EBU4375: SIGNALS AND SYSTEMS

TOPIC 3-2: FOURIER SERIES OF DISCRETE-TIME PERIODIC SIGNALS

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ACKNOWLEDGMENT

These slides are partially from lectures prepared by

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AGENDA

1. Quick review
2. The notion of frequency in discrete-time signals
3. Fourier series representation of discrete-time periodic signals
4. Important properties of Fourier series

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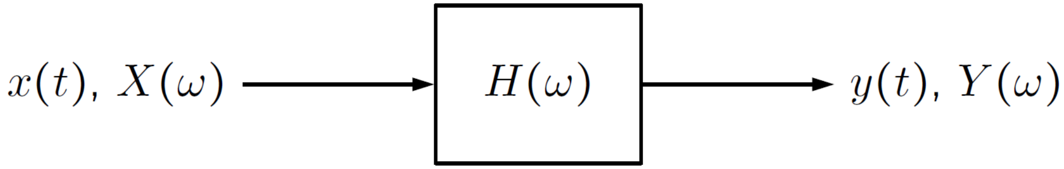
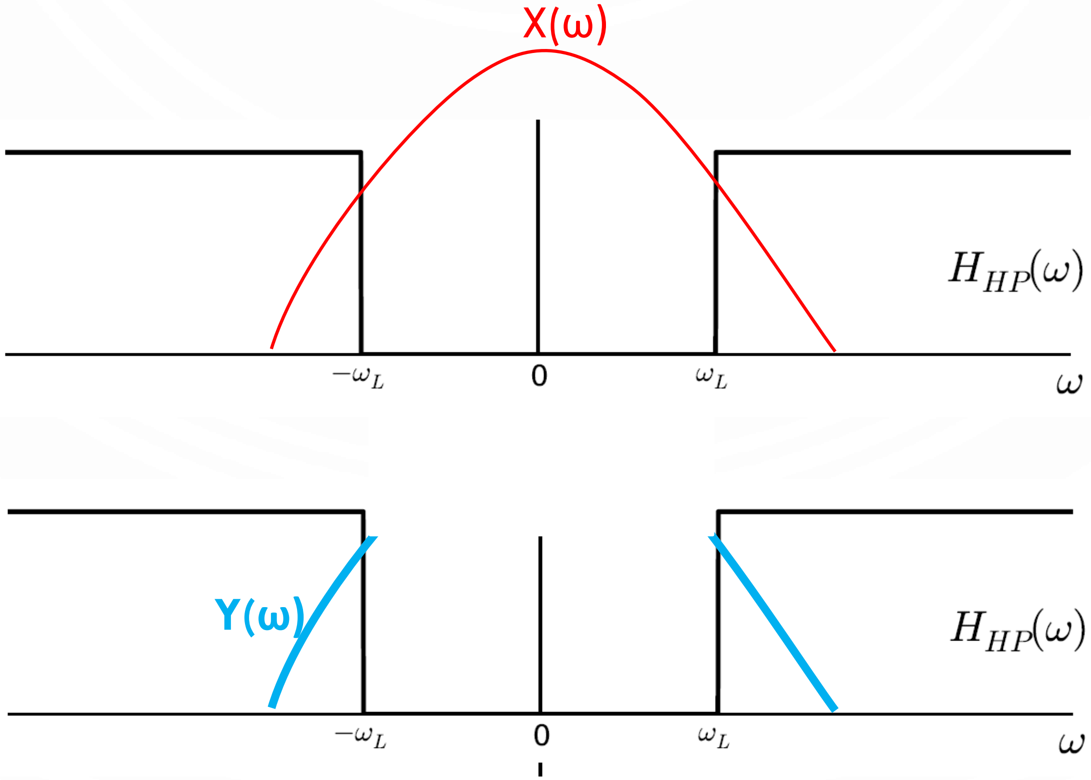
1: QUICK REVIEW

- Please put the recording on hold and login to QM+
- Go to Topic 3
- Take 10 minutes to answer the questions in T3-Q3
- You can retry as many times as you wish.
- Take 5 minutes to understand your mistakes and discuss with your friends
- You are still unsure? Post your question on the MS Teams channel or QM+ forum.

IDEAL HIGH PASS FILTER



$$y(t) = x(t) \star h(t) \overset{FT}{\Longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$$



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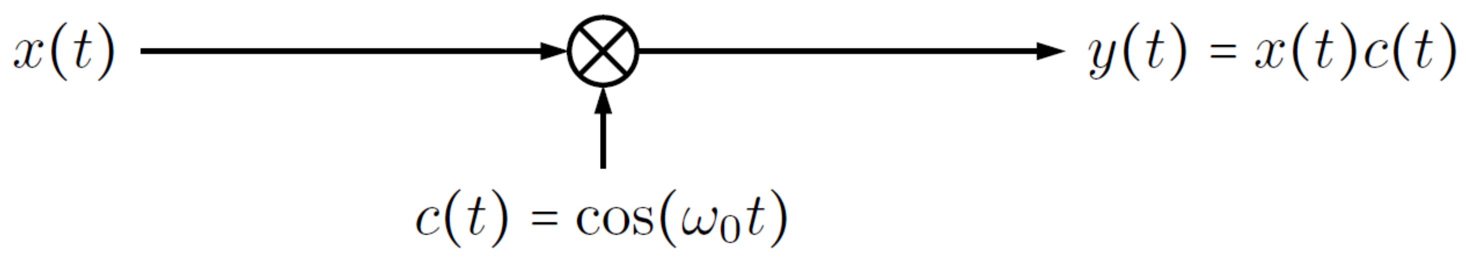

