Chapter 8 Graph Theory 图论

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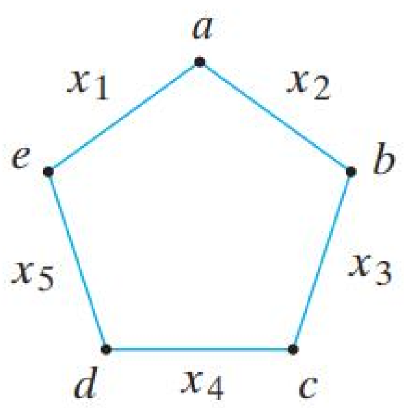
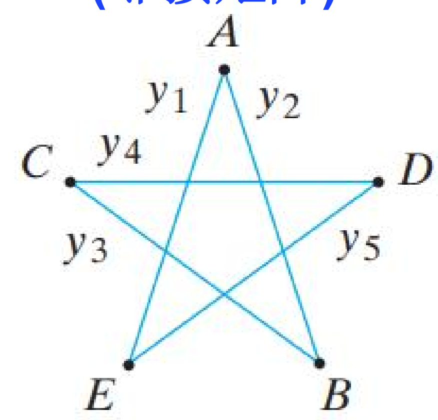
8.6 Isomorphisms of Graphs 图的同构

Definition 8.6.1 G_1 and G_2 are **isomorphic (同构的)** if there exist a one-to-one, onto functions f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2. The pair of functions f and g is called an **isomorphism of G_1 onto G_2 (G_1到G_2上的同构映射)**.

8.6 Isomorphisms of Graphs 图的同构

Theorem 8.6.4 Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their **adjacency matrices (邻接矩阵)** are equal.





$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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Theorem 8.6.4 Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their **adjacency matrices (邻接矩阵)** are equal.

Corollary 8.6.5 Let G_1 and G_2 be **simple graphs**. The following are equivalent:
(a) G_1 and G_2 are isomorphic.
(b) There is a one-to-one, onto function f from the vertex set of G_1 to the vertex set of G_2 satisfying the following: Vertices v and w are adjacent in G_1 if and only if the vertices $f(v)$ and $f(w)$ are adjacent in G_2.

8.6 Isomorphisms of Graphs 图的同构

How to prove that two simple graphs Graphs G_1 and G_2 are not isomorphic?

Find a property of G_1 that G_2 does not have but that G_2 would have if G_1 and G_2 were isomorphic. Such a property is called an **invariant (不变量)**.

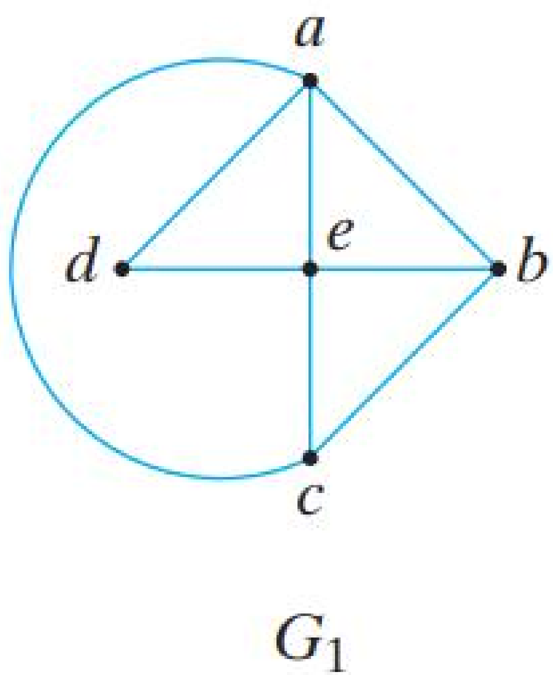
A property P is an invariant if whenever G_1 and G_2 are isomorphic graphs:

