EBU4375: SIGNALS AND SYSTEMS

TOPIC 3-4: DISCRETE TIME SIGNALS IN THE FREQUENCY DOMAIN

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AGENDA

1. Can you guess the Fourier transform of $x[n] = e^{j\Omega_0 n}$?
2. From Fourier series to Fourier Transform
3. Summary

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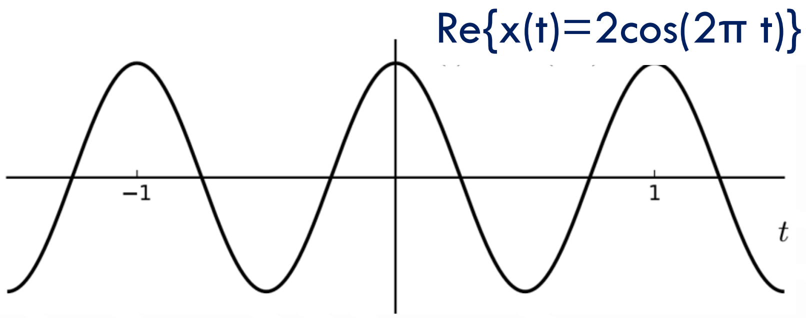
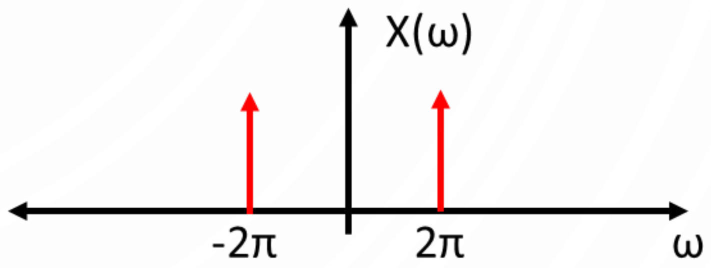
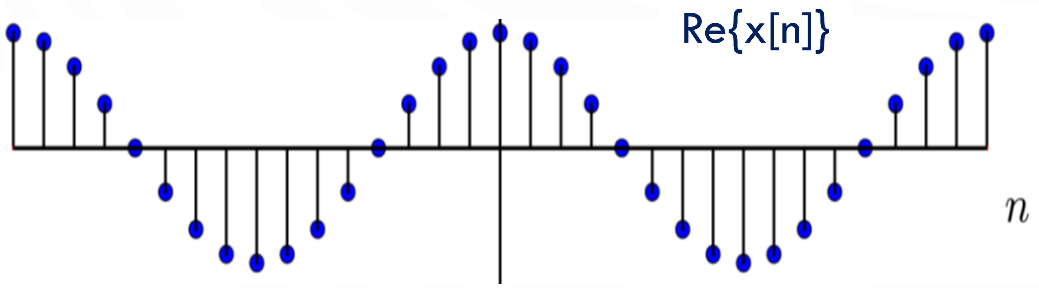
1: LET'S BRAIN STORM…

- We know that: $x(t) = e^{j\omega_0 t} \overset{FT}{\Longleftrightarrow} X(\omega) = \delta(\omega - \omega_0)$



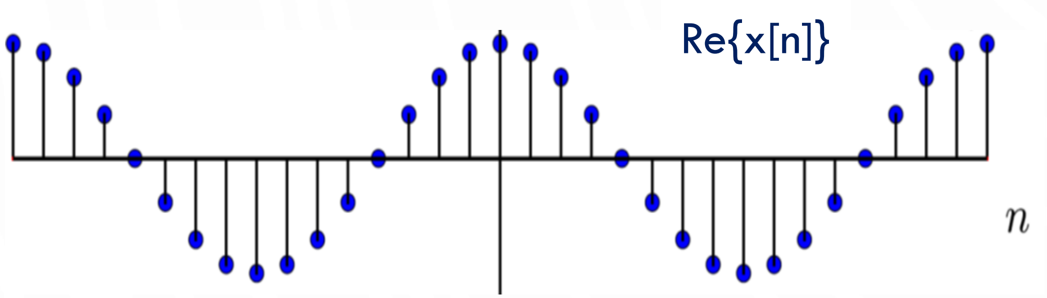


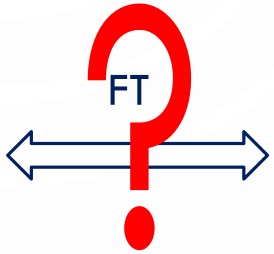
- What do you think would be the FT of $x_N[n] = e^{j\Omega_0 n}$?

