EBU4375: SIGNALS AND SYSTEMS

LECTURE 13: PART 1

Queen Mary
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Orthogonal Signal Space and Fourier Series

Orthogonal Signal Space and Fourier Series

1) **Periodic continuous-time signals** can be expressed as a weighted sum of sinusoids (or a weighted sum of complex exponential functions). In this case, the frequency spectrum can be generated by computing the ***Fourier series***

2) The resulting representations are referred to as the **continuous-time Fourier series (CTFS)**

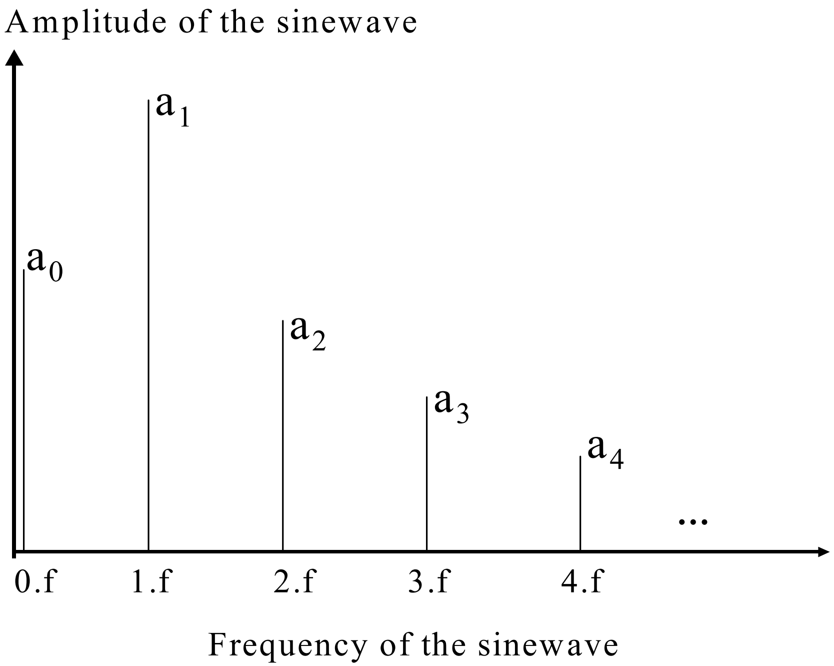
3) The Fourier series is named after the French physicist **Jean Baptiste Fourier** (1768-1830), who was the first one to propose that periodic waveforms could be represented by **a sum of sinusoids (or complex exponentials)**

Orthogonal Signal Space and Fourier Series

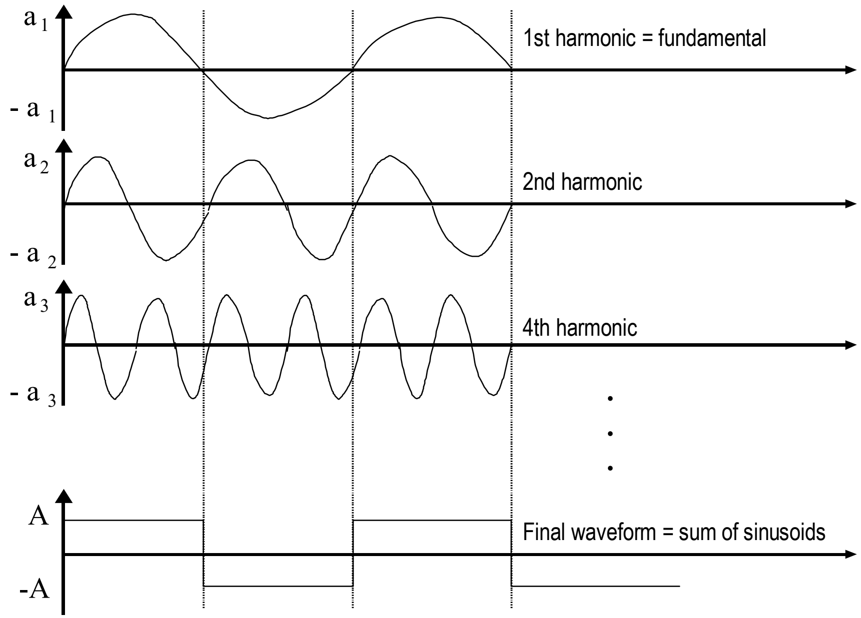
1) **Periodic continuous-time** signals $\longrightarrow$ We use **Continuous-Time Fourier series (CTFS)** to decompose such signals into their frequency components

2) **Aperiodic continuous-time** signals $\longrightarrow$ We use **Continuous-Time Fourier Transform (CTFT)** to decompose such signals into their frequency components