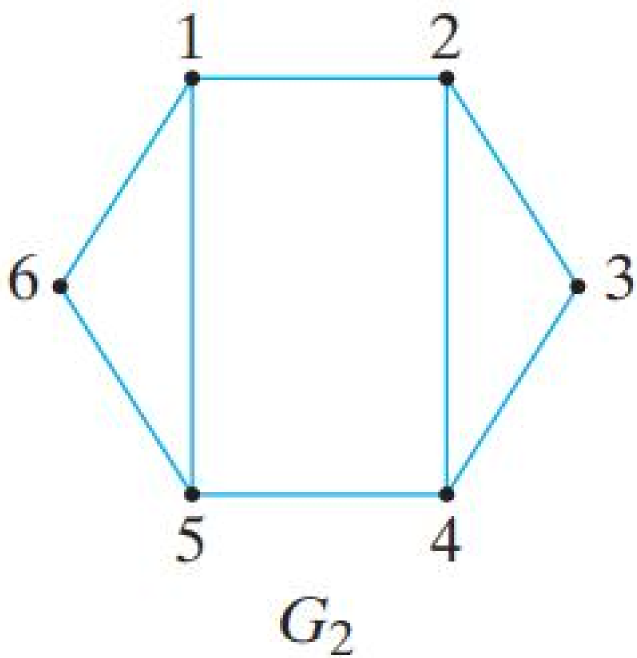
If G_1 has property P, G_2 also has property P.

- If G_1 and G_2 are isomorphic, then G_1 and G_2 have the same number of edges and the same number of vertices.
- If k is a positive integer, “has a vertex of degree k” is an invariant.
- If l is a positive integer, “has a simple cycle of length l” is an invariant.

8.6 Isomorphisms of Graphs 图的同构

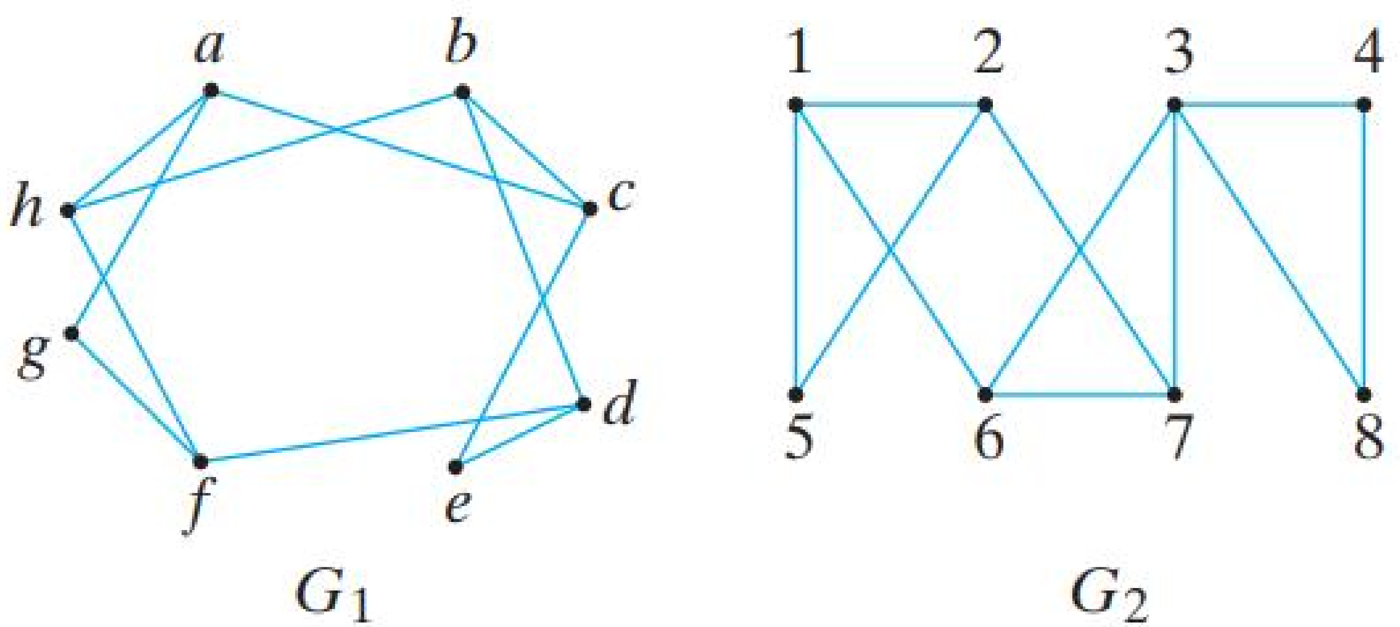
Determine whether the graphs G_1 and G_2 are isomorphic.





