3.1

(a) $\int_0^{+\infty}\int_0^{\infty} ce^{-ax}e^{-by}\,dxdy = 1$

$\cancel{\frac{c}{ab}\, e^{-ax}\Big|_0^{\infty} \cdot e^{-by}\Big|_0^{\infty} = \frac{1}{ab}(-2)}$

$= -\frac{c}{a}\int_0^{\infty} e^{-ax-by}\Big|_0^{\infty} dy$

$= \frac{c}{a}\int_0^{+\infty} e^{-by}\,dy = \frac{c}{ab} = 1 \Rightarrow c = ab$

(b) $P(Y \le X) = \int_0^{+\infty}\int_0^{x} f(x,y)\,dydx$

$= \int_0^{+\infty}\int_0^{x} ab\,e^{-ax-by}\,dydx$

$= -a\int_0^{\infty} e^{-ax-by}\Big|_0^{x} dx = -a\int_0^{+\infty} \left(e^{-ax-bx} - e^{-ax}\right)dx$

$= -a\cdot\left[-\frac{1}{a+b}e^{-(a+b)x} + \frac{1}{a}e^{-ax}\right]\Big|_0^{+\infty}$

$= -a\left(\frac{1}{a+b} - \frac{1}{a}\right) = \frac{b}{a+b}$

(c). if $X \le 0$ $Y \le 0$ $F(X,Y) = 0$

if $x > 0$ $Y > 0$

$F(X,Y) = \int_0^{x}\int_0^{y} \cancel{f(u,v)}\, ab\,e^{-au-bv}\,dvdu$

$= -a\int_0^{x} e^{-au-bv}\Big|_0^{y} du = -a\int_0^{x}\left(e^{-au-by} - e^{-au}\right)du$

$= e^{-au-by}\Big|_0^{x} - e^{-au}\Big|_0^{x} = e^{-ax-by} - e^{-by} - e^{-ax} + 1$

3.9

(a) $U^2 - 4V \ge 0$ $U, V = \pm 1$

when $V = -1$ $U^2 - 4V > 0$ $\Rightarrow$ $P = \frac{1}{2}$

(b)

V \ U	1	-1	
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
-1	$\frac{1}{3}$	$\frac{1}{6}$	
	$\frac{1}{2}$		

$x = \frac{-U \pm \sqrt{U^2 - 4V}}{2}$

$\begin{cases} U = 1 & V = -1 \ \checkmark \\ U = -1 & V = -1 \ \checkmark \end{cases}$

$E(X) = \left(\frac{\sqrt{5}-1}{2}\times\frac{1}{3} + \frac{\sqrt{5}+1}{2}\times\frac{1}{6}\right)\times 2 = \frac{3\sqrt{5}-1}{6}$

(c) $(U+V)^2 - 4(U+V) \ge 0$

$\Rightarrow (U+V-2)^2 \ge 4$ $|U+V-2| \ge 2$

$\therefore P(X) = \frac{1}{6}$

$x^2(1-3x+3x^2-x^3)$ $\quad -\frac{3}{20}+\frac{1}{6} \quad \frac{10-9}{60}$

3.11

$\because X, Y$ are independent

$\therefore f_{XY}(xy) = f_X(x)\, f_Y(y) = \begin{cases} 8y, & 0<x<1 \text{ and } 0<y<\frac{1}{2} \\ 0, & \text{other} \end{cases}$

$\therefore P(X>Y) = \int_0^{\frac{1}{2}} 8y\, dy \int_y^1 dx = \int_0^{\frac{1}{2}} 8y(1-y)\, dy = 4y^2 - \frac{8}{3}y^3 \Big|_0^{\frac{1}{2}} = \frac{2}{3}$

3.12

(a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^{1-x^2} \frac{15}{4}x^2\, dy = \frac{15}{4}x^2(1-x^2) \qquad -1 \le x \le 1$

$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4}x^2\, dx = \frac{5}{2}(1-y)^{\frac{3}{2}} \qquad 0 \le y \le 1$

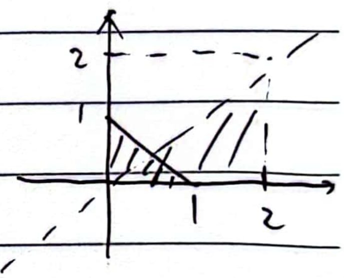
(b) $\because f_X(x) \cdot f_Y(y) \neq f(x,y)$

$\therefore X, Y$ are not independent

3.14

(a) $f_X(x) = \begin{cases} \frac{3}{8}x^2 & 0 \le x \le 2 \\ 0 & \text{other} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{3}{8}y^2 & 0 \le y \le 2 \\ 0 & \text{other} \end{cases}$