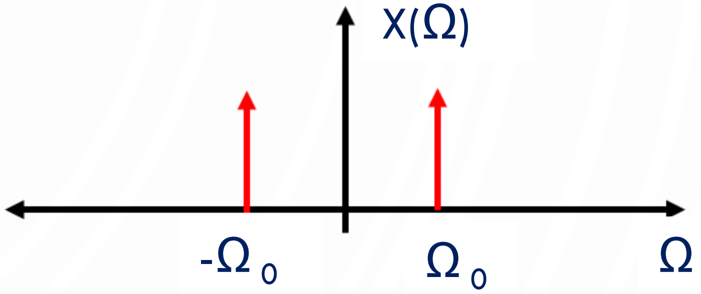


?

1: BRAIN STORMING STILL…



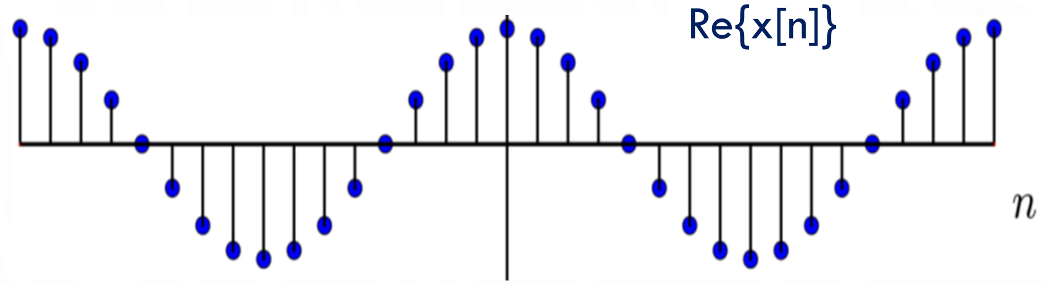


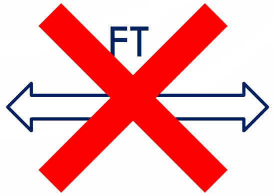


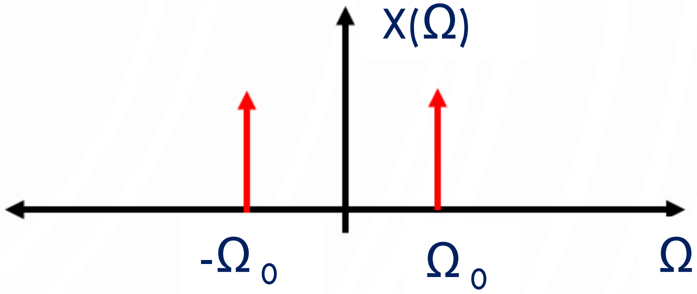
N=16 => Ω_0= 2π/N= π/8

Let’s go to Mentimeter!!!

