Graphs of Functions

Even and Odd Functions

Terminology	Definition	Illustration	Type of symmetry of graph
f is an *even* **function.**	$f(-x) = f(x)$ for every x in the domain.	$y = f(x) = x^2$	with respect to the y-axis
f is an *odd* **function.**	$f(-x) = -f(x)$ for every x in the domain.	$y = f(x) = x^3$	with respect to the origin

EXAMPLE 1 Determining whether a function is even or odd

Determine whether f is even, odd, or neither even nor odd.

(a) $f(x) = 3x^4 - 2x^2 + 5$ **(b)** $f(x) = 2x^5 - 7x^3 + 4x$

(c) $f(x) = x^3 + x^2$

SOLUTION In each case the domain of f is $\mathbb{R}$. To determine whether f is even or odd, we begin by examining $f(-x)$, where x is any real number.

(a)
$$\begin{aligned} f(-x) &= 3(-x)^4 - 2(-x)^2 + 5 && \text{substitute } -x \text{ for } x \text{ in } f(x) \\ &= 3x^4 - 2x^2 + 5 && \text{simplify} \\ &= f(x) && \text{definition of } f \end{aligned}$$

Since $f(-x) = f(x)$, f is an even function.

(b)
$$\begin{aligned} f(-x) &= 2(-x)^5 - 7(-x)^3 + 4(-x) && \text{substitute } -x \text{ for } x \text{ in } f(x) \\ &= -2x^5 + 7x^3 - 4x && \text{simplify} \\ &= -(2x^5 - 7x^3 + 4x) && \text{factor out } -1 \\ &= -f(x) && \text{definition of } f \end{aligned}$$

Since $f(-x) = -f(x)$, f is an odd function.

(c)
$$\begin{aligned} f(-x) &= (-x)^3 + (-x)^2 && \text{substitute } -x \text{ for } x \text{ in } f(x) \\ &= -x^3 + x^2 && \text{simplify} \end{aligned}$$

Since $f(-x) \neq f(x)$, and $f(-x) \neq -f(x)$ (note that $-f(x) = -x^3 - x^2$), the function f is neither even nor odd.

EXAMPLE 2 Sketching the graph of the absolute value function

Let $f(x) = |x|$.

(a) Determine whether f is even or odd.

(b) Sketch the graph of f.

(c) Find the intervals on which f is increasing or is decreasing.

SOLUTION

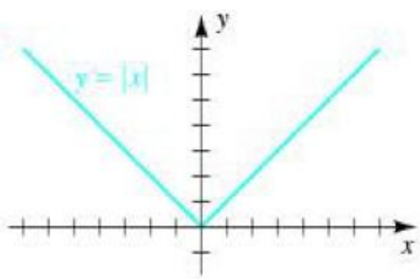
(a) The domain of f is $\mathbb{R}$, because the absolute value of x exists for every real number x. If x is in $\mathbb{R}$, then

$$f(-x) = |-x| = |x| = f(x).$$

Thus, f is an even function, since $f(-x) = f(x)$.

(b) Since f is even, its graph is symmetric with respect to the y-axis. If $x \geq 0$, then $|x| = x$, and therefore the first quadrant part of the graph coincides with the line $y = x$. Sketching this half-line and using symmetry gives us Figure 1.

Figure 1



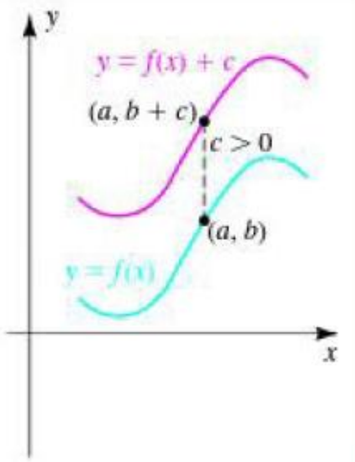
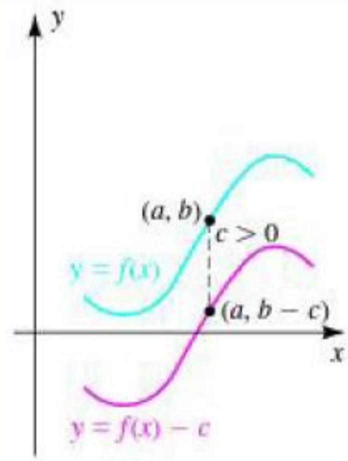
(c) Referring to the graph, we see that f is decreasing on $(-\infty, 0]$ and is increasing on $[0, \infty)$.