Orthogonal Signal Space and Fourier Series

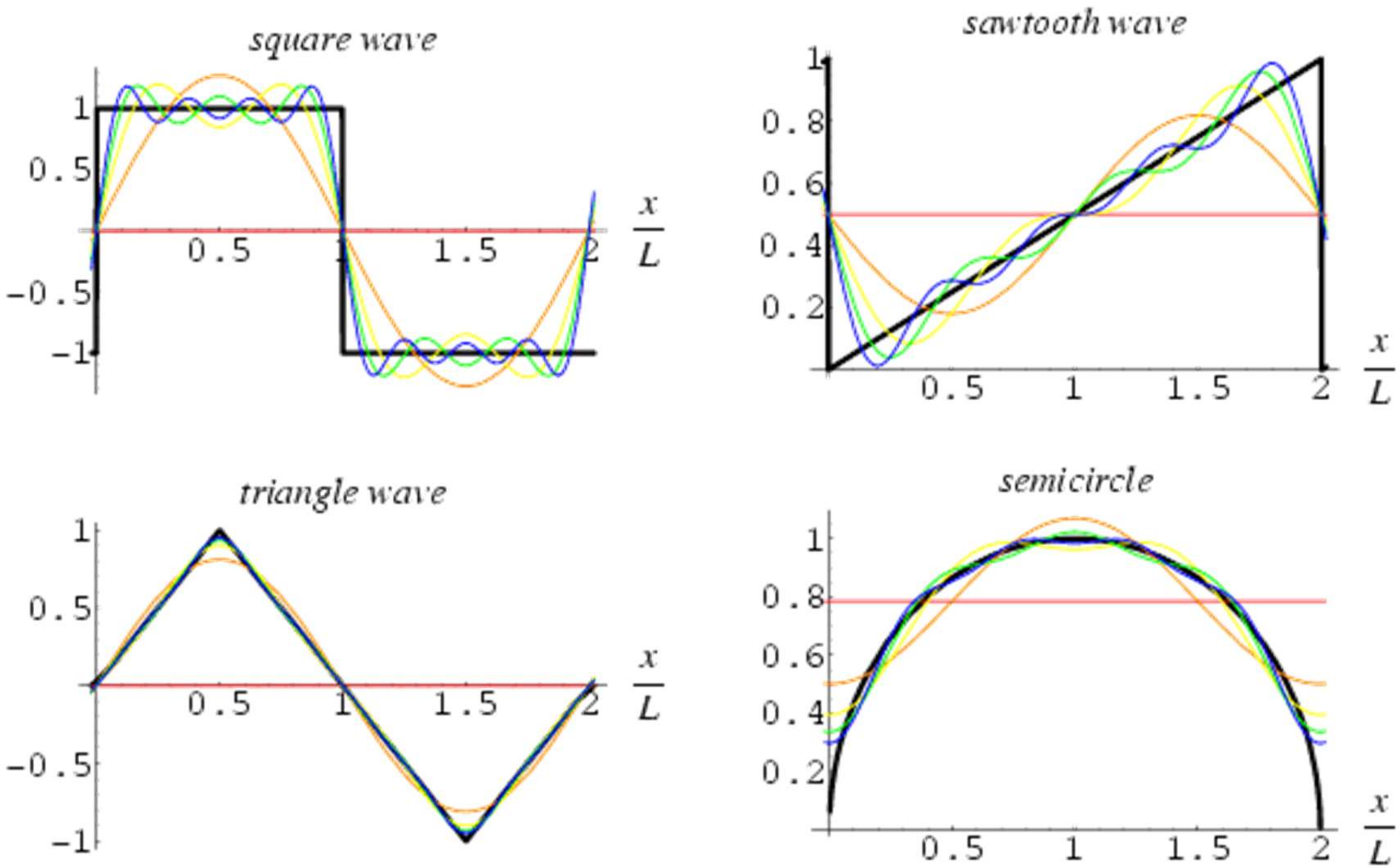


This diagram represents the frequency domain



This diagram represents the time domain. NOTE: the bottom line is not the sum of the first 3 lines.

Orthogonal Signal Space and Fourier Series



Orthogonal Signal Space

Definition *Two non-zero signals $p(t)$ and $q(t)$ are said to be orthogonal over interval $t = [t_1,\ t_2]$ if*

$$\int_{t_1}^{t_2} p(t)q^*(t)\mathrm{d}t = \int_{t_1}^{t_2} p^*(t)q(t)\mathrm{d}t = 0,$$

where the superscript $*$ denotes the complex conjugation operator. In addition if both signals $p(t)$ and $q(t)$ also satisfy the unit magnitude property:

$$\int_{t_1}^{t_2} p(t)p^*(t)\mathrm{d}t = \int_{t_1}^{t_2} q(t)q^*(t)\mathrm{d}t = 1,$$

they are said to be orthonormal to each other over the interval $t = [t_1,\ t_2]$.

Orthogonal Signal Space

Example

Show that

(i) functions $\cos(2\pi t)$ and $\cos(3\pi t)$ are orthogonal over interval $t = [0,\ 1]$;
(ii) functions $\exp(\mathrm{j}2t)$ and $\exp(\mathrm{j}4t)$ are orthogonal over interval $t = [0, \pi]$;
(iii) functions $\cos(t)$ and t are orthogonal over interval $t = [-1, 1]$.

