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Section 6.3 Power Spectral Density of Stationary Processes

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$$F(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-i t \omega}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{F(\omega)\, e^{+i t \omega}}_{\text{spectral density.}}\, d\omega$$

$$F(\omega) \Leftrightarrow x(t)$$

Q=average power: 平均功率.

$$Q = \frac{I_{(t)}^2 R \cancel{t}}{\cancel{t}} \xrightarrow{R=1} I_{(t)}^2$$

$$\frac{\frac{U_{(t)}^2}{R} \cancel{t}}{t} \xrightarrow{R=1} U_{(t)}^2$$

$$Q = E\left(X^2(t)\right) = R_X(0)$$

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)\, d\omega$$

$$\boxed{S_X(\omega) \longleftrightarrow R_X(\tau)}$$

$$\frac{2A\beta}{\omega^2+\beta^2} \longleftrightarrow A e^{-\beta|\tau|}$$

Power Spectral Density of Stationary Processes

If x_t is the voltage across a $R = 1\Omega$ resistor, then the total energy in x_t is

$$E = \int_{-\infty}^{+\infty} x_t^2 dt.$$

But for communications signals, the energy is effectively infinite (the signals are of unlimited duration), so we usually work with Power quantities.

When the voltage is random, how about its power? In this section, we will introduce the definitions of the average power and power spectral density of a stationary process. The properties of power spectral density are also reached.

Contents

Average Power and Power Spectral Density

Theorem

Suppose that x_t $(t \in \mathbb{R})$ *is a deterministic real-valued function satisfying the Dirichlet condition. Then*
(i) *the Fourier transform of* x_t *exists and we can compute it by the integral*

$$F(\omega) = \int_{-\infty}^{+\infty} x_t e^{-i\omega t} dt.$$

The Fourier transform $F(\omega)$ *of* x_t *is often called frequency spectrum.*
(ii) *The inverse Fourier transform exists, which equals* x_t *at all points of continuity,*

$$x_t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega.$$

Average Power and Power Spectral Density

Theorem

(i) *$F(\omega)$ is a complex-valued function and*

$$F(-\omega) = \int_{-\infty}^{+\infty} x_t e^{i\omega t} dt = \overline{F(\omega)}.$$

(ii) *The Parseval equality holds, that is,*

$$\int_{-\infty}^{+\infty} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega.$$

$|F(\omega)|^2$ denotes the distribution of the total energy of x_t and it is often called the **energy spectral density**. Thus the Parseval equality can be seen as the spectral representation of the total energy.

Average Power and Power Spectral Density

In many cases, the total energy does not exists. Then we always need compute the average power of x_t. The standard steps to find the average power are as follows.

S1. Given a function x_t ($t \in \mathbb{R}$). Consider the finite support segment,

$$x_T(t) = \begin{cases} x_t & \text{if } |t| \leqslant T, \\ 0 & \text{if } |t| > T. \end{cases}$$

S2. Compute the Fourier transform of $x_T(t)$

$$F_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-i\omega t} dt = \int_{-T}^{T} x_T(t) e^{-i\omega t} dt.$$

S3. The Parseval equality holds, that is,

$$\int_{-\infty}^{+\infty} x_T^2(t) \mathsf{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F_T(\omega)|^2 d\omega.$$

Average Power and Power Spectral Density

S4. Dividing by $2T$, we get

$$\frac{1}{2T}\int_{-T}^{T} x_t^2 dt = \frac{1}{2T}\int_{-\infty}^{+\infty} x_T^2(t)dt = \frac{1}{4\pi T}\int_{-\infty}^{+\infty} |F_T(\omega)|^2 d\omega.$$

S5. Let $T \to +\infty$. We get the average power

$$Q = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to+\infty} \frac{1}{2T}|F_T(\omega)|^2 d\omega. \tag{1}$$

Denote the integrand in the right hand of equation (1) by $S_X(\omega)$, i.e.,

$$S_X(\omega) = \lim_{T\to+\infty} \frac{1}{2T}|F_T(\omega)|^2,$$

which is called the **power spectral density**(PSD) of x_t.

