不学: 5.3.2, 5.4
6.1.3, 6.2, 6.3.3, 6.4
$(Q = R_X(0))$

Chapter 6 Stationary Processes

必考! μ ✓
R ✓

School of Sciences, BUPT

Stationary Processes

$\forall \vec{t} \xrightarrow[\forall h]{\text{shift}} \vec{t+h}$

严平稳

Many important processes have the property that their finite-dimensional distributions are invariant under $\boxed{\text{time shifts}}$ (or space shifts). These processes are called *strict-sense stationary processes*.

很难满足!

Another kind of stationary is *wide-sense stationary* while it has constant mean and its autocorrelation function is invariant under time shifts.

求其次.

(宽)平稳,

$$\begin{cases} \mu_X(t) = E(X_t) = \cdots = C. \quad \text{is constant.} \\ R_X(t, t+\tau) = E(X_t \cdot X_{t+\tau}) = \cdots = R_X(\tau) \end{cases}$$

Contents

depends only on τ.

Strict Stationary Processes

Definition

A ***strict-sense stationary process*** *(SSS) is a stochastic process* $\{X_t\}_{t\in T}$ *with the property that for any constant* h*, for any positive integer* n*, for any* $t_1, \cdots, t_n \in T$ *and for any* $t_1 + h, \cdots, t_n + h \in T$,

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$$

and

$$(X_{t_1+h}, X_{t_2+h}, \cdots, X_{t_n+h})$$

have the same nth-order probability distribution function, i.e.,

$$F_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = F_X(x_1, \cdots, x_n; t_1 + h, \cdots, t_n + h)$$

for all $x_1, \cdots, x_n \in \mathbb{R}$.

Strict Stationary Processes

$d=1.$ $X_t \overset{d.}{=} X_{t+h}$

$\mu_X(t) = E(X_t) = E(X_0) = \text{const.}$

esp. $h = -t.$ $X_t \overset{d.}{=} X_0, \forall t.$

Remark

For a SSS process, the first-order distribution is independent of t. To see this, note that from the definition of stationarity, for any t, the distribution of $X(t)$ is as same as $X(t+h)$.

The second-order distribution — the distribution of any two r.v.s $X(s)$ and $X(t)$ depends only on $\tau = t - s$. To see this, take $h = -s$, the joint distribution of $X(s)$ and $X(t)$ is the same as the joint distribution of $X(0)$ and $X(t-s)$.

$d=2.$ $(X_t, X_{t+\tau}) \overset{d.}{=} (X_{t+h}, X_{t+\tau+h})$

$R_X(t, t+\tau) = R_X(0,\tau) = R_X(\tau)$

esp. $h = -t.$ $(X_t, X_{t+\tau}) \overset{d.}{=} (X_0, X_\tau)$

Strict Stationary Processes

二阶矩过程

Definition

A process $\{X_t\}_{t\in T}$ *is said to be* $\boxed{\textit{of second order}}$ *if*

$$E(X_t^2) < \infty \quad \text{for all } t \in T.$$

$$E(X_s X_t)^2 \le E(X_s^2) \cdot E(X_t^2)$$

Theorem

Suppose that the process $\{X_t\}_{t\in T}$ *is strict-sense stationary and of second order. Then*
(i) *for all* $t \in T$, $\mu_X(t) = E(X_t) =$ *constant;*
(ii) $R_X(s,t) = E(X_s X_t)$ *depends only on the time difference* $t - s$ *for all* $t, s \in T$.

Proof. (i) By the Cauchy-Schwarz inequality, we have

$$[E(X_t)]^2 \leqslant E(X_t^2) < +\infty,$$

thus $E(X_t)$ exists.
If we let $h = -t$ in the definition of strict-sense stationary process, then we conclude that X_t and X_0 have the same distribution function. Hence,

$$\mu_X(t) = E(X_t) = E(X_{t-t}) = E(X_0)$$

is a constant.

