Chapter 1 Sets and Logic

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1.1 Sets

Set = a collection of objects 一些对象的全体

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Example: $A = \{1, 2, 3, 4\}$

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Elements 元素

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Example: A = {1, 2, 3, 4}

Elements

A set is determined by its elements and not by any particular order.

A = {1, 2, 4, 3}

1.1 Sets

Set = a collection of objects

Example: $B = \{x \mid x = 2k + 1, 0 < k < 3\}$

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How to determine a set

- **Listing**
  $A = \{1, 2, 3, 4\}$

- **Describing Property**
  $B = \{x \mid x = 2k + 1, 0 < k < 3\}$

1.1 Sets

Set = a collection of objects

How to determine a set

- **Listing** $\longrightarrow$ a set is finite and not too large

  $A = \{1, 2, 3, 4\}$

- **Describing Property** $\longrightarrow$ a set is a large finite set or an infinte set

  $B = \{x \mid x = 2k + 1, 0 < k < 3\}$

1.1 Sets

Finite sets

Examples:

A = {1, 2, 3, 4}

B = {x | x is an integer, $1 \le x \le 4$}

Infinite sets

Examples:

Z = {integers} = {…, -3, -2, -1, 0, 1, 2, 3,…}

S={x| x is a real number and $1 \le x \le 5$} = [1, 5]

1.1 Sets

A set may contain any kind of element

Examples:

{1, 2, Jason}

{1, 5, {3.5, 17}, Jason }

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Sets of Numbers

Examples:

Z : Integers

Q: Rational numbers

R: Real numbers

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Sets of Numbers

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Z : Integers

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Guess: What are $\mathbf{Z}^{+}$, $\mathbf{Z}^{-}$, and $\mathbf{Z}^{nonneg}$**?**

1.1 Sets

Cardinality of a set A (in symbols $|A|$) 集合的势

$|A|$=the number of elements in A

Examples:

If $A = \{1, 2, 3\}$ then $|A| = 3$

If $B = \{x \mid x \text{ is a natural number and } 1 \leq x \leq 9\}$, then $|B| = 9$

1.1 Sets

Cardinality of a set $\mathbf{A}$ (in symbols $|\mathrm{A}|$)

$|\mathrm{A}|$=the number of elements in A

An element x **is in a set** X**:** $x \in X$

An element x **is not in a set** X**:** $x \notin X$

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Empty Set $\varnothing$

The set with no elements is call the empty set

or null set
or void set

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

- For every x, if $x \in A$, then $x \in B$,

and

- For every x, if $x \in B$, then $x \in A$.

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

Examples:

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2}
- A = {dog, cat, horse},
  B = {cat, horse, squirrel, dog}
- A = {dog, cat, horse},
  B = {cat, horse, dog, dog}

Set Equality

Sets A and B are **equal** if and only if they contain exactly the same elements.

Examples1.1.3

- A = $\{x \mid x^2 + x - 6 = 0\}$, B = {2, -3}, A=B?

Subsets

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- A is a **proper subset** of B if $A \subseteq B$ but does not equal B and write $A \subset B$
- Observation: $\varnothing$ is a subset of every set

Power set of A 集合A的幂集

The set of all subsets (proper or not) of a set A.
(in symbols $\mathcal{P}(A)$)

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If A={a, b, c}, the members of $\mathcal{P}$(A) are

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If A={a, b, c}, the members of $\mathcal{P}(A)$ are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

The power set of a set with n elements has ? elements

Set Operations

Given two sets X and Y

- The union (并集) of X and Y is defined as the set

$$\mathbf{X} \cup \mathbf{Y} = \{ \mathbf{x} \mid \mathbf{x} \in \mathbf{X} \text{ or } \mathbf{x} \in \mathbf{Y} \}$$

