Chapter 4 Algorithm

Su

北京邮电大学

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What is an algorithm?

- *An algorithm is a* **step-by-step method** *of* **solving some problem.**

This is a rather vague definition. We will get to know a more precise and mathematically useful definition in Chapter 12.

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- **Input(输入)** *The algorithm receives input.*
- **Output(输出)** *The algorithm produces output.*
- **Precision(准确性)** *The steps are precisely stated.*
- **Determinism(决定性)** *The intermediate results of each step of execution are unique and are determined only by the inputs and the results of the preceding steps.*
- **Finiteness(有限性)** *The algorithm terminates; that is, it stops after finitely many instructions have been executed.*
- **Correctness(正确性)** *The output produced by the algorithm is correct; that is, the algorithm correctly solves the problem.*
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if (c > large)
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4. large = b
5. if (c > large)
6. large = c
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8. }
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Proposed algorithm: *checks whether Goldbach's conjecture is true*

1. Let n = 4.

2. If n is not the sum of two primes,
output "no" and stop.

3. Else increase n by 2
and continue with step 2.

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- *Which properties of an algorithm—input, output, precision, determinism, finiteness, correctness, generality—does this proposed algorithm have?*

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Proposed algorithm: *checks whether Goldbach's conjecture is true*

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- *Which properties of an algorithm—input, output, precision, determinism, finiteness, correctness, generality—does this proposed algorithm have?*
- *Do any of them depend on the truth of Goldbach's conjecture?*

4.1 Introduction

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- *In computer science,* **pseudocode** *is a plain language description of the steps in an algorithm or another system.*

An algorithm (written in pseudocodes) always consist of

- *a title,*
- *a brief description of the algorithm,*
- *the input to and output from the algorithm,*
- *the functions containing the instructions of the algorithm.*

4.2 Examples of Algorithm

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In this section, we give examples of several useful algorithms.

Searching

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- *Looking for specified text in a document when running a word processor is an example of a searching problem.*
- *We discuss an algorithm to solve the text-searching problem.*

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- *If p does not occur in t, it returns* 0.

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- *Suppose that we are given text t (e.g. of length n).*
- *We want to find the first occurrence of pattern p (of length m) in t or determine that p does not occur in t.*
- **Idea**: *compare $t_i \ldots t_{i+m-1}$ and $p_1 \ldots p_m$, if equal then return i, if not, then $i = i + 1$.*

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Input: p *(indexed from* 1 *to* m*)*, m, t *(indexed from* 1 *to* n*)*, n;

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$$\text{text search}(p, m, t, n)\ \{$$

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$$j = 1$$

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The following figures show a trace of the text search algorithm where we are searching for the pattern "001" in the text "010001".

$j = 1$
$\downarrow$
```
  001
  010001
```
$\uparrow$
$i = 1$

(1)

$j = 2$
$\downarrow (\times)$
```
  001
  010001
```
$\uparrow$
$i = 1$

(2)

$j = 1$
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(3)

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$$\begin{array}{ccc}
j = 1 & j = 2 & j = 3 \\
\downarrow & \downarrow & \downarrow (\times) \\
\texttt{\ \ 001} & \texttt{\ 001} & \texttt{001} \\
\texttt{010001} & \texttt{010001} & \texttt{010001} \\
\uparrow & \uparrow & \uparrow \\
i = 3 & i = 3 & i = 3 \\
(4) & (5) & (6)
\end{array}$$

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```
  j = 1
    ↓
   001
010001
   ↑
 i = 4
```

(7)

```
    j = 2
    ↓
   001
010001
   ↑
 i = 4
```

(8)

