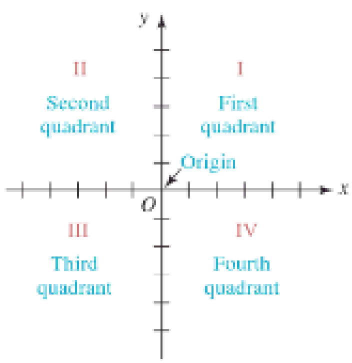
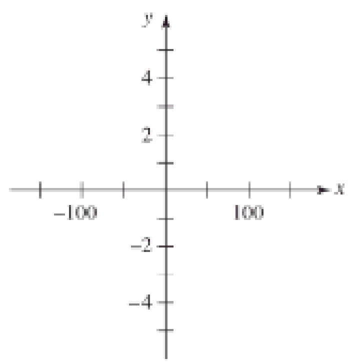
1.3 The Rectangular Coordinate System

Coordinate Plane A **rectangular coordinate system** is formed by two perpendicular number lines that intersect at the point corresponding to the number 0 on each line. This point of intersection is called the **origin** and is denoted by the symbol O. The horizontal and vertical number lines are called the ***x*-axis** and the ***y*-axis**, respectively. These axes divide the plane into four regions, called **quadrants**, which are numbered as shown in FIGURE 1.3.1(a). As we can see in FIGURE 1.3.1(b), the scales on the x- and y-axes need not be the same. Throughout this text, if tick marks are *not* labeled on the coordinates axes, as in Figure 1.3.1(a), then you may assume that one tick corresponds to one unit. A plane containing a rectangular coordinate system is called an ***xy*-plane**, a **coordinate plane**, or simply **2-space**.



(a) Four quadrants



(b) Different scales on x- and y-axes

FIGURE 1.3.1 Coordinate plane

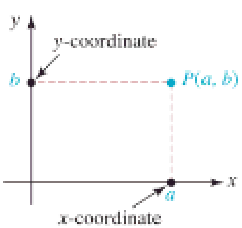


FIGURE 1.3.2 Point with coordinates (a, b)

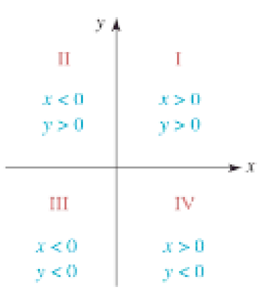


FIGURE 1.3.3 Algebraic signs of coordinates in the four quadrants

The rectangular coordinate system and the xy-plane are also called the **Cartesian coordinate system** and the **Cartesian plane** after the famous French mathematician and philosopher **René Descartes** (1596–1650).

Coordinates of a Point Let P represent a point in the coordinate plane. We associate an ordered pair of real numbers with P by drawing a vertical line from P to the x-axis and a horizontal line from P to the y-axis. If the vertical line intersects the x-axis at the number a and the horizontal line intersects the y-axis at the number b, we associate the **ordered pair** of real numbers (a, b) with the point. Conversely, to each ordered pair (a, b) of real numbers there corresponds a point P in the plane. This point lies at the intersection of the vertical line through a on the x-axis and the horizontal line passing through b on the y-axis. Hereafter we will refer to an ordered pair as a **point** and denote it by either $P(a, b)$ or (a, b).* The number a is the ***x*-coordinate** of the point and the number b is the ***y*-coordinate** of the point and we say that P has **coordinates** (a, b). For example, the coordinates of the origin are $(0, 0)$. See FIGURE 1.3.2.

The algebraic signs of the x-coordinate and the y-coordinate of any point (x, y) in each of the four quadrants are indicated in FIGURE 1.3.3. Points on either of the two axes are not considered to be in any quadrant. Because a point on the x-axis has the form $(x, 0)$, an equation that describes the x-axis is $y = 0$. Similarly, a point on the y-axis has the form $(0, y)$ and so an equation of the y-axis is $x = 0$. When we locate a point in the coordinate plane corresponding to an ordered pair of numbers and represent it using a solid dot, we say that we **plot** or **graph** the point.