If we know the graph of $y = f(x)$, it is easy to sketch the graphs of

$$y = f(x) + c \qquad \text{and} \qquad y = f(x) - c$$

for any positive real number c. As in the next chart, for $y = f(x) + c$, we add c to the y-coordinate of each point on the graph of $y = f(x)$. This *shifts* the graph of f *upward* a distance c. For $y = f(x) - c$ with $c > 0$, we subtract c from each y-coordinate, thereby shifting the graph of f a distance c *downward*. These are called **vertical shifts** of graphs.

Vertically Shifting the Graph of $y = f(x)$

Equation	$y = f(x) + c$ with $c > 0$	$y = f(x) - c$ with $c > 0$
Effect on graph	The graph of f is shifted vertically upward a distance c.	The graph of f is shifted vertically downward a distance c.
Graphical interpretation		

EXAMPLE 3 Vertically shifting a graph

Sketch the graph of f:

(a) $f(x) = x^2$ **(b)** $f(x) = x^2 + 4$ **(c)** $f(x) = x^2 - 4$

SOLUTION We shall sketch all graphs on the same coordinate plane.

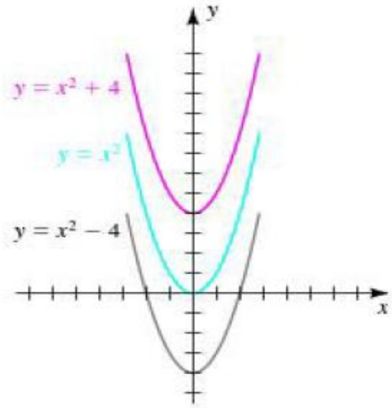
(a) Since

$$f(-x) = (-x)^2 = x^2 = f(x),$$

the function f is even, and hence its graph is symmetric with respect to the y-axis. Several points on the graph of $y = x^2$ are $(0, 0)$, $(1, 1)$, $(2, 4)$, and $(3, 9)$. Drawing a smooth curve through these points and reflecting through the y-axis gives us the sketch in Figure 2. The graph is a parabola with vertex at the origin and opening upward.

(b) To sketch the graph of $y = x^2 + 4$, we add 4 to the y-coordinate of each point on the graph of $y = x^2$; that is, we shift the graph in part (a) upward 4 units, as shown in the figure.

(c) To sketch the graph of $y = x^2 - 4$, we decrease the y-coordinates of $y = x^2$ by 4; that is, we shift the graph in part (a) downward 4 units.

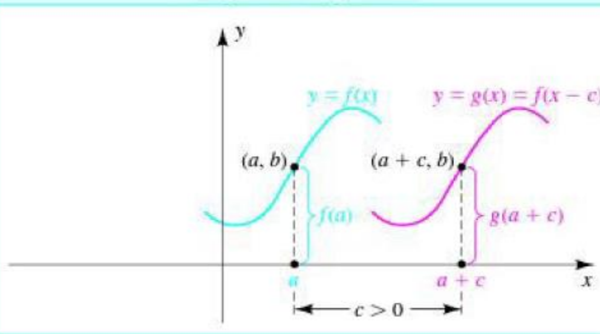
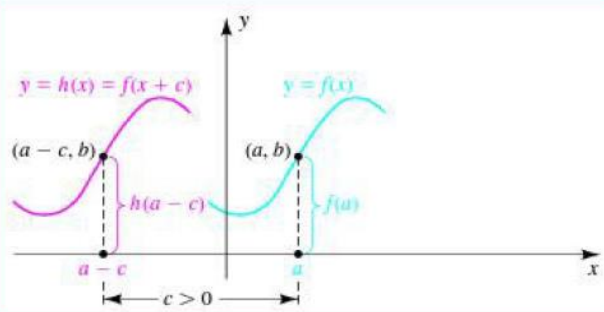


We can also consider **horizontal shifts** of graphs. Specifically, if $c > 0$, consider the graphs of $y = f(x)$ and $y = g(x) = f(x - c)$ sketched on the same coordinate plane, as illustrated in the next chart. Since

$$g(a + c) = f([a + c] - c) = f(a),$$

we see that the point with x-coordinate a on the graph of $y = f(x)$ has the same y-coordinate as the point with x-coordinate $a + c$ on the graph of $y = g(x) = f(x - c)$. This implies that the graph of $y = g(x) = f(x - c)$ can be obtained by shifting the graph of $y = f(x)$ *to the right* a distance c. Similarly, the graph of $y = h(x) = f(x + c)$ can be obtained by shifting the graph of f *to the left* a distance c, as shown in the chart.