1: BRAIN STORMING STILL…







$N=16 \Rightarrow \Omega_0 = 2\pi/N = \pi/8$

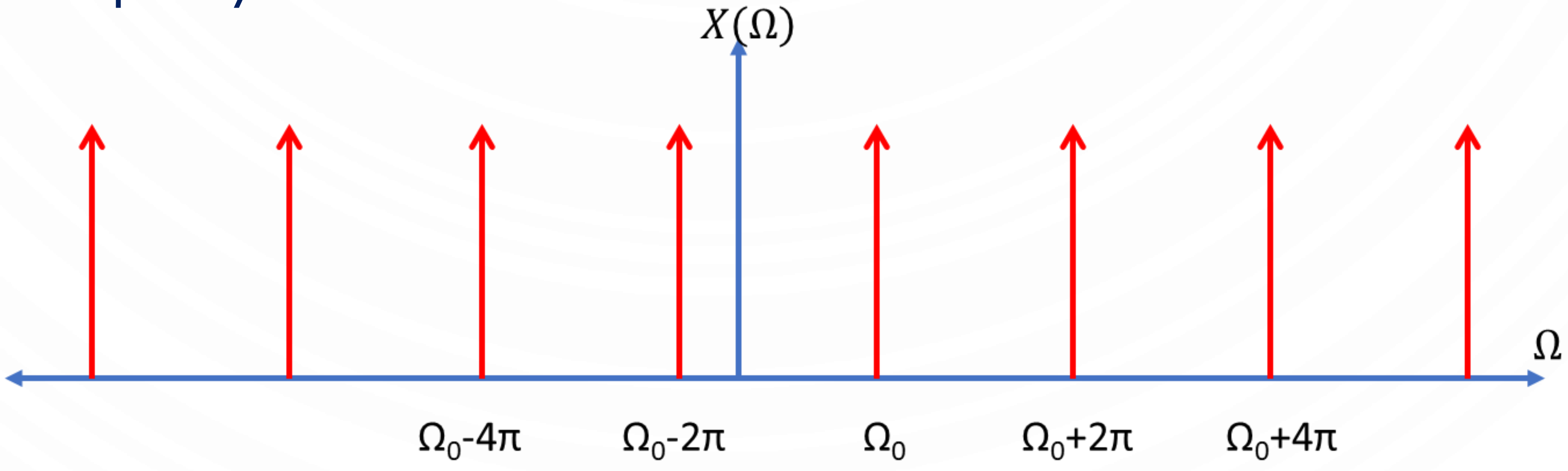
The Fourier Transform of a DT signal is always **periodic** with period **2π**!!!

How?

1: PROPOSITION

I say that the FT of $x_N[n] = e^{j\Omega_0 n}$ should be $X(\Omega) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)$

- It is periodic with period 2π.
- It reflects the relation between complex exponential in time domain and Dirac function in Frequency domain.



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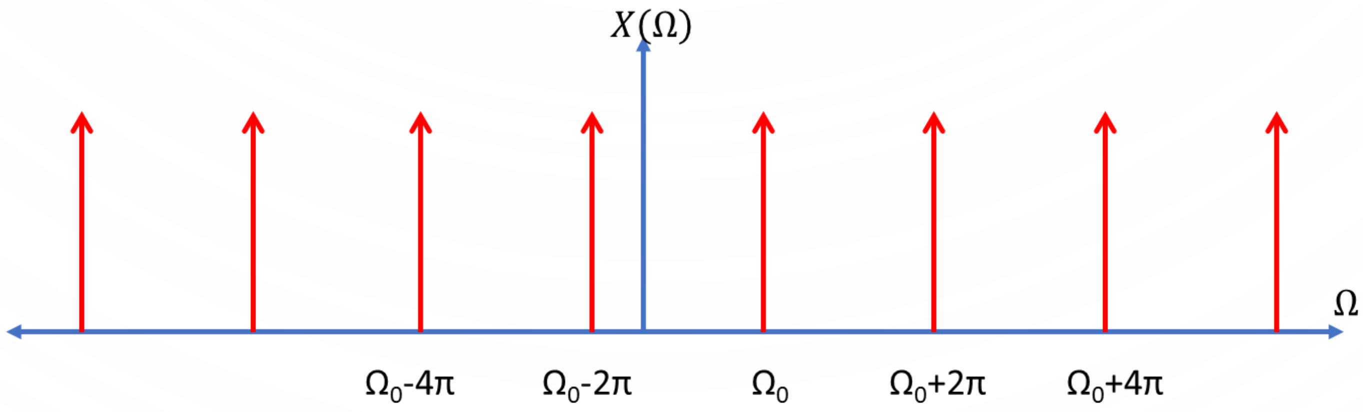
1: VALIDATION

- In order to validate my claim, let's use the synthesis equation to get x[n] from X(Ω), as follows:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)\, e^{j\Omega n}d\Omega$$



$$x[n] = \int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)\, e^{j\Omega n} d\Omega$$

1: VALIDATION

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)\, e^{j\Omega n} d\Omega$$

- For any interval 2π, there is exactly a single impulse function in the summation!
- Say the impulse is at $\Omega_0 + 2\pi r$
- Then,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)\, e^{j\Omega n} d\Omega = e^{j(\Omega_0 + 2\pi r)n} = e^{j\Omega_0 n}$$