Strict Stationary Processes

(ii) By the Cauchy-Schwarz inequality, we have

$$[E(X_s X_t)]^2 \leqslant E(X_s^2)E(X_t^2) < +\infty.$$

Thus $E(X_s X_t)$ exists.
If we let $h = -s$, then (X_s, X_t) and (X_0, X_{t-s}) have the same distribution function. It follows that

$$E(X_s X_t) = E(X_0 X_{t-s}),$$

that is,

$$R_X(s, s+\tau) = E(X_0 X_\tau) = R_X(\tau),$$

where $\tau = t - s$. Therefore, $R_X(s,t)$ depends only on the time difference $t - s$. $\square$

Strict Stationary Processes

$Cov(X,Y) = E(XY) - \mu_X \cdot \mu_Y$

Furthermore,

$$C_X(t, t+\tau) = \cancel{E[(X_t - \mu_X)(X_{t+\tau} - \mu_X)]}$$

$$\cancel{= E(X_t X_{t+\tau}) - E(X_t \mu_X) - E(\mu_X X_{t+\tau}) + E(\mu_X^2)}$$

$$= R_X(\tau) - \mu_X^2 = C_X(\tau)$$

depends only on τ and

$$\sigma_X^2(t) = C_X(0) = \boxed{R_X(0)} - \mu_X^2$$

is a constant.

$R_X(0) = E(X_t^2), \forall t.$

$$R_X^2(\tau) = E(X_t \cdot X_{t+\tau})^2$$

C-S. Ineq.

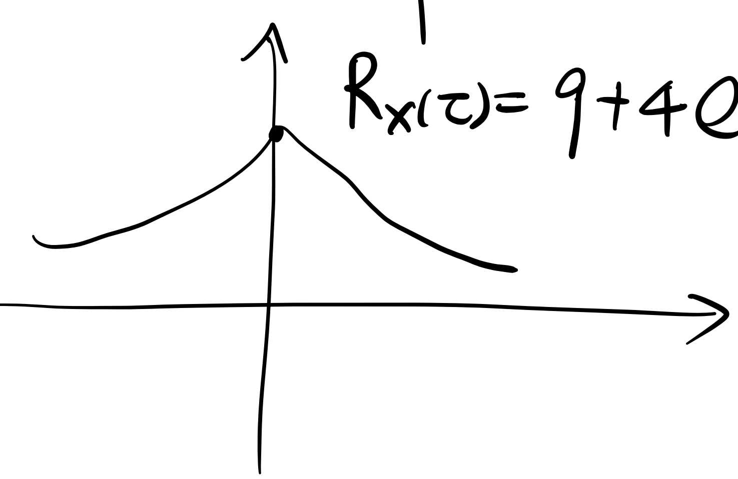
$$\le E(X_t^2) \cdot E(X_{t+\tau}^2)$$

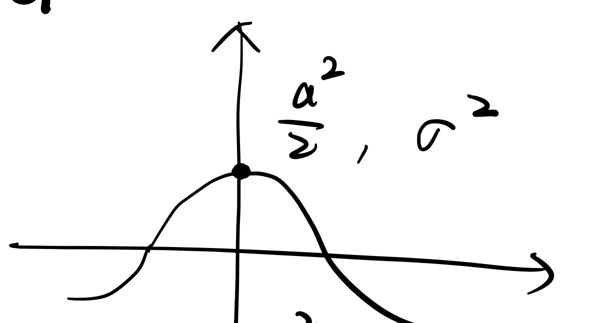
$$= R_X(0) \cdot R_X(0)$$

$$\left[\int fg\,dx\right]^2 \le \int f^2dx \cdot \int g^2dx$$

$$\Rightarrow \quad |R_X(\tau)| \le |R_X(0)|$$

$R_X(\tau) = 9 + 4e^{-0.2|\tau|}$





$R_X(\tau) = \frac{a^2}{2}\cos\omega\tau$.

Example

Suppose that $\{X_n\}_{n\geqslant 0}$ are IID random sequences, each of which has a uniform distribution in the interval $(0,1)$. Is $\{X_n\}_{n\geqslant 0}$ a s-tationary process in the strict sense? Determine the value $E(X_n)$ and $E(X_nX_m)$, $n,m=0,1,2,\cdots$.