```
     j = 3
     ↓
   001
010001
   ↑
 i = 4
```

(9)

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text search(p, m, t, n) {
for $(i = 1;\ i \leq nm + 1;\ i = i + 1)\{$
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- *i is the index in t of the first character of the substring*
- *to compare with p, and j is the index in p*
- *the while loop compares $t_i \ldots t_{i+m1}$ and $p_1 \ldots p_m$*

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- *Many sorting algorithms have been devised.*
- *We discuss bubble sort (冒泡排序).*

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- *The algorithm is named for the way the larger elements "bubble" up to the top of the list.*

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Input: l *(a list of numbers)*, n *(length of the list)*; **Output**: l, t

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for $(i = 0;\ i < n - 1;\ i = i + 1)$ {

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if (swapped == false)
break; }
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- *An operating system (e.g., Windows), for example, is better thought of as a program that never terminates rather than as a finite program with input and output.*
- *As an illustration, we present an example — a* **randomized algorithm**.

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- *A* **randomized algorithm** *does not require that the intermediate steps be* **uniquely defined** *and depend only on the inputs and the middle results.*
- **Description.** *A randomized algorithm executes, at some points it makes random choices.*

Randomized Algorithms

A useful function: $\mathrm{rand}(i, j)$

- *It returns a random integer between the integers i and j, inclusive.*

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shuffle (a,n){
for $(i = 0;\ i < n - 1;\ i = i + 1)$ {
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This algorithm shuffles the values in the sequence $(a_1, \ldots, a_n)$.

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shuffle (a,n){
for (i = 0; i < n − 1; i = i + 1) {
swapp(a_i, a_rand(i,n))
}
```

Notice that the output (i.e., the rearranged sequence) depends on the random choices made by the random number generator.

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The space estimations depend so much on the hardware, so we just estimate the time in this section.

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- *Instead of dealing directly with the input, we use the size of the input.*

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  we might **count the number of iterations of the loop**.

So the function is not so precise, but the growing order is.

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We can ask for the maximum/minimum/average time function to execute the algorithm.

4.3 Analysis of Algorithms

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Best-case time = minimum time *needed to execute the algorithm for inputs of size n.*
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- **Average-case time**: *Very hard to compute,*
  comparisons $= \frac{1}{2}\left(n^2 - n \cdot \ln n - (\gamma + \ln(2) - 1) \cdot n\right) + \mathcal{O}(\sqrt{n})$,
  swaps $= \frac{1}{4}(n^2 - n)$.

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Definition. *($O($_$)$)*

Let f and g be functions with domain $\{1, 2, 3, \ldots\}$. *We write*

$$f(n) = O(g(n))$$

and say that $f(n)$ *is* **of order at most** $g(n)$ *or* $f(n)$ *is* **big oh** *of* $g(n)$ *if there exists a positive constant* C_1 *such that*

$$|f(n)| \leq C_1 |g(n)|$$

for all but finitely many positive integers n.

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Definition. $(\Omega(\_))$

Let f and g be functions with domain $\{1, 2, 3, \ldots\}$. *We write*

$$f(n) = \Omega(g(n))$$

and say that $f(n)$ *is of* **order at least** $g(n)$ *or* $f(n)$ *is* **omega** *of* $g(n)$ *if there exists a positive constant* C_2 *such that*

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A better idea to consider the time functions is to introduce the following **Asymptotic Notations**.

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and say that $f(n)$ *is* **of order** $g(n)$ *or* $f(n)$ *is* **theta** *of* $g(n)$ *if* $f(n) = O(g(n))$ *and* $f(n) = \Omega(g(n))$

Asymptotic Notations.

Definition. $(\boldsymbol{O}(\_), \Omega(\_), \Theta(\_))$

Let f and g be functions with domain $\{1, 2, 3, \ldots\}$.

- $f(n) = O(g(n))$: *exists* $C_1 > 0$ *s.t.* $|f(n)| \leq C_1|g(n)|$ *for n sufficiently large.*
- $f(n) = \Omega(g(n))$: *exists* $C_2 > 0$ *s.t.* $|f(n)| \geq C_2|g(n)|$ *for n sufficiently large.*
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When we analyze the growth of complexity functions, $f(n)$ *and* $g(n)$ *are always positive. Therefore, we can simplify the big-O requirement to* $f(n) \leq C_1 g(n)$ *for n sufficiently large.*

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- **Best-case time**: $comparisons = O(n)n$, $swaps = O(1)$;
- **Worst-case time**: $comparisons = O(n^2)$, $swaps = O(n^2)$;
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Proposition.

Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ *be a polynomial in* n *of degree* k*, where each* a_i *is nonnegative. Then*

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$$p(n) = \Theta\left(n^k\right).$$

Proof. *When* n *is sufficiently large, we have* $\frac{|a_k|}{2} n^k \leq p(n) \leq 2|a_k| n^k$.