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2- Fourier series of DT periodic signals

- We have shown that a periodic signal $x_N[n]$ with period N can be expressed as a sum of N complex exponentials:

$$x_N[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} = \sum_{k=<N>} z_k$$

- where, $z_k[n] = a_k e^{jk\Omega_0 n}$ with $Z_k(\Omega) = \sum_{l=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0 - 2\pi l))$
- Using the linearity property, the Fourier Transform can be written as:

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - \Omega_0 k)$$

Remember that the Fourier series coefficients a_k are periodic with period N, so that $2\pi a_0 = 2\pi a_N = 2\pi a_{-N}$.

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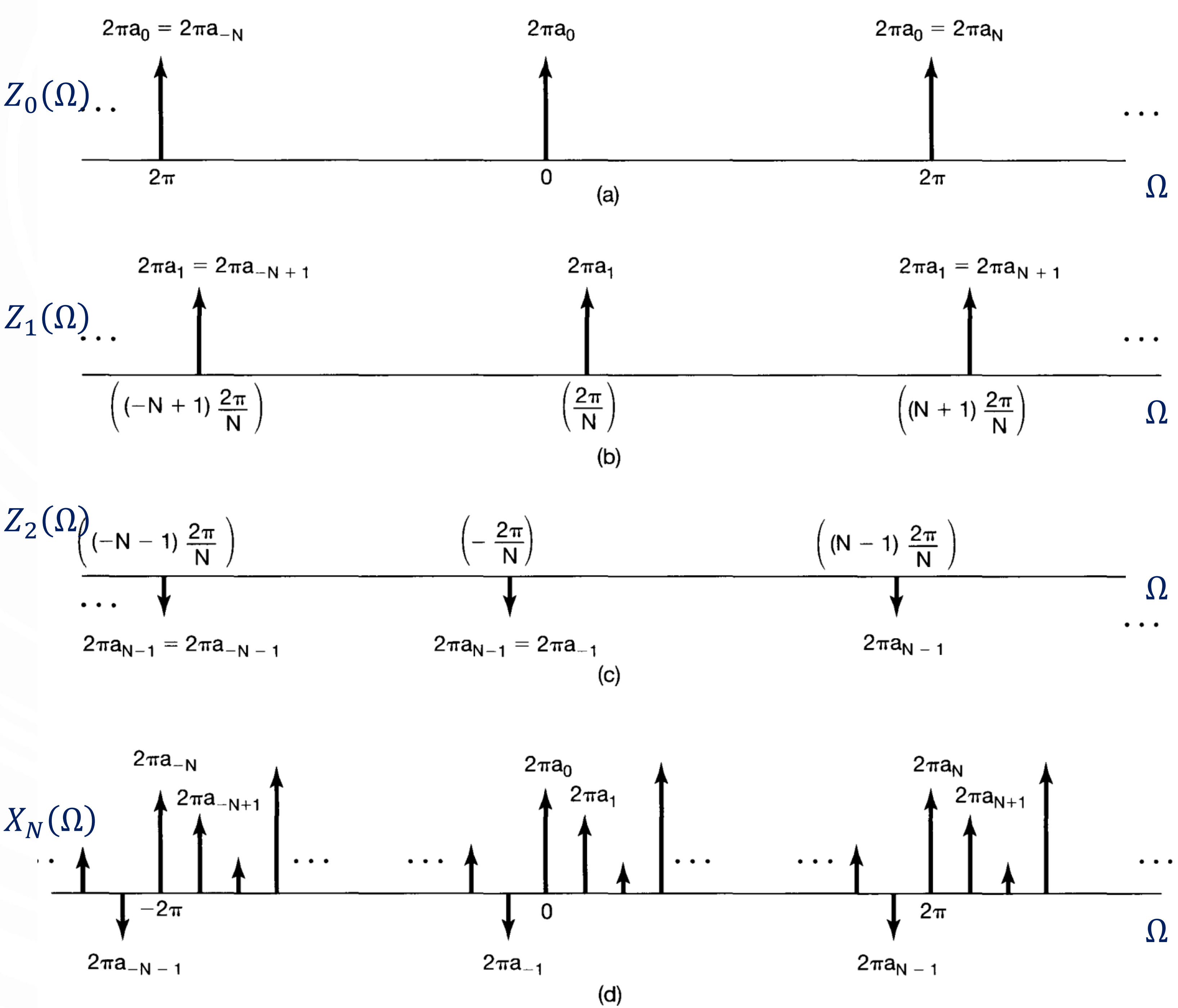
$$x_N[n] = \sum_{k=<N>} z_k$$