Horizontally Shifting the Graph of $y = f(x)$

Equation	Effect on graph	Graphical interpretation
$y = g(x)$ $= f(x - c)$ with $c > 0$	The graph of f is shifted horizontally to the *right* a distance c.	
$y = h(x)$ $= f(x + c)$ with $c > 0$	The graph of f is shifted horizontally to the *left* a distance c.	

EXAMPLE 4 Horizontally shifting a graph

Sketch the graph of f:

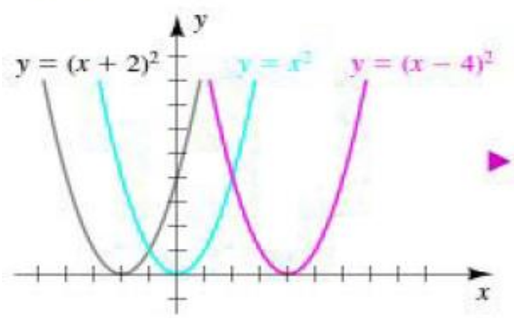
(a) $f(x) = (x - 4)^2$ **(b)** $f(x) = (x + 2)^2$

SOLUTION The graph of $y = x^2$ is sketched in Figure 3.

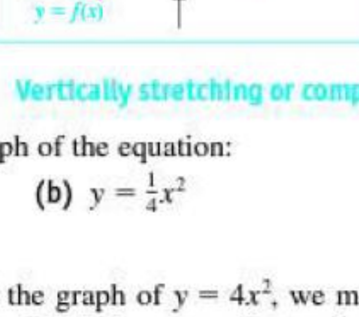
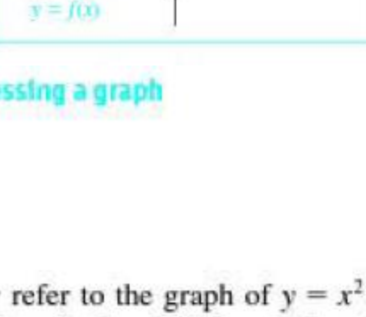
(a) Shifting the graph of $y = x^2$ to the right 4 units gives us the graph of $y = (x - 4)^2$, shown in the figure.

(b) Shifting the graph of $y = x^2$ to the left 2 units leads to the graph of $y = (x + 2)^2$, shown in the figure.

Figure 3



Vertically Stretching or Compressing the Graph of $y = f(x)$

Equation	$y = cf(x)$ with $c > 1$	$y = cf(x)$ with $0 < c < 1$
Effect on graph	The graph of f is stretched vertically by a factor c.	The graph of f is compressed vertically by a factor $1/c$.
Graphical interpretation		

EXAMPLE 5 Vertically stretching or compressing a graph

Sketch the graph of the equation:

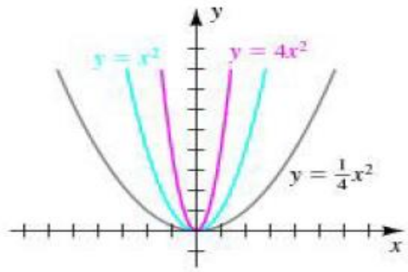
(a) $y = 4x^2$ **(b)** $y = \frac{1}{4}x^2$

SOLUTION

(a) To sketch the graph of $y = 4x^2$, we may refer to the graph of $y = x^2$ in Figure 4 and multiply the y-coordinate of each point by 4. This stretches the graph of $y = x^2$ vertically by a factor 4 and gives us a narrower parabola that is sharper at the vertex, as illustrated in the figure.

(b) The graph of $y = \frac{1}{4}x^2$ may be sketched by multiplying the y-coordinates of points on the graph of $y = x^2$ by $\frac{1}{4}$. This compresses the graph of $y = x^2$ vertically by a factor $1/\frac{1}{4} = 4$ and gives us a wider parabola that is flatter at the vertex, as shown in Figure 4.