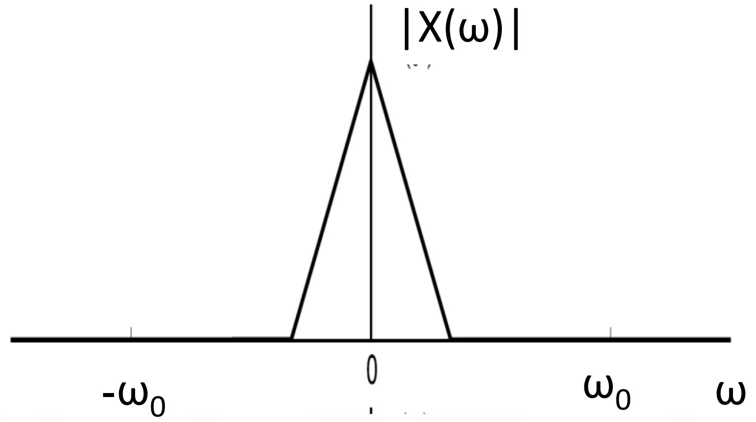

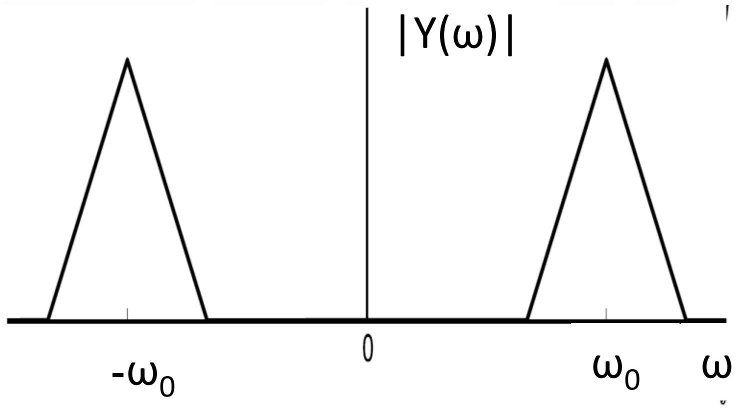

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