Solution

$$\int_0^1 \cos(2\pi t)\cos(3\pi t)\mathrm{d}t = \frac{1}{2}\int_0^1 [\cos(\pi t) + \cos(5\pi t)]\mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\sin(\pi t) + \frac{1}{5\pi}\sin(5\pi t)\right]_0^1 = 0.$$

Therefore, the functions $\cos(2\pi t)$ and $\cos(3\pi t)$ are orthogonal over interval $t = [0, 1]$.

Orthogonal Signal Space

(ii)

$$\int_0^{\pi} e^{j2t} e^{-j4t} dt = \int_0^{\pi} e^{-j2t} dt = \frac{1}{-2j}[e^{-j2t}]_0^{\pi} = -\frac{1}{2j}[e^{-j2\pi} - 1]_0^{\pi} = 0,$$

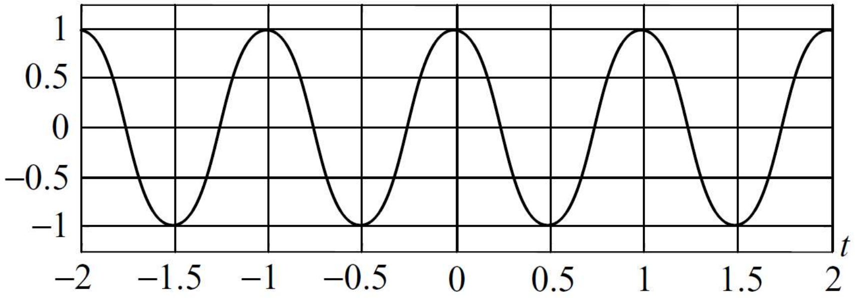
implying that the functions $\exp(j2t)$ and $\exp(j4t)$ are orthogonal over interval $t = [0, \pi]$.

(iii)

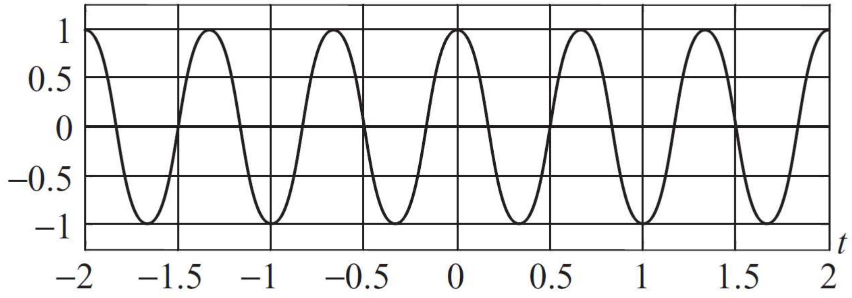
$$\int_{-1}^{1} t\cos(t)dt = [t\sin(t) + \cos(t)]_{-1}^{1} = [1 \cdot \sin(1) + \cos(1)] - [(-1) \cdot \sin(-1) + \cos(-1)] = 0,$$

implying that the functions $\cos(t)$ and t are orthogonal over interval $t = [-1, 1]$.

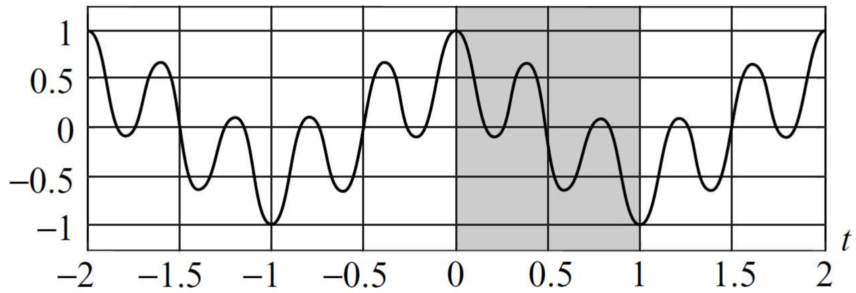
Orthogonal Signal Space



(a)



(b)



Graphical illustration of the orthogonality condition for the functions $\cos(2\pi t)$ and $\cos(3\pi t)$

(a) Waveform for $\cos(2\pi t)$.

(b) Waveform for $\cos(3\pi t)$.

(c) Waveform for $\cos(2\pi t) \times \cos(3\pi t)$.

Orthogonal Signal Space

Example

Show that the set $\{1, \cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots\}$, consisting of all possible harmonics of sine and cosine waves with fundamental frequency of ω_0, is an orthogonal set over any interval $t = [t_0, t_0 + T_0]$, with duration $T_0 = 2\pi/\omega_0$.

Solution

It may be noted that the set $\{1, \cos(\omega_0 t), \cos(2\omega_0 t), \cos(3\omega_0 t), \ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t), \ldots\}$ contains three types of functions: 1, $\{\cos(m\omega_0 t)\}$, and $\{\sin(n\omega_0 t)\}$ for arbitrary integers $m, n \in Z^+$, where Z^+ is the set of positive integers. We will consider all possible combinations of these functions.

Orthogonal Signal Space

Case 1 The following proof shows that functions $\{\cos(m\omega_0 t),\ m \in Z^+\}$ are orthogonal to each other over interval $t = [t_0,\ t_0 + T_0]$ with $T_0 = 2\pi/\omega_0$.