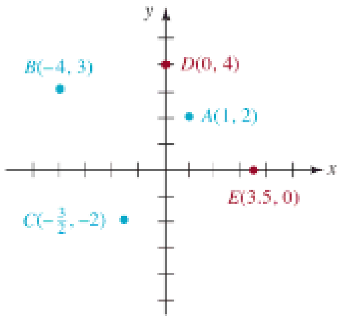


FIGURE 1.3.4 Plots of five points in Example 1

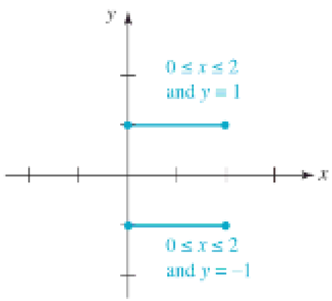


FIGURE 1.3.5 Set of points in Example 2

EXAMPLE 1 Plotting Points

Plot the points $A(1, 2)$, $B(-4, 3)$, $C\left(-\frac{3}{2}, -2\right)$, $D(0, 4)$, and $E(3.5, 0)$. Specify the quadrant in which each point lies.

Solution The five points are plotted in the coordinate plane in FIGURE 1.3.4. Point A lies in the first quadrant (quadrant I), B in the second quadrant (quadrant II), and C is in the third quadrant (quadrant III). Points D and E, which lie on the y- and x-axes, respectively, are not in any quadrant. ≡

EXAMPLE 2 Plotting Points

Sketch the set of points (x, y) in the xy-plane whose coordinates satisfy both $0 \leq x \leq 2$ and $|y| = 1$.

Solution First, recall that the absolute-value equation $|y| = 1$ implies that $y = -1$ or $y = 1$. Thus the points that satisfy the given conditions are the points whose coordinates (x, y) *simultaneously* satisfy the conditions: each x-coordinate is a number in the closed interval $[0, 2]$ and each y-coordinate is either $y = -1$ or $y = 1$. For example, $(1, 1)$, $\left(\frac{1}{2}, -1\right)$, $(2, -1)$ are a few of the points that satisfy the two conditions. Graphically, the set of all points satisfying the two conditions are points on the two parallel line segments shown in FIGURE 1.3.5. ≡

EXAMPLE 3 Regions Defined by Inequalities

Sketch the set of points (x, y) in the xy-plane whose coordinates satisfy each of the following conditions. **(a)** $xy < 0$ **(b)** $|y| \geq 2$

Solution (a) From (*ii*) of the sign properties of products in Section 1.1, we know that a product of two real numbers x and y is negative when one of the numbers is positive and the other is negative. Thus, $xy < 0$ when $x > 0$ and $y < 0$ *or* when $x < 0$ and $y > 0$. We see from Figure 1.3.3 that $xy < 0$ for all points (x, y) in the second and fourth quadrants. Hence we can represent the set of points for which $xy < 0$ by the shaded regions in FIGURE 1.3.6. The coordinate axes are shown as dashed lines to indicate that the points on these axes are not included in the solution set.

(b) In Section 1.2 we saw that $|y| \geq 2$ means that either $y \geq 2$ or $y \leq -2$. Since x is not restricted in any way it can be any real number, and so the points (x, y) for which

$$y \geq 2 \quad \text{and} \quad -\infty < x < \infty \qquad \text{or} \qquad y \leq -2 \quad \text{and} \quad -\infty < x < \infty$$

can be represented by the two shaded regions in FIGURE 1.3.7. We use solid lines to represent the boundaries $y = -2$ and $y = 2$ of the region to indicate that the points on these boundaries are included in the solution set.

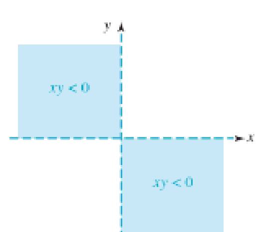


FIGURE 1.3.6 Region in the xy-plane satisfying the condition in (a) of Example 3

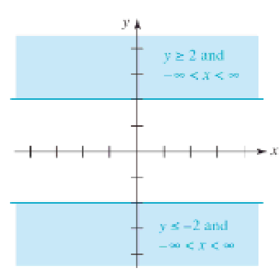


FIGURE 1.3.7 Region in the xy-plane satisfying the condition in (b) of Example 3

Distance Formula Suppose $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two distinct points in the xy-plane that are not on a vertical line or on a horizontal line. As a consequence, P_1, P_2, and $P_3(x_1, y_2)$ are vertices of a right triangle, as shown in FIGURE 1.3.8. The length of the side P_3P_2 is $|x_2 - x_1|$ and the length of the side P_1P_3 is $|y_2 - y_1|$. If we denote the length of P_1P_2 by d, then

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 \tag{1}$$

by the Pythagorean theorem. Since the square of any real number is equal to the square of its absolute value, we can replace the absolute value signs in (1) with parentheses. The distance formula given next follows immediately from (1).