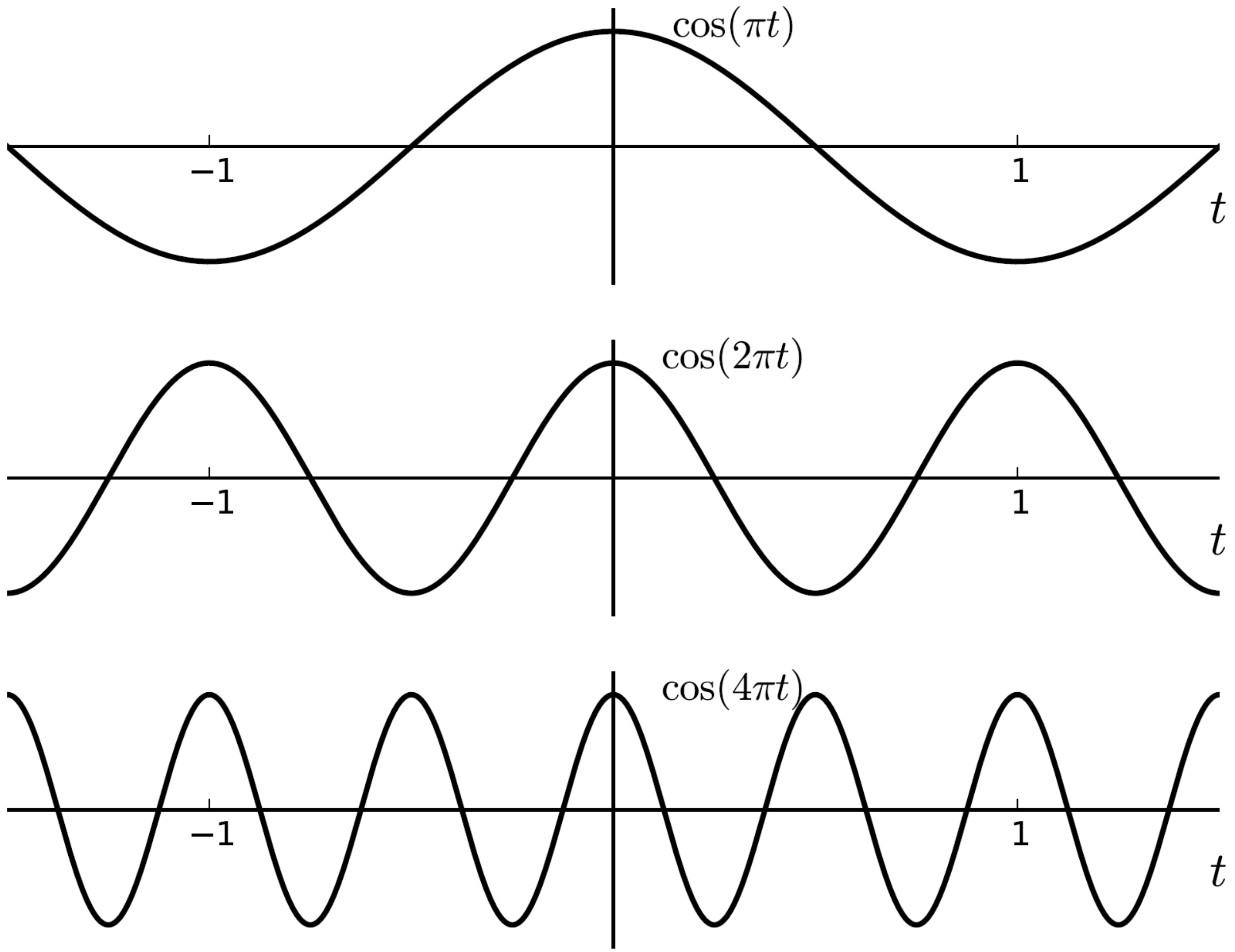
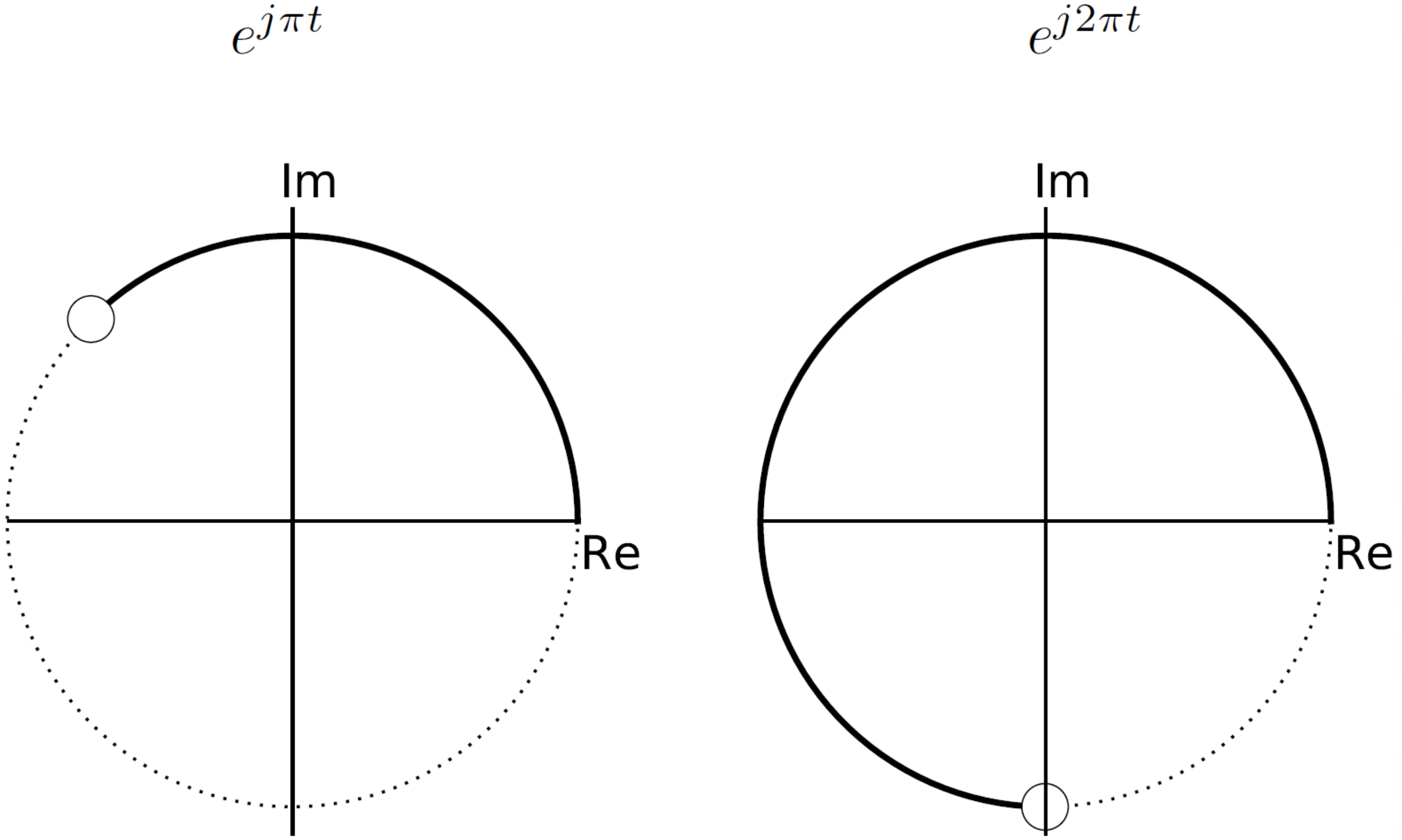
2: CT sinusoidal signals: PERIODICITY