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t \quad \text{for any arbitrary } t_0.$$

Using the trigonometric identity $\cos(m\omega_0 t)\cos(n\omega_0 t) = (1/2)[\cos((m - n)\omega_0 t) + \cos((m + n)\omega_0 t)]$, the above integral reduces as follows:

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \begin{cases} \left[\dfrac{\sin(m-n)\omega_0 t}{2(m-n)\omega_0} + \dfrac{\sin(m+n)\omega_0 t}{2(m+n)\omega_0}\right]_{t_0}^{t_0+T_0} & m \neq n \\ \left[\dfrac{t}{2} + \dfrac{\sin 2m\omega_0 t}{4m\omega_0}\right]_{t_0}^{t_0+T_0} & m = n, \end{cases}$$

Orthogonal Signal Space

or

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\cos(n\omega_0 t)\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n, \end{cases}$$

for $m, n \in Z^+$.

Case 2 By following the procedure outlined in case 1, it is straightforward to show that

$$\int_{\langle T_0 \rangle} \sin(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n, \end{cases}$$

for $m, n \in Z^+$.

Orthogonal Signal Space

Case 3 To verify that functions $\{\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$ are mutually orthogonal, consider the following:

$$\int_{\langle T_0 \rangle} \cos(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t = \int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\sin(n\omega_0 t)\mathrm{d}t$$

$$= \begin{cases} \frac{1}{2}\int_{t_0}^{t_0+T_0} [\sin((m+n)\omega_0 t) - \sin((m-n)\omega_0 t)]\mathrm{d}t & m \neq n \\ \frac{1}{2}\int_{t_0}^{t_0+T_0} [\sin(2m\omega_0 t)\mathrm{d}t & m = n \end{cases}$$

$$= \begin{cases} -\frac{1}{2}\left[\frac{\cos((m+n)\omega_0 t)}{(m+n)\omega_0}\right]_{t_0}^{t_0+T_0} + \frac{1}{2}\left[\frac{\cos((m-n)\omega_0 t)}{(m-n)\omega_0}\right]_{t_0}^{t_0+T_0} & m \neq n \\ -\frac{1}{2}\left[\frac{\cos(2n\omega_0 t)}{2m\omega_0}\right]_{t_0}^{t_0+T_0} & m = n \end{cases}$$

$$= \begin{cases} 0 & m \neq n \\ 0 & m = n, \end{cases}$$

for $m, n \in Z^+$, which proves that $\{\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$ are orthogonal over interval $t = [t_0, t_0 + T_0]$ with $T_0 = 2\pi/\omega_0$.

Orthogonal Signal Space

Case 4 The following proof demonstrates that the function "1" is orthogonal to $\cos(m\omega_0 t)\}$ and $\{\sin(n\omega_0 t)\}$:

$$\int_{\langle T_0 \rangle} 1 \cdot \cos(m\omega_0 t)\mathrm{d}t = \left[\frac{\sin(m\omega_0 t)}{m\omega_0}\right]_{t_0}^{t_0+T_0}$$

$$= \left[\frac{\sin(m\omega_0 t_0 + 2m\pi) - \sin(m\omega_0 t_0)}{m\omega_0}\right] = 0$$

and

$$\int_{\langle T_0 \rangle} 1 \cdot \sin(m\omega_0 t)\mathrm{d}t = \left[-\frac{\cos(m\omega_0 t)}{m\omega_0}\right]_{t_0}^{t_0+T_0}$$

$$= -\left[\frac{\cos(m\omega_0 t_0 + 2m\pi) - \cos(m\omega_0 t_0)}{m\omega_0}\right] = 0$$

Trigonometric CTFS

Definition *An arbitrary periodic function $x(t)$ with fundamental period T_0 can be expressed as follows:*

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)),$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of $x(t)$ and coefficients a_0, a_n, and b_n are referred to as the trigonometric CTFS coefficients. The coefficients are calculated as follows:

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) \mathrm{d}t,$$

$$a_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \cos(n\omega_0 t) \mathrm{d}t,$$

and

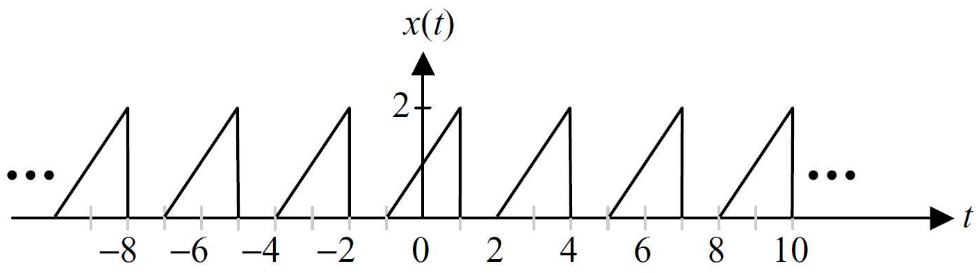
$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t) \mathrm{d}t.$$

