Definition of Function

The notion of **correspondence** occurs frequently in everyday life. Some examples are given in the following illustration.

ILLUSTRATION Correspondence

- To each book in a library there corresponds the number of pages in the book.
- To each human being there corresponds a birth date.
- If the temperature of the air is recorded throughout the day, then to each instant of time there corresponds a temperature.

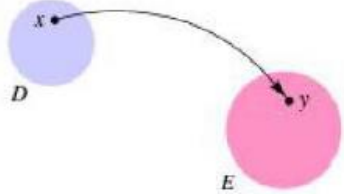
Each correspondence in the previous illustration involves two sets, D and E. In the first illustration, D denotes the set of books in a library and E the set of positive integers. To each book x in D there corresponds a positive integer y in E—namely, the number of pages in the book.

We sometimes depict correspondences by diagrams of the type shown in Figure 1, where the sets D and E are represented by points within regions in a plane. The curved arrow indicates that the element y of E corresponds to the element x of D. The two sets may have elements in common. As a matter of fact, we often have $D = E$. It is important to note that *to each x in D there corresponds exactly one y in E.* However, the same element of E may correspond to different elements of D. For example, two books may have the same number of pages, two people may have the same birthday, and the temperature may be the same at different times.

In most of our work, D and E will be sets of numbers. To illustrate, let both D and E denote the set $\mathbb{R}$ of real numbers, and to each real number x let us assign its square x^2. This gives us a correspondence from $\mathbb{R}$ to $\mathbb{R}$.

Each of our illustrations of a correspondence is a *function,* which we define as follows.

Figure 1



Definition of Function

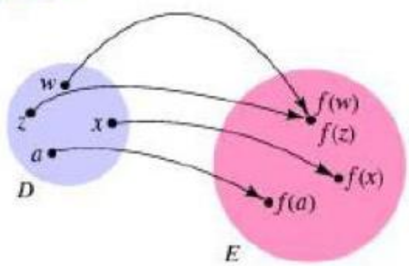
A **function** f from a set D to a set E is a correspondence that assigns to each element x of D exactly one element y of E.

For many cases, we can simply remember that the **domain** is the set of x-values and the **range** is the set of y-values.

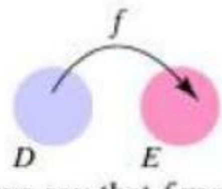
The element x of D is the **argument** of f. The set D is the **domain** of the function. The element y of E is the **value** of f at x (or the **image** of x under f) and is denoted by $f(x)$, read "f of x." The **range** of f is the subset R of E consisting of all possible values $f(x)$ for x in D. Note that there may be elements in the set E that are not in the range R of f.

Consider the diagram in Figure 2. The curved arrows indicate that the elements $f(w)$, $f(z)$, $f(x)$, and $f(a)$ of E correspond to the elements w, z, x, and a of D. *To each element in D there is assigned exactly one function value in E;* however, different elements of D, such as w and z in Figure 2, may have the same value in E.

Figure 2



The symbols

$D \xrightarrow{f} E$, $\quad f: D \to E$, $\quad$ and 

signify that f is a function from D to E, and we say that f **maps** D *into* E. Initially, the notations f and $f(x)$ may be confusing. Remember that f is used to represent the function. It is neither in D nor in E. However, $f(x)$ is an element of the range R—the element that the function f assigns to the element x, which is in the domain D.

Two functions f and g from D to E are **equal,** and we write

$$f = g \text{ provided } \quad f(x) = g(x) \quad \text{for every } x \text{ in } D.$$

For example, if $g(x) = \frac{1}{2}(2x^2 - 6) + 3$ and $f(x) = x^2$ for every x in $\mathbb{R}$, then $g = f$.

EXAMPLE 1 Finding function values

Let f be the function with domain $\mathbb{R}$ such that $f(x) = x^2$ for every x in $\mathbb{R}$.

(a) Find $f(-6)$, $f(\sqrt{3})$, $f(a + b)$, and $f(a) + f(b)$, where a and b are real numbers.

(b) What is the range of f?

SOLUTION

Note that, in general,

$$f(a + b) \neq f(a) + f(b).$$

If a function is defined as in Example 1, the symbols used for the function and variable are immaterial; that is, expressions such as $f(x) = x^2$, $f(s) = s^2$, $g(t) = t^2$, and $k(r) = r^2$ all define the same function. This is true because if a is any number in the domain, then the same value a^2 is obtained regardless of which expression is employed.

