Primera asignación

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Sección 1.57 (Ejercicios 2a, 2d y 2f) Considere que:

- $r = x\hat{i} + y\hat{i} + z\hat{i}$.
- $a = a(r) = a(x, y, z) = a^i(x, y, z)\hat{i}_i$ y $b = b(r) = b(x, y, z) = b^i(x, y, z)\hat{i}_i$.
- $\phi = \phi(r) = \phi(x, y, z)$ y $\psi = \psi(r) = \psi(x, y, z)$.

Utilizando la notación de índices demuestre que:

2a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Demostración:

$\nabla(\phi\psi) = \partial^i(\phi(x^j)\psi(x^j))\hat{i}_i = [\partial^i(\phi(x^j))\psi(x^j) + \phi(x^j)\partial^i(\psi(x^j))]\hat{i}_i = [\phi(x^j)\partial^i(\psi(x^j)) + \partial^i(\phi(x^j))\psi(x^j)]\hat{i}_i = \phi\nabla(\psi) + \psi\nabla(\phi)$.

2b) ¿Qué se puede decir de $\nabla \cdot (\nabla \times a)$ y de $\nabla \times (\nabla \cdot a)$?

- $\nabla \cdot (\nabla \times a)$ está bien definido, es el producto punto entre dos vectores.
- $\nabla \times (\nabla \cdot a)$ está mal definido, pues el producto cruz se realiza entre vectores.
- En notación de indices $\nabla \cdot (\nabla \times a) = \partial_m \epsilon^{mjk} \partial_j a_k$.

2f) $\nabla \times (\nabla \times a)$=$\nabla(\nabla \cdot a) - \nabla^2 a$

Demostración:

$\nabla \times (\nabla \times a) = \epsilon^{ijk}\partial_j \epsilon^{kmn}\partial_m a_n = \epsilon_{kmn}\epsilon^{ijk}\partial_j\partial_m a_n = \epsilon_{mnk}\epsilon^{ijk}\partial_j\partial_m a_n = (\delta^i_m\delta^j_n - \delta^j_m\delta^i_n)\partial_j\partial_m a_n = \delta^i_m\delta^j_n\partial_j\partial_m a_n - \delta^j_m\delta^i_n\partial_j\partial_m a_n = \partial_j\partial_i a_i - \partial_j\partial_j a_i = \nabla(\nabla \cdot a) - \nabla^2 a$

Sección 1.66 (Ejercicios 2,5,6) 2) Demuestre que:

- $cos(3\alpha) = cos^3(\alpha) - 3cos(\alpha)sen^2(\alpha)$.
- $sen(3\alpha) = 3cos^2(\alpha)sen(\alpha) - sen^3(\alpha)$.

Demostración:

Por la formula de Euler $[cos(\theta) + isen(\theta)]^n = [e^{i\theta}]^n = e^{in\theta} = cos(n\theta) + isen(n\theta)$.

Veamos ahora que: $cos(3\alpha)+isen(3\alpha) = [cos(\alpha)+isen(\alpha)]^3 = cos^3(\alpha)+3icos^2(\alpha)sen(\alpha)-3cos(\alpha)sen^2(\alpha)-isen^3(\alpha) = (cos^3(\alpha) - 3cos(\alpha)sen^2(\alpha)) + i(3cos^2(\alpha)sen(\alpha) - sen^3(\alpha))$.

Igualando las partes reales e imaginarias se tiene:

- $cos(3\alpha) = cos^3(\alpha) - 3cos(\alpha)sen^2(\alpha)$.
- $sen(3\alpha) = 3cos^2(\alpha)sen(\alpha) - sen^3(\alpha)$.

5) Encuentre las raices de:

- $(2i)^{\frac{1}{2}}$.

$(2i) = 2(cos(\frac{\pi}{2}) + isen(\frac{\pi}{2})) = 2e^{i\frac{\pi}{2}}$.

$(2i)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}(cos(\frac{\pi}{4}) + isen(\frac{\pi}{4})) = \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 1 + i$.

- $(1-\sqrt{3}i)^{\frac{1}{2}}$.

$|1-\sqrt{3}i| = 2$ y $\theta = 2\pi - tan^{-1}(\sqrt{3}) = \frac{5\pi}{3}$. Entonces:

$(1-\sqrt{3}i)^{\frac{1}{2}} = [2(cos(\frac{5\pi}{3}) + isen(\frac{5\pi}{3}))]^{\frac{1}{2}} = (2e^{i\frac{5\pi}{3}})^{\frac{1}{2}} = \sqrt{2}e^{i\frac{5\pi}{6}} = \sqrt{2}(cos(\frac{5\pi}{6}) + isen(\frac{5\pi}{6})) = \sqrt{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i.$

- $(-1)^{\frac{1}{3}}$.

$(-1) = cos(\pi) + isen(\pi) = e^{i\pi}$. De lo cual,

$(-1)^{\frac{1}{3}} = e^{i\frac{\pi}{3}} = cos(\frac{\pi}{3}) + isen(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$

- $(8)^{\frac{1}{6}}$.

$(8)^{\frac{1}{6}} = (8^{\frac{1}{3}})^{\frac{1}{2}} = \sqrt{2}$

- $(-8-8\sqrt{3}i)^{\frac{1}{4}}$.

$|-8-8\sqrt{3}i| = 16$ y $\theta = 2\pi - tan^{-1}(\sqrt{3}) = \frac{5\pi}{3}$. Entonces:

$(-8-8\sqrt{3}i)^{\frac{1}{4}} = [16(cos(\frac{5\pi}{3}) + isen(\frac{5\pi}{3}))]^{\frac{1}{4}} = (16e^{i\frac{5\pi}{3}})^{\frac{1}{4}} = 2e^{i\frac{5\pi}{6}} = 2(cos(\frac{5\pi}{6}) + isen(\frac{5\pi}{6})) = 2(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = \sqrt{3} - i.$

6) Demuestre que

- $Log(-ie) = 1 - \frac{\pi}{2}i.$
- $Log(1-i) = \frac{1}{2}ln(2) - \frac{\pi}{4}i.$
- $Log(e) = 1 + 2n\pi i.$
- $Log(i) = (2n + \frac{1}{2})\pi i.$

Observación:

$Log(z) \equiv ln|r| + i(\theta + 2\pi n)$. Sí $z \neq 0$, entonces $e^{Log(z)} = z$.
Llamaremos valor principal de $Log(z)$ al valor que se obtiene cuando \$n=0 \$ y se denota $Ln(z) = ln|r| + i\theta$.

Podemos ver que cuando z es un número real positivo, es decir, $z = re^{i0}$,entonces recobramos la función logarítmica usual: $Ln(z) = ln(r)$.

Demostración:

- $Log(-ie) = Log(ee^{i(-\frac{\pi}{2})}) = ln(e) - \frac{\pi}{2}i = 1 - \frac{\pi}{2}i.$
- $Log(1-i)$; $|1-i| = \sqrt(2)$ y $\theta = -\frac{\pi}{4}$. Así, $Log(1-i) = Log(\sqrt{2}e^{i(-\frac{\pi}{4})})) = ln(\sqrt{2}) - \frac{\pi}{4}i = \frac{1}{2}ln(2) - \frac{\pi}{2}i.$
- $Log(e)$; $|e| = e$ y $\theta = 0$. Así, $Log(e) = ln(e) + i(\theta + 2n\pi) = 1 + 2n\pi i$
- $Log(i) = Log(e^{\frac{\pi}{2}}) = ln(1) + (\frac{\pi}{2} + 2n\pi)i = (2n + \frac{1}{2})\pi i.$