Trigonometric CTFS

Example

Calculate the trigonometric CTFS coefficients of the periodic signal $x(t)$ defined over one period $T_0 = 3$ as follows:

$$x(t) = \begin{cases} t+1 & -1 \le t \le 1 \\ 0 & 1 < t < 2. \end{cases}$$



$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t)\mathrm{d}t = \frac{1}{3} \int_{-1}^{1} (t+1)\mathrm{d}t = \frac{1}{3}\left[\frac{1}{2}t^2 + t\right]_{-1}^{1} = \frac{2}{3}$$

Trigonometric CTFS

The CTFS coefficients a_n are given by

$$a_n = \frac{2}{T_0}\int_{\langle T_0\rangle} x(t)\cos(n\omega_0 t)\mathrm{d}t = \frac{2}{3}\int_{-1}^{1}(t+1)\cos(n\omega_0 t)\mathrm{d}t$$

$$= \frac{2}{3}\int_{-1}^{1}\underbrace{t\cos(n\omega_0 t)}_{\text{odd function}}\mathrm{d}t + \frac{2}{3}\int_{-1}^{1}\underbrace{\cos(n\omega_0 t)}_{\text{even function}}\mathrm{d}t.$$

Since the integral of odd functions within the limit $[-t_0, t_0]$ is zero,

$$\int_{-1}^{1} t\cos(n\omega_0 t)\mathrm{d}t = 0,$$

and the value of a_n is given by

$$a_n = \frac{2}{3}\int_{-1}^{1}\cos(n\omega_0 t)\mathrm{d}t = \frac{4}{3}\int_{0}^{1}\cos(n\omega_0 t)\mathrm{d}t = \frac{4}{3}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_0^1 = \frac{4\sin(n\omega_0)}{3n\omega_0}$$

Trigonometric CTFS

Substituting $\omega_0 = 2\pi/3$, we obtain

$$a_n = \begin{cases} 0 & n = 3k \\ \dfrac{\sqrt{3}}{n\pi} & n = 3k+1 \\ -\dfrac{\sqrt{3}}{n\pi} & n = 3k+2, \end{cases}$$

for $k \in Z$. Similarly, the CTFS coefficients b_n are given by

$$b_n = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t) \sin(n\omega_0 t) \mathrm{d}t = \frac{2}{3} \int_{-1}^{1} (t+1) \sin(n\omega_0 t) \mathrm{d}t$$

$$= \frac{2}{3} \int_{-1}^{1} \underbrace{t \sin(n\omega_0 t)}_{\text{even function}} \mathrm{d}t + \frac{2}{3} \int_{-1}^{1} \underbrace{\sin(n\omega_0 t)}_{\text{odd function}} \mathrm{d}t.$$

Trigonometric CTFS

Since the integral of odd functions within the limits $[-t_0, t_0]$ is zero,

$$\int_{-1}^{1} \sin(n\omega_0 t)\mathrm{d}t = 0,$$

and the value of b_n is given by

$$b_n = \frac{2}{3}\int_{-1}^{1} t\,\sin(n\omega_0 t)\mathrm{d}t = \frac{4}{3}\int_{0}^{1} t\,\sin(n\omega_0 t)\mathrm{d}t$$

$$= \frac{4}{3}\left[-t\frac{\cos(n\omega_0 t)}{n\omega_0} + \frac{\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^1 = -\frac{4\cos(n\omega_0)}{3n\omega_0} + \frac{4\sin(n\omega_0)}{3(n\omega_0)^2}.$$

Substituting $\omega_0 = 2\pi/3$, we obtain

$$b_n = \begin{cases} -\dfrac{2}{n\pi} & n = 3k \\[2ex] \dfrac{1}{n\pi} + \dfrac{3\sqrt{3}}{2(n\pi)^2} & n = 3k+1 \\[2ex] \dfrac{1}{n\pi} - \dfrac{3\sqrt{3}}{2(n\pi)^2} & n = 3k+2, \end{cases}$$

Trigonometric CTFS

for $k \in Z$. The periodic signal $x(t)$ is therefore expressed as follows:

$$x(t) = \underbrace{\frac{2}{3}}_{x_{\mathrm{av}}(t)} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{3}t\right)}_{\mathrm{Ev}\{x(t)-a_0\}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{3}t\right)}_{\mathrm{Odd}\{x(t)-a_0\}},$$

Coefficient a_0 represents the average value of signal $x(t)$, referred to as $x_{av}(t)$. The cosine terms collectively represent the zero-mean even component of signal $x(t)$, denoted by $\mathrm{Ev}\{x(t) - a_0\}$, while the sine terms collectively represent the zero-mean odd component of $x(t)$, denoted by $\mathrm{Odd}\{x(t) - a_0\}$.

