Chapter 3 Functions, Sequences, and Relations
函数、序列、和关系

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3.1 Functions 函数

Definition 3.1.1 Let X and Y be sets. A function f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.

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Definition 3.1.21 A function f from X to Y is said to be **one-to-one (or injective) (单射的)** if **for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$**.

Definition 3.1.29 A function f from X to Y is said to be **onto** Y **(or surjective) (满射的)** if **for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$**.

Definition 3.1.35 A function that is **both one-to-one and onto** is called a **bijection (双射)**.

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Pigeonhole Principle (First Form)
If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons.

Pigeonhole Principle (Second Form)
If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$.

A function from a larger set to a smaller set cannot be injective.
(There must be at least two elements in the domain that have the same image in the codomain.)

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Exercise 1 In any group of n people there are at least two persons having the same number friends. (It is assumed that if a person x is a friend of y then y is also a friend of x.)

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Exercise 2 Given n integers $a_1, a_2, \dots, a_n$, not necessarily distinct, there exist integers k and l with $0 \le k < l \le n$ such that the sum $a_{k+1} + a_{k+2} + \dots + a_l$ is a multiple of n.

Pigeonhole Principle 鸽巢原理 (in Sec 6.8)

Exercise 1 In any group of n people there are at least two persons having the same number friends. (It is assumed that if a person x is a friend of y then y is also a friend of x.)

Proof. The number of friends of a person x is an integer k with $0 \leq k \leq n-1$. If there is a person y whose number of friends is $n-1$, then everyone is a friend of y, that is, no one has 0 friend. This means that 0 and $n-1$ can not be simultaneously the numbers of friends of some people in the group. The pigeonhole principle tells us that there are at least two people having the same number of friends. $\square$

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Proof. Consider the n integers

$$a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \quad \ldots, \quad a_1 + a_2 + \cdots + a_n.$$

Dividing these integers by n, we have

$$a_1 + a_2 + \cdots + a_i = q_i n + r_i, \quad 0 \leq r_i \leq n - 1, \quad i = 1, 2, \ldots, n.$$

If one of the remainders $r_1, r_2, \ldots, r_n$ is zero, say, $r_k = 0$, then $a_1 + a_2 + \cdots + a_k$ is a multiple of n. If none of $r_1, r_2, \ldots, r_n$ is zero, then two of them must the same (since $1 \leq r_i \leq n - 1$ for all i), say, $r_k = r_l$ with $k < l$. This means that the two integers $a_1 + a_2 + \cdots + a_k$ and $a_1 + a_2 + \cdots + a_l$ have the same remainder. Thus $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n. □

3.2 Sequences and Strings 序列和串

Definition 3.2.1 A **sequence (序列)** s is a function whose domain D is a subset of integers.

The notation s_n is typically used instead of the more general function notation $s(n)$. The term n is called the **index(下标)** of the sequence.

If D is a finite set, we call s a finite sequence **(有限序列)**;
otherwise, s is an infinite sequence **(无限序列)**.

Important Types of Sequences

- **Increasing Sequences (递增序列)**

A sequence s is **increasing** if for all i and j in the domain of s,

if $i < j$, then $s_i < s_j$.

- **Decreasing Sequences (递减序列)**

A sequence s is **decreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i > s_j$.

- **Nonincreasing Sequences (非增序列)**

A sequence s is **nonincreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \geq s_j$.

- **Nondecreasing Sequences (非减序列)**

A sequence s is **nondecreasing** if for all i and j in the domain of s,

if $i < j$, then $s_i \leq s_j$.

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

- Finite sequences are also called strings.
- The string with no elements is called **null string (空串)** and is denoted λ.
- Let X^* denote the set of all strings over X.
- Let X^+ denote the set of all nonnull strings over X.

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Example 3.2.5 Let $X = \{a, b\}$. Some elements in X^* are λ, a, b, $abab$ and $b^2a^{50}ba$.

String

Definition 3.2.23 A string over X, where X is a finite set, is a finite sequence of elements from X.

● The **length (长度)** of a string α is the number of elements in α. The length of α is denoted $|\alpha|$.

Example 3.2.26 If $\alpha = aabab$ and $\beta = a^3b^4a^{32}$, then $|\alpha| = 5$ and $|\beta| = 39$.

3.3 Relations 关系

Definition 3.3.2 A **(binary) relation (二元关系)** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y.

If $X = Y$, we call R a (binary) relation on X.