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- The difference (or relative complement) (差集) of X and Y

  $$\mathbf{X} - \mathbf{Y} = \{ \mathbf{x} \mid \mathbf{x} \in \mathbf{X} \text{ and } \mathbf{x} \notin \mathbf{Y} \}$$

Set Operations

Example 1.1.15 If A={1, 3, 5} and B={4, 5, 6} then

$A \cup B =$

$A \cap B =$

$A - B=$

Set Operations

Sets X and Y are called **disjoint** (不相交) if their intersection is empty, that is, they share no elements:

$$X \cap Y = \varnothing$$

A collection of sets S (集族) is said to be **pairwise disjoint** （两两不相交） if, whenever X and Y are distinct sets in S, X and Y are disjoint

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Example 1.1.17

{1, 4, 3} and {2, 6}

S={{1, 3, 5}, {2, 8}, {4}}

Univeral Set and Complement

Universal set (全集/域): When all of the consided sets are subset of a set U, we called U a universal set or a universe

Complement of X(余/补集): U-X

Venn Diagrams

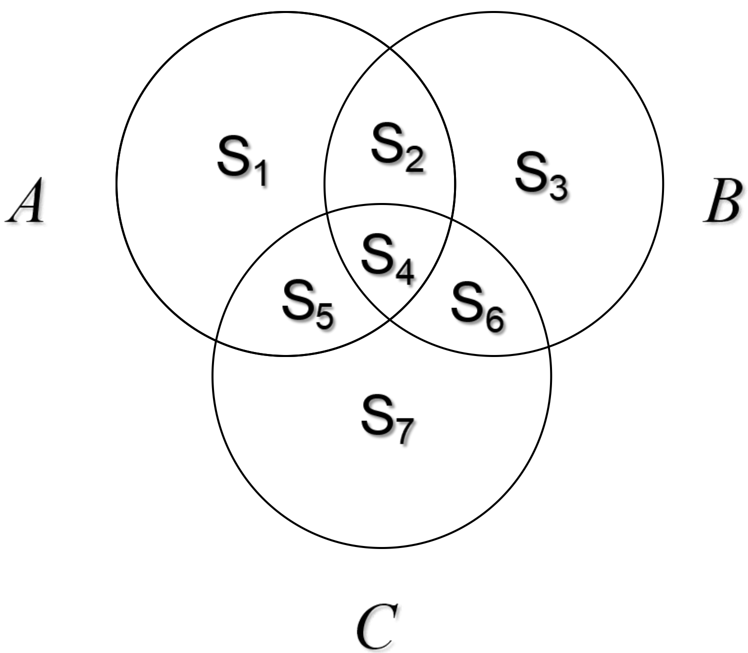
- A Venn diagram provides a graphic view of sets.
- Set union, intersection, difference, symmetric difference and complements can be identified.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive Law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

We can also verify this law more carefully



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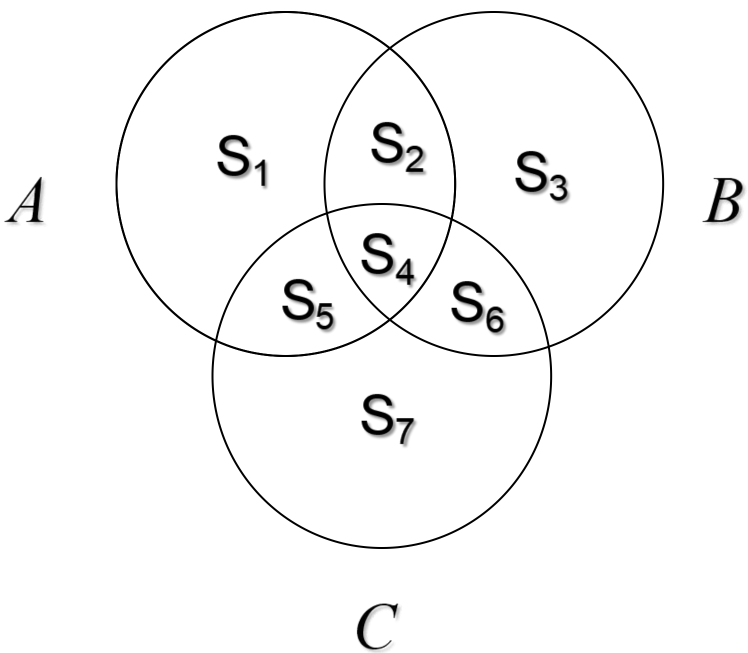