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LECTURE 13: PART 2

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Exponential CTFS

Definition *An arbitrary periodic function $x(t)$ with a fundamental period T_0 can be expressed as follows:*

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

where the exponential CTFS coefficients D_n are calculated as

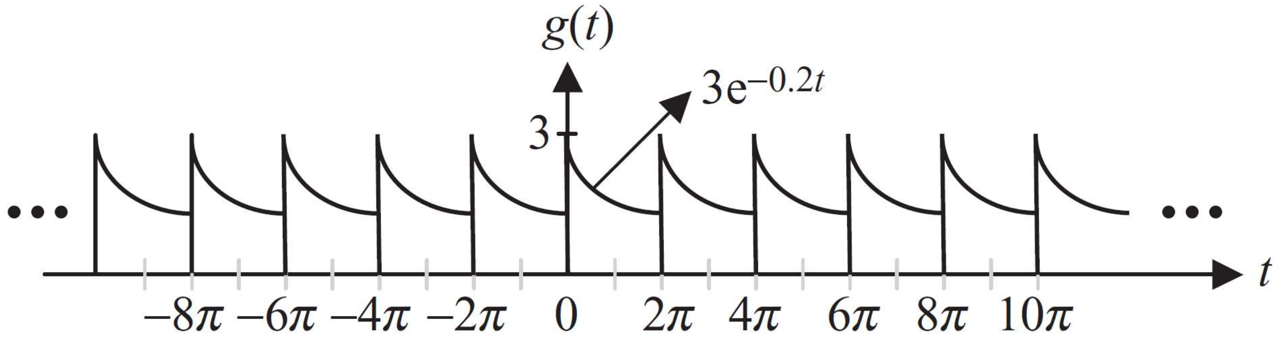
$$D_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t} dt,$$

ω_0 being the fundamental frequency given by $\omega_0 = 2\pi/T_0$.

Exponential CTFS

Example

Calculate the exponential CTFS coefficients for the periodic function $g(t)$



Solution

By inspection, the fundamental period $T_0 = 2\pi$, which gives the fundamental frequency $\omega_0 = 2\pi/2\pi = 1$. The exponential CTFS coefficients D_n are given by

$$D_n = \frac{1}{T_0}\int_{\langle T_0\rangle} g(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 3e^{-0.2t}e^{-jn\omega_0 t}\,dt = \frac{3}{2\pi}\int_0^{2\pi} e^{-(0.2+jn\omega_0)\,t}\,dt$$

Exponential CTFS

or

$$D_n = -\frac{3}{2\pi}\left[\frac{e^{-(0.2+jn\omega_0)t}}{(0.2+jn\omega_0)}\right]_0^{2\pi} = \frac{3}{2\pi}\frac{1}{(0.2+jn\omega_0)}\left[1-e^{-(0.2+jn\omega_0)2\pi}\right].$$

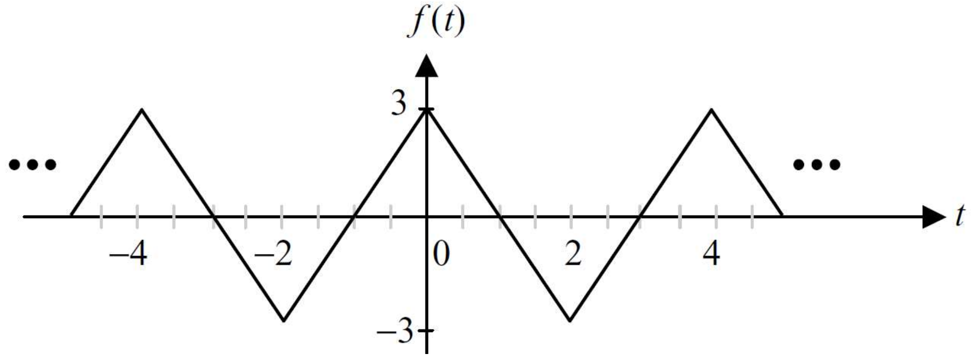
Substituting $\omega_0 = 1$, we obtain the following expression for the exponential CTFS coefficients:

$$\begin{aligned} D_n &= \frac{3}{2\pi(0.2+jn)}\left[1-e^{-(0.2+jn)2\pi}\right] \\ &= \frac{3}{2\pi(0.2+jn)}\left[1-e^{-0.4\pi}\right] \approx \frac{0.3416}{(0.2+jn)}. \end{aligned}$$

Exponential CTFS

Example

Calculate the exponential CTFS coefficients for $f(t)$ as shown



Solution

Since the fundamental period $T_0 = 4$, the angular frequency $\omega_0 = 2\pi/4 = \pi/2$. The exponential CTFS coefficients D_n are calculated directly from the definition as follows:

$$D_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} f(t) e^{-jn\omega_0 t} dt = \frac{1}{4} \int_{-2}^{2} f(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{4} \int_{-2}^{2} \underbrace{f(t)\cos(n\omega_0 t)}_{\text{even function}} dt - j\frac{1}{4} \int_{-2}^{2} \underbrace{f(t)\sin(n\omega_0 t)}_{\text{odd function}} dt.$$