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- **The relationship between Function, Sequence and Relation**

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Special case

sequence $\longleftarrow$ **function**

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- **The relationship between Function, Sequence and Relation**

A function f from X to Y is a relation from X to Y having the properties:
(a) The domain of f is equal to X.
(b) For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$.

$$\text{sequence} \xleftarrow{\text{Special case}} \text{function} \xleftarrow{\text{Special case}} \text{relation}$$

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A relation can be defined by

- simply specifying which ordered pairs belong to the relation

$$R = \{(\text{Bill}, \text{CompSci}), (\text{Mary}, \text{Math}), (\text{Bill}, \text{Art}), (\text{Beth}, \text{History}), (\text{Beth}, \text{CompSci}), (\text{Dave}, \text{Math})\}$$

TABLE 3.3.1 Relation of Students to Courses

Student	*Course*
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

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- defining a relation by giving a rule for membership in the relation

$$f = \{(x, x^2) \mid x \in \mathbf{Z}\}$$

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Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \quad \text{if } x \text{ divides } y,$$

we obtain $R = ?$

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Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = ?$

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A relation on a set

draw its **digraph (有向图)**

Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

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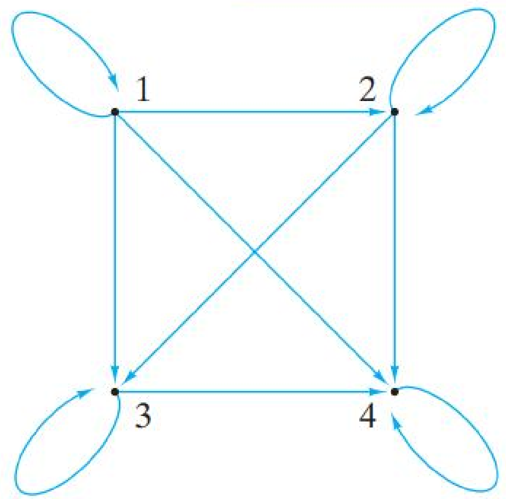

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- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

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How many different relations can we define on a set X with n elements?

A relation on a set X is a subset of $X \times X$.
How many elements are in $X \times X$?
There are n^2 elements in $X \times X$, so how many subsets (= relations on X) does $X \times X$ have?
Therefore, 2^{n^2} subsets can be formed out of $X \times X$.

Answer: We can define 2^{n^2} different relations on X.

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Definition 3.3.6 A relation R on a set X is **reflexive (自反的)** if $(x, x) \in R$ for every $x \in X$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$

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$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

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Definition 3.3.9 A relation R on a set X is **symmetric (对称的)** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Definition 3.3.12 A relation R on a set X is **antisymmetric (反对称的)** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

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for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

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Example 3.3.4 Let $X = \{1, 2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

symmetric? antisymmetric?

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not symmetric = antisymmetric?

$R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$.

Both symmetric and antisymmetric!

3.3 Relations 关系

Definition 3.3.17 A relation R on a set X is **transitive (传递的)** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Exercise Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$

$R = \{(1, 3), (3, 2), (2, 1)\}$

$R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$

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The relation $R = \emptyset$

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$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

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not reflexive: if there exists $x \in X$, such that $(x, x) \notin R$.

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not symmetric: if there exists x and y, such that $(x, y) \in R$ and $(y, x) \notin R$.

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if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

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3.3 Relations 关系

Exercise Give examples of relations on $\{1, 2, 3, 4\}$ having the properties specified as follows.

(a) Reflexive, symmetric, and not transitive

(b) Reflexive, not symmetric, and not transitive

(c) Reflexive, antisymmetric, and not transitive

(d) Not reflexive, symmetric, not antisymmetric, and transitive

(e) Not reflexive, not symmetric, and transitive

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If R is a partial order on a set X, the notation $x \preccurlyeq y$ is sometimes used to indicate that $(x, y) \in R$.

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We say that

- x and y are **comparable (可比的)**: If $x, y \in X$ and either $x \preccurlyeq y$ or $y \preccurlyeq x$.
- x and y are **incomparable (不可比的)**: If $x, y \in X$ and either $x \npreceq y$ or $y \npreceq x$.

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If every pair of elements in X is comparable, we call R a **total order (全序)**.

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Example ?

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Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

3.3 Relations 关系

Definition 3.3.20 A relation R on a set X is a **partial order (偏序)** if R is **reflexive, antisymmetric, and transitive**.

If every pair of elements in X is comparable, we call R a **total order (全序)**.

Example 3.3.4 Let $X = \{2, 3, 4\}$ definded by$(x, y) \in R$ if $x \leq y$, $x, y \in X$.
Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)(4,4)\}$.