$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{9}{64}x^2y^2, & 0 \le x \le 2,\ 0 \le y \le 2 \\ 0, & \text{other} \end{cases}$



(b) $P(X=Y) = 0$

(c) $P(X>Y)$

$= \cancel{\int_0^2 \int_y^2 \frac{9}{64}x^2y^2\, dx\, dy} \int_0^2 \frac{9}{64}x^2 dx \int_0^x y^2\, dy$

$= \frac{9}{64} \int_0^2 x^2 \cdot \frac{x^3}{3}\, dx$

$= \frac{1}{128} x^6 \Big|_0^2 = \frac{2^6}{128} = \frac{1}{2}$

(d) $P(X+Y \le 1) = \int_0^1 \frac{9}{64}x^2 dx \int_0^{1-x} y^2 dy = \frac{3}{64}\int_0^1 x^2 \cdot (1-x)^3\, dx$

$= \frac{3}{64}\left(\frac{x^3}{3} - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{1}{6}x^6\right)\Big|_0^1 = \frac{3}{64} \cdot \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{3}{64} \times \frac{1}{60} = \frac{1}{1280}$

3.15

$f_X(x) = \int_0^{+\infty} xe^{-(x+y)}\, dy = xe^{-x}\int_0^{+\infty} e^{-y}\, dy = xe^{-x}$

$f_Y(y) = \int_0^{+\infty} xe^{-(x+y)}\, dx = e^{-y}\int_0^{+\infty} xe^{-x}\, dx = e^{-y}\left(1 - xe^{-x}\Big|_0^{+\infty}\right) = e^{-y}$

$f_X(x)\, f_Y(y) = f(x,y)$

$\therefore X$ and Y are independent.

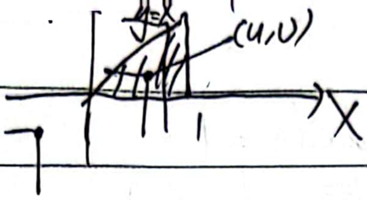
$f_Y(y) = \int_0^y 2\, dx = 2y \qquad f_X(x) = \int_x^1 2\, dy = 2 - 2x$

$f_X(x) \cdot f_Y(y) \neq f(x,y)$

$\therefore X$ and Y are not independent.

HW 8

Eg 3.1.8 $F(x,y)$



$0<y<x<1$

1° $0<y<x<1$

$F(x,y)=\int_0^y dv\int_v^x 8uv\,du = 2x^2y^2-y^4$

2° $0<x<1$ $y\geq x$

$F(x,y)=\int_0^x du\int_0^u 8uv\,dv = \int_0^x 4u^3\,du = x^4$

3° $x\geq 1$ $0\leq y<1$

$F(x,y)=\int_0^y dv\int_v^1 8uv\,du = 2y^2-y^4$

4° $x\geq 1$ $y\geq 1$ $F(x,y)=1$

$$\therefore F(x,y)=\begin{cases} 0 & x\leq 0,\ y\leq 0 \\ 2x^2y^2-y^4 & 0<x<y<1 \\ x^4 & 0<x<1 \quad \cancel{y<x}\ y\geq x \\ 2y^2-y^4 & x\geq 1 \quad 0\leq y<1 \\ 1 & x\geq 1 \quad y>1 \end{cases}$$

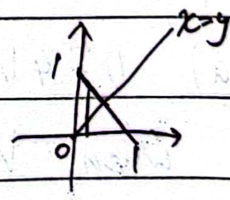
Ex

3.4

(a) $\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1 \Rightarrow \int_0^\infty\int_0^\infty ce^{-(x+y)}\,dx\,dy=1$

$= c\int_0^\infty -e^{-(x+y)}\Big|_0^\infty dy = c\int_0^\infty e^{-y}dy$

$= c\cdot(-e^{-y})\Big|_0^\infty = c = 1$



(b) $P(X+Y>1) = 1-P(X+Y\leq 1)$

$=1-\int_0^1\int_0^{1-x} e^{-(x+y)}\,dy\,dx = 1-\int_0^1 -e^{-(x+y)}\Big|_0^{1-x} dx$

$=1-\int_0^1 (e^{-x}-e^{-1})\,dx = 1-(-e^{-x}-e^{-1}x)\Big|_0^1 = \frac{2}{e}$

(c) $P(X<Y)$

$=\int_0^y\int_0^{+\infty} f(x,y)\,dy\,dx = \int_0^{+\infty}\int_0^y e^{-(x+y)}\,dx\,dy$

$=\int_0^{+\infty} -e^{(x+y)}\Big|_0^y dy = \int_0^{+\infty}(e^{-y}-e^{-2y})\,dy = (-e^{-y}+\frac{1}{2}e^{-2y})\Big|_0^{+\infty}$

$= 1-\frac{1}{2} = \frac{1}{2}$