Exponential CTFS

Since the integration of an odd function within the limits $[t_0, -t_0]$ is zero,

$$D_n = \frac{1}{4}\int_{-2}^{2} f(t)\cos(n\omega_0 t)\mathrm{d}t = \frac{1}{2}\int_{0}^{2} (3-3t)\cos(n\omega_0 t)\mathrm{d}t,$$

which simplifies to

$$\begin{aligned} D_n &= \frac{1}{2}\left[(3-3t)\frac{\sin(n\omega_0 t)}{n\omega_0} - 3\frac{\cos(n\omega_0 t)}{(n\omega_0)^2}\right]_0^2 \\ &= \frac{3}{2}\left[-\frac{\sin(2n\omega_0)}{n\omega_0} - \frac{\cos(2n\omega_0)}{(n\omega_0)^2} + \frac{1}{(n\omega_0)^2}\right]. \end{aligned}$$

Substituting $\omega_0 = \pi/2$, we obtain

$$D_n = \frac{3}{2}\left[-\frac{\sin(n\pi_0)}{0.5n\pi} - \frac{\cos(n\pi)}{(0.5n\pi)^2} + \frac{1}{(0.5n\pi)^2}\right] = \frac{6}{(n\pi)^2}[1-(-1)^n]$$

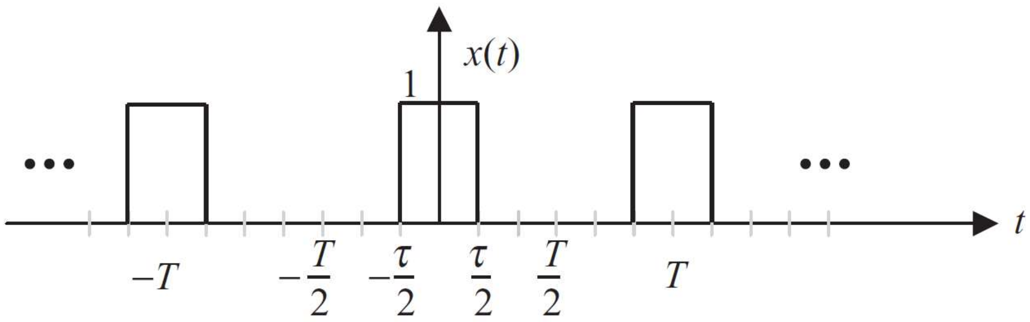
or

$$D_n = \begin{cases} 0 & n \text{ is even} \\ \dfrac{12}{(n\pi)^2} & n \text{ is odd.} \end{cases}$$

Exponential CTFS

Example

Calculate the exponential Fourier series of the signal $x(t)$



Case I For $n = 0$, the exponential CTFS coefficients are given by

$$D_n = \frac{1}{T}[t]_{-\tau/2}^{\tau/2} = \frac{\tau}{T}.$$

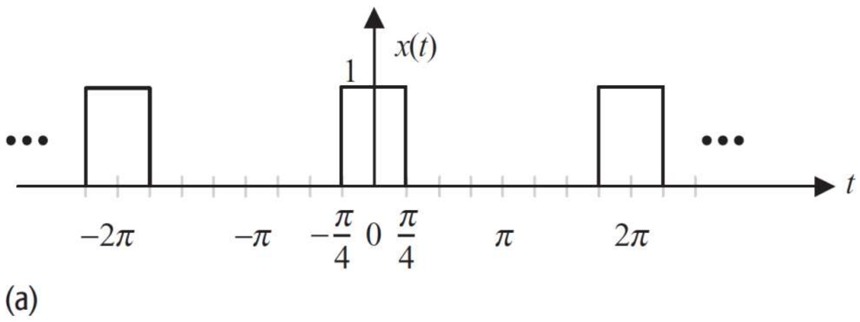
Case II For $n \neq 0$, the exponential CTFS coefficients are given by

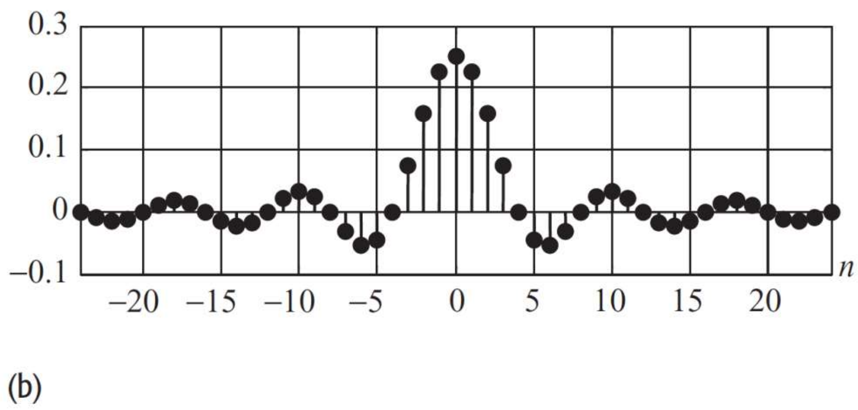
$$D_n = -\frac{1}{\mathrm{j}n\omega_0 T}[\mathrm{e}^{-\mathrm{j}n\omega_0 t}]_{-\tau/2}^{\tau/2} = \frac{1}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)$$