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \ \text{ if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

3.3 Relations 关系

Definition 3.3.23 Let R be a relation from X to Y. The **inverse of R (R的逆)**, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \text{ if } x \text{ divides } y,$$

we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

$R^{-1} =$

3.3 Relations 关系

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 =$

3.3 Relations 关系

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Example 3.3.26 The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is $R_2 \circ R_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

(2) $R \cap S$

(3) $R \circ S$

3.3 Relations 关系

Example 3.3.27 Suppose that R and S are transitive relations on a set X. Determine whether each of $R \cup S$, $R \cap S$, or $R \circ S$ must be transitive.

(1) $R \cup S$

$$R = \{(1, 2)\}, S = \{(2, 3)\}, R \cup S = \{(1, 2), (2, 3)\}.$$

(2) $R \cap S$

If $\{x, y\}, \{y, z\} \in R \cap S$, then $\{x, z\} \in R \cap S$.

(3) $R \circ S$

$$R = \{(5, 2), (6, 3)\}, S = \{(1,5), (2, 6)\}, R \circ S = \{(1, 2), (2, 3)\}.$$

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Example Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Theorem 3.4.1 Let $\mathcal{S}$ be a partition of a set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

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- A **partition (划分)**:

- A **collection of sets (集族)**： A set $\mathcal{S}$ whose elements are sets.

Example: $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$.

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- A **partition (划分)**： A collection $\mathcal{S}$ of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of $\mathcal{S}$.
- A **collection of sets (集族)**： A set $\mathcal{S}$ whose elements are sets.

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- A **collection of sets (集族)**：A set $\mathcal{S}$ whose elements are sets.

Example: $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$.

In other words, the collection of subsets A_i, $i \in I$, forms a **partition** of X if and only if (i) $A_i \neq \emptyset$ for $i \in I$ (ii) $A_i \cap A_j = \emptyset$, if $i \neq j$ (iii) $\bigcup_{i \in I} A_i = X$

3.4 Equivalence Relations 等价关系

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Theorem 3.4.1 Let $\mathcal{S}$ be a partition of a set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

- A **partition (划分)**： A collection $\mathcal{S}$ of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of $\mathcal{S}$.

Example

$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is a partition of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

In other words, the collection of subsets A_i, $i \in I$, forms a **partition** of X if and only if (i) $A_i \neq \emptyset$ for $i \in I$ (ii) $A_i \cap A_j = \emptyset$, if $i \neq j$ (iii) $\bigcup_{i \in I} A_i = X$

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- reflexive
- symmetric
- transitive

3.4 Equivalence Relations 等价关系

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Theorem 3.4.1 Let $\mathcal{S}$ be a partition of a set X. Define xRy to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

Example 3.4.2 Consider the partition $\mathcal{S} = \{\{1, 3, 5\}, \{2. 6\}, \{4\}\}$ of $X = \{1, 2, 3, 4, 5, 6\}$. The relation R on X is given by Theorem 3.4.1. Then

$R =$

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Example Consider the relation $R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}$ on $\{1,2,3,4,5\}$. Is it an equivalence relation?

Diagraph?

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Exercise Which of the following relation is an equivalence relation?

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$.

The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Exercise Which of the following relation is an equivalence relation?

The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$. No.

The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$. No.

3.4 Equivalence Relations 等价关系

Definition 3.4.3 A relation that is **reflexive, symmetric, and transitive** on a set X is called an **equivalence relation (等价关系)** on X.

Definition 3.4.9 Let R be an equivalence relation on a set X. For $\forall\ a \in X$, let $[a] = \{x \in X \mid xRa\}$. The sets $[a]$ is called the **equivalence classes (等价类)** of X given by the relation R.

3.4 Equivalence Relations 等价关系

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Theorem 3.4.8 Let R be an equivalence relation on a set X.
Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.

3.4 Equivalence Relations 等价关系

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Theorem 3.4.8 Let R be an equivalence relation on a set X.
Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.

Example 3.4.2 Consider the relation
$R = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 6), (6, 2), (6, 6), (4, 4)\}$
on $\{1, 2, 3, 4, 5, 6\}$.

3.4 Equivalence Relations 等价关系

Theorem 3.4.16 Let R be an equivalence relation on a finite set X. If each equivalence class has r elements, there are $|X|/r$ equivalence classes.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is ?

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The matrix of a relation from X to Y is dependent on the orderings of X and Y.

3.5 Matrices of Relations 关系矩阵

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Example 3.5.2 The matrix of the above relation R relative to the orderings $2, 3, 4, 1$ and d, b, a, c is ?