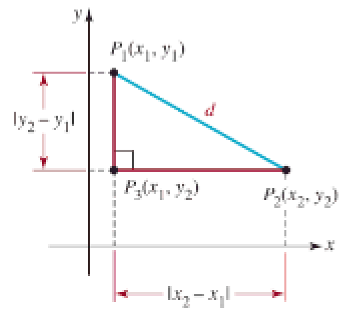


FIGURE 1.3.8 Distance between points P_1 and P_2

THEOREM 1.3.1 Distance Formula

The **distance** between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the xy-plane is given by

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{2}$$

Although we derived this equation for two points not on a vertical or horizontal line, (2) holds in these cases as well. Also, because $(x_2 - x_1)^2 = (x_1 - x_2)^2$, it makes no difference which point is used first in the distance formula, that is, $d(P_1, P_2) = d(P_2, P_1)$.

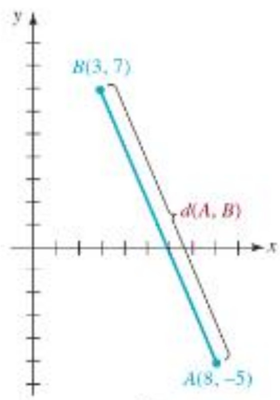


FIGURE 1.3.9 Distance between two points in Example 4

EXAMPLE 4 Distance Between Two Points

Find the distance between the points $A(8, -5)$ and $B(3, 7)$.

Solution From (2), with A and B playing the parts of P_1 and P_2:

$$\begin{aligned} d(A, B) &= \sqrt{(3 - 8)^2 + (7 - (-5))^2} \\ &= \sqrt{(-5)^2 + (12)^2} = \sqrt{169} = 13. \end{aligned}$$

The distance d is illustrated in FIGURE 1.3.9. ≡

EXAMPLE 5 Three Points Form a Triangle

Determine whether the points $P_1(7, 1)$, $P_2(-4, -1)$, and $P_3(4, 5)$ are the vertices of a right triangle.

Solution From plane geometry we know that a triangle is a right triangle if and only if the sum of the squares of the lengths of two of its sides is equal to the square of the length of the remaining side. Now, from the distance formula (2), we have

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(-4 - 7)^2 + (-1 - 1)^2} \\ &= \sqrt{121 + 4} = \sqrt{125}, \\ d(P_2, P_3) &= \sqrt{(4 - (-4))^2 + (5 - (-1))^2} \\ &= \sqrt{64 + 36} = \sqrt{100} = 10, \\ d(P_3, P_1) &= \sqrt{(7 - 4)^2 + (1 - 5)^2} \\ &= \sqrt{9 + 16} = \sqrt{25} = 5. \end{aligned}$$

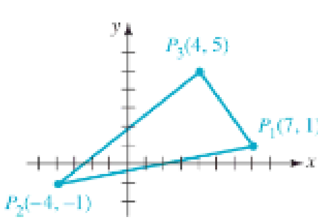


FIGURE 1.3.10 Triangle in Example 5

Since

$$[d(P_3, P_1)]^2 + [d(P_2, P_3)]^2 = 25 + 100 = 125 = [d(P_1, P_2)]^2,$$

we conclude that P_1, P_2, and P_3 are the vertices of a right triangle with the right angle at P_3. See FIGURE 1.3.10. ≡

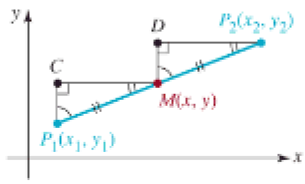


FIGURE 1.3.11 M is the midpoint of the line segment joining P_1 and P_2

Midpoint Formula In Section 1.2 we saw that the midpoint of a line segment between two numbers a and b on the number line is the average, $(a + b)/2$. In the xy-plane, each coordinate of the midpoint M of a line segment joining two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ shown in FIGURE 1.3.11 is the average of the corresponding coordinates of the endpoints of the intervals $[x_1, x_2]$ and $[y_1, y_2]$.

To prove this, we note in Figure 1.3.11 that triangles P_1CM and MDP_2 are congruent because corresponding angles are equal and $d(P_1, M) = d(M, P_2)$. Hence, $d(P_1, C) = d(M, D)$, or $y - y_1 = y_2 - y$. Solving the last equation for y gives $y = \dfrac{y_1 + y_2}{2}$. Similarly, $d(C, M) = d(D, P_2)$, so that $x - x_1 = x_2 - x$, and therefore $x = \dfrac{x_1 + x_2}{2}$. We have proved the following result.