2: CT complex exponentials: Periodicity

$$t = \frac{3}{4}$$



1: The frequency domain and the Fourier transform

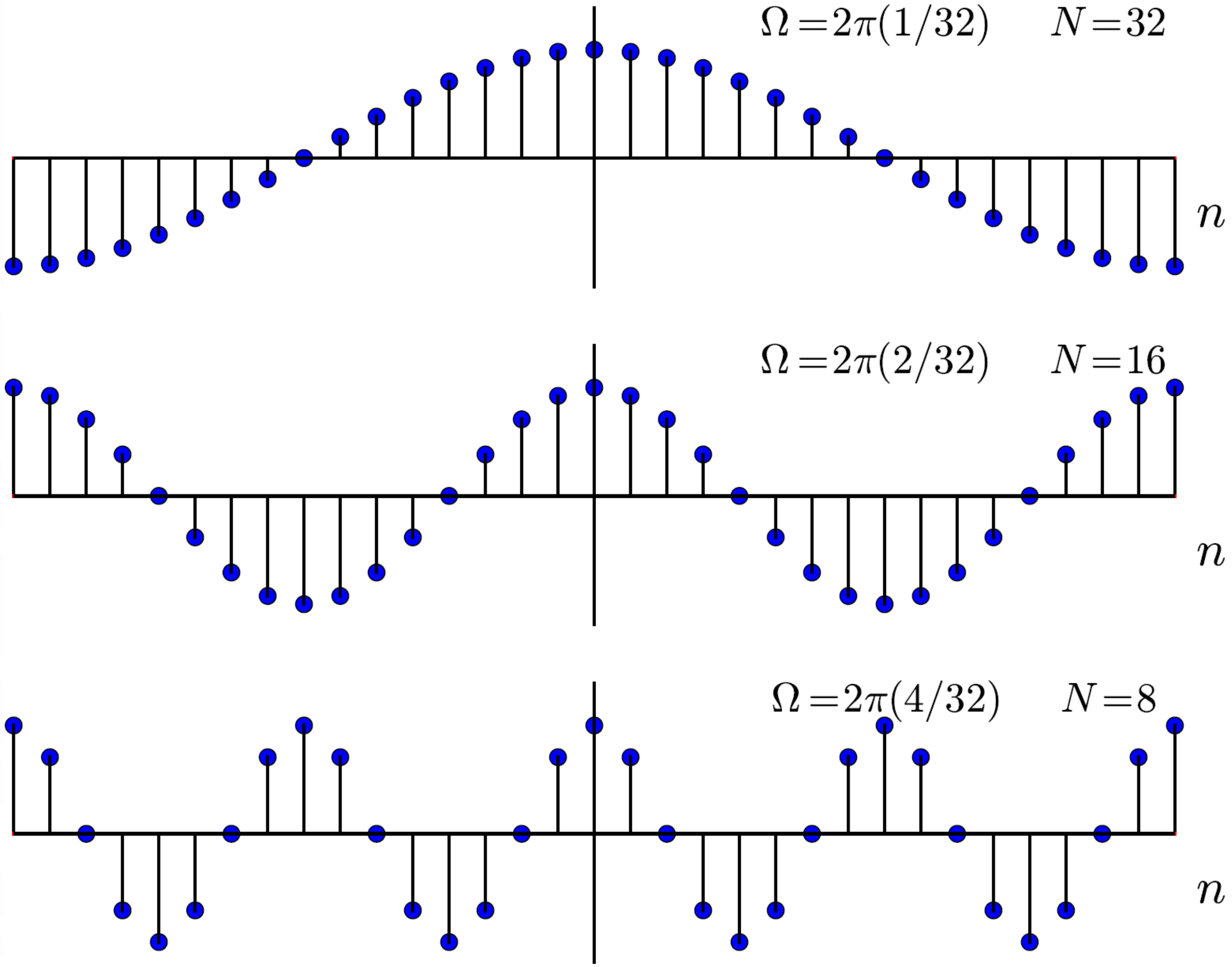
- Consider the discrete time complex exponentials $x[n] = e^{i\Omega n}$.
- If x[n] is periodic with period N then: $x[n]=x[n+N]$
- Not every value of Ω produces a periodic signal. If we assume that $x[n] = e^{i\Omega n}$ is periodic, then:

$$X[n+N]= e^{i\Omega(n+N)=} e^{i\Omega n} .e^{i\Omega N}$$

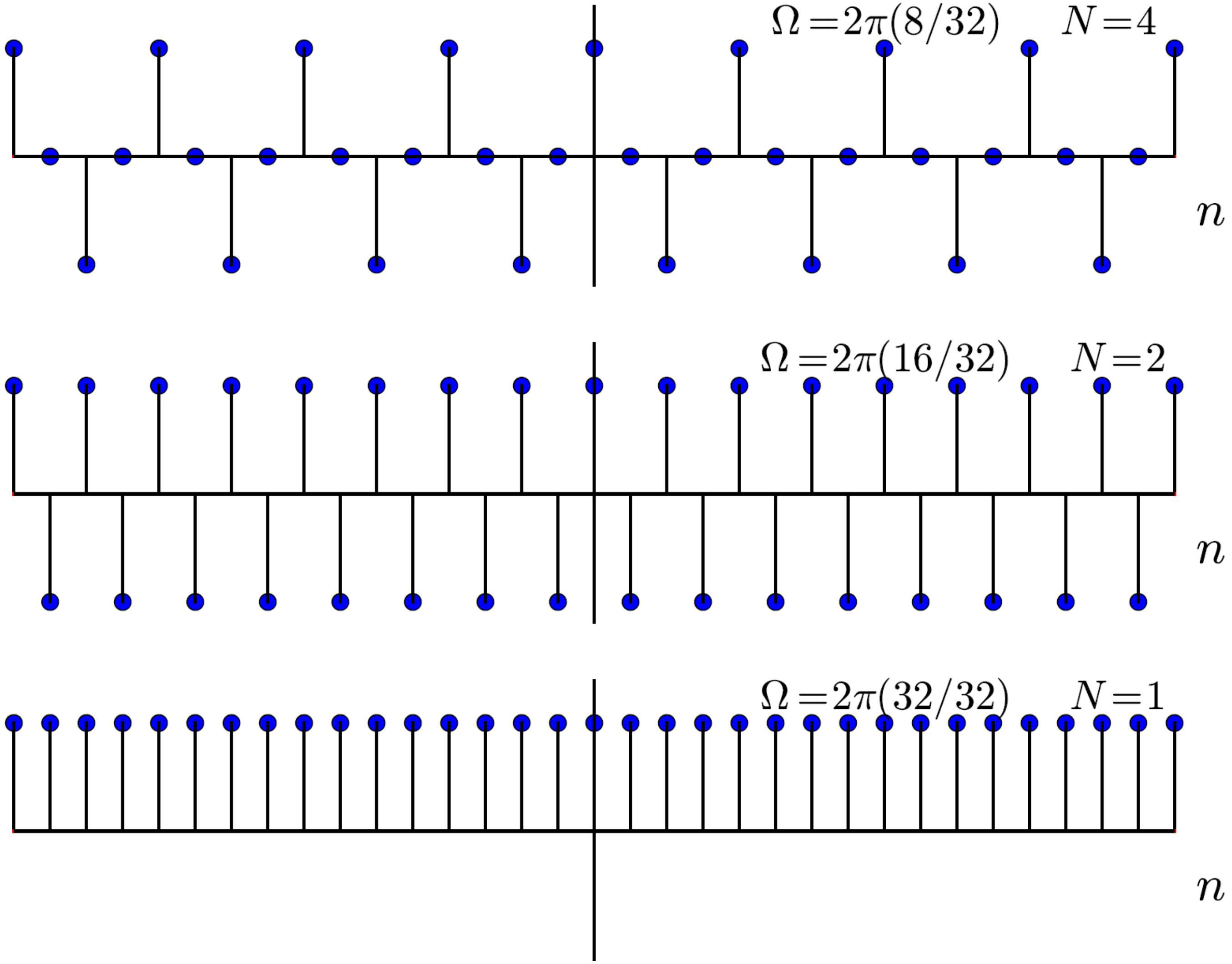
- If periodic, then $x[n+N]=x[n] \rightarrow e^{i\Omega n} .e^{i\Omega N}= e^{i\Omega n} \rightarrow e^{i\Omega N}=1$.