The matrix of a relation from X to Y is dependent on the orderings of X and Y.

3.5 Matrices of Relations 关系矩阵

Example 3.5.1 The relation $R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings $1, 2, 3, 4$ and a, b, c, d is

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Example 3.5.2 The matrix of the above relation R relative to the orderings $2, 3, 4, 1$ and d, b, a, c is

$$\begin{array}{cc} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

Example 3.5.4 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$ on $\{a, b, c, d\}$ relative to the ordering a, b, c, d is ?

When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

3.5 Matrices of Relations 关系矩阵

The matrix of the relation R from X to Y

Lable the rows with the elements of X (in some arbitrary order), and label the columns with the elements of Y (again, in some arbitrary order). Set the entry in row x and column y to 1 if xRy and to 0 otherwise.

A relation on a set

- reflexive 自反的
- symmetric 对称的
- antisymmetric 反对称的
- transitive 传递的

When we write the matrix of a relation R on a set X (i.e., from X to X), we use the same ordering for the rows as we do for the columns.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

The product of these matrices is $A_1 A_2 =$?

设矩阵 $\boldsymbol{A} = (\boldsymbol{a}_{ij})_{m\times s}$, $\boldsymbol{B} = (\boldsymbol{b}_{ij})_{s\times n}$, 令 $\boldsymbol{C} = (\boldsymbol{c}_{ij})_{m\times n}$ 是由下面的 $\boldsymbol{m} \times \boldsymbol{n}$ 个元素 $\boldsymbol{c}_{ij} = \boldsymbol{a}_{i1}\boldsymbol{b}_{1j} + \boldsymbol{a}_{i2}\boldsymbol{b}_{2j} + \ldots + \boldsymbol{a}_{is}\boldsymbol{b}_{sj}$ $(\boldsymbol{i} = 1, 2, \ldots, \boldsymbol{m}; \boldsymbol{j} = 1, 2, \ldots, \boldsymbol{n})$ 构成的 $\boldsymbol{m}$ 行 $\boldsymbol{n}$ 列矩阵. 称矩阵 $\boldsymbol{C}$ 为矩阵 $\boldsymbol{A}$ 与矩阵 $\boldsymbol{B}$ 的乘积，记为 $\boldsymbol{C} = \boldsymbol{AB}$.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

$$A_1 = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

$$A_2 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is $A_1 A_2 =$?

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

设矩阵 $\boldsymbol{A} = (a_{ij})_{m\times s}$，$\boldsymbol{B} = (b_{ij})_{s\times n}$，令 $\boldsymbol{C} = (c_{ij})_{m\times n}$ 是由下面的 $m \times n$ 个元素 $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{is}b_{sj}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ 构成的 m 行 n 列矩阵. 称矩阵 $\boldsymbol{C}$ 为矩阵 $\boldsymbol{A}$ 与矩阵 $\boldsymbol{B}$ 的乘积，记为 $\boldsymbol{C} = \boldsymbol{AB}$.

3.5 Matrices of Relations 关系矩阵

Example 3.5.5 Let R_1 be the relation from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and
let R_1 be the relation from Y to $Z = \{x, y, z\}$ defined by
$R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

The matrix of R_1 relative to the orderings $1, 2, 3$ and a, b is $A_1 =$?

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

The matrix of R_2 relative to the orderings a, b and x, y, z is $A_2 =$?

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is $A_1A_2 =$?

$$A_1A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $R_2 \circ R_1$, is the relation from X to Z defined by
$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by 1.

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Definition 3.3.25 Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition of R_1 and R_2**, denoted $\boldsymbol{R_2 \circ R_1}$, is the relation from X to Z defined by
$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}$.

3.5 Matrices of Relations 关系矩阵

Theorem 3.5.6 Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y, and Z. Let A_1 be the matrix of R_1 and let A_1 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by 1.

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$ on $\{a, b, c, d\}$ is transitive?

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Example 3.5.7 The relation $R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, b)\}$ on $\{a, b, c, d\}$ is transitive?

The relation R is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many relations are there on X?
- How many reflexive relations are there on X?
- How many symmetric relations are there on X?
- How many antisymmetric relations are there on X?

3.5 Matrices of Relations 关系矩阵

Let X be an n-element set.

- How many reflexive and symmetric relations are there on X?

- How many reflexive and antisymmetric relations are there on X?

- How many aymmetric and antisymmetric relations are there on X?

- How many reflexive, symmetric, and antisymmetric relations are there on X?