In the remainder of our work, the phrase *f is a function* will mean that the domain and range are sets of real numbers. If a function is defined by means of an expression, as in Example 1, and the domain D is not stated, then we will consider D to be the totality of real numbers x such that $f(x)$ is real. This is sometimes called the **implied domain** of f. To illustrate, if $f(x) = \sqrt{x - 2}$, then the implied domain is the set of real numbers x such that $\sqrt{x - 2}$ is real—that is, $x - 2 \geq 0$, or $x \geq 2$. Thus, the domain is the infinite interval $[2, \infty)$. If x is in the domain, we say that *f is defined at x* or that *$f(x)$ exists.* If a set S is contained in the domain, *f is defined on S.* The terminology *f is undefined at x* means that x is not in the domain of f.

EXAMPLE 2 Finding function values

Let $g(x) = \dfrac{\sqrt{4 + x}}{1 - x}$.

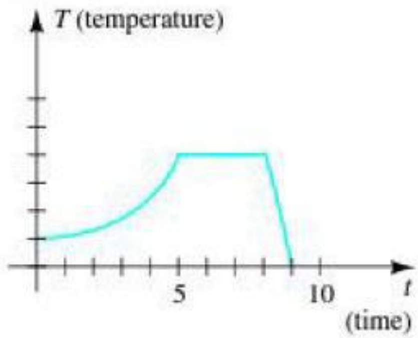
(a) Find the domain of g.

(b) Find $g(5)$, $g(-2)$, $g(-a)$, and $-g(a)$.

SOLUTION

Graphs are often used to describe the variation of physical quantities. For example, a scientist may use the graph in Figure 5 to indicate the temperature T of a certain solution at various times t during an experiment. The sketch shows that the temperature increased gradually for time $t = 0$ to time $t = 5$, did not change between $t = 5$ and $t = 8$, and then decreased rapidly from $t = 8$ to $t = 9$.

Figure 5



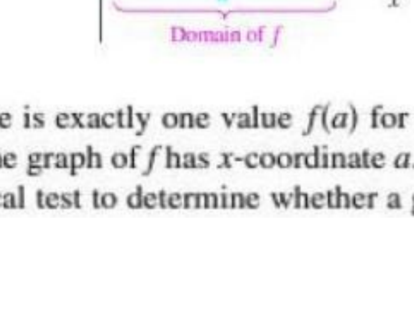
Similarly, if f is a function, we may use a graph to indicate the change in $f(x)$ as x varies through the domain of f. Specifically, we have the following definition.

Definition of Graph of a Function

The **graph of a function** f is the graph of the equation $y = f(x)$ for x in the domain of f.

We often attach the label $y = f(x)$ to a sketch of the graph. If $P(a, b)$ is a point on the graph, then the y-coordinate b is the function value $f(a)$, as illustrated in Figure 6 on the next page. The figure displays the domain of f (the set of possible values of x) and the range of f (the corresponding values of y). Although we have pictured the domain and range as closed intervals, they may be infinite intervals or other sets of real numbers.

Figure 6



Since there is exactly one value $f(a)$ for each a in the domain of f, only *one* point on the graph of f has x-coordinate a. In general, we may use the following graphical test to determine whether a graph is the graph of a function.

Vertical Line Test

> The graph of a set of points in a coordinate plane is the graph of a function if every vertical line intersects the graph in at most one point.

Thus, *every vertical line intersects the graph of a function in at most one point*. Consequently, the graph of a function cannot be a figure such as a circle, in which a vertical line may intersect the graph in more than one point.

The x-intercepts of the graph of a function f are the solutions of the equation $f(x) = 0$. These numbers are called the **zeros** of the function. The y-intercept of the graph is $f(0)$, if it exists.

EXAMPLE 3 Sketching the graph of a function

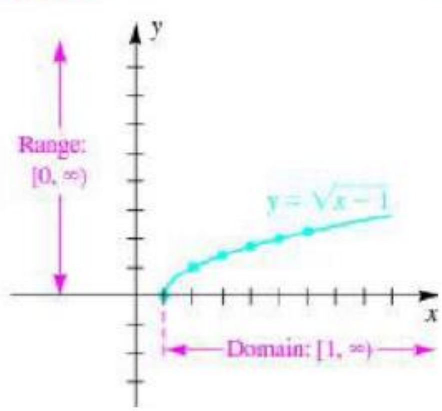
Let $f(x) = \sqrt{x - 1}$.