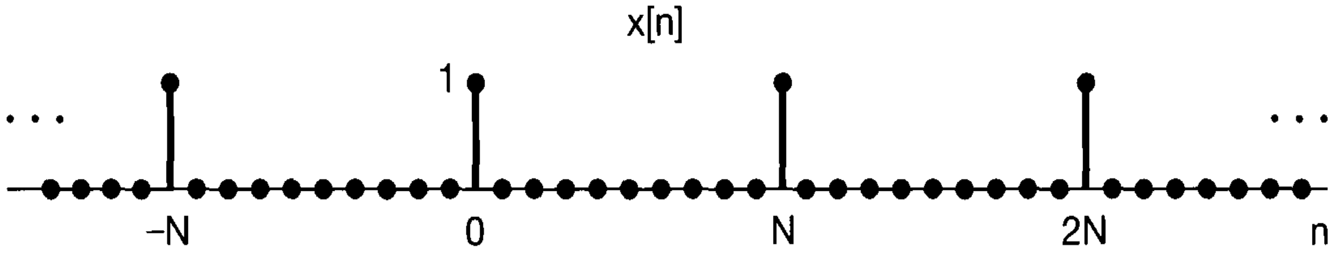
$$z_k[n] = a_k e^{jk\Omega_0 n}$$

$$Z_k(\Omega) = \sum_{l=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0 - 2\pi l))$$



2- EXAMPLE

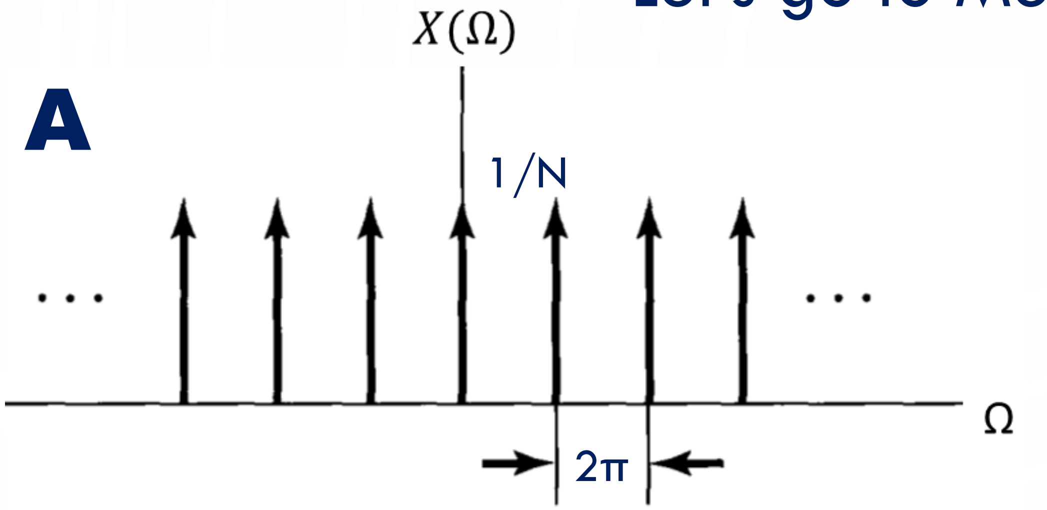
- $x_N[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$
- Find $X_N(\Omega)$



$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - \Omega_0 k)$$

Let's go to Mentimeter!!!