Exponential CTFS

or

$$D_n = \frac{\tau}{T}\frac{\sin\left(\pi\dfrac{n\tau}{T}\right)}{\left(\pi\dfrac{n\tau}{T}\right)} = \frac{\tau}{T}\operatorname{sinc}\left(\frac{n\tau}{T}\right)$$

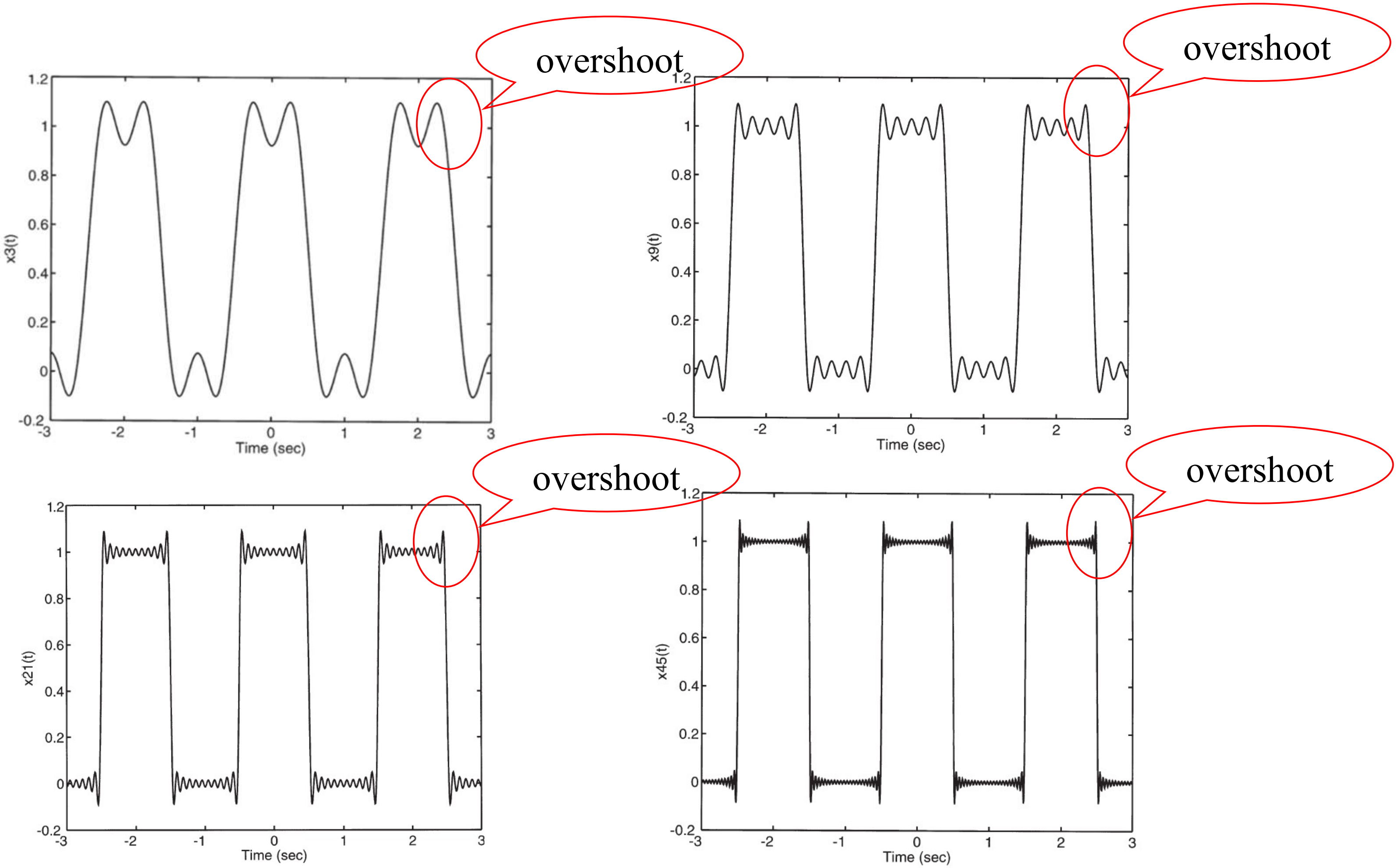




Exponential CTFS coefficients for the signal $x(t)$ with $\tau = \pi/2$ and $T = 2\pi$. (a) Waveform for $x(t)$. (b) Exponential CTFS coefficients.

$$\longrightarrow \quad D_n = \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right)$$

Gibbs Phenomenon



Gibbs Phenomenon

- **The overshoot at the corners is still present even in the limit as N approaches to infinity. This characteristic was first discovered by Josiah Willard Gibbs (1893-1903), and this overshoot is referred to as the *Gibbs phenomenon***

- **Now let x(t) be an arbitrary periodic signal. As a consequence of the Gibbs phenomenon, the Fourier series representation of x(t) is not actually equal to the true value of x(t) at any points where x(t) is discontinuous**

- **If x(t) is discontinuous at $t = t_1$, the Fourier series representation is off by approximately 9% at t_1^+ and t_1^-**