(a) Sketch the graph of f.

(b) Find the domain and range of f.

SOLUTION

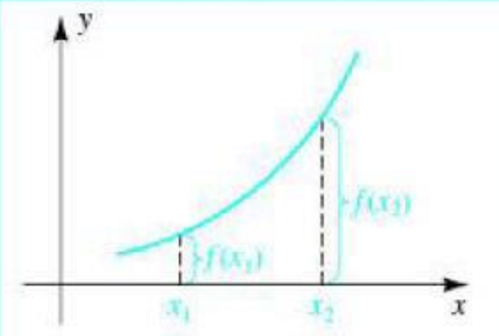
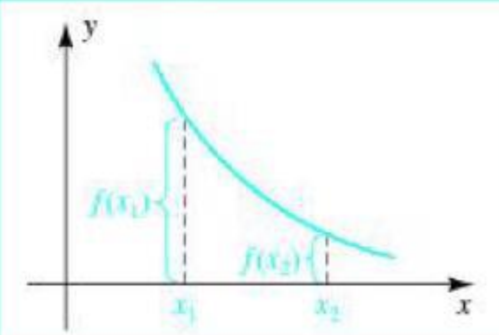
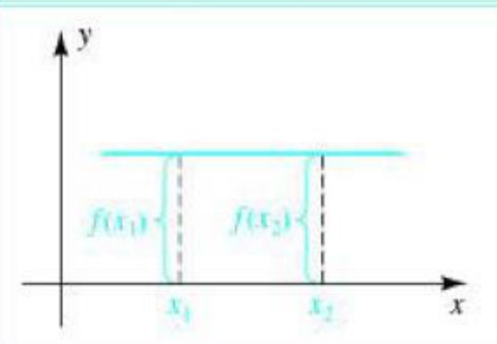
Figure 7



The **square root function,** defined by $f(x) = \sqrt{x}$, has a graph similar to the one in Figure 7, but the endpoint is at (0, 0). The y-value of a point on this graph is the number displayed on a calculator when a square root is requested. This graphical relationship may help you remember that $\sqrt{9}$ is 3 and that $\sqrt{9}$ is *not* ± 3. Similarly, $f(x) = x^2$, $f(x) = x^3$, and $f(x) = \sqrt[3]{x}$ are often referred to as the **squaring function,** the **cubing function,** and the **cube root function,** respectively.

In Example 3, as x increases, the function value $f(x)$ also increases, and we say that the graph of f *rises* (see Figure 7). A function of this type is said to be *increasing*. For certain functions, $f(x)$ decreases as x increases. In this case the graph *falls,* and f is a *decreasing* function. In general, we shall consider functions that increase or decrease on an interval I, as described in the following chart, where x_1 and x_2 denote numbers in I.

Increasing, Decreasing, and Constant Functions

Terminology	Definition	Graphical interpretation
f is **increasing** on an interval I	$f(x_1) < f(x_2)$ whenever $x_1 < x_2$	
f is **decreasing** on an interval I	$f(x_1) > f(x_2)$ whenever $x_1 < x_2$	
f is **constant** on an interval I	$f(x_1) = f(x_2)$ for every x_1 and x_2	

An example of an *increasing function* is the **identity function,** whose equation is $f(x) = x$ and whose graph is the line through the origin with slope 1. An example of a *decreasing function* is $f(x) = -x$, an equation of the line through the origin with slope -1. If $f(x) = c$ for every real number x, then f is called a *constant function.*

EXAMPLE 4 **Using a graph to find domain, range, and where a function increases or decreases**

Let $f(x) = \sqrt{9 - x^2}$.

(a) Sketch the graph of f.

(b) Find the domain and range of f.

(c) Find the intervals on which f is increasing or is decreasing.

SOLUTION

Definition of Linear Function

A function f is a **linear function** if

$$f(x) = ax + b,$$

where x is any real number and a and b are constants.

The graph of f in the preceding definition is the graph of $y = ax + b$, which, by the slope-intercept form, is a line with slope a and y-intercept b.

Thus, *the graph of a linear function is a line.* Since $f(x)$ exists for every x, the domain of f is $\mathbb{R}$. As illustrated in the next example, if $a \neq 0$, then the range of f is also $\mathbb{R}$.

