Inverse Functions

A function f may have the same value for different numbers in its domain. For example, if $f(x) = x^2$, then $f(2) = 4$ and $f(-2) = 4$, but $2 \neq -2$. For *the inverse of a function* to be defined, it is essential that different numbers in the domain *always* give different values of f. Such functions are called *one-to-one functions*.

Definition of One-to-One Function

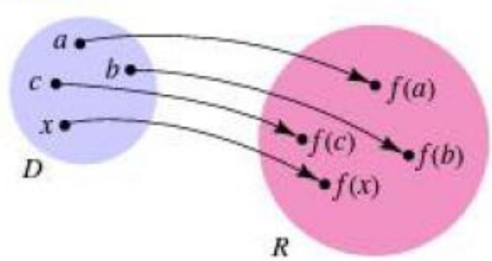
A function f with domain D and range R is a **one-to-one function** if either of the following equivalent conditions is satisfied:

(1) Whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

(2) Whenever $f(a) = f(b)$ in R, then $a = b$ in D.

The arrow diagram in Figure 1 illustrates a one-to-one function. Note that each function value in the range R corresponds to *exactly one* element in the domain D. The function illustrated in Figure 2 of Section 2.4 is not one-to-one, since $f(w) = f(z)$, but $w \neq z$.

Figure 1



EXAMPLE 1 Determining whether a function is one-to-one

(a) If $f(x) = 3x + 2$, prove that f is one-to-one.

(b) If $g(x) = x^2 - 3$, prove that g is not one-to-one.

SOLUTION

(a) We shall use condition 2 of the preceding definition. Thus, suppose that $f(a) = f(b)$ for some numbers a and b in the domain of f. This gives us

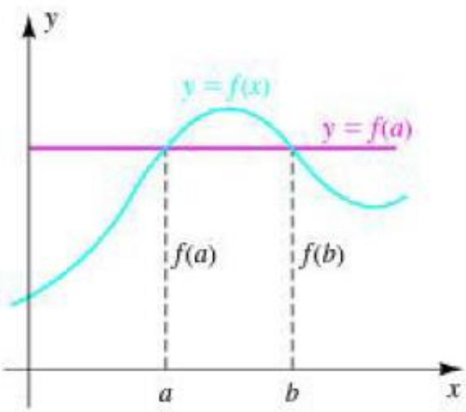
$$\begin{aligned} 3a + 2 &= 3b + 2 && \text{definition of } f(x) \\ 3a &= 3b && \text{subtract 2} \\ a &= b && \text{divide by 3} \end{aligned}$$

Since we have concluded that a must equal b, f is one-to-one.

(b) Showing that a function *is* one-to-one requires a *general* proof, as in part (a). To show that g is *not* one-to-one we need only find two distinct real numbers in the domain that produce the same function value. For example, $-1 \neq 1$, but $g(-1) = g(1)$. In fact, since g is an even function, $g(-a) = g(a)$ for every real number a.

If we know the graph of a function f, it is easy to determine whether f is one-to-one. For example, the function whose graph is sketched in Figure 2 is not one-to-one, since $a \neq b$, but $f(a) = f(b)$. Note that the horizontal line $y = f(a)$ (or $y = f(b)$) intersects the graph in more than one point. In general, we may use the following graphical test to determine whether a function is one-to-one.

Figure 2



Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f in at most one point.

EXAMPLE 2 Using the horizontal line test

Use the horizontal line test to determine if the function is one-to-one.

(a) $f(x) = 3x + 2$

(b) $g(x) = x^2 - 3$

SOLUTION

(a) The graph of $f(x) = 3x + 2$ is a line with y-intercept 2 and slope 3, as shown in Figure 3. We see that any horizontal line intersects the graph of f in at most one point. Thus, f is one-to-one.