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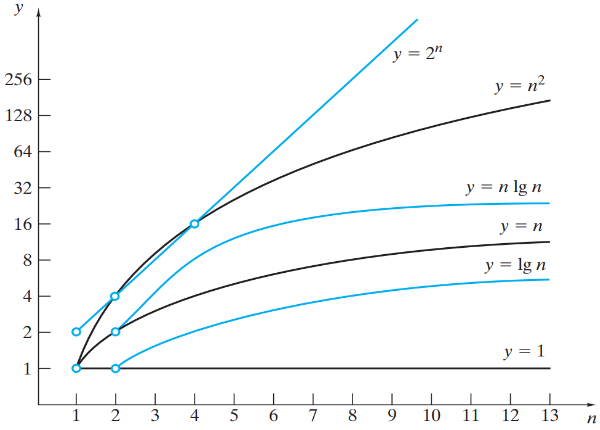

Figure 4.3.1 Growth of some common functions.

TABLE 4.3.3 Common Growth Functions

Theta Form	*Name*
$\Theta(1)$	Constant
$\Theta(\lg \lg n)$	Log log
$\Theta(\lg n)$	Log
$\Theta(n)$	Linear
$\Theta(n \lg n)$	n log n
$\Theta(n^2)$	Quadratic
$\Theta(n^3)$	Cubic
$\Theta(n^k), k \geq 1$	Polynomial
$\Theta(c^n), c > 1$	Exponential
$\Theta(n!)$	Factorial

Certain growth functions occur so often that they are given special names, as shown in Table 4.3.3.

TABLE 4.3.3 ■ **Common Growth Functions**

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$$6n^6 + n + 4 \qquad\qquad =$$

4.3 Analysis of Algorithms

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\ln(n!) &= O(n\ln n)
\end{aligned}$$

By Stirling's approximation,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

$$\ln(n!) = n\ln n - n + O(\ln n).$$

- *A problem that has a worst-case* **polynomial-time** *algorithm is considered to have a "good" algorithm; the interpretation is that such a problem has an efficient solution.*

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- *Such problems are called* **feasible** *or* **tractable.**
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  Given an arbitrary program and a set of inputs, will the program eventually halt?
- *A large number of solvable problems have undetermined status. We don't know whether they are intractable.*

4.3 Problem-Solving Tips

Apply the techniques learned in the calculus courses.

4.4 Recursive Algorithms

Recursive function *(pseudocode) is a function that invokes itself.*
A **recursive algorithm** *is an algorithm that contains a recursive function.*

Example. *Fibonacci sequence*

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Definition

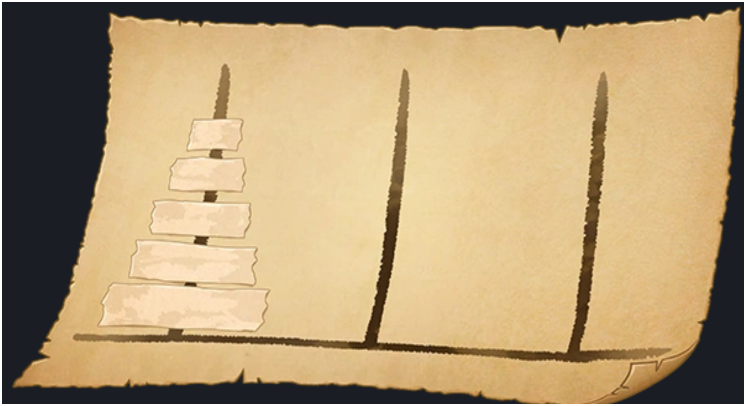
The Fibonacci sequence $\{f_n\}$ *is defined by the recursive formula*

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3.$$

Example.