Solution. Suppose that the distribution function of X_n is $F(x)$. Then for all $k,h\in\mathbb{N}$ and $0<n_1<n_2<\cdots<n_k$, the joint distribution functions of $(X_{n_1},X_{n_2},\ldots,X_{n_k})$ and $(X_{n_1+h},X_{n_2+h},\ldots,X_{n_k+h})$ are both

$$
\begin{aligned}
F(x_1,x_2,\ldots,x_k) &= P(X_{n_1}\le x_1, X_{n_2}\le x_2,\cdots,X_{n_k}\le x_k)\\
&= P(X_{n_1}\le x_1)P(X_{n_2}\le x_2)\cdots P(X_{n_k}\le x_k)\\
&= F(x_1)F(x_2)\cdots F(x_k).
\end{aligned}
$$

Thus $\{X_n\}_{n\geqslant 0}$ is a strict-sense stationary process.

Strict Stationary Processes

Since $\{X_n\}_{n\geqslant 0}$ are IID random sequences, each of which has a uniform distribution in the interval $(0,1)$, we get $E(X_n) = \frac{1}{2}$,

$$E(X_n X_m) = E(X_n)E(X_m) = \frac{1}{4} \qquad \text{for} \quad m \neq n,$$

and

$$E(X_n X_m) = E(X_n^2) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \qquad \text{for} \quad m = n.$$

Thus,

$$\boxed{E(X_n X_m) = \begin{cases} \frac{1}{3}, & \text{for } m = n, \\ \frac{1}{4}, & \text{for } m \neq n. \end{cases}}$$

$\tau = |m-n|$

$\square$

Contents

Wide Stationary Processes

A weaker form of stationarity exists which does not directly constrain the nth-order probability density functions, but rather just the first- and second-order moments. This property, which is easier to check, is called wide-sense stationarity and will be quite useful.

Test.

Wide Stationary Processes

Definition

A process $\{X_t\}_{t\in T}$ *is said to be* ***wide-sense stationary*** *(WSS), if it is of second order and satisfies*
(i) *for all* $t \in T$, $\mu_X(t) = E(X_t) =$ *constant;*
(ii) *for all* $t,\ t+\tau \in T$, $R_X(t, t+\tau) = R_X(\tau)$ *depends only on the time difference* τ.

In particular, a random sequence $\{X_n\}_{n\geqslant 0}$ *is said to be a stationary random sequence (or time sequence) in the wide sense if it is of second order and satisfies*
(i) *for all* $n \geqslant 0$, $E(X_n) =$ *constant,*
(ii) *for all* $m, n \geqslant 0$, $R_X(n, n+m) = R_X(m)$ *depends only on* m.

Wide Stationary Processes

Clearly if SSS is of second order, it must imply WSS. The converse is not necessarily true.

Example (HW)

Suppose that $\{X_n\}_{n\geqslant 1}$ *are IID random sequences, each of which has a standard normal distribution, and* $\{Y_n\}_{n\geqslant 1}$ *are also IID random sequences, while each of which has a uniform distribution in the interval* $(-\sqrt{3}, \sqrt{3})$. *Suppose that* $\{X_n\}_{n\geqslant 1}$ *and* $\{Y_n\}_{n\geqslant 1}$ *are independent. Let*

$$Z_n = \begin{cases} X_n & \text{if n is odd,} \\ Y_n & \text{if n is even.} \end{cases}$$

Prove $\{Z_n\}_{n\geqslant 1}$ *is WSS, but not SSS.*

Wide Stationary Processes

Let X_t be a WSS process and relabel $R_X(t, t+\tau)$ as $R_X(\tau)$. We can see $R_X(\tau)$ has the following properties.

Proposition

(i) $R_X(\tau)$ *is even, i.e.,* $R_X(-\tau) = R_X(\tau)$ *for all* τ.
(ii) $R_X(0) \geqslant 0$.
(iii) $|R_X(\tau)| \leqslant R_X(0)$. (CS) $R_X(0) = E(X_t^2)$
(iv) *If* X_t *is a periodic process with period* T *(i.e.,* $X(t+T) = X(t)$*), then its autocorrelation function is also periodic with period* T*. That is,* $R_X(\tau + T) = R_X(\tau)$.

The above properties of $R_X(\tau)$ are necessary but not sufficient for a function to qualify as an autocorrelation function for a WSS process.

