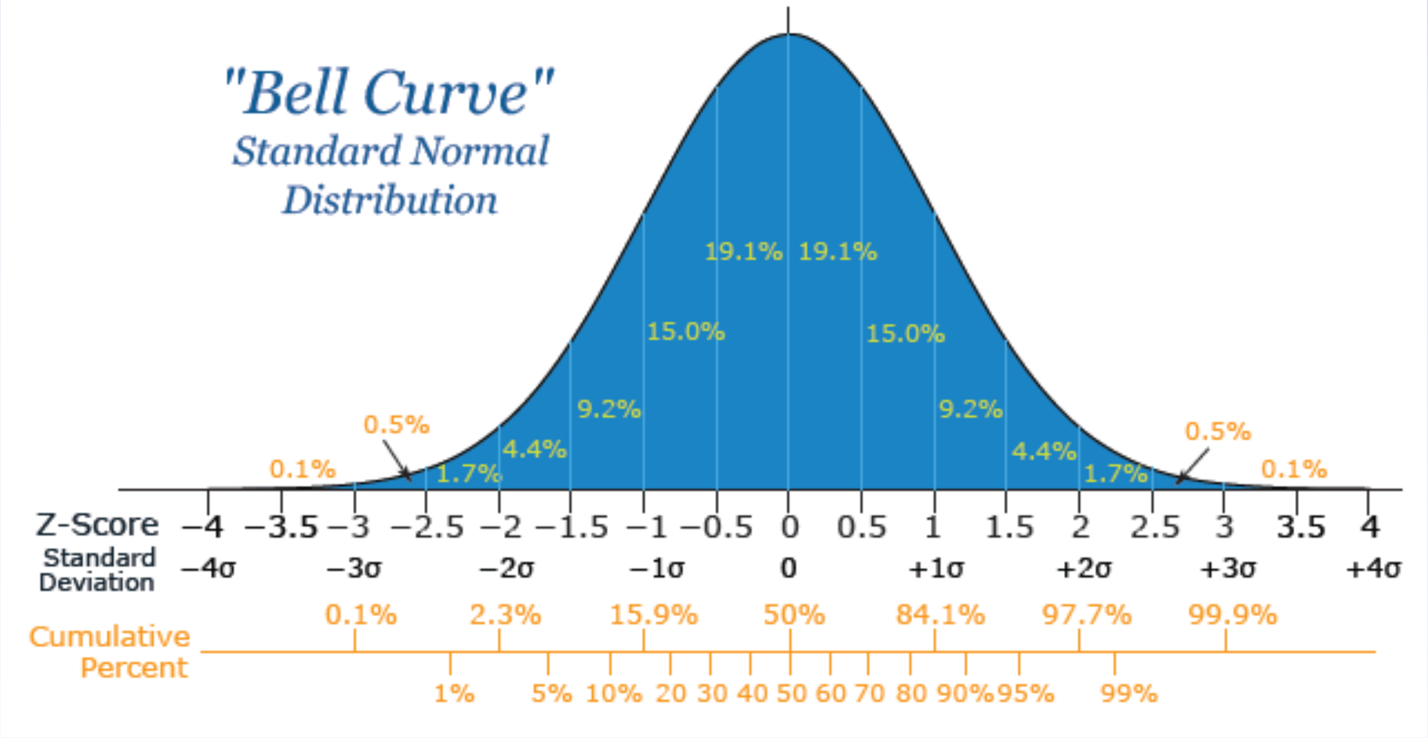
Similarly,

$$P(|X-\mu| \leqslant 2\sigma) = P(-2 \leqslant Z \leqslant 2) = 0.9544,$$

$$P(|X-\mu| \leqslant 3\sigma) = P(-3 \leqslant Z \leqslant 3) = 0.9974.$$

Normal Distribution

3σ-principle



Normal Distribution

Proposition

Suppose that $X \sim N(\mu, \sigma^2)$. *Let* $Y = aX + b$, $(a \neq 0)$. *Then* Y *has a normal distribution with parameters* $a\mu + b$ *and* $a^2\sigma^2$.

For example,

If $X \sim N(0,1)$ and $Y = 2X + 3$, then $Y \sim N(3,4)$.

If $X \sim N(1,4)$ and $Y = 5X + 2$, then $Y \sim N(7,100)$.

$X \sim N(\mu, \sigma^2)$, $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$

Example

Suppose that $X \sim N(3,4)$. Let $Y = 2X + 1$.

(a) Find the value of $P(7 < Y < 9)$.

(b) Let $Y = aX + 4$, find the value a such that $P(Y \leqslant 7) = 1/2$.