Figure 3

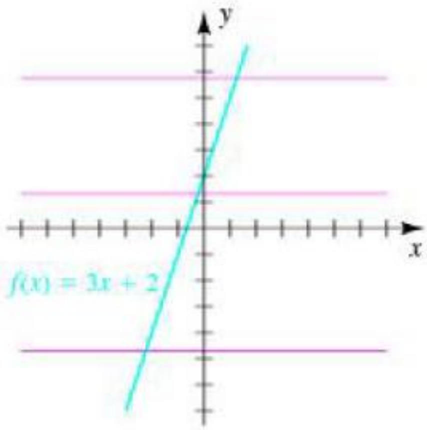
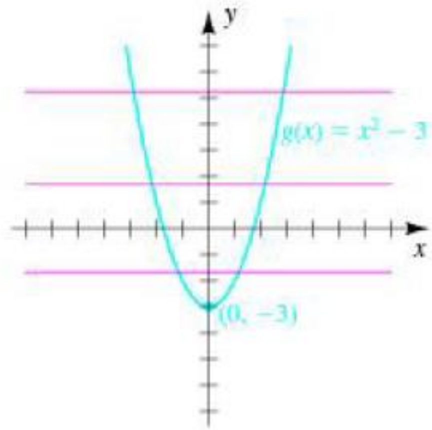


Figure 4



(b) The graph of $g(x) = x^2 - 3$ is a parabola opening upward with vertex $(0, -3)$, as shown in Figure 4. In this case, any horizontal line with equation $y = k$, where $k > -3$, will intersect the graph of g in two points. Thus, g is *not* one-to-one.

Theorem: Increasing or Decreasing Functions Are One-to-One

(1) A function that is increasing throughout its domain is one-to-one.

(2) A function that is decreasing throughout its domain is one-to-one.

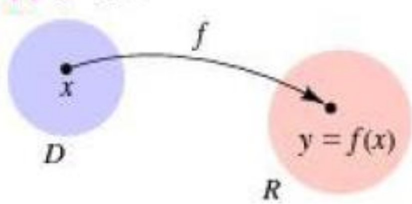
Let f be a one-to-one function with domain D and range R. Thus, for each number y in R, there is *exactly one* number x in D such that $y = f(x)$, as illustrated by the arrow in Figure 5(a). We may, therefore, define a function g from R to D by means of the following rule:

$$x = g(y)$$

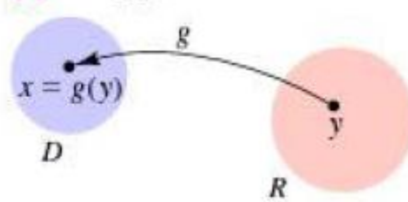
As in Figure 5(b), *g reverses the correspondence given by f*. We call g the *inverse function* of f, as in the next definition.

Figure 5

(a) $y = f(x)$



(b) $x = g(y)$



Definition of Inverse Function

Let f be a one-to-one function with domain D and range R. A function g with domain R and range D is the **inverse function** of f, provided the following condition is true for every x in D and every y in R:

$$y = f(x) \qquad \text{if and only if} \qquad x = g(y)$$

Theorem on Inverse Functions

Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f if and only if both of the following conditions are true:

(1) $g(f(x)) = x$ for every x in D

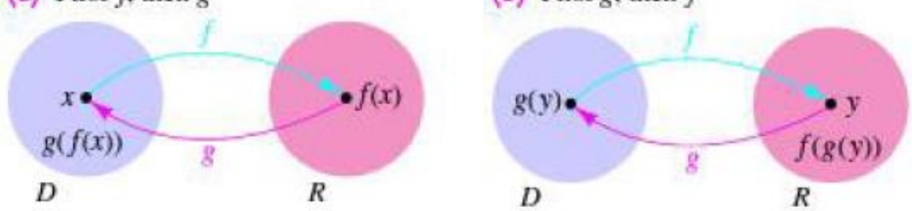
(2) $f(g(y)) = y$ for every y in R

Conditions 1 and 2 of the preceding theorem are illustrated in Figure 6(a) and (b), respectively, where the blue arrow indicates that f is a function from D to R and the red arrow indicates that g is a function from R to D.