Figure 4



Replacing y with $-y$ reflects the graph of $y = f(x)$ through the x-axis.

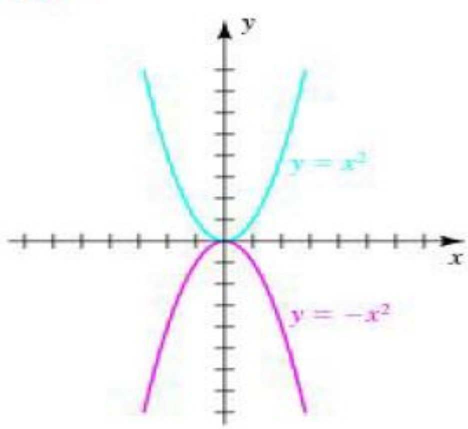
We may obtain the graph of $y = -f(x)$ by multiplying the y-coordinate of each point on the graph of $y = f(x)$ by -1. Thus, every point (a, b) on the graph of $y = f(x)$ that lies above the x-axis determines a point $(a, -b)$ on the graph of $y = -f(x)$ that lies below the x-axis. Similarly, if (c, d) lies below the x-axis (that is, $d < 0$), then $(c, -d)$ lies above the x-axis. The graph of $y = -f(x)$ is a **reflection** of the graph of $y = f(x)$ through the x-axis.

EXAMPLE 6 **Reflecting a graph through the x-axis**

Sketch the graph of $y = -x^2$.

SOLUTION The graph may be found by plotting points; however, since the graph of $y = x^2$ is familiar to us, we sketch it as in Figure 5 and then multiply the y-coordinates of points by -1. This procedure gives us the reflection through the x-axis indicated in the figure.

Figure 5



Functions are sometimes described by more than one expression, as in the next examples. We call such functions **piecewise-defined functions.**

EXAMPLE 8 Sketching the graph of a piecewise-defined function

Sketch the graph of the function f if

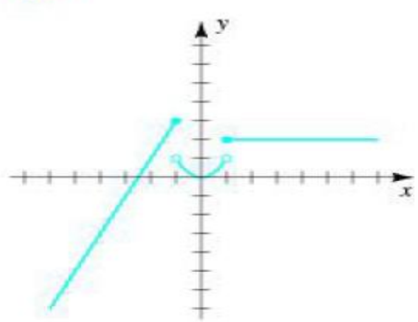
$$f(x) = \begin{cases} 2x + 5 & \text{if } x \le -1 \\ x^2 & \text{if } |x| < 1 \\ 2 & \text{if } x \ge 1 \end{cases}$$

SOLUTION If $x \le -1$, then $f(x) = 2x + 5$ and the graph of f coincides with the line $y = 2x + 5$ and is represented by the portion of the graph to the left of the line $x = -1$ in Figure 7. The small dot indicates that the point $(-1, 3)$ is on the graph.

If $|x| < 1$ (or, equivalently, $-1 < x < 1$), we use x^2 to find values of f, and therefore this part of the graph of f coincides with the parabola $y = x^2$, as indicated in the figure. Note that the points $(-1, 1)$ and $(1, 1)$ are *not* on the graph.

Finally, if $x \ge 1$, the values of f are always 2. Thus, the graph of f for $x \ge 1$ is the horizontal half-line in Figure 7.

Figure 7



Exercises

Exer. 1–2: Suppose f is an even function and g is an odd function. Complete the table, if possible.

1

x	-2	2
$f(x)$		7
$g(x)$		-6

2

x	-3	3
$f(x)$		-5
$g(x)$		15

Exer. 3–12: Determine whether f is even, odd, or neither even nor odd.

3 $f(x) = 5x^3 + 2x$

▶ 4 $f(x) = |x| - 3$

5 $f(x) = 3x^4 + 2x^2 - 5$

6 $f(x) = 7x^5 - 4x^3$

7 $f(x) = 8x^3 - 3x^2$

8 $f(x) = 12$

9 $f(x) = \sqrt{x^2 + 4}$

10 $f(x) = 3x^2 - 5x + 1$

11 $f(x) = \sqrt[3]{x^3 - x}$

12 $f(x) = x^3 - \dfrac{1}{x}$

Exer. 13–26: Sketch, on the same coordinate plane, the graphs of f for the given values of c. (Make use of symmetry, shifting, stretching, compressing, or reflecting.)