8.6 Isomorphisms of Graphs 图的同构

Determine whether the graphs G_1 and G_2 are isomorphic.

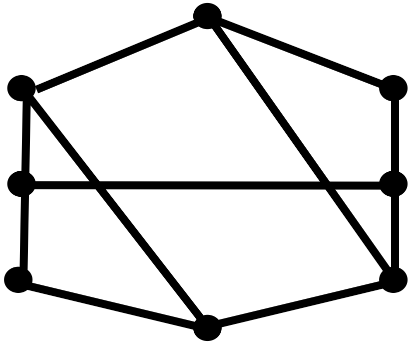


8.7 Planar Graphs 平面图

Definition 8.7.1 A graph is **planar (平面图)** if it can be drawn in the plane without its edges crossing.

Application: In designing printed circuits it is desirable to have as few lines cross as possible; thus the designer of printed circuits faces the problem of planarity.

Exercise 1 Show that the following graph is planar by redrawing it so that no edges cross.

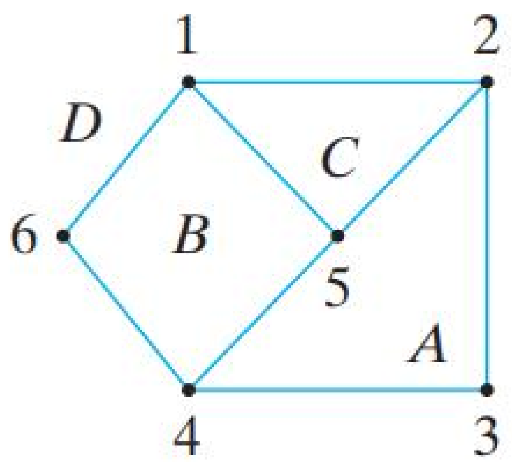


8.7 Planar Graphs 平面图

Definition 8.7.1 A graph is **planar (平面图)** if it can be drawn in the plane without its edges crossing.

If a connected, planar graph is drawn in the plane, the plane is divided into contiguous regions called **faces (面)**.

- A face is characterized by the cycle that forms its boundary.

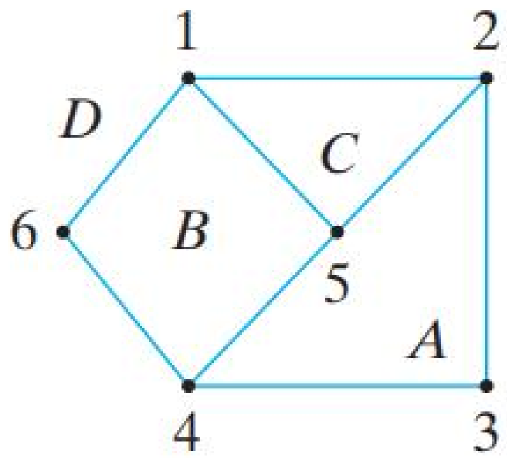


Face A is bounded by the cycle (5, 2, 3, 4, 5).

8.7 Planar Graphs 平面图

Theorem 8.7.9 Euler's Formula for Graphs (图的欧拉公式)
If G is a connected, planar graph with e edges, v vertices, and f faces, then

$$f = e - v + 2.$$



The graph has f =4 faces,
$e = 8$ edges, and $v = 6$ vertices.

$$f = e - v + 2.$$

Face A is bounded by the cycle (5, 2, 3, 4, 5).