Figure 6

(a) First f, then g

(b) First g, then f



Note that in Figure 6(a) we first apply f to the number x in D, obtaining the function value $f(x)$ in R, and then apply g to $f(x)$, obtaining the number $g(f(x))$ in D. Condition 1 of the theorem states that $g(f(x)) = x$ for every x; that is, g *reverses* the correspondence given by f.

In Figure 6(b) we use the opposite order for the functions. We first apply g to the number y in R, obtaining the function value $g(y)$ in D, and then apply f to $g(y)$, obtaining the number $f(g(y))$ in R. Condition 2 of the theorem states that $f(g(y)) = y$ for every y; that is, f *reverses* the correspondence given by g.

If a function f has an inverse function g, we often denote g by f^{-1}. The -1 used in this notation should not be mistaken for an exponent; that is,

$$f^{-1}(y) \textit{ does not mean } 1/[f(y)].$$

The reciprocal $1/[f(y)]$ may be denoted by $[f(y)]^{-1}$. It is important to remember the following facts about the domain and range of f and f^{-1}.

Domain and Range of f and f^{-1}

$$\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}$$

When we discuss functions, we often let x denote an arbitrary number in the domain. Thus, for the inverse function f^{-1}, we may wish to consider $f^{-1}(x)$, *where x is in the domain R of f^{-1}*. In this event, the two conditions in the theorem on inverse functions are written as follows:

(1) $f^{-1}(f(x)) = x$ for every x in the domain of f
(2) $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

Guidelines for Finding f^{-1} in Simple Cases

1 Verify that f is a one-to-one function throughout its domain.

2 Solve the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.

3 Verify the following two conditions:

(a) $f^{-1}(f(x)) = x$ for every x in the domain of f

(b) $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

EXAMPLE 3 Finding the inverse of a function

Let $f(x) = 3x - 5$. Find the inverse function of f.

SOLUTION

Guideline 1 The graph of the linear function f is a line of slope 3, and hence f is increasing throughout $\mathbb{R}$. Thus, f is one-to-one and the inverse function f^{-1} exists. Moreover, since the domain and range of f are $\mathbb{R}$, the same is true for f^{-1}.

Guideline 2 Solve the equation $y = f(x)$ for x:

$$\begin{aligned} y &= 3x - 5 && \text{let } y = f(x) \\ x &= \frac{y+5}{3} && \text{solve for } x \text{ in terms of } y \end{aligned}$$

We now formally let $x = f^{-1}(y)$; that is,

$$f^{-1}(y) = \frac{y+5}{3}.$$

Since the symbol used for the variable is immaterial, we may also write

$$f^{-1}(x) = \frac{x + 5}{3},$$

where x is in the domain of f^{-1}.

Guideline 3 Since the domain and range of both f and f^{-1} are $\mathbb{R}$, we must verify conditions (a) and (b) for every real number x. We proceed as follows:

(a)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x - 5) && \text{definition of } f \\ &= \frac{(3x - 5) + 5}{3} && \text{definition of } f^{-1} \\ &= x && \text{simplify} \end{aligned}$$

(b)
$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{x + 5}{3}\right) && \text{definition of } f^{-1} \\ &= 3\left(\frac{x + 5}{3}\right) - 5 && \text{definition of } f \\ &= x && \text{simplify} \end{aligned}$$

These verifications prove that the inverse function of f is given by

$$f^{-1}(x) = \frac{x + 5}{3}.$$

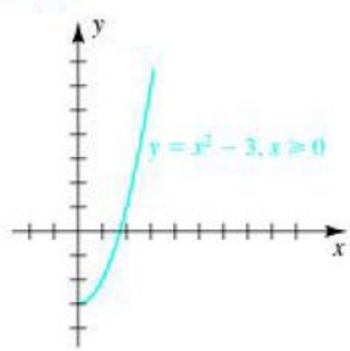
EXAMPLE 4 Finding the inverse of a function

Let $f(x) = x^2 - 3$ for $x \geq 0$. Find the inverse function of f.

