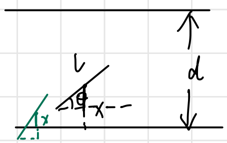
Example 1.3.8

Given a needle of length l dropped on a plane ruled with parallel lines d (l < d) units apart, see Fig. 1.3(a). What is the probability that the needle will cross a line?



x is the distance from the center of needle to the nearest line.

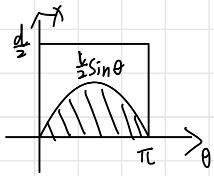
$0 \le x \le \frac{d}{2}$

$0 \le \theta \le \pi$

$\Omega = \{(x,\theta) \mid 0 \le x \le \frac{d}{2}, \ 0 \le \theta \le \pi\}$

when the needle intersects with one line, $x \le \frac{l \sin\theta}{2}$

$A = \{(x,\theta) \mid 0 \le x \le \frac{d}{2}, \ 0 \le \theta \le \pi, \ x \le \frac{l\sin\theta}{2}\}$



$$P(A) = \frac{L(A)}{L(\Omega)} = \frac{\int_0^{\pi} \frac{l\sin\theta}{2} d\theta}{\frac{d\pi}{2}} = \frac{2l}{\pi d}.$$

1.5 Let E, F, G be three events. Find expressions for the events that of E, F, G

(a) only F occurs,

(b) both E and F but not G occur,

(c) at least one event occurs,

(d) at least two events occur,

(e) all three events occur,

(f) none occurs,

(g) at most one occurs,

(h) at most two occur.

(a) $F \cap \bar{E} \cap \bar{G}$

(b) $E \cap F \cap \bar{G}$

(c) $\overline{\bar{E} \cap \bar{F} \cap \bar{G}} = E \cup F \cup G$

(d) $(E \cap F) \cup (E \cap G) \cup (F \cap G)$

(e) $E \cap F \cap G$

(f) $\bar{E} \cap \bar{F} \cap \bar{G} = \overline{E \cup F \cup G}$

(g) $(\bar{E} \cap \bar{F}) \cup (\bar{E} \cap \bar{G}) \cup (\bar{F} \cap \bar{G})$

(h) $\bar{E} \cup \bar{F} \cup \bar{G}$

1.7 Suppose that a coin is tossed ten times. Let A denote the event that a head is obtained on the first toss, and let B denote the event that a head is obtained on the sixth toss. Are A and B disjoint?

Solution: Events A and B are not disjoint. Disjoint events are events that cannot occur simultaneously. In this case, it's possible for both events A and B to occur simultaneously. Therefore, they are not disjoint. They are independent. The probability of their occurrence is both one-half.

1.9 You roll two dice. What is the probability of the events:

(a) They show the same?

(b) Their sum is seven or eleven?

(c) They have no common factor greater than unity?

(d) The sum of the numbers is 2, 3, or 12?

(e) The sum is odd?

(f) The difference is odd?

(g) The product is odd?

(h) One number divides the other?

(i) The first die shows a smaller number than the second?

(j) Different numbers are shown and the smaller of the two numbers is $r, 1 \leqslant r \leqslant 6$?

(a) $\#\Omega = C_6^1 \cdot C_6^1 = 36$

$\#A = C_6^1 = 6$

$P(A) = \frac{\#A}{\#\Omega} = \frac{1}{6}$

(b) $\#B = 8$

$P(B) = \frac{\#B}{\#\Omega} = \frac{2}{9}$

(c) $\#C = 23$

$P(C) = \frac{\#C}{\#\Omega} = \frac{23}{36}$

(d) $\#D = 4$

$P(D) = \frac{\#D}{\#\Omega} = \frac{1}{9}$

(e) $\#E = 18$ (an odd and an even)

$P(E) = \frac{\#E}{\#\Omega} = \frac{1}{2}$

(f) $\#F = 18$ (an odd and an even)

$P(F) = \frac{\#F}{\#\Omega} = \frac{1}{2}$

(g) $\#G = 9$ (two odd)

$P(G) = \frac{\#G}{\#\Omega} = \frac{1}{4}$

(h) $\#H = 22$

$P(H) = \frac{\#H}{\#\Omega} = \frac{11}{18}$

(i) $\#I = 15$

$P(I) = \frac{\#I}{\#\Omega} = \frac{5}{12}$

(j) $\#J = \#\Omega - C_6^1 = 30$

$P(J) = \frac{\#J}{\#\Omega} = \frac{5}{6}$

1.12 A point (x, y) is to be selected from the square A containing all points (x, y) such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Suppose that the probability that the selected point belong to each specified subset of A is equal to the area of that subset. Find the probability of each of the following subsets:

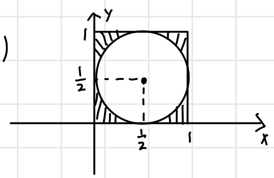
(a) the subset of points such that $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \geqslant \frac{1}{4}$;

(b) the subset of points such that $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 \geqslant \frac{1}{4}$;

(c) the subset of points such that $y \leqslant 1 - x^2$;

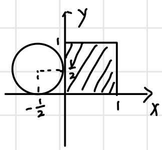
(d) the subset of points such that $y = x$.

(a)



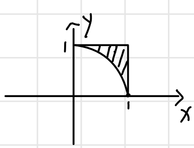
$P(A) = 1 - \frac{1}{4}\pi$

(b)



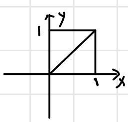
$P(B) = 1$

(c)



$P(C) = \frac{\#C}{\#\Omega} = \frac{\int_0^1 (1-x^2)\,dx}{1} = \frac{2}{3}$

(d)



$P(D) = 0$

1.16 Suppose that A and B are events in sample space Ω. Suppose that $P(A) = 1$ and $P(B) = 0$. Is A an inevitable event? Is B an impossible event?

Solution: Probability is indeed the the stable value of a frequency. When the stable value of a frequency approaches 1, we say the probability is 1, but it doesn't necessarily mean it will definitely occur; it just means the likelihood of its occurrence is very high. Similarly, when the probability is 0, it meas the likelihood of the event occuring is very low, but it doesn't necessarily mean it will definitely not occur.

3.2 ✓