8.7 Planar Graphs 平面图

Theorem 8.7.9 Euler's Formula for Graphs (图的欧拉公式)
If G is a connected, planar graph with e edges, v vertices, and f faces, then

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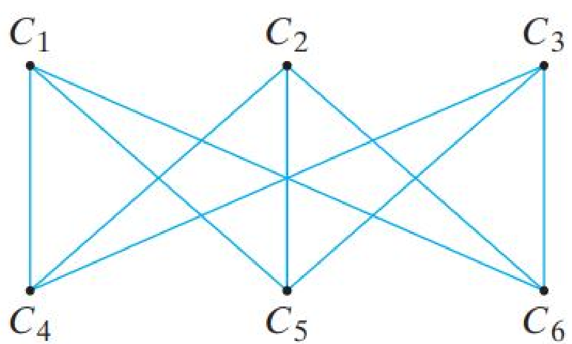
Whether graph $K_{3,3}$ is planar?

$$\boldsymbol{2e \geq 4f}$$



$K_{3,3}$

8.7 Planar Graphs 平面图

Definition 8.7.3 If a graph G has a vertex v of degree 2 and edges (v, v_1) and (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in **series (串联的)**. A **series reduction (串联约减)** consists of deleting the vertex v from the graph G and replacing the edges (v, v_1) and (v, v_2) by the edge (v_1, v_2). The resulting graph G' is said to be obtained from G by a series reduction. By convention, G is said to be obainable from itself by a series reduction.

8.7 Planar Graphs 平面图

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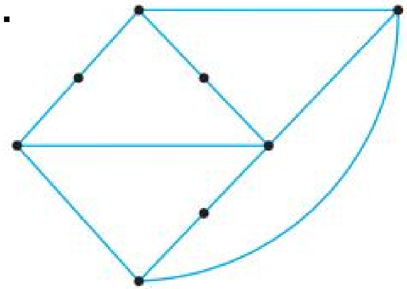
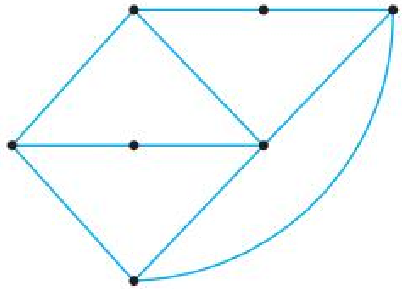
Definition 8.7.5 Graph G_1 and G_2 are **homeomorphic (同胚的)** if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.