- hence we need that $\Omega N = 2\pi k \rightarrow \Omega = 2\pi k/N$.
- Can you find the condition for sinusoidal signals to be periodic?

1: Discrete time Sinusoidal signals



1: Discrete time Sinusoidal signals



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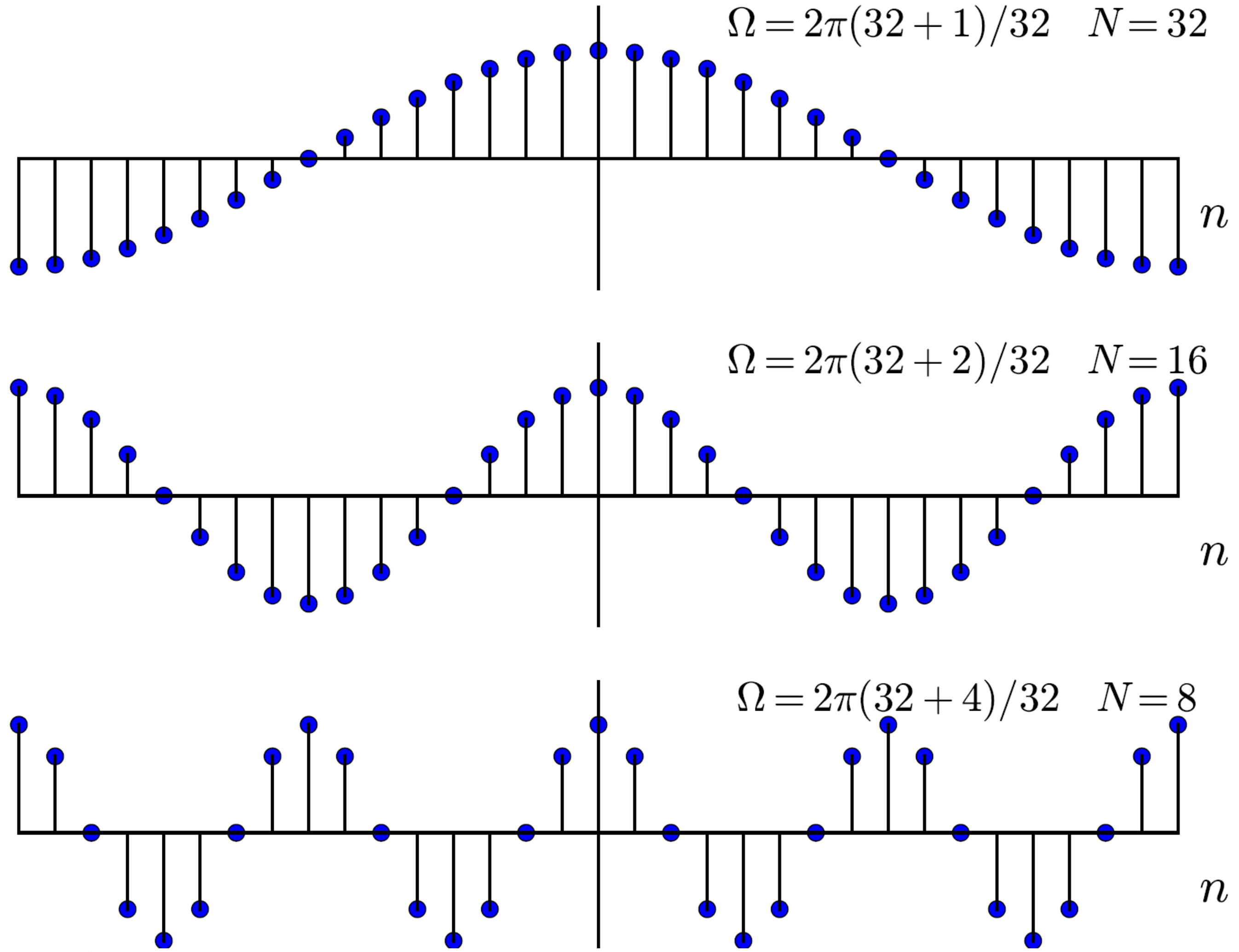
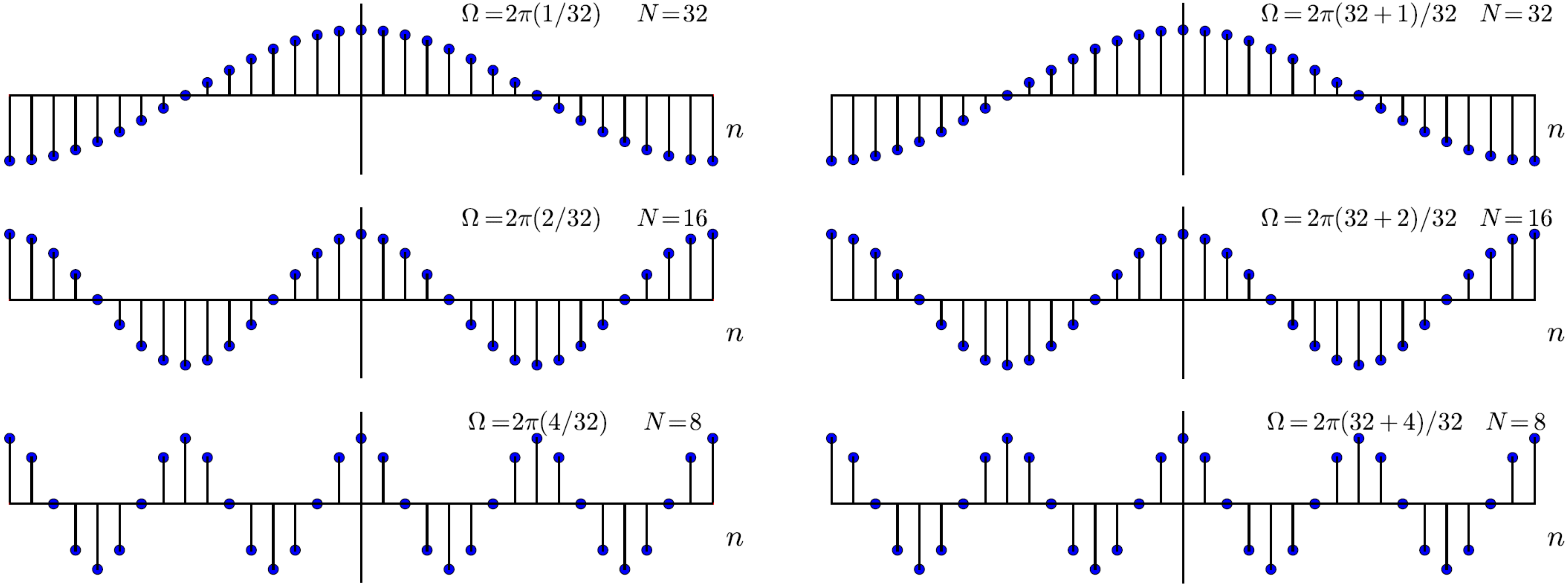
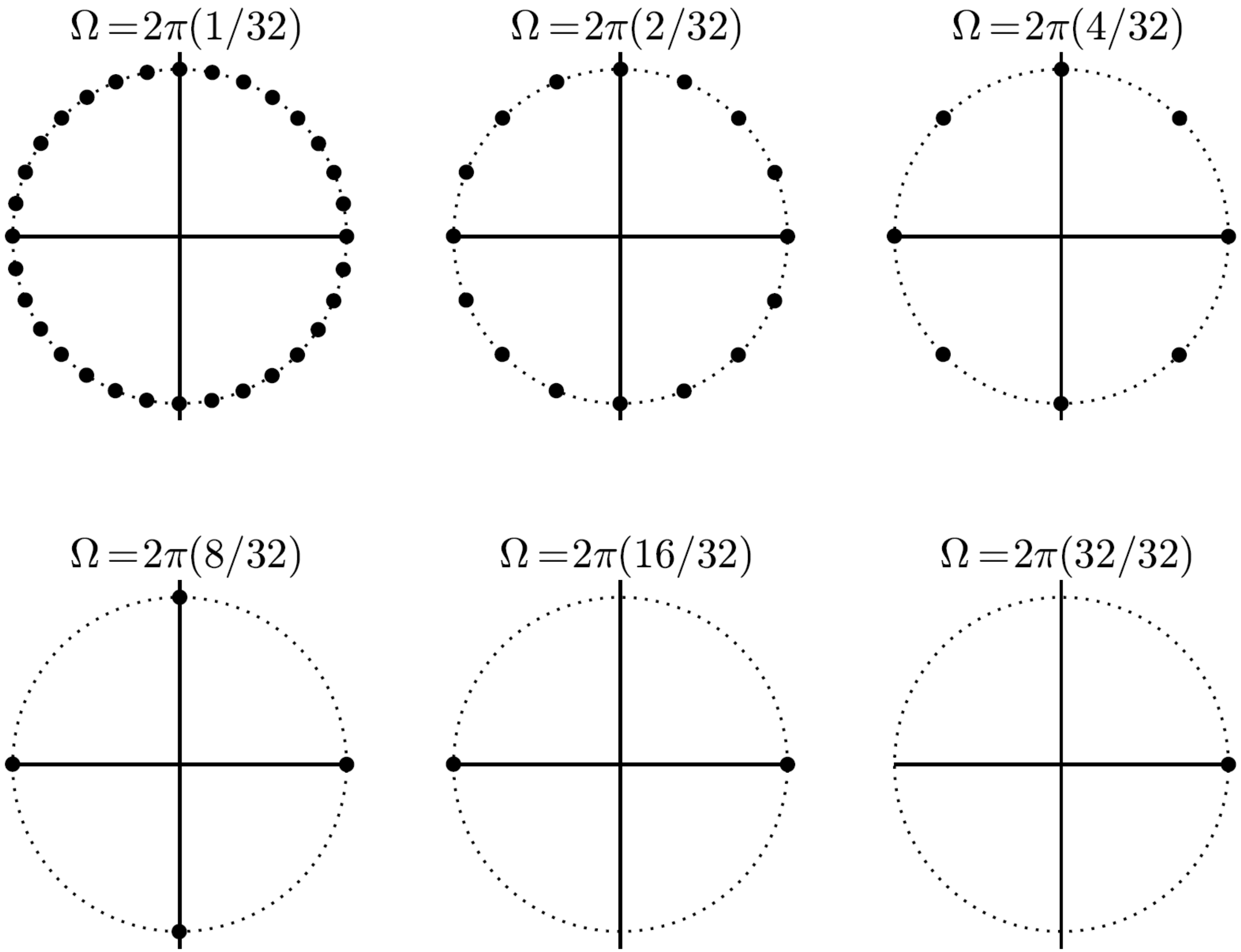