13 $f(x) = |x| + c$; $\quad c = -3, 1, 3$

14 $f(x) = |x - c|$; $\quad c = -3, 1, 3$

15 $f(x) = -x^2 + c$; $\quad c = -4, 2, 4$

16 $f(x) = 2x^2 - c$; $\quad c = -4, 2, 4$

17 $f(x) = 2\sqrt{x} + c$; $\quad c = -3, 0, 2$

18 $f(x) = \sqrt{9 - x^2} + c$; $\quad c = -3, 0, 2$

19 $f(x) = \frac{1}{2}\sqrt{x - c}$; $\quad c = -2, 0, 3$

Exer. 27–32: If the point P is on the graph of a function f, find the corresponding point on the graph of the given function.

27 $P(0, 5)$; $\quad y = f(x + 2) - 1$

28 $P(3, -1)$; $\quad y = 2f(x) + 4$

29 $P(3, -2)$; $\quad y = 2f(x - 4) + 1$

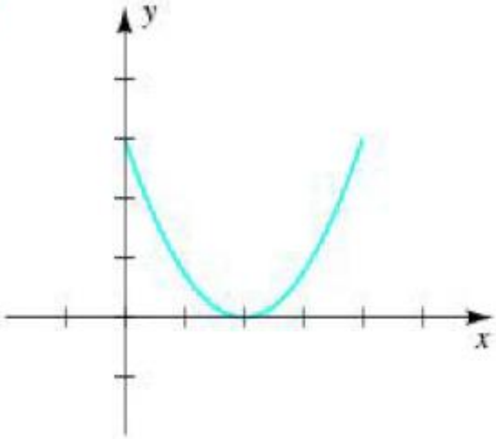
30 $P(-2, 4)$; $\quad y = \frac{1}{2}f(x - 3) + 3$

31 $P(3, 9)$; $\quad y = \frac{1}{3}f\left(\frac{1}{2}x\right) - 1$

32 $P(-2, 1)$; $\quad y = -3f(2x) - 5$

Exer. 41–42: The graph of a function f with domain [0, 4] is shown in the figure. Sketch the graph of the given equation.

41



(a) $y = f(x + 3)$

(b) $y = f(x - 3)$

(c) $y = f(x) + 3$

(d) $y = f(x) - 3$

(e) $y = -3f(x)$

(f) $y = -\frac{1}{3}f(x)$

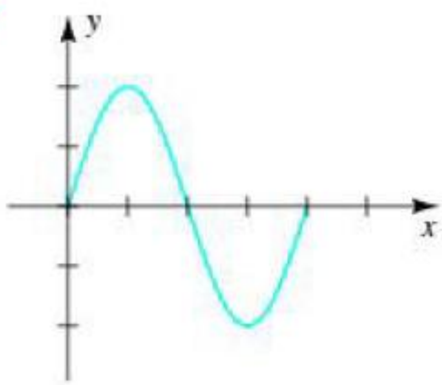
(g) $y = f\left(-\frac{1}{2}x\right)$

(h) $y = f(2x)$

(i) $y = -f(x + 2) - 3$

(j) $y = f(x - 2) + 3$

42



(a) $y = f(x - 2)$

(b) $y = f(x + 2)$

(c) $y = f(x) - 2$

(d) $y = f(x) + 2$

(e) $y = -2f(x)$

(f) $y = -\frac{1}{2}f(x)$

Exer. 47–52: Sketch the graph of f.

47 $f(x) = \begin{cases} 3 & \text{if } x \le -1 \\ -2 & \text{if } x > -1 \end{cases}$

48 $f(x) = \begin{cases} -1 & \text{if } x \text{ is an integer} \\ -2 & \text{if } x \text{ is not an integer} \end{cases}$

49 $f(x) = \begin{cases} 3 & \text{if } x < -2 \\ -x + 1 & \text{if } |x| \leq 2 \\ -3 & \text{if } x > 2 \end{cases}$

50 $f(x) = \begin{cases} -2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ -2 & \text{if } x \geq 1 \end{cases}$

51 $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^3 & \text{if } |x| < 1 \\ -x + 3 & \text{if } x \geq 1 \end{cases}$

52 $f(x) = \begin{cases} x - 3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x + 4 & \text{if } x \geq 1 \end{cases}$