Average Power and Power Spectral Density

In fact, for deterministic process, we only consider the time average in equation (1).

When the process X_t we meet is random, what does this term

$$\int_{-T}^{T} X^2(t)dt$$

mean? How to define the average power and the corresponding power spectral density?

Definition

*(Continuity in Mean) A stochastic process X_t is **continuous in mean square** at t_0 if*

$$\lim_{\varepsilon \to 0} E\left[(X_{t_0+\varepsilon} - X_{t_0})^2\right] = 0$$

Denote it by

$$\text{l.i.m.}_{t \to t_0} X_t = X_{t_0},$$

where l.i.m. *denotes limit in mean. The stochastic process $\{X_t\}$ is called a mean square continuous process if it is continuous in mean square for all t.*

Theorem

A stochastic process X_t is a mean square continuous process if and only if its autocorrelation function $R_X(t_1, t_2)$ is continuous.

Definition

Suppose that X_t is a mean square continuous process. We call X_t has a ***mean square derivative*** *X_t' if*

$$\mathrm{l.i.m.}_{\varepsilon\to 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = X_t',$$

that is

$$\lim_{\varepsilon\to 0} E\left[\left(\frac{X_{t+\varepsilon} - X_t}{\varepsilon} - X_t'\right)^2\right] = 0.$$

Stochastic Analysis

Theorem

Suppose that the mean and the autocorrelation function of X_t *are* $\mu_X(t)$ *and* $R_X(s,t)$ *respectively. Then*

$$\mu_{X'}(t) = E\left(X'(t)\right) = \frac{d}{dt}E(X_t) = \mu_X'(t),$$

and

$$R_{X'}(s,t) = E(X_s'X_t') = \frac{\partial^2 R_X(s,t)}{\partial s \partial t}.$$

Definition

A mean square integral of a random process X_t is defined by

$$Y_t = \int_{t_0}^{t} X_s ds = \text{l.i.m.}_{\triangle s_i \to 0} \sum_i X(s_i) \triangle s_i,$$

where $t_0 < s_1 < \cdots < t$ and $\triangle s_i = s_{i+1} - s_i$.

Stochastic Analysis

Theorem

Let $\mu_Y(t)$ *and* $R_Y(t_1, t_2)$ *be the mean and the autocorrelation function of*

$$Y_t = \int_{t_0}^{t} X_s ds$$

respectively. Then

$$\mu_Y(t) = \int_{t_0}^{t} \mu_X(s) ds,$$

and

$$R_Y(t_1, t_2) = \int_{t_0}^{t_1} \int_{t_0}^{t_2} R_X(s_1, s_2) ds_1 ds_2.$$

Theorem

Suppose that X_t is a mean square continuous process, and $f(t)$ and $g(t)$ are real-valued continuous functions. Then

$$E\left(\int_{t_0}^{t} f(s)X_s ds\right) = \int_{t_0}^{t} f(s)E\big(X_s\big)ds,$$

and

$$E\left(\int_{t_0}^{t_1} f(s_1)X(s_1)ds_1 \int_{t_0}^{t_2} g(s_2)X(s_2)ds_2\right)$$
$$= \int_{t_0}^{t_1}\int_{t_0}^{t_2} f(s_1)g(s_2)E\big(X_{s_1}X_{s_2}\big)ds_1 ds_2.$$

Average Power and Power Spectral Density

Theorem

If X_t is a WSS process, then it is mean square continuous if and only if its autocorrelation function $R_X(\tau)$ is continuous for $\tau = 0$.

For random process X_t, our earlier results can be extended to cover X_t by the inclusion of an extra averaging or expectation step over all possible values of X_t since the integral in equation (1) will be a mean-square integral.