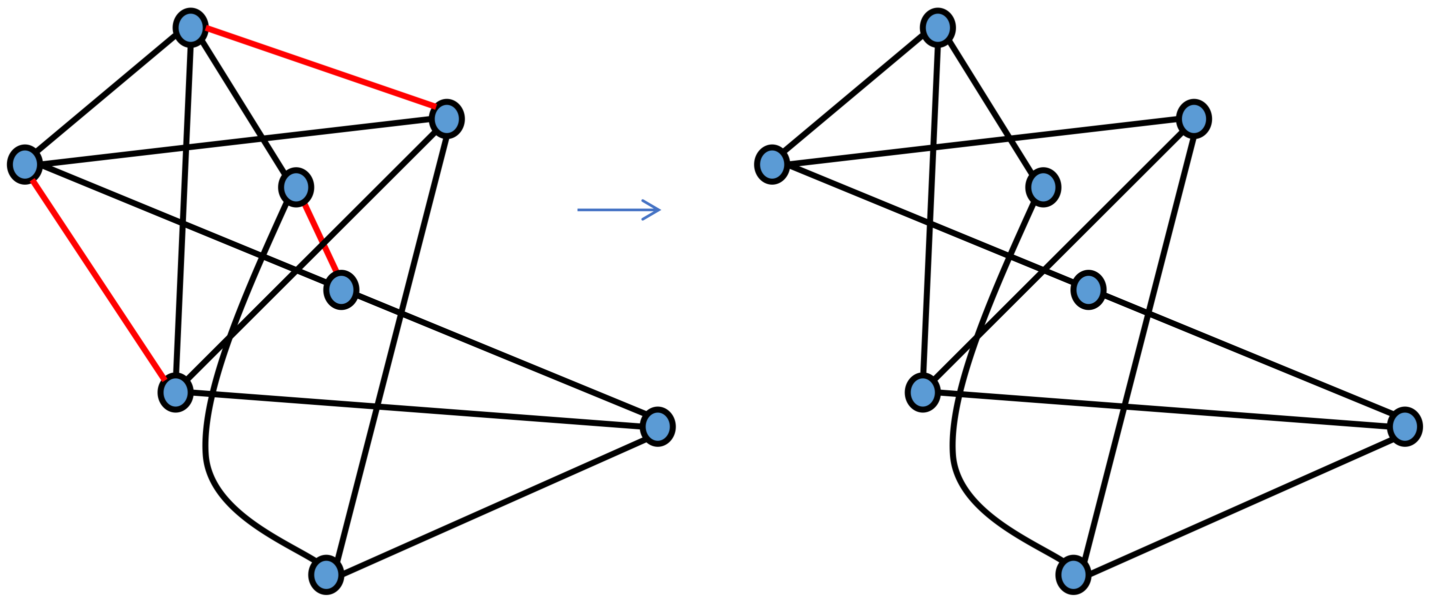




8.7 Planar Graphs 平面图

Theorem 8.7.7 Kuratowski's Theorem 库拉托夫斯基定理
A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 and $K_{3,3}$.

Example 8.7.8



8.7 Planar Graphs 平面图

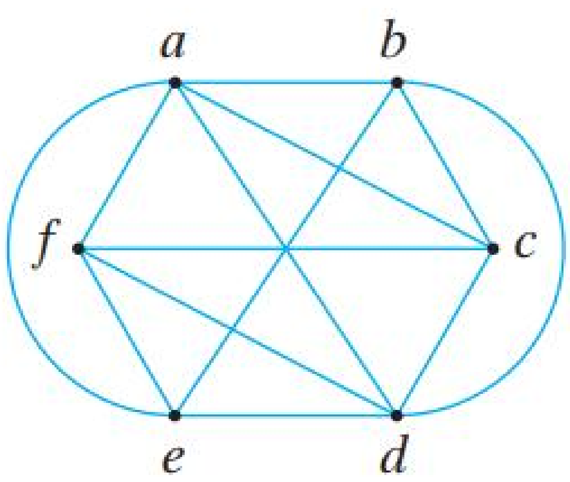
Exercise 1 Show that in any simple, connected, planar graph, $e \leq 3v - 6$.

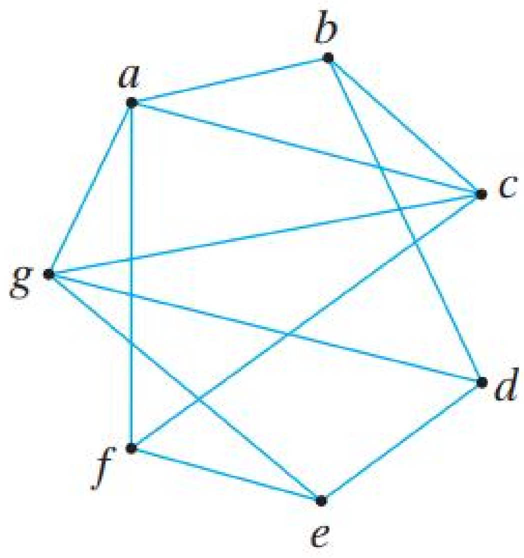
8.7 Planar Graphs 平面图

Exercise 2 Give an example of a simple, connected, nonplanar graph for which $e \leq 3v - 6$.

8.7 Planar Graphs 平面图

Show that each graph is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$.





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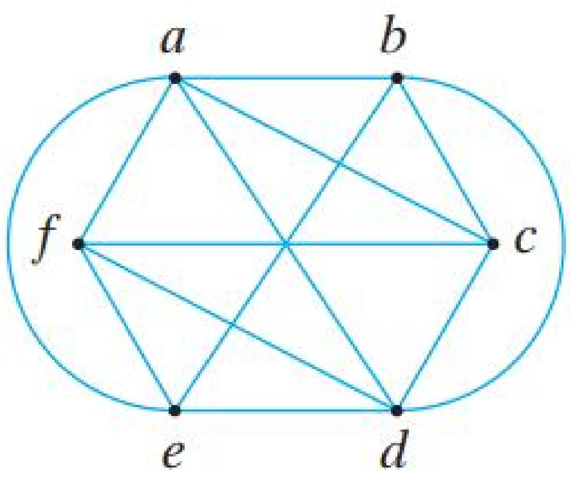

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Show that each graph is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$.