Solution. Let $F(x)$ be the d.f. of X.
(a)

$$\begin{aligned} P(7 < Y < 9) &= P(3 < X < 4) \\ &= F(4) - F(3) \\ &= \Phi\left(\frac{4-3}{2}\right) - \Phi\left(\frac{3-3}{2}\right) \\ &= 0.69146 - 0.5 = 0.19146. \end{aligned}$$

$$X \sim N(3,4)$$

$$Y = 2X+1 \sim N(7,16)$$

$$P(7 < Y < 9) = P\left(0 < \cancel{\frac{Y-7}{4}} < \frac{1}{2}\right)$$

$$= \Phi(0.5) - \frac{1}{2}$$

$$P(Y \le 7) = \frac{1}{2}, \quad \Rightarrow \underbrace{\mu_Y}_{u_{0.5}} = 7 = 3a+4$$

$$\Rightarrow a = 1.$$

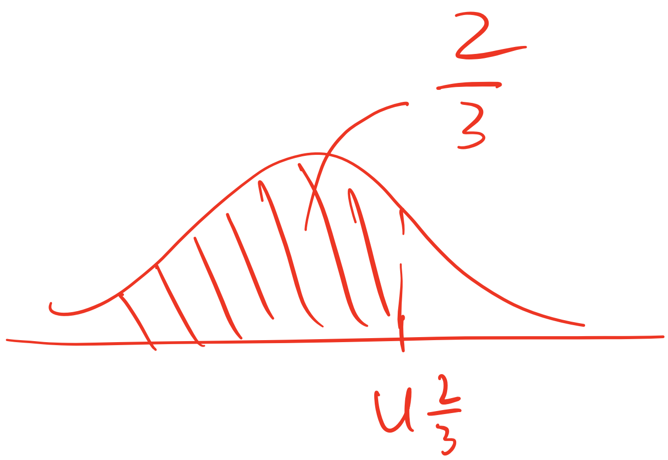
$$P(Y \le 7) = \frac{2}{3} \Rightarrow$$

$$Y = aX+4 \sim N(3a+4, 4a^2)$$

$$P(Y \le 7) = P\left(\cancel{\frac{Y-(3a+4)}{2a}} \le \frac{7-(3a+4)}{2a}\right)$$

$$= \Phi\left(\frac{7-(3a+4)}{2a}\right) = \frac{2}{3}$$

$$\frac{7-(3a+4)}{2a} = u_{\frac{2}{3}}$$



Normal Distribution

Example

Suppose that $X \sim N(3,4)$. Let $Y = 2X + 1$.

(a) Find the value of $P(7 < Y < 9)$.

(b) Let $Y = aX + 4$, find the value a such that $P(Y \leqslant 7) = 1/2$.

Solution.
(b) Since $P(Y \leqslant 7) = 1/2$ and $F_Y(E(Y)) = 0.5$,

$$E(Y) = 7.$$

We have $E(Y) = E(aX + 4) = aE(X) + 4 = 3a + 4$. Thus

$$a = 1.$$

$\square$

Normal Distribution

Examples

- reaction time for an in-traffic response
- the breakdown voltage of a randomly chosen diode
- length of human pregnancy
- stock price
- height, weight, IQ-score, $\cdots$

History

- 1733 De Moivre, an approximation distribution
- 1783 Laplace, describe the distribution of errors
- 1809 Gauss, analyze astronomical data

Normal Distribution

Distribution	p.f. or p.d.f.	Parameters
Bernoulli	$p(x) = p^x(1-p)^{1-x}, x = 0, 1$	p
Binomial	$p(x) = \binom{n}{x} p^x(1-p)^{n-x}, x = 0, 1, \cdots$	n and p
Geometric	$p(x) = (1-p)^{x-1} \cdot p,\ x = 1, 2, \cdots$	p
Poisson	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, \cdots .$	λ
Uniform	$f(x) = \frac{1}{b-a},\ a \leqslant x \leqslant b$	$[a, b]$
Exponential	$f(x) = \lambda \exp^{-\lambda x}, x \geqslant 0,$	λ
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty$	μ and σ

Table: some important distributions

The end

Thank you for your patience !

X	p.f or p.d.f	EX	$Var(X)$
$B(n,p)$		np	$np(1-p)$
$P(\lambda)$		λ	λ
$Geom(p)$		$\frac{1}{p}$	$\frac{1}{p^2}$
$U(a,b)$		$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Exp(\lambda)$		$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$N(\mu,\sigma^2)$		μ	σ^2

$X \sim Geom(p)$

$$E(X) = \sum_{k=1}^{\infty} k \cdot p(k)$$

$$= \sum_{k=1}^{\infty} \underline{k \cdot (1-p)^{k-1}} \cdot p.$$

$$= \sum_{k=1}^{\infty} \underline{\left((1-p)^k\right)'} \, p$$

$$= \left[\sum_{k=1}^{\infty} (1-p)^k\right]' \cdot p.$$

$$E(X) = 1 \times p + (1-p)\left[1 + EX\right]$$

首步分解 $E(X) = \frac{1}{p}.$