Average Power and Power Spectral Density

Assume X_t is a mean-square continuous stationary process. Similarly to above steps,

we construct a truncated function $X_T(t)$,

compute the Fourier transform of $X_T(t)$,

find the total energy for $X_T(t)$, dividing by $2T$,

find the expectations of both sides and let $T \to +\infty$, then we get the average power

$$\lim_{T \to +\infty} E\left(\frac{1}{2T}\int_{-T}^{T} X_t^2 dt\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T \to +\infty} \frac{1}{2T} E\left(|F_T(\omega)|^2\right) d\omega. \quad (2)$$

Average Power and Power Spectral Density

Definition

Suppose that X_t is a mean-square continuous stationary process.

$$Q := \lim_{T \to +\infty} E\left(\frac{1}{2T} \int_{-T}^{T} X_t^2 dt \right)$$

is defined as the average power of the process X_t.

The power spectral density of X_t is defined to be

$$S_X(\omega) = \lim_{T \to +\infty} \frac{1}{2T} E\big(|F_T(\omega)|^2\big).$$

$$\boxed{Q = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{S_X(\omega)} d\omega.}$$

Average Power and Power Spectral Density

Since X_t is stationary, we have

$$Q = \lim_{T \to +\infty} E\left(\frac{1}{2T}\int_{-T}^{T} X_t^2 dt\right) = \lim_{T \to +\infty} \frac{1}{2T}\int_{-T}^{T} E(X_t^2)dt = R_X(0). \tag{3}$$

Average Power and Power Spectral Density

Example

Find the average power for the sine wave with random phase

$$X_t = a\sin(\omega_0 t + \Theta),$$

where a and ω_0 are constants and Θ has a uniform distribution in the interval $(0, 2\pi)$.

Solution. Since $R_X(\tau) = \frac{a^2}{2}\cos\omega_0\tau$, we have the average power

$$Q = R_X(0) = \frac{a^2}{2}.$$

$\square$

Contents

Power Spectral Density and Autocorrelation Function

$$S_X(\omega) = \lim_{T \to +\infty} \frac{1}{2T} E\left(|F_T(\omega)|^2\right).$$

The power spectral density for a signal is a measure of its power distribution as a function of frequency.

It is a useful concept which allows us to determine the bandwidth required of a transmission system.

Power Spectral Density and Autocorrelation Function

Proposition

(i) $S_X(\omega)$ *is nonnegative for all* ω.
(ii) $S_X(\omega)$ *is a real and even function, that is* $\overline{S}_X(\omega) = S_X(\omega)$ *and* $S_X(\omega) = S_X(-\omega)$.
(iii) *If* $S_X(\omega)$ *is a rational function, then it must have the following form:*

$$S_X(\omega) = S_0 \frac{\omega^{2n} + a_{2n-2}\omega^{2n-2} + \cdots + a_0}{\omega^{2m} + b_{2m-2}\omega^{2m-2} + \cdots + b_0},$$

where a_{2n-i}, b_{2n-j} $(i = 2, \cdots, 2n,\ j = 2, 4, \cdots, 2m)$ *are constants and* $S_0 > 0$, $m > n$ *and the denominator has no real root.*

Power Spectral Density and Autocorrelation Function

Theorem

(**Wiener-Khintchine theorem**) *Suppose that $R_X(\tau)$ is the autocorrelation function of the wide-sense stationary process $\{X_t\}$ and $S_X(\omega)$ is the power spectral density. If*

$$\int_{-\infty}^{+\infty} |R_X(\tau)| d\tau < +\infty, \tag{4}$$

then

$S_X(\omega) \longleftrightarrow R_X(\tau)$

(i)

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-i\omega\tau} d\tau. \tag{5}$$

(ii) *Under condition (4), the inverse Fourier transform is given by*

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_X(\omega) e^{i\omega\tau} d\omega. \tag{6}$$

Power Spectral Density and Autocorrelation Function

Remark

$R_X(\tau)$ is the frequency response of X_t. Its Fourier Transformation is just the average power density over frequency.