EXAMPLE 6 **Sketching the graph of a linear function**

Let $f(x) = 2x + 3$.

(a) Sketch the graph of f.

(b) Find the domain and range of f.

(c) Determine where f is increasing or is decreasing.

SOLUTION

EXAMPLE 7 **Finding a linear function**

If f is a linear function such that $f(-2) = 5$ and $f(6) = 3$, find $f(x)$, where x is any real number.

SOLUTION

Many formulas that occur in mathematics and the sciences determine functions. For instance, the formula $A = \pi r^2$ for the area A of a circle of radius r assigns to each positive real number r exactly one value of A. This determines a function f such that $f(r) = \pi r^2$, and we may write $A = f(r)$. The letter r, which represents an arbitrary number from the domain of f, is called an **independent variable.** The letter A, which represents a number from the range of f, is a **dependent variable,** since its value depends on the number assigned to r. If two variables r and A are related in this manner, we say that *A is a function of r*. In applications, the independent variable and dependent variable are sometimes referred to as the **input variable** and **output variable,** respectively. As another example, if an automobile travels at a uniform rate of 50 mi/hr, then the distance d (miles) traveled in time t (hours) is given by $d = 50t$, and hence *the distance d is a function of time t.*

Exercises

1 If $f(x) = -x^2 - x - 4$, find $f(-2)$, $f(0)$, and $f(4)$.

2 If $f(x) = -x^3 - x^2 + 3$, find $f(-3)$, $f(0)$, and $f(2)$.

3 If $f(x) = \sqrt{x-4} - 3x$, find $f(4)$, $f(8)$, and $f(13)$.

4 If $f(x) = \dfrac{x}{x-3}$, find $f(-2)$, $f(0)$, and $f(3)$.

Exer. 5–10: If a and h are real numbers, find

(a) $f(a)$ **(b)** $f(-a)$ **(c)** $-f(a)$ **(d)** $f(a+h)$

(e) $f(a) + f(h)$ **(f)** $\dfrac{f(a+h) - f(a)}{h}$, **if** $h \neq 0$

5 $f(x) = 5x - 2$

6 $f(x) = 3 - 4x$

7 $f(x) = -x^2 + 4$

8 $f(x) = 3 - x^2$

9 $f(x) = x^2 - x + 3$

10 $f(x) = 2x^2 + 3x - 7$

Exer. 11–14: If a is a positive real number, find

(a) $g\left(\dfrac{1}{a}\right)$ **(b)** $\dfrac{1}{g(a)}$ **(c)** $g\left(\sqrt{a}\right)$ **(d)** $\sqrt{g(a)}$

11 $g(x) = 4x^2$

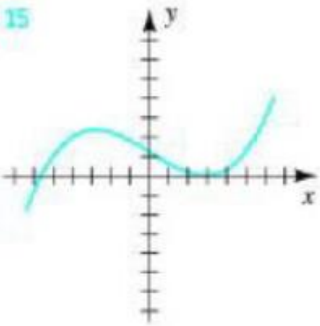
12 $g(x) = 2x - 5$

13 $g(x) = \dfrac{2x}{x^2 + 1}$

14 $g(x) = \dfrac{x^2}{x + 1}$

Exer. 15–16: Explain why the graph is or is not the graph of a function.

15



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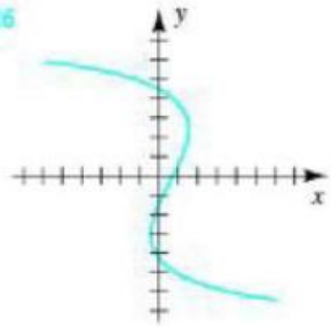
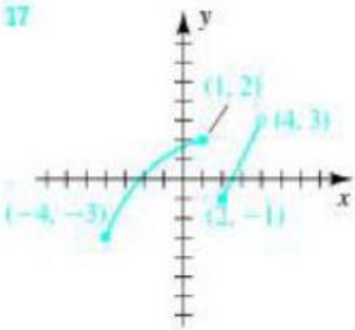


Exer. 17–18: Determine the domain D and range R of the function shown in the figure.