L.H.S

$$A = S_1 \cup S_2 \cup S_4 \cup S_5$$

$$B \cap C = S_4 \cup S_6$$

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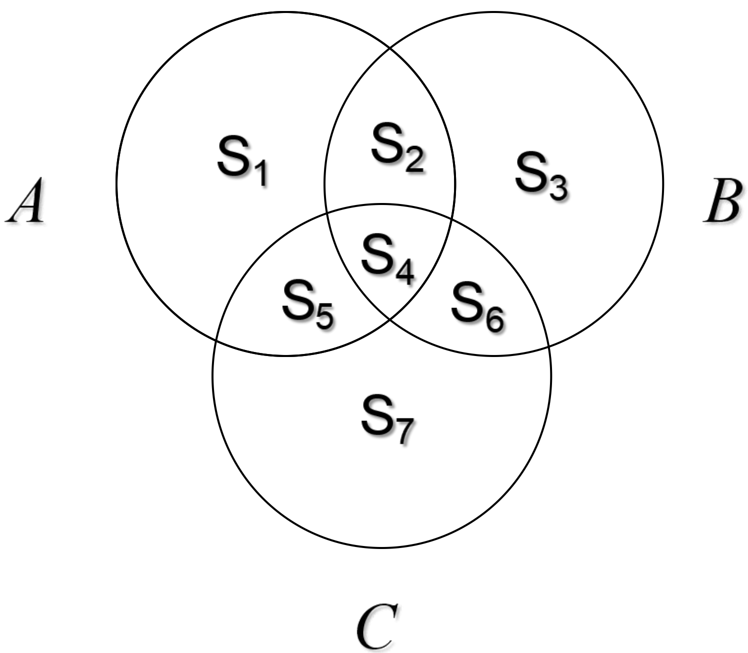

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$$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$$

$$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$$

$$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

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Venn Diagrams

Example 1.1.21

Among a group of 165 students,

- 8 are taking calculus, psychology, and computers science;
- 33 are taking calculus and computer science;
- 20 are taking calculus and psychology
- 24 are taking psychology and computer science：
- 79 are taking calculus;
- 83 are taking psychology;
- 63 are taking computer science.

How many are taking none of the three subjects?

Venn Diagrams

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- 8 are taking calculus, psychology, and computers science;
- 33 are taking calculus and computer science;
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Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

a) Associative law (结合律）:

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

b) Commutative law (交换律)

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

c) Distributive laws (分配律)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Distributive Law:

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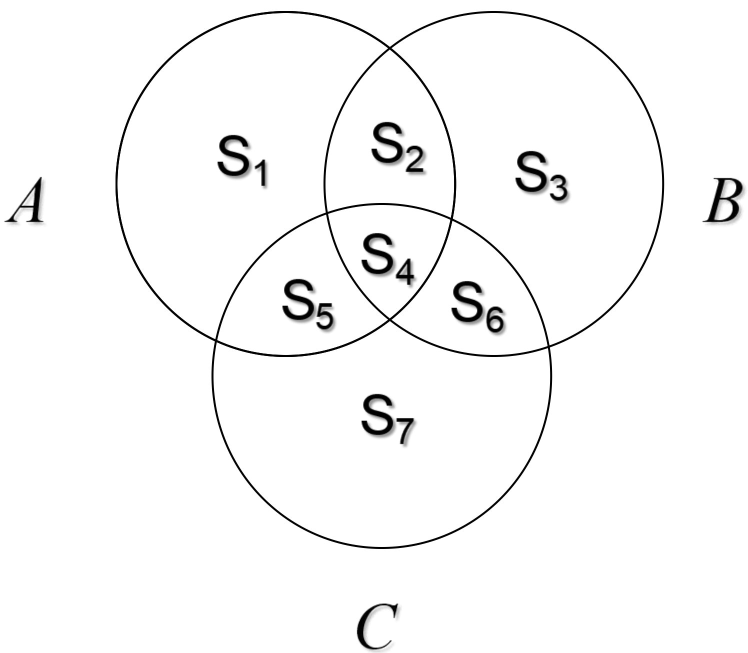