SOLUTION

Guideline 1 The graph of f is sketched in Figure 7. The domain of f is $[0, \infty)$, and the range is $[-3, \infty)$. Since f is increasing, it is one-to-one and hence has an inverse function f^{-1} with domain $[-3, \infty)$ and range $[0, \infty)$.

Figure 7



Guideline 2 We consider the equation

$$y = x^2 - 3$$

and solve for x, obtaining

$$x = \pm\sqrt{y + 3}.$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ and let

$$f^{-1}(y) = \sqrt{y+3} \qquad \text{or, equivalently,} \qquad f^{-1}(x) = \sqrt{x+3}.$$

(Note that if the function f had domain $x \le 0$, we would choose the function $f^{-1}(x) = -\sqrt{x+3}$.)

Guideline 3 We verify conditions (a) and (b) for x in the domains of f and f^{-1}, respectively.

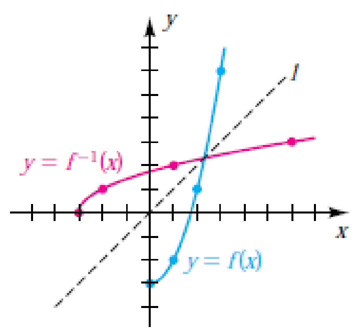
(a) $f^{-1}(f(x)) = f^{-1}(x^2 - 3)$
$= \sqrt{(x^2 - 3) + 3} = \sqrt{x^2} = x$ for $x \ge 0$

(b) $f(f^{-1}(x)) = f(\sqrt{x+3})$
$= (\sqrt{x+3})^2 - 3 = (x + 3) - 3 = x$ for $x \ge -3$

Thus, the inverse function is given by

$$f^{-1}(x) = \sqrt{x+3} \quad \text{for } x \ge -3.$$

FIGURE 9



Note that the graphs of f and f^{-1} intersect on the line $y = x$.

EXAMPLE 5

Let $f(x) = x^3$. Find the inverse function f^{-1} of f, and sketch the graphs of f and f^{-1} on the same coordinate plane.

SOLUTION The graph of f is sketched in Figure 10. Note that f is an odd function, and hence the graph is symmetric with respect to the origin.

Guideline 1 Since f is increasing throughout its domain, $\mathbb{R}$, it is one-to-one and hence has an inverse function f^{-1}.

Guideline 2 We consider the equation

$$y = x^3$$

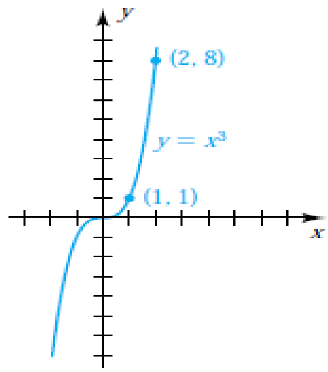
and solve for x by taking the cube root of each side, obtaining

$$x = y^{1/3} = \sqrt[3]{y}.$$

We now let

$$f^{-1}(y) = \sqrt[3]{y} \qquad \text{or, equivalently,} \qquad f^{-1}(x) = \sqrt[3]{x}.$$

FIGURE 10

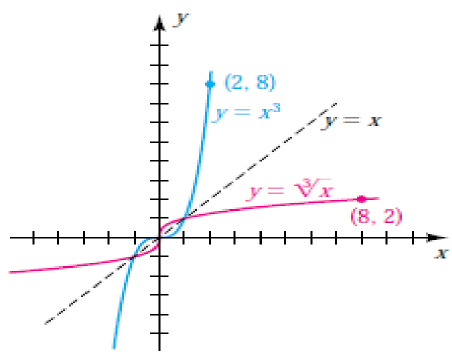


Guideline 3 We verify conditions (a) and (b):

