Chapter 2 Random Variable

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$$\begin{tikzcd} \omega\in\Omega\ (\Omega,\mathcal{F}) \arrow[rr, "P(\cdot)"] \arrow[dr, "X(\omega)"'] & & {[0,1]} \\ & \mathbb{R} \arrow[ur, "F_X(x)"'] & \end{tikzcd}$$

random variable

distribution function

$$\boxed{\text{Event} \iff \text{Value of } X}$$

$$I \subset \mathbb{R}$$

$$\underbrace{\{\omega : X(\omega) \in I\}}_{\in \mathcal{F}} = \{X \in I\}$$

$$P(X \in I) = P(\{\omega : X(\omega) \in I\})$$

$$\Omega = \{ HH, HT, TH, TT \}.$$

$$X = \# H.$$

$$X(\omega) = \begin{cases} 0, & \omega = TT \\ 1, & \omega = HT, TH \\ 2, & \omega = HH. \end{cases}$$

$$\{ X = 5 \} = \phi.$$

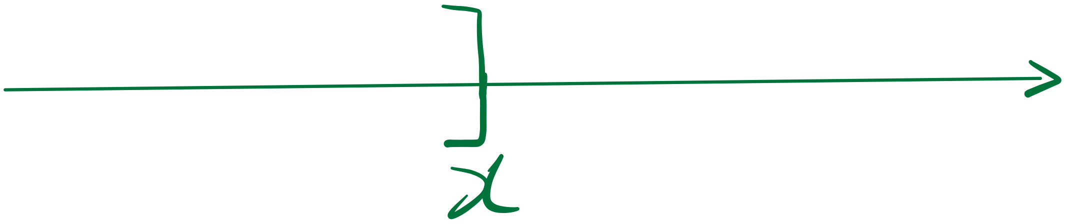
$$P(\{ X = 2 \}) = P(\{\omega = HH\}) = \frac{1}{4}$$

$$P(X \leq 1.5) = \frac{3}{4}$$

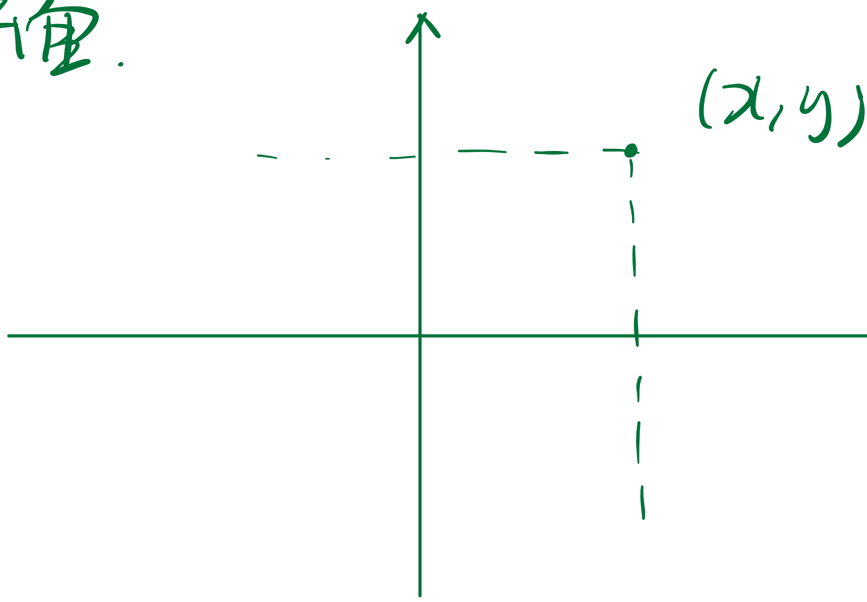
$$P(X \leq 0) = \frac{1}{4}.$$

$$\boxed{\text{Value of } X \iff \text{Distribution func. of } X}$$

半直线: $(-\infty, x]$



左下平面.