The Solution and Foundation of "Pagoda Stack"
Faruzan's Hangout Event

4.4 Recursive Algorithms



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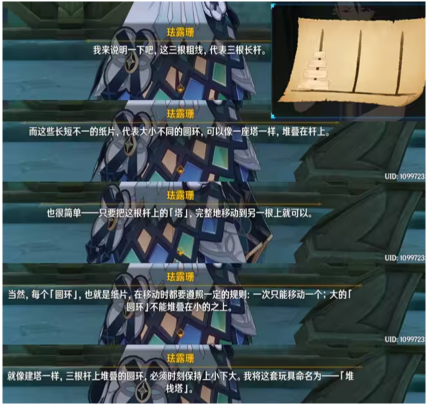

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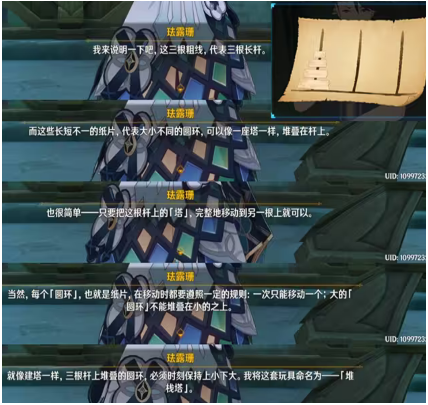

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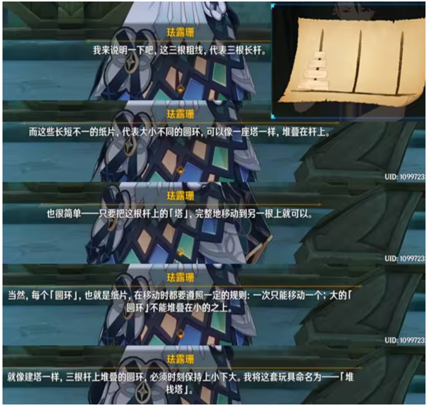

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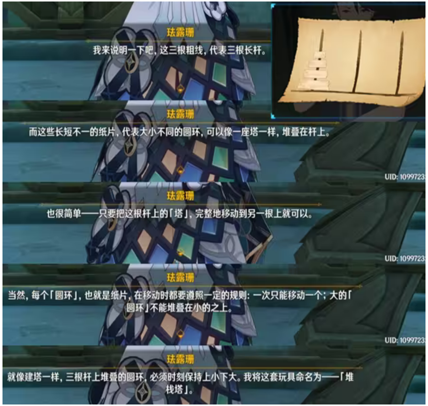

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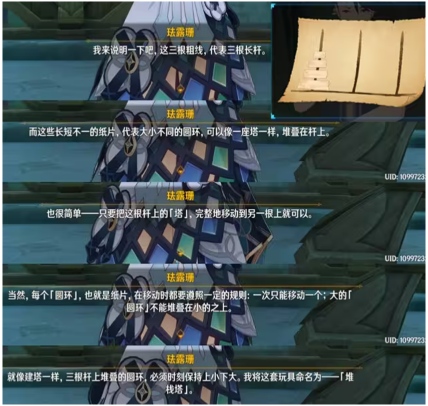

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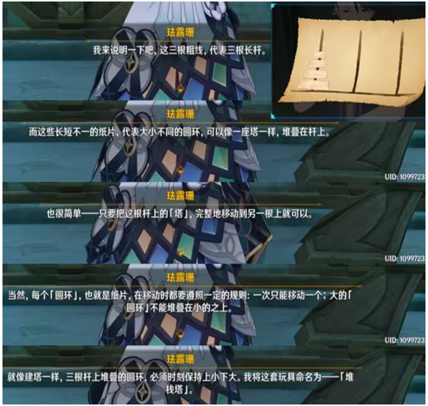

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- *High school method:* $p_n + 1 = 2(p_{n-1} + 1)$, $p_n = 2^n - 1.$

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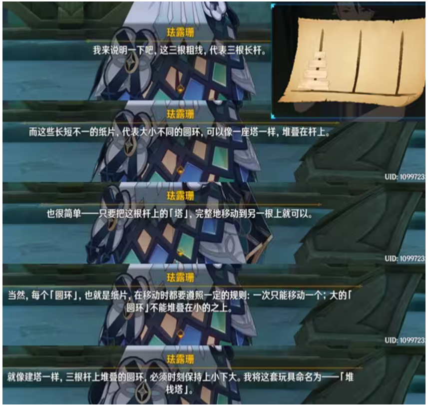

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- *In particular,* $p_7 = 127.$

4.4 Recursive Algorithms

The End