(a) $f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x$ for every x in $\mathbb{R}$

(b) $f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$ for every x in $\mathbb{R}$

FIGURE 11



Exercises

Exer. 1–2: If possible, find
(a) $f^{-1}(5)$ (b) $g^{-1}(6)$

1

x	2	4	6
$f(x)$	3	5	9

x	1	3	5
$g(x)$	6	2	6

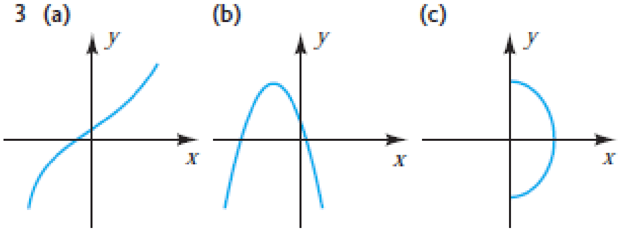
2

t	0	3	5
$f(t)$	2	5	6

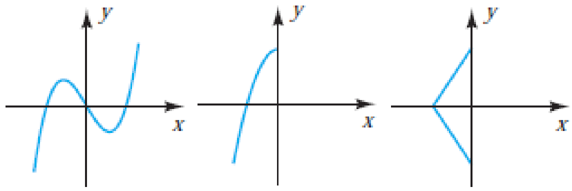
t	1	2	4
$g(t)$	3	6	6

Exer. 3–4: Determine if the graph is a graph of a one-to-one function.

3 (a) (b) (c)



4 (a) (b) (c)



Exer. 5–16: Determine whether the function f is one-to-one.

5 $f(x) = 2x + 5$

6 $f(x) = \dfrac{1}{x - 2}$

7 $f(x) = x^2 - 5$

8 $f(x) = x^2 + 3$

9 $f(x) = \sqrt{x}$

10 $f(x) = \sqrt[3]{x}$

11 $f(x) = |x|$

12 $f(x) = 3$

13 $f(x) = \sqrt{4 - x^2}$

14 $f(x) = 2x^3 - 4$

15 $f(x) = \dfrac{1}{x}$

16 $f(x) = \dfrac{1}{x^2}$

Exer. 19–22: Use the theorem on inverse functions to prove that f and g are inverse functions of each other, and sketch the graphs of f and g on the same coordinate plane.

19 $f(x) = 3x - 2;$ $\quad g(x) = \dfrac{x + 2}{3}$

20 $f(x) = x^2 + 5,\ x \le 0;$ $\quad g(x) = -\sqrt{x - 5},\ x \ge 5$

21 $f(x) = -x^2 + 3,\ x \ge 0;$ $\quad g(x) = \sqrt{3 - x},\ x \le 3$

22 $f(x) = x^3 - 4;$ $\quad g(x) = \sqrt[3]{x + 4}$

Exer. 23–26: Determine the domain and range of f^{-1} for the given function without actually finding f^{-1}. *Hint:* First find the domain and range of f.

23 $f(x) = -\dfrac{2}{x - 1}$

24 $f(x) = \dfrac{5}{x + 3}$

25 $f(x) = \dfrac{4x + 5}{3x - 8}$

26 $f(x) = \dfrac{2x - 7}{9x + 1}$

Exer. 27–48: Find the inverse function of f.

27 $f(x) = 3x + 5$

28 $f(x) = 7 - 2x$

29 $f(x) = \dfrac{3}{2x - 5}$

30 $f(x) = \dfrac{1}{x + 3}$

31 $f(x) = \dfrac{3x + 2}{2x - 5}$

32 $f(x) = \dfrac{4x}{x - 2}$

33 $f(x) = 2 - 3x^2,\ x \le 0$

34 $f(x) = 5x^2 + 2,\ x \ge 0$