Equations (5) and (6) illustrate the relationship between the property in the time domain and the property in the frequency domain of the stationary process. Using equation (6) we find

$$R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)d\omega. \tag{7}$$

Power Spectral Density and Autocorrelation Function

Corralary

If $R_X(\tau)$ and $S_X(\omega)$ are both even, then we have the following equalities

$$S_X(\omega) = 2\int_0^{+\infty} R_X(\tau)\cos\omega\tau d\tau, \tag{8}$$

$$R_X(\tau) = \frac{1}{\pi}\int_0^{+\infty} S_X(\omega)\cos\omega\tau d\omega. \tag{9}$$

Example

Find the power spectral density for the following exponential autocorrelation function $\boxed{R_X(\tau) = Ae^{-\beta|\tau|}}$ with parameters $A > 0$ and $\beta > 0$.

$S_X(\omega) = \dfrac{2A\beta}{\omega^2 + \beta^2}$

Power Spectral Density and Autocorrelation Function

Example

Let $\{X_t\}$ be a stationary process with autocorrelation function

$$R_X(\tau) = e^{-a|\tau|} \cos \omega_0 \tau,$$

where $a > 0$ and ω_0 are constants. Find the power spectral density $S_X(\omega)$.

Power Spectral Density and Autocorrelation Function

If $X_t = aY_t + bZ_t$, then

$$S_X(\omega) = aS_Y(\omega) + bS_Z(\omega) \quad \text{and} \quad R_X(\tau) = aR_Y(\tau) + bR_Z(\tau).$$

Example

Assume that the power spectral density of stochastic process X_t is

$$S_X(\omega) = \frac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9}.$$

Find the corresponding autocorrelation function $R_X(\tau)$.

White noise process

One of the very important random processes is the white noise process. Noises in many practical situations are approximated by the white noise process. Most importantly, the white noise plays an important role in modeling of WSS signals.

Definition

(**White noise processes**)
A white noise process W_t is defined by

$$S_W(\omega) = \frac{N_0}{2}, \quad -\infty < \omega < +\infty$$

where N_0 is a real constant and called the intensity of the white noise.

White noise process

$$(R_X(\tau) = 1 \to S_X(\omega) = 2\pi\delta(\omega), \quad R_X(\tau) = \delta(\tau) \to S_X(\omega) = 1.)$$

Since the power spectral density of white noise is

$$S_W(\omega) = \frac{N_0}{2}, \quad -\infty < \omega < +\infty$$

the corresponding autocorrelation function is given by

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta.

The average power of white noise

$$Q = E[W^2(t)] = \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{N_0}{2} d\omega \to \infty.$$

White noise process

The term white noise is analogous to white light which contains all visible light frequencies.

A white noise is a mathematical abstraction; it cannot be physically realized since it has infinite average power.

Example

Find the power spectral density for the following stochastic process $X_t = a\cos(\omega_0 t + \Theta)$, where a and ω_0 are constants and Θ has a uniform distribution in the interval $(0, 2\pi)$.

Power Spectral Density and Autocorrelation Function

Theorem

Suppose that $\{X_n\}$ is a stationary random sequence with the mean function $E(X_n) = 0$ and the autocorrelation function $R_X(m) = E(X_n X_{n+m})$. If

$$\sum_{m=-\infty}^{+\infty} |R_X(m)| < +\infty,$$

then the power spectral density of $\{X_n\}$ is the Fourier transform of $R_X(m)$, that is,

$$S_X(\omega) = \sum_{m=-\infty}^{+\infty} R_X(m) e^{-im\omega}, \quad for - \pi \leqslant \omega \leqslant \pi.$$

The inverse Fourier transform is given as

$$R_X(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) e^{im\omega} d\omega.$$

Power Spectral Density and Autocorrelation Function

Example

Suppose that $\{X_n, n = 0, \pm 1, \pm 2, \ldots,\}$ is a stationary random sequence with the mean function 0 and the autocorrelation function

$$R_X(n) = \begin{cases} \sigma^2 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Find the power spectral density $S_X(\omega)$.

The end

Thank you for your patience !