$$F_X(x) = P(\underbrace{X \le x}_{\uparrow \, 0,1,2})$$

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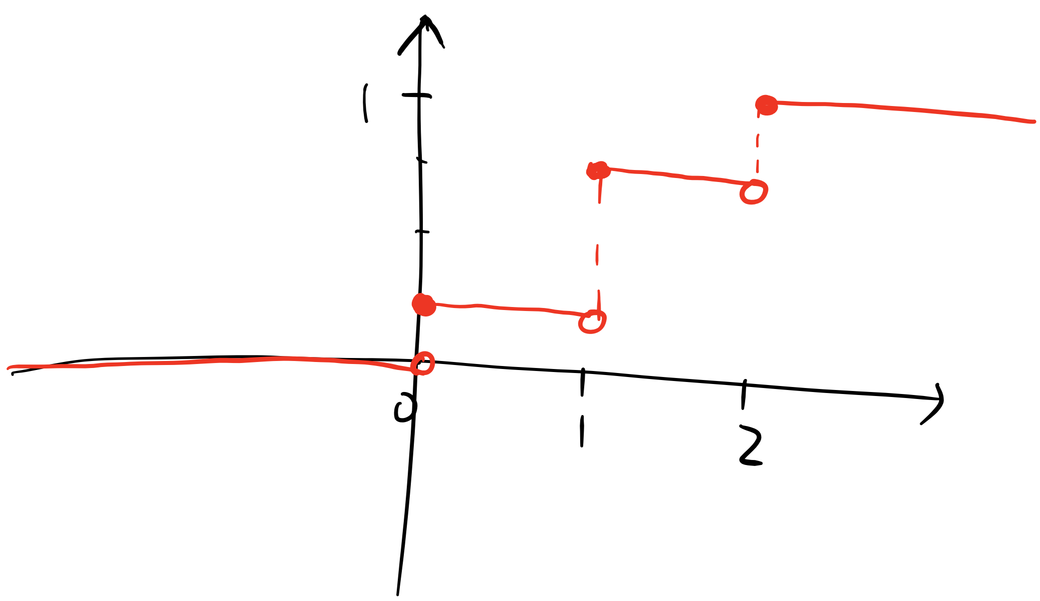
$$F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{4} + \frac{2}{4}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

$[x_k, x_{k+1})$

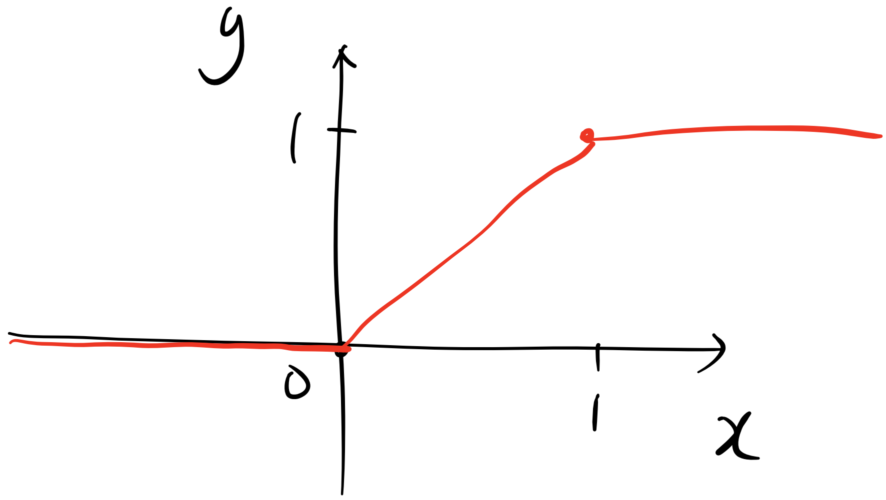


$$\Omega = [0,1] \xrightarrow{X(\omega)} [0,1]$$

$$X(\omega) = \omega$$

$$F(x) = P(X \le x)$$

$$= \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \geq 1 \end{cases}$$



Random Variable

The elements of a sample space may take diverse forms: real numbers, brands of components, colors, "good" or "defective" and so on.

In this chapter we transform all the elementary outcomes into numerical values, by means of random variables.

Contents

1. Random Variable
2. The Distribution Function of a Random Variable

The Definition of a Random Variable

Definition

*A **random variable** is a function that assigns a real number to each outcome in the sample space.*