Wide Stationary Processes

Proposition

The necessary and sufficient condition for a function $R_X(\tau)$ to be an autocorrelation function for a WSS process is that it be even and nonnegative definite, that is, for any $n \in \mathbb{N}$, for any $t_1 < t_2 < \cdots < t_n$, and for any real vector $(a_1, a_2, \ldots, a_n)$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R_X(t_j - t_i) \geqslant 0.$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R_X(t_j - t_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j E(X_{t_i} X_{t_j}) = E\Big[\Big(\sum_{i=1}^{n} a_i X_{t_i}\Big)^2\Big]$$

Wide Stationary Processes

Example

Suppose that $\{X_n, n = 0, \pm 1, \pm 2, \dots\}$ is a pairwisely uncorrelated time sequence (i.e., $C_X(n, m) = 0$ for any $n \neq m$) and $E(X_n) = 0$, $Var(X_n) = \sigma^2 > 0$. Then $\{X_n\}$ is wide-sense stationary.

Solution. Since $E(X_n) = 0$ and

$$E(X_m X_n) = \begin{cases} \sigma^2 & \text{for } m = n, \\ 0 & \text{for } m \neq n, \end{cases}$$

$\{X_n\}$ is a wide-sense stationary time sequence. □

We call a random process $\{X_t\}$ a uncorrelated process if $C_X(s, t) = 0$ for any $s \neq t$.

Wide Stationary Processes

Example

Let X_t be strict-sense stationary with finite second moment, and let $Y_t = a + bt + X_t$, $W_t = t[X_t - E(X_t)]$ and

$$Z_t = \begin{cases} \alpha + X_t & \text{if } t \leqslant t_0, \\ \beta + X_t & \text{if } t > t_0. \end{cases}$$

Are Y_t, W_t and Z_t stationary?

Recall a sinusoid with random phase $X_t = a\sin(\omega t + \Theta)$, the mean and the autocorrelation function of X_t are $\mu_X(t) = 0$ and $R_X(t, t+\tau) = \frac{a^2}{2}\cos\tau$ respectively. Thus X_t is wide-sense stationary. In fact we have a general results as follows.

Example

Suppose that S_t is a deterministic function with period T and the random variable Θ has a uniform distribution in the interval $(0, T)$. Let $X_t = S_{t+\Theta}$. Discuss the stationarity of X_t.

Wide Stationary Processes

Solution. Since the p.d.f. of Θ is

$$f(\theta) = \begin{cases} \dfrac{1}{T} & \text{for } 0 < \theta < T, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} E(X_t) = E(S_{t+\Theta}) &= \frac{1}{T}\int_0^T S_{t+\theta}\,d\theta = \frac{1}{T}\int_t^{t+T} S_\phi\,d\phi \\ &= \frac{1}{T}\int_0^T S_\phi\,d\phi = \text{constant}, \end{aligned}$$

Wide Stationary Processes

and

$$\begin{aligned}
R_X(t, t+\tau) &= E[S_{t+\Theta} \cdot S_{t+\tau+\Theta}] \\
&= \frac{1}{T}\int_0^T S_{t+\theta} S_{t+\tau+\theta} d\theta \\
&= \frac{1}{T}\int_t^{t+T} S_\phi S_{\tau+\phi} d\phi.
\end{aligned}$$

It follows from the periodicity of $S_\phi S_{\tau+\phi}$ (with respect to ϕ) that

$$R_X(t, t+\tau) = \frac{1}{T}\int_0^T S_\phi S_{\tau+\phi} d\phi$$

depends only on τ. Thus X_t is a wide-sense stationary process.

Wide Stationary Processes

Example

Suppose $\{X_t\}$ is a stationary process and

$$Y_t = X_t \cos(\omega_0 t + \Theta),$$

where X_t and Θ are independent, $\Theta \sim U(0, 2\pi)$, and ω_0 is a constant. Prove that Y_t is stationary.

Pf. Let $\mu_X = E(X_t) = \text{const}.$

$R_X(\tau) = E(X_t \cdot X_{t+\tau})$ depends only on τ.