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$$A = S_1 \cup S_2 \cup S_4 \cup S_5$$

$$B \cap C = S_4 \cup S_6$$

$$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

R.H.S.

$$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$$

$$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$$

$$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$$

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d) Identity laws (同一律)

$$A \cap U = A$$

$$A \cup \varnothing = A$$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

e) Complement laws (补余律)

$A \cup A^c = U$

$A \cap A^c = \varnothing$

f) Idempotent laws (等幂律)

$A \cup A = A$

$A \cap A = A$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

g) Bound laws (零律)

$$A \cup U = U$$

$$A \cap \varnothing = \varnothing$$

h) Absorption laws (吸收律)

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Theorem 1.1.22

Let U be a universal set and let A, B and C be subsets of U. The following properties hold.

i) Involution law (对合律)

$$(A^c)^c = A$$

j) 0/1 laws (0/1律)

$$\varnothing^c = U,\ U^c = \varnothing$$

k) De Morgan’s laws for sets (德·摩根定律)

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

We define the union of a collection of sets S to be those elements x belonging to at least one set X in S. Formally,

$\cup S = \{x \mid x \in X \text{ for some } X \in S\}$.

Similarly, we define the intersection of a collection of sets S to be those elements x belonging to every set X in S. Formally,

$\cap S = \{x \mid x \in X \text{ for all } X \in S\}$.

If $S = \{A_1, A_2, \ldots, A_n\}$, we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{n} A_i,$$

If $S = \{A_1, A_2, \ldots A_n \ldots,\}$, we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Set partition

A partition of a set *X* divides *X* into nonoverlapping subsets.

More formally, a collection *S* of nonempty subsets of *X* is said to be a **partition** of the set *X* if every element in *X* belongs to exactly one member of *S*.

Example 1.1.25

Since each element of X={1,2,3,4,5,6,7,8} is in exactly one member of S={{1, 4, 5}, {2, 6}, {3, 8}, {7}}, S is a partition of X

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Notice that if *S* is a partition of *X*, *S* is pairwise disjoint and $\cup S = X$.

Cartesian Product 笛卡尔积

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Example: $A = \{x, y\}$, $B = \{a, b, c\}$

$A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product 笛卡尔积

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

$|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } 1 \leq i \leq n\}$

Problem-Solving Tips

To verify that two sets A and B are equal, written $A = B$, show that for every x, if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.

To verify that two sets A and B are not equal, written $A \neq B$, find at least one element that is in A but not in B, or find at least one element that is in B but not in A. One or the other conditions suffices; you need not (and may not be able to) show both conditions.

Problem-Solving Tips

To verify that A is a subset of B, written $A \subseteq B$, show that for every x, if $x \in A$, then $x \in B$. Notice that if A is a subset of B, it is possible that $A = B$.

To verify that A is not a subset of B, find at least one element that is in A but not in B.

Problem-Solving Tips

To verify that A is a proper subset of B, written $A \subset B$, verify that A is a subset of B as described previously, and that $A = B$, that is, that there is at least one element that is in B but not in A.

To visualize relationships among sets, use a Venn diagram. A Venn diagram can suggest whether a statement about sets is true or false.

A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n-tuples take order into account.