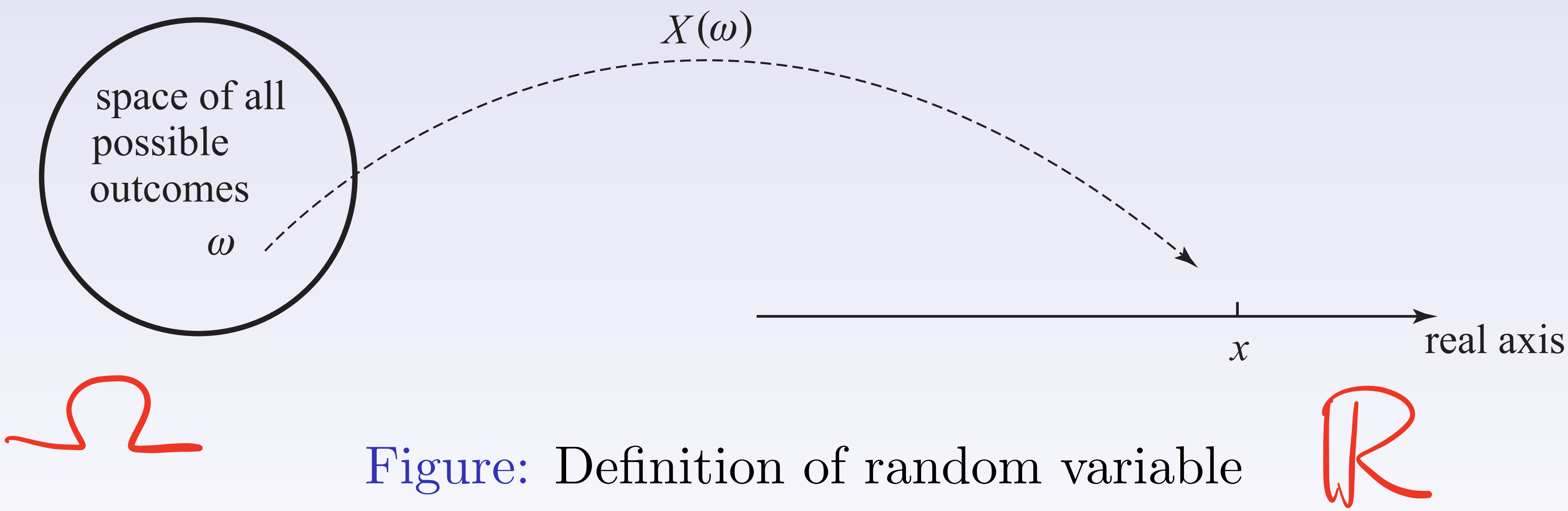


Figure: Definition of random variable

The Definition of a Random Variable

Example

In the experiment of tossing a coin, we could get the outcome "Head" or "Tail". Let "Head" = 1 and "Tail" = 0. Then we can get a random variable "X" defined on $\Omega = \{$ Head, Tail $\}$:

$$\begin{array}{ccc} \text{Random} & \text{Possible} & \text{Random} \\ \text{Variable} & \text{Values} & \text{Events} \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$X = X(\omega) = \begin{cases} 1, & \omega_1 = \text{Head}, \\ 0, & \omega_2 = \text{Tail}. \end{cases}$$

If we toss n coins, let Y be the total number of heads shown by the n coins. Clearly, Y is a random variable defined on $\Omega = \{0, 1, 2, \cdots, n\}$.

The Definition of a Random Variable

Example

Suppose that our experiment consists of tossing three fair coins. Let X denote the number of heads appearing. Then X is a random variable defined on

$$\Omega = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

and it takes on one of the values 0, 1, 2 and 3. That is,

$$X(TTT) = 0, \qquad X(TTH) = X(THT) = X(HTT) = 1,$$

$$X(THH) = X(HTH) = X(HHT) = 2, \quad X(HHH) = 3,$$

The Definition of a Random Variable

Example

Flip a fair coin until the rth head appears. Let X be the number of flips required.Then X is a random variable defined on $\Omega = \{r, r+1, r+2, \cdots\}$ and $X(n) = n$, $n = r, r+1, r+2, \cdots$.

Example

Let (x) denote a life aged x, where $x \geqslant 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x). Then T_x is a random variable defined on $\Omega = [0, L - x)$, where L is the limiting age.

$$P(X=n) = \underset{r-1\text{次}}{B(n-1,p)} * \underset{\substack{= \\ r^{th}}}{p}$$

第n次

$$P(Y=r-1)$$

The Definition of Distribution Function

Example

Suppose that our experiment consists of tossing two fair coins. Let X denote the number of heads appearing. Then X is a random variable taking on one of the values 0, 1, 2 with respective probabilities

$$P(X=0) = P(\omega \mid X(\omega)=0) = P(\{TT\}) = 1/4,$$
$$P(X=1) = P(\omega \mid X(\omega)=1) = P(\{TH, HT\}) = 2/4,$$
$$P(X=2) = P(\omega \mid X(\omega)=2) = P(\{HH\}) = 1/4.$$

The Definition of Distribution Function

Now let us calculate the probability of $A = \{X \leqslant 1.5\}$

$$\begin{aligned} P(A) &= P(X \leqslant 1.5) = P(X \in (\infty, 1.5]) \\ &= P(\omega \mid X(\omega) \leqslant 1.5) = P(\{TT, TH, HT\}) \\ &= P(\{X = 0\} \cup \{X = 1\}) \\ &= P(X = 0) + P(X = 1) = 3/4. \end{aligned}$$

For $A = (-\infty, x]$,

$$P(X \leqslant x) = P(X \in (-\infty, x]) = \sum_{x_k \leqslant x} P(X = x_k)$$

Contents

1. Random Variable

2. The Distribution Function of a Random Variable

The Definition of Distribution Function

Definition

The function $F(x)$ that associates with each real number x the probability $P(X \leqslant x)$ that the random variable X takes on a value smaller than or equal to this number is called the ***distribution function*** of X. *That is*

$$F(x) = P(X \leqslant x), \qquad \forall\, x \in \mathbb{R}. \tag{1}$$

The abbreviation for distribution function is d.f.. Some authors use the term ***cumulative distribution function***, instead of distribution function, and use the abbreviation c.d.f..

The end

Thank you for your patience !