SCHOOL CHOICE PROBLEM
VERSION V1-101016

1. Introduction

Stable matchings is a very active and successful branch of cooperative game theory commonly used to analyze two-sided markets and their mechanisms. It was originated by Gale and Shapley and further extended by Roth (and others). Their results revealed unknown properties of existing allocation mechanisms and provided a powerful methodology to design new mechanisms with a great impact on our daily lives. As examples of applications, there are matching mechanisms assigning students to colleges, interns to hospital (the US National Resident Matching Program has been partly designed by Roth) or the US national kidney exchange program that gives hospitals the incentive to pool the organs in order to create exchange cycles increasing the number of transplants. Roth and Shapley obtained the Nobel prize in Economics in 2012 "for the theory of stable allocations and the practice of market design".

In this project, we study three mechanisms for the assignment of students to schools. Students are traditionally assigned to their neighborhood schools but some countries (like US) have progressively allowed parents to emit preferences over the schools they want for their children. In this case, an assignment mechanism is required to take also into account schools' priorities, like the student's home address or other characteristics related e.g. to affirmative action. For such a mechanism to be successful (i.e., accepted by all players), there are some desirable properties, like feasibility, elimination of justified envy or Pareto efficiency.

2. Model

Definition 1. *A* **school choice problem** *is a five-tuple* $(I, S, q, \succ_S, \succ_I)$*, where* I *is a finite set of students,* S *is a finite set of schools,* $q = (q_s)_{s\in S}$ *is a capacity profile for schools, where* q_s *is the number of available seats at school* $s \in S$*,* $\succ_S = (\succ_s)_{s\in S}$ *is a profile of strict priorities for schools, and* $\succ_I = (\succ_i)_{i\in I}$ *is a profile of strict preferences for students. For school* $s \in S$*,* $\succ_s$ *is a complete, irreflexive and transitive binary relation over* $I \cup \varnothing$*. For student* $i \in I$*,* $\succ_i$ *is a complete, irreflexive and transitive binary relations over* $S \cup \varnothing$*. Here* $\varnothing$ *represents the fact of remaining unmatched.*

The school choice problem is closely related to the *college admission problem* introduced by Gale and Shapley in 1962. The key difference lies in the fact that in the school choice problem, schools are not strategic agents with preferences. Instead, they have priorities determined by exogenous factors like student's home address. As a results, schools can be seen as resources to be allocated to students and only the student welfare matters in mechanism design. Note that we have assumed strict preferences and priorities, i.e., there is no indifference. In practical cases, ties can be randomly broken.

Definition 2. *A* **matching** *of students to schools for the school choice problem is a mapping* $\mu : I \cup S \to 2^{I \cup S}$ *such that:*

(1) $\mu(i) \subset S$ *with* $|\mu(i)| \leq 1$ *for all* $i \in I$*; and*
(2) $\mu(s) \subset I$ *with* $|\mu(s)| \leq q_s$ *for all* $s \in S$*; and*
(3) $s \in \mu(i)$ *if and only if* $i \in \mu(s)$ *for all* $i \in I$ *and* $s \in S$*.*

Definition 3. *A* **student assignment mechanism** *is a systematic procedure that selects a matching for each school choice problem.*

We now define some appealing properties for matchings and student assignment mechanisms.

Definition 4. *A matching* μ *is* **feasible** *if* $i \succ_s \varnothing$ *for all* $i \in \mu(s)$ *and* $s \in S$*.*

Definition 5. *A matching* μ **eliminates justified envy** *if there is no unmatched student-school pair* (i, s)*, where student* i *prefers* s *to his assignment and he has higher priority than some other student who is assigned a seat at school* s*.*

Note that the elimination of justified envy is equivalent to the notion of stability in the college admission problems if priorities are interpreted as preferences.

Definition 6. *A matching is* **Pareto efficient** *if there is no other matching, which assigns each student a weakly better school and at least one student a strictly better school.*

Definition 7. *A student assignment mechanism is a* **direct mechanism** *if it requires students to reveal their preferences over schools and selects a matching based on these submitted preferences and student priorities. A student assignment mechanism* **eliminates justified envy** *if it always selects a matching that eliminates justified envy. A student assignment mechanism is* **Pareto efficient** *if it always selects a Pareto efficient matching. A direct mechanism is* **strategy-proof** *if no student can ever benefit by unilaterally misrepresenting his preferences.*

3. Mechanisms

3.1. **Boston Mechanism.** This direct mechanism has been in use in Boston until 2005. It works as follows:

(1) Each student submits a preference ranking of the schools.
(2) In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.
(3) In Round k, consider the remaining students. Only the k-th choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k-th choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her k-th choice.