17



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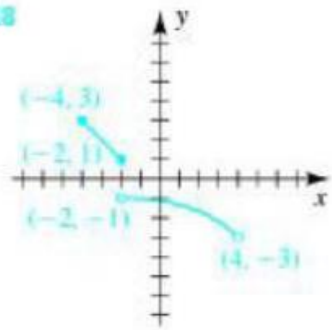


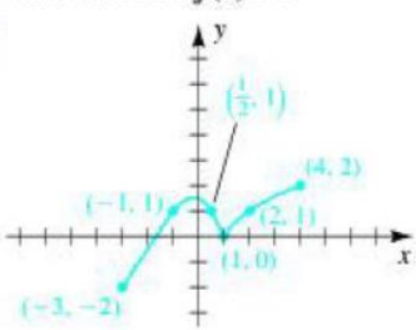
Exer. 19–20: For the graph of the function f sketched in the figure, determine

(a) the domain **(b)** the range **(c)** $f(1)$

(d) all x such that $f(x) = 1$

(e) all x such that $f(x) > 1$

19



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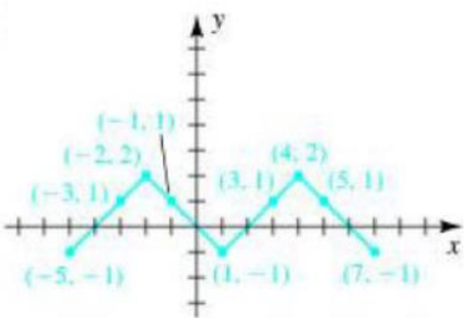

Exer. 21–32: Find the domain of *f*.

21 $f(x) = \sqrt{2x + 7}$

22 $f(x) = \sqrt{8 - 3x}$

23 $f(x) = \sqrt{9 - x^2}$

24 $f(x) = \sqrt{x^2 - 25}$

25 $f(x) = \dfrac{x + 1}{x^3 - 4x}$

26 $f(x) = \dfrac{4x}{6x^2 + 13x - 5}$

27 $f(x) = \dfrac{\sqrt{2x - 3}}{x^2 - 5x + 4}$

28 $f(x) = \dfrac{\sqrt{4x - 3}}{x^2 - 4}$

29 $f(x) = \dfrac{x - 4}{\sqrt{x - 2}}$

30 $f(x) = \dfrac{1}{(x - 3)\sqrt{x + 3}}$

31 $f(x) = \sqrt{x + 2} + \sqrt{2 - x}$

32 $f(x) = \sqrt{(x - 2)(x - 6)}$

Exer. 33–34: (a) Find the domain *D* and range *R* of *f*. (b) Find the intervals on which *f* is increasing, is decreasing, or is constant.

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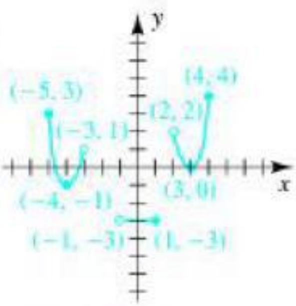

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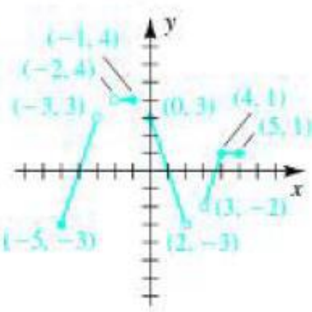

35 Sketch the graph of a function that is increasing on $(-\infty, -3]$ and $[2, \infty)$ and is decreasing on $[-3, 2]$.

36 Sketch the graph of a function that is decreasing on $(-\infty, -2]$ and $[1, 4]$ and is increasing on $[-2, 1]$ and $[4, \infty)$.

Exer. 37–46: (a) Sketch the graph of f. (b) Find the domain D and range R of f. (c) Find the intervals on which f is increasing, is decreasing, or is constant.

37 $f(x) = 3x - 2$

38 $f(x) = -2x + 3$

39 $f(x) = 4 - x^2$

40 $f(x) = x^2 - 1$

41 $f(x) = \sqrt{x + 4}$

42 $f(x) = \sqrt{4 - x}$

43 $f(x) = -2$

44 $f(x) = 3$

45 $f(x) = -\sqrt{36 - x^2}$

46 $f(x) = \sqrt{16 - x^2}$

Exer. 53–54: If a linear function f satisfies the given conditions, find $f(x)$.

53 $f(-3) = 1$ and $f(3) = 2$

54 $f(-2) = 7$ and $f(4) = -2$