Then. $\mu_Y(t) = E(Y_t)$

$$= E(X_t \cdot \cos(\omega_0 t + \Theta))$$

$$= E(X_t) \cdot E(\cos(\omega_0 t + \Theta))$$

$$= \mu_X \cdot \int_0^{2\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$$

$$= 0$$

$\therefore$ $\mu_Y(t)$ is constant.

$$R_Y(t, t+\tau) = E(Y_t \cdot Y_{t+\tau})$$

$$= E(X_t \cdot \cos(\omega_0 t + \Theta) \cdot X_{t+\tau} \cdot \cos(\omega_0(t+\tau) + \Theta))$$

$$= E(X_t \cdot X_{t+\tau}) \cdot E(\cos(\omega_0 t + \Theta) \cdot \cos(\omega_0(t+\tau) + \Theta))$$

$$= R_X(\tau) \cdot \frac{1}{2} \cos(\omega_0 \tau)$$

depends only on τ.

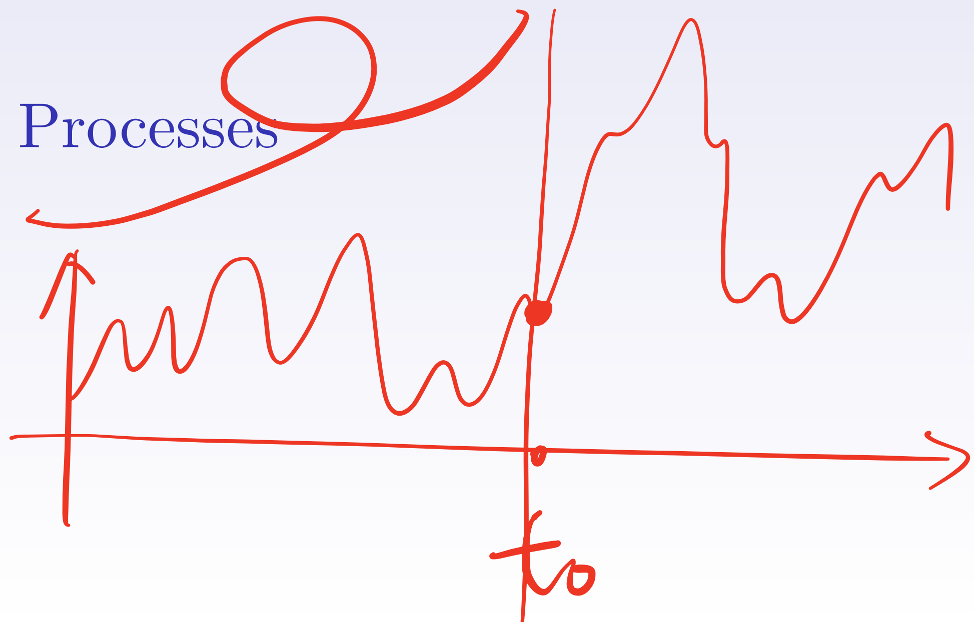
Thus. $\{Y_t\}$ is stationary.

Contents

1. Strict Stationary Processes
2. Wide Stationary Processes
3. Ergodicity of Stationary Processes

$X_{t_0} \in \{x_1, x_2 \ldots x_{100}\}$

遍历性.



Definition

(*Continuity in Mean*) *A stochastic process X_t is* ***continuous in mean square*** *at t_0 if*

$$\lim_{\varepsilon \to 0} E\left[(X_{t_0+\varepsilon} - X_{t_0})^2\right] = 0$$

Denote it by

$$\mathrm{l.i.m.}_{t \to t_0} X_t = X_{t_0},$$

where l.i.m. *denotes limit in mean. The stochastic process $\{X_t\}$ is called a mean square continuous process if it is continuous in mean square for all t.*

Definition

A mean square integral of a random process X_t is defined by

$$Y_t = \int_{t_0}^{t} X_s ds = \text{l.i.m.}_{\triangle s_i \to 0} \sum_i X(s_i) \triangle s_i,$$

where $t_0 < s_1 < \cdots < t$ and $\triangle s_i = s_{i+1} - s_i$.