1: Discrete time Sinusoidal signals



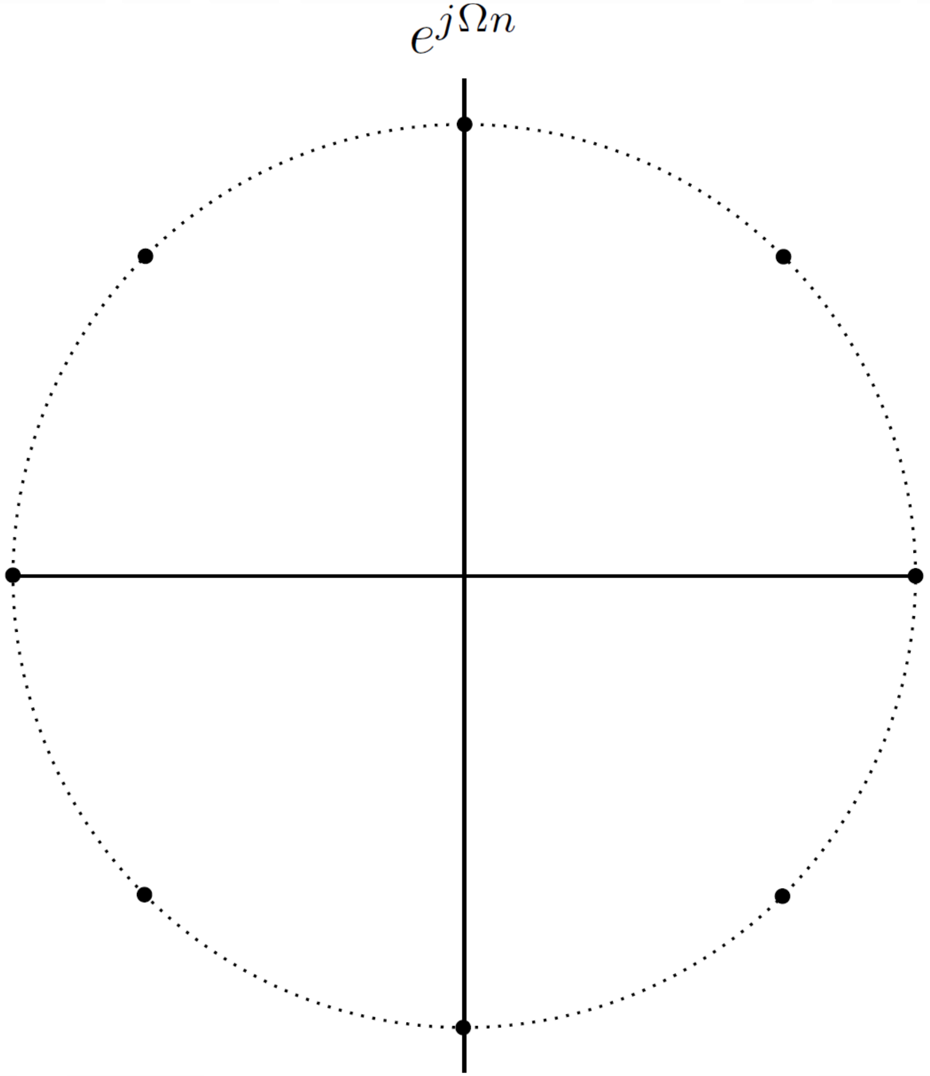
We have found some sinusoidal signals with **different frequencies** that are **identical**. *Why is that?*

1: Discrete time Complex Exponentials



1: Discrete time Complex Exponentials

As with sinusoidal signals, have found some Complex Exponentials with **different frequencies** that are **identical**.



Ω
$2\pi\frac{1}{8}$
$2\pi\left[\frac{1}{8}+1\right]$
$2\pi\left[\frac{1}{8}+2\right]$
$2\pi\left[\frac{1}{8}+3\right]$
$2\pi\left[\frac{1}{8}+4\right]$
$\dots$

1: Discrete time Complex Exponentials

- Consider Two discrete time complex exponentials $x_1[n] = e^{-j\Omega_1 n}$ and $x_2[n] = e^{-j\Omega_2 n}$, where $\Omega_2 = \Omega_1 + 2\pi$. The angular frequency of $x_2[n]$ is higher than the angular frequency of $x_1[n]$, $\Omega_2 > \Omega_1$.
- However, $x_1[n]$ and $x_2[n]$ are the same signal:

$$\begin{aligned} x_2[n] &= e^{j\Omega_2 n} \\ &= e^{j(\Omega_1+2\pi)n} \\ &= e^{j\Omega_1 n} e^{j2\pi n} \\ &= e^{j\Omega_1 n} \\ &= x_1[n] \end{aligned}$$

- In discrete-time, all the possible complex exponentials that can be generated are within any interval of frequencies of size 2π, for instance [− π; π].

1: Discrete time Complex Exponentials

We have seen that:

- In order for a complex exponential to be periodic, its angular frequency Ω must be such that $\Omega=2\pi k/N$, where N is the period (whenever k and N have no factors in common).
- The frequencies Ω_1 and $\Omega_2 = \Omega_1 + 2\pi$ produce the same signal, since they visit the same points in the complex plane.