A



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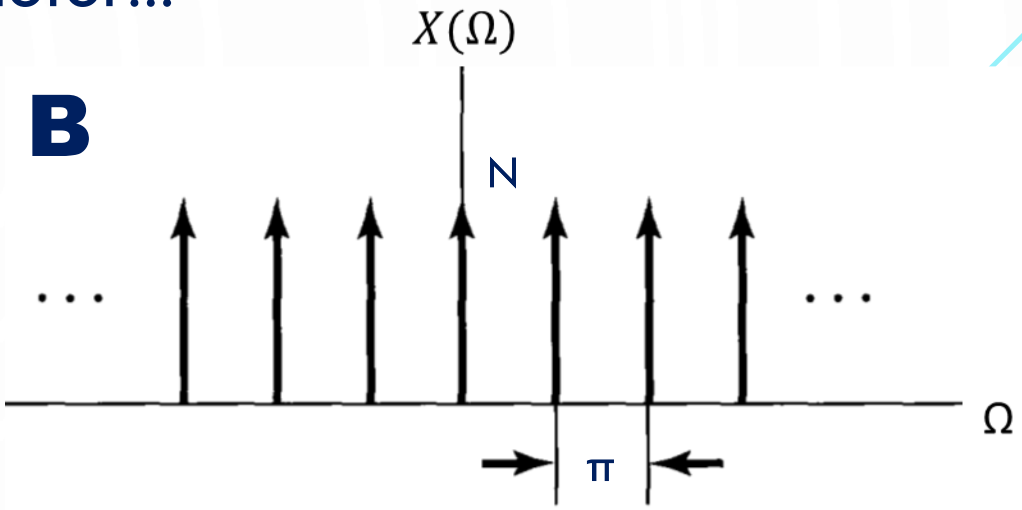

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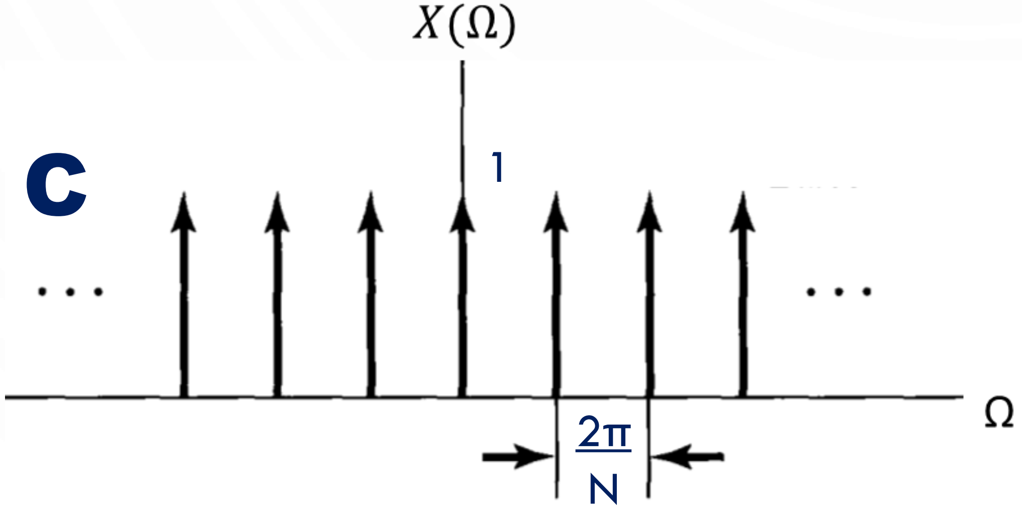

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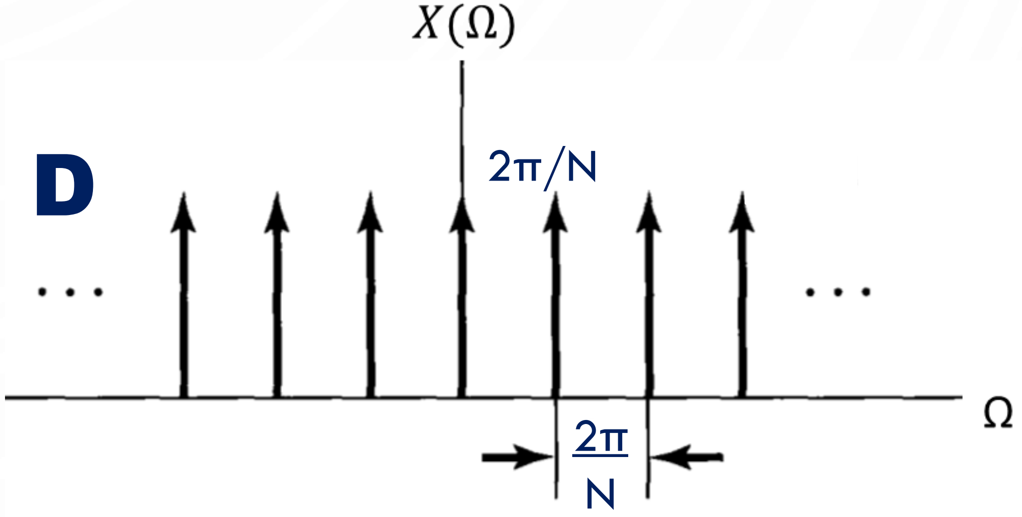