Ergodicity of Stationary Process

Definition

The time average of a WSS stochastic process $\{X_t\}_{t\in\mathbb{R}}$ *is defined by*

$$\langle X_t\rangle = \mathrm{l.i.m.}_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} X_t dt.$$

Definition

The time correlation function of a WSS stochastic process $\{X_t\}_{t\in\mathbb{R}}$ *is defined by*

$$\langle X_t X_{t+\tau}\rangle = \ \mathrm{l.i.m.}_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} X_t X_{t+\tau} dt.$$

Ergodicity of Stationary Process

Definition

A WSS stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is ergodic in the mean if the time average converges to the ensemble average μ_X, that is,

$$\lim_{T\to\infty} E\left[\left(\frac{1}{2T}\int_{-T}^{T} X_t \mathsf{d}t - \mu_X\right)^2\right] = 0. \tag{1}$$

Definition

A WSS stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is ergodic in correlation for shift (lag) τ if

$$\lim_{T\to\infty} E\left[\left(\frac{1}{2T}\int_{-T}^{T} X_t X_{t+\tau} dt - R_X(\tau)\right)^2\right] = 0. \tag{2}$$

If this condition is true for all τ, we say X_t is ergodic in correlation.

Ergodicity of Stationary Process

Theorem

A mean-square continuous and WSS stochastic process X_t is ergodic in the mean iff its covariance function, that is, $R_X(\tau) - \mu_X^2$ satisfies

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) \left[R_X(\tau) - \mu_X^2\right] d\tau = 0. \tag{3}$$

Corralary

A mean-square continuous and WSS stochastic process $\{X_t\}$ is ergodic in the mean iff

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^{2T} (1 - \frac{\tau}{2T}) R_X(\tau) d\tau = \mu_X^2.$$

Ergodicity of Stationary Process

Theorem

Suppose that $\{X_t\}$ *is a mean-square continuous and WSS stochastic process, and for each* τ, $\{X_t X_{t+\tau}\}$ *is also a mean-square continuous and WSS stochastic process. Then* $\{X_t\}$ *is ergodic in correlation iff*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau_1}{2T}\right) \left[B(\tau_1) - R_X^2(\tau)\right] d\tau_1 = 0,$$

where $B(\tau_1) = E(X_t X_{t+\tau} X_{t+\tau_1} X_{t+\tau+\tau_1})$.

Ergodicity of Stationary Process

Definition

Suppose that $\{X_t\}_{t\in\mathbb{R}}$ *is a WSS process. If it is ergodic in the mean and the correlation, then we say it is an ergodic process.*

Example

Consider the WSS process

$$X_t = \sin(\omega_0 t + \Theta),\ \ t \in (-\infty, +\infty),$$

where ω_0 is a constant and $\Theta \sim U(0, 2\pi)$. Discuss the ergodicity of this process.

Ergodicity of Stationary Process

Solution. We have known that $E(X_t) = \mu_X = 0$ and $R_X(\tau) = \frac{1}{2}\cos\tau$. On the other hand,

$$\begin{aligned}\langle X_t \rangle &= \text{l.i.m.}_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 t + \Theta)dt \\ &= \text{l.i.m.}_{T\to+\infty} \frac{\sin(\omega_0 T)\sin\Theta}{T\omega_0} = 0 = \mu_X.\end{aligned}$$

Thus $\{X_t\}$ is ergodic in the mean. Furthermore,

$$\begin{aligned}\langle X_t X_{t+\tau} \rangle &= \text{l.i.m.}_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 t + \Theta)\sin[\omega_0(t+\tau) + \Theta]dt \\ &= \frac{1}{2}\cos\omega_0\tau = R_X(\tau).\end{aligned}$$

That means, $\{X_t\}$ is ergodic in correlation. Hence, $\{X_t\}$ is an ergodic process. $\square$

Ergodicity of Stationary Process

$$\mu_X = \mu_X(t) = \int_{-\infty}^{\infty} xf(x,t)dx = \int_{-\infty}^{\infty} xf(x)dx,$$

$$R_X(\tau) = R_X(t,t+\tau) = \iint_{\mathbb{R}^2} xyf(x,y;t,t+\tau)dxdy$$

$$= \iint_{\mathbb{R}^2} xyf(x,y;\tau)dxdy.$$

$$\mu_X \approx \frac{1}{N}\sum_{k=1}^{n} x_t^{(k)} \quad \text{and} \quad R_X(\tau) \approx \frac{1}{N}\sum_{k=1}^{n} x_t^{(k)} x_{t+\tau}^{(k)},$$

What we usually use is the time average

$$\frac{1}{2T}\int_{-T}^{T} x(t)dt \quad \text{and} \quad \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt.$$

Thank you for your patience !