We can conclude that **there only exist N different complex exponentials of period N,** namely:

$$0, \quad 2\pi\frac{1}{N}, \quad 2\pi\frac{2}{N}, \quad \ldots \quad , \quad 2\pi\frac{N-1}{N}$$

- For instance, $\Omega=2\pi(2N+2)/N$ produces the same signal as $\Omega=2\pi(2)/N$
- and $\Omega=2\pi(-2)/N$ produces the same signal as $\Omega=2\pi(N-2)/N$

1: SUMMARY

- CT complex exponentials
- Always periodic
- Different frequencies produce different signals
- There exist infinite complex exponentials with period T, namely those of frequencies

$$\frac{2\pi}{T}, 2\frac{2\pi}{T}, 3\frac{2\pi}{T}, \ldots$$

- DT complex exponentials
- Only periodic for $\Omega = \frac{2\pi k}{N}$; k, N integers
- Frequencies within an interval of size 2π produce different signals
- There only exist N complex exponentials with period N, namely those of frequencies

$$\frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, \ldots, N\frac{2\pi}{N}$$

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3: Fourier series in Continuous Time

- A Fourier series is a representation of a periodic signal as a linear combination of harmonically related complex exponentials.
- By harmonically related we mean that their frequencies can be expressed as an integer multiple of the fundamental frequency.
- For instance, in continuous-time, a periodic signal x_T (t) with period T can be expressed as:

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

where $\omega_0 = {}^{2\pi}/_{T}$ is the fundamental frequency, $k\omega_0$ are its harmonics and a_k are its coefficients.

3: What is different in Discrete Time?

- Just the fact that there are only N different complex exponentials with period N!
- Hence, the Fourier series representation of a periodic discrete-time signal $x_N[n]$ with period N is:

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

where $\Omega_0 = {}^{2\pi}/_{N}$ is the fundamental frequency, $k\Omega_0$ are its harmonics and a_k are its coefficients.

- This equation is, of course, a **synthesis equation**.

FOURIER SERIES: Determining the coefficients (1)

- How do we obtain the coefficients of the Fourier series of a discrete-time periodic signal $x_N[n]$?
- One approach would be to solve the following system of N equations and N unknowns (a_k):

$$
\begin{aligned}
x_N[0] &= \sum_{k=\langle N\rangle} a_k \\
x_N[1] &= \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0} \\
x_N[2] &= \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 2} \\
&\dots \\
x_N[N-1] &= \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 (N-1)}
\end{aligned}
$$

FOURIER SERIES: Determining the coefficients (1)

- In matrix form, the resulting system of linear equations is:

$$\begin{bmatrix} x_N[0] \\ x_N[1] \\ x_N[2] \\ \vdots \\ x_N[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\Omega_0} & e^{j2\Omega_0} & & e^{j(N-1)\Omega_0} \\ 1 & e^{j\Omega_0 2} & e^{j2\Omega_0 2} & & e^{j(N-1)\Omega_0 2} \\ \vdots & & & \ddots & \vdots \\ 1 & e^{j\Omega_0(N-1)} & e^{j2\Omega_0(N-1)} & \cdots & e^{j(N-1)\Omega_0(N-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

FOURIER SERIES: Determining the coefficients (2)

- Another option is using an analysis equation.
- Analysis equations use the fact that **harmonically related exponentials** are **orthogonal**, so that:

$$\sum_{n=\langle N\rangle} e^{jk_1\Omega_0 n}\left(e^{jk_2\Omega_0 n}\right)^* = \sum_{n=\langle N\rangle} e^{jk_1\Omega_0 n} e^{-jk_2\Omega_0 n}$$

$$= \sum_{n=\langle N\rangle} e^{j(k_1-k_2)\Omega_0 n}$$

$$= \begin{cases} N & \text{if } k_1 - k_2 = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

FOURIER SERIES: Determining the coefficients (2)

- So we know that:

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n},$$

$$\sum_{n=\langle N \rangle} e^{jk_1\Omega_0 n}\left(e^{jk_2\Omega_0 n}\right)^* = \begin{cases} N & \text{if } k_1 - k_2 = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- Let us calculate:

$$\begin{aligned} \sum_{n=\langle N \rangle} x_N[n]\left(e^{jm\Omega_0 n}\right)^* &= \sum_{n=\langle N \rangle} \left[\sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \right] e^{-jm\Omega_0 n} \\ &= a_m N \end{aligned}$$

- Hence: $a_k = \dfrac{1}{N} \displaystyle\sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}$

3: SUMMARY

Continuous-time, $\omega_0 = \frac{2\pi}{T}$

Discrete-time, $\Omega_0 = \frac{2\pi}{N}$

Analysis

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x_T(t)e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x_N[n]e^{-jk\Omega_0 n}$$

Synthesis

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

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PARSEVAL'S RELATION

- The average power of a periodic signal $x_N[n]$ can be calculated both in the time domain and by using the coefficients of its Fourier series:

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \sum_{k=\langle N\rangle}|a_k|^2$$

- The coefficient a_k of the Fourier series of $x_N[n]$ tells us how much of the harmonic frequency $k\Omega_0$ there is in the signal.

LINEARITY AND TIME-SHIFTING

- <u>Linearity</u>: Consider two periodic signals $x_N[n]$ and $y_N[n]$ with period N and Fourier coefficients a_k and b_k, respectively. Then:
  - The signal $z_N[n] = Ax_N[n] + By_N[n]$ is periodic with period N
  - and its Fourier coefficients are $c_k = Aa_k + Bb_k$.
- <u>Time-shifting</u>: The signal $v_N[n] = x_N[n - n_0]$ is:
  - periodic with period N
  - and its Fourier coefficients are $v_k = a_k e^{-jk\Omega_0 n_0}$.

FOURIER SERIES AND LTI SYSTEMS

$$x_N(n) \longrightarrow \boxed{H(\Omega)} \longrightarrow y_N(n)$$

$$x_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \longrightarrow y_N[n] = \sum_{k=\langle N\rangle} H(k\Omega_0) a_k e^{jk\Omega_0 n}$$

Hence, the Fourier coefficients of $y_N[n]$ are $b_k = a_k H(k\Omega_0)$.
In words, they are a filtered version of the coefficients a_k.