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3: COMPLEX EXPONENTIALS

- CT complex exponentials
- Always periodic
- Different frequencies produce different signals
- There exist infinite complex exponentials with period T, namely those of frequencies

$$\frac{2\pi}{T}, 2\frac{2\pi}{T}, 3\frac{2\pi}{T}, \ldots$$

- DT complex exponentials
- Only periodic for $\Omega = \frac{2\pi k}{N}$; k, N integers
- Frequencies within an interval of size 2π produce different signals
- There only exist N complex exponentials with period N, namely those of frequencies

$$\frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, \ldots, N\frac{2\pi}{N}$$

3: FOURIER SERIES OF PERIODIC SIGNALS

Continuous-time, $\omega_0 = \frac{2\pi}{T}$

Discrete-time, $\Omega_0 = \frac{2\pi}{N}$

Analysis

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x_T(t)e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x_N[n]e^{-jk\Omega_0 n}$$

Synthesis

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x_N[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$$

3- FOURIER TRANSFORM OF DT SIGNALS

- The Fourier transform describes a signal in the frequency domain, i.e. describes its frequency components.
- Because of the nature of DT complex exponentials, we only need an interval of frequencies of size 2π.
- If we want to consider all the frequencies, we only need to replicate the Fourier transform in the original interval. Consequently, the **Fourier transform of DT signals can be seen as a periodic function.**
- Finally, **low frequency** components are located around the frequencies
  - $\Omega = 0; \pm 2\pi; \pm 4\pi; \ldots; \pm k2\pi; \ldots,$
- whereas **high frequencies** are around
  - $\Omega = \pm\pi; \pm 3\pi; \ldots; \pm(2k+1)\pi; \ldots.$

3- SOME IMPORTANT PROPERTIES (I)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$
$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$
$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Periodicity: $X(\Omega + 2\pi) = X(\Omega)$

- Linearity: $Ax_1[n] + Bx_2[n] \overset{FT}{\Longleftrightarrow} AX_1(\Omega) + BX_2(\Omega)$

3- SOME IMPORTANT PROPERTIES (II)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$

$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$

$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Time shift:

$$x[n-n_0] \overset{FT}{\Longleftrightarrow} e^{-j\Omega n_0} X(\Omega)$$

- Frequency shift:

$$e^{j\Omega_0 n} x[n] \overset{FT}{\Longleftrightarrow} X(\Omega - \Omega_0)$$

3- SOME IMPORTANT PROPERTIES (III)

- Reflexion: $$x[-n] \overset{FT}{\Longleftrightarrow} X(-\Omega_0)$$

- Real signals $$x[n] \text{real} \quad \Longrightarrow \quad X(\Omega_0) = X^*(-\Omega_0)$$
$$\Longrightarrow \quad |X(\Omega_0)| = |X(-\Omega_0)|$$
$$\Longrightarrow \quad \angle X(\Omega_0) = -\angle X(-\Omega_0)$$

3- SOME IMPORTANT PROPERTIES (IV)

- Consider:

$$x[n] \overset{FT}{\Longleftrightarrow} X(\Omega)$$
$$x_1[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)$$
$$x_2[n] \overset{FT}{\Longleftrightarrow} X_2(\Omega)$$

