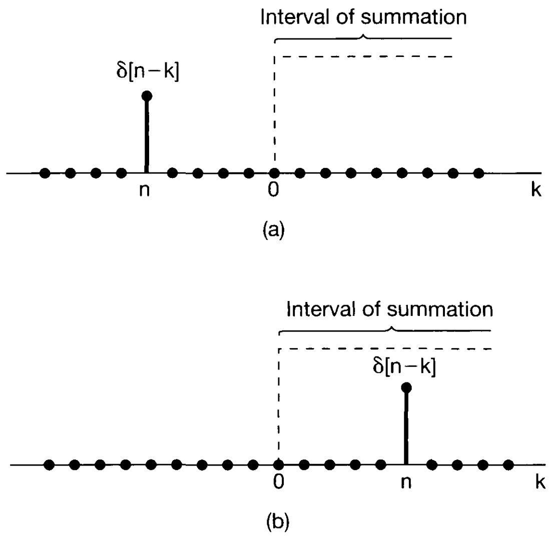

Figure 1.31 Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

Equation (1.67) is illustrated in Figure 1.31. In this case the nonzero value of $\delta[n-k]$ is at the value of k equal to n, so that again we see that the summation in eq. (1.67) is 0 for $n < 0$ and 1 for $n \geq 0$.

An interpretation of eq. (1.67) is as a superposition of delayed impulses; i.e., we can view the equation as the sum of a unit impulse $\delta[n]$ at $n = 0$, a unit impulse $\delta[n-1]$ at $n = 1$, another, $\delta[n-2]$, at $n = 2$, etc. We will make explicit use of this interpretation in Chapter 2.

The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]. \tag{1.68}$$

More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]. \tag{1.69}$$

This sampling property of the unit impulse will play an important role in Chapters 2 and 7.

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time *unit step function* $u(t)$ is defined in a manner similar to its discrete-time counterpart. Specifically,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}, \tag{1.70}$$

as is shown in Figure 1.32. Note that the unit step is discontinuous at $t = 0$. The continuous-time *unit impulse function* $\delta(t)$ is related to the unit step in a manner analogous

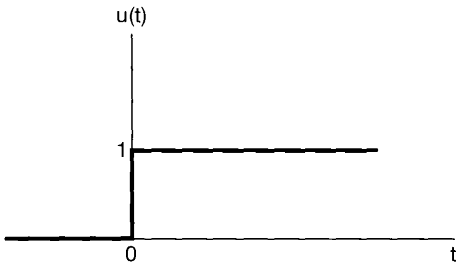

Figure 1.32 Continuous-time unit step function.

to the relationship between the discrete-time unit impulse and step functions. In particular, the continuous-time unit step is the *running integral* of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau. \tag{1.71}$$

This also suggests a relationship between $\delta(t)$ and $u(t)$ analogous to the expression for $\delta[n]$ in eq. (1.65). In particular, it follows from eq. (1.71) that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}. \tag{1.72}$$

In contrast to the discrete-time case, there is some formal difficulty with this equation as a representation of the unit impulse function, since $u(t)$ is discontinuous at $t = 0$ and consequently is formally not differentiable. We can, however, interpret eq. (1.72) by considering an approximation to the unit step $u_\Delta(t)$, as illustrated in Figure 1.33, which rises from the value 0 to the value 1 in a short time interval of length Δ. The unit step, of course, changes values instantaneously and thus can be thought of as an idealization of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. Formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$. Let us now consider the derivative

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}, \tag{1.73}$$

as shown in Figure 1.34.

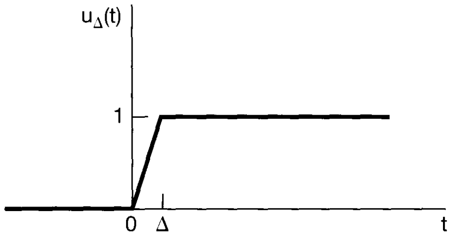

Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

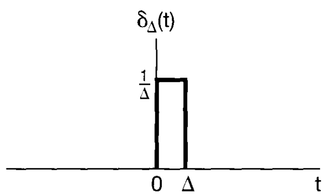

Figure 1.34 Derivative of $u_\Delta(t)$.

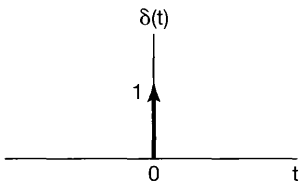

Figure 1.35 Continuous-time unit impulse.

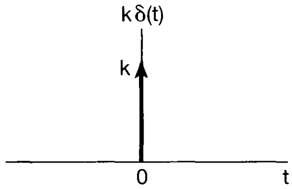

Figure 1.36 Scaled impulse.

Note that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \rightarrow 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area. Its limiting form,

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_\Delta(t), \tag{1.74}$$

can then be thought of as an idealization of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant. Since $\delta(t)$ has, in effect, no duration but unit area, we adopt the graphical notation for it shown in Figure 1.35, where the arrow at $t = 0$ indicates that the area of the pulse is concentrated at $t = 0$ and the height of the arrow and the "1" next to the arrow are used to represent the *area* of the impulse. More generally, a scaled impulse $k\delta(t)$ will have an area k, and thus,

$$\int_{-\infty}^{t} k\delta(\tau)\,d\tau = ku(t).$$

A scaled impulse with area k is shown in Figure 1.36, where the height of the arrow used to depict the scaled impulse is chosen to be proportional to the area of the impulse.

As with discrete time, we can provide a simple graphical interpretation of the running integral of eq. (1.71); this is shown in Figure 1.37. Since the area of the continuous-time unit impulse $\delta(\tau)$ is concentrated at $\tau = 0$, we see that the running integral is 0 for $t < 0$ and 1 for $t > 0$. Also, we note that the relationship in eq. (1.71) between the continuous-time unit step and impulse can be rewritten in a different form, analogous to the discrete-time form in eq. (1.67), by changing the variable of integration from τ to $\sigma = t - \tau$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma),$$

or equivalently,

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\,d\sigma. \tag{1.75}$$

The graphical interpretation of this form of the relationship between $u(t)$ and $\delta(t)$ is given in Figure 1.38. Since in this case the area of $\delta(t-\sigma)$ is concentrated at the point $\sigma = t$, we again see that the integral in eq. (1.75) is 0 for $t < 0$ and 1 for $t > 0$. This type of graphical interpretation of the behavior of the unit impulse under integration will be extremely useful in Chapter 2.