At each round, every assignment is final and the algorithm terminates when no more students are assigned.

Question 1. Does the Boston mechanism eliminate justified envy?

Question 2. Implement the Boston mechanism and run the mechanism on the test-example (see section 4).

Question 3. Modify the test-example (see section 4) to show that the Boston mechanism is not strategy-proof.

3.2. **Gale-Shapley Student Optimal Stable Mechanism (SOSM).** As the school choice problem is closely related to the college admission problem, a possible approach is to interpret school priorities as preferences and run the student optimal deferred acceptance algorithm as follows:

(1) Step 1: Each student proposes to his first choice. Each school tentatively assigns its seats to its proposers one at a time following its priority order. Any remaining proposers are rejected.
(2) Step k: Each student who was rejected in the previous step proposes to his next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

The algorithm terminates when no student proposal is rejected and each student is assigned his final tentative assignment.

Question 4. Does SOSM mechanism eliminate justified envy?

Proposition 1. *SOSM Pareto dominates any other mechanism that eliminates justified envy.*

Proposition 2. *SOSM is strategy-proof.*

Question 5. Implement SOSM and run the mechanism on the test-example (see section 4).

Question 6. Illustrate through an example the result of Proposition 2.

Example 1. We consider the following school choice problem. There are three students and three schools, each one with a single seat. We have the following school priorities:

$$s_1 : \ i_1 \succ i_3 \succ i_2$$

$$s_2 : \ i_2 \succ i_1 \succ i_3$$

$$s_3 : \ i_2 \succ i_1 \succ i_3$$

and students' preferences:

$$i_1 : \ s_2 \succ s_1 \succ s_3$$

$$i_2 : \ s_1 \succ s_2 \succ s_3$$

$$i_3 : \ s_1 \succ s_2 \succ s_3$$

Question 7. Show on this example that the stable matching obtained by SOSM is not Pareto efficient.

We thus see that Pareto efficiency and the elimination of justified envy may be two conflicting goals.

3.3. **Top Trading Cycles Mechanism (TTCM).** We now introduce a mechanism that does not completely eliminates justified envy but is Pareto efficient. It is based on the Top Trading Cycles Algorithm, which has been used to find the unique core allocation in the housing markets. It works as follows:

(1) Step 1: Assign a counter for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools. Each student points to his favorite school under his announced preferences. Each school points to the student who has the highest priority for the school. Since the number of students and schools are finite, there is at least one cycle. A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, ..., s_k, i_k)$ where s_1 points to i_1, i_1 points to s_2, ..., s_k points to i_k, i_k points to s_1. Moreover, each school can be part of at most one cycle. Similarly, each student can be part of at most one cycle. Every student in a cycle is assigned a seat at the school he points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools are unchanged.

(2) Step k: Each remaining student points to his favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned a seat at the school that he points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero the school is also removed. Counters of all other schools are unchanged.

The algorithm terminates when all students are assigned a seat.

Proposition 3. *TTCM is Pareto efficient.*

Proposition 4. *TTCM is strategy-roof.*

Question 8. Implement TTCM and run the mechanism on the test-example (see section 4).

4. Test Example

In this example, there are 8 students (i_1,...,i_8) and four schools (s_1,...,s_4). We have the following capacities: $q_1 = q_2 = 2$ and $q_3 = q_4 = 3$. School priorities are:

$$s_1 : \ i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5 \succ i_6 \succ i_7 \succ i_8$$
$$s_2 : \ i_3 \succ i_5 \succ i_4 \succ i_8 \succ i_7 \succ i_2 \succ i_1 \succ i_6$$
$$s_3 : \ i_5 \succ i_3 \succ i_1 \succ i_7 \succ i_2 \succ i_8 \succ i_6 \succ i_4$$
$$s_4 : \ i_6 \succ i_8 \succ i_7 \succ i_4 \succ i_2 \succ i_3 \succ i_5 \succ i_1$$

Students' preferences are:

$$\begin{array}{ll} i_1 : \ s_2 \succ s_1 \succ s_3 \succ s_4 & i_2 : \ s_1 \succ s_2 \succ s_3 \succ s_4 \\ i_3 : \ s_3 \succ s_2 \succ s_1 \succ s_4 & i_4 : \ s_3 \succ s_4 \succ s_1 \succ s_2 \\ i_5 : \ s_1 \succ s_3 \succ s_4 \succ s_2 & i_6 : \ s_4 \succ s_1 \succ s_2 \succ s_3 \\ i_7 : \ s_1 \succ s_2 \succ s_3 \succ s_4 & i_8 : \ s_1 \succ s_2 \succ s_4 \succ s_3 \end{array}$$