- Convolution:

$$x_1[n] * x_2[n] \overset{FT}{\Longleftrightarrow} X_1(\Omega)X_2(\Omega)$$

- Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\Omega)|^2 d\Omega$$

3-Discrete-Time LTI systems and Fourier transform

$$x[n],\ X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n],\ Y(\Omega)$$

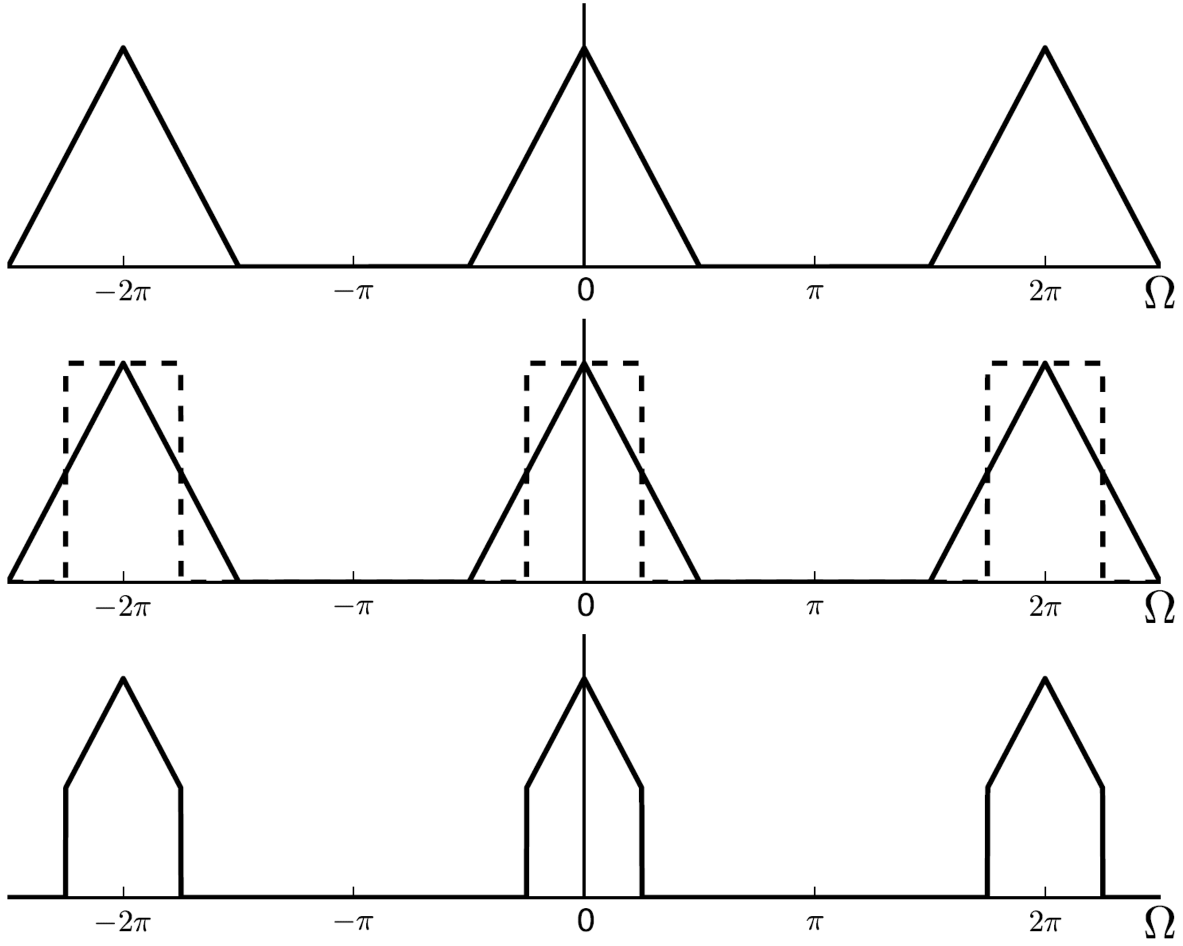
$$y[n] = x[n] \star h[n] \quad \overset{FT}{\Longleftrightarrow} \quad Y(\Omega) = X(\Omega)H(\Omega)$$

$$x[n] \quad \overset{FT}{\Longleftrightarrow} \quad X(\Omega)$$

$$h[n] \quad \overset{FT}{\Longleftrightarrow} \quad H(\Omega)$$

$$y[n] \quad \overset{FT}{\Longleftrightarrow} \quad Y(\Omega)$$

3-Discrete-Time LTI systems and Fourier transform



3-Discrete-Time Filters