THEOREM 1.3.2 Midpoint Formula

The coordinates of the **midpoint** M of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \tag{3}$$

EXAMPLE 6 Midpoint of a Line Segment

Find the coordinates of the midpoint of the line segment joining $A(-2, 5)$ and $B(4, 1)$.

Solution From the midpoint formula (3), the coordinates of the midpoint M are given by

$$\left(\frac{-2 + 4}{2}, \frac{5 + 1}{2}\right) \qquad \text{or} \qquad (1, 3).$$

This point is indicated in color in FIGURE 1.3.12. ≡

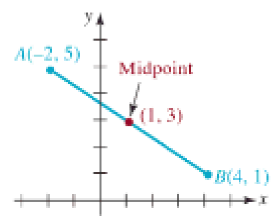


FIGURE 1.3.12 Midpoint of line segment in Example 6

1.3 Exercises

In Problems 1–4, plot the given points.

1. $(2, 3), (4, 5), (0, 2), (-1, -3)$
2. $(1, 4), (-3, 0), (-4, 2), (-1, -1)$
3. $(-\frac{1}{2}, -2), (0, 0), (-1, \frac{4}{3}), (3, 3)$
4. $(0, 0.8), (-2, 0), (1.2, -1.2), (-2, 2)$

In Problems 5–16, determine the quadrant in which the given point lies if (a, b) is in quadrant I.

5. $(-a, b)$
6. $(a, -b)$
7. $(-a, -b)$
8. (b, a)
9. $(-b, a)$
10. $(-b, -a)$
11. (a, a)
12. $(b, -b)$
13. $(-a, -a)$
14. $(-a, a)$
15. $(b, -a)$
16. $(-b, b)$

17. Plot the points given in Problems 5–16 if (a, b) is the point shown in FIGURE 1.3.13.

18. Give the coordinates of the points shown in FIGURE 1.3.14.

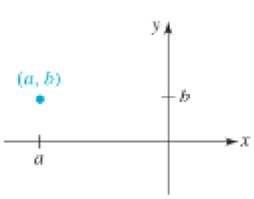


FIGURE 1.3.13 Point (a, b) in Problem 17

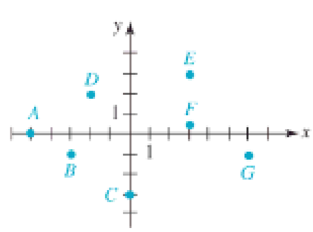


FIGURE 1.3.14 Points A–G in Problem 18

19. The points $(-2, 0)$, $(-2, 6)$, and $(3, 0)$ are vertices of a rectangle. Find the fourth vertex.

20. Describe the set of all points (x, x) in the coordinate plane. The set of all points $(x, -x)$.

In Problems 21–26, sketch the set of points (x, y) in the xy-plane whose coordinates satisfy the given conditions.

21. $xy = 0$

22. $xy > 0$

23. $|x| \leq 1$ and $|y| \leq 2$

24. $x \leq 2$ and $y \geq -1$

25. $|x| > 4$

26. $|y| \leq 1$

In Problems 27–32, find the distance between the given points.

27. $A(1, 2)$, $B(-3, 4)$

28. $A(-1, 3)$, $B(5, 0)$

29. $A(2, 4)$, $B(-4, -4)$

30. $A(-12, -3)$, $B(-5, -7)$

31. $A\left(-\frac{3}{2}, 1\right)$, $B\left(\frac{5}{2}, -2\right)$

32. $A\left(-\frac{5}{3}, 4\right)$, $B\left(-\frac{2}{3}, -1\right)$

In Problems 33–36, determine whether the points A, B, and C are vertices of a right triangle.

33. $A(8, 1)$, $B(-3, -1)$, $C(10, 5)$

34. $A(-2, -1)$, $B(8, 2)$, $C(1, -11)$

35. $A(2, 8)$, $B(0, -3)$, $C(6, 5)$

36. $A(4, 0)$, $B(1, 1)$, $C(2, 3)$

37. Determine whether the points $A(0, 0)$, $B(3, 4)$, and $C(7, 7)$ are vertices of an isosceles triangle.

38. Find all points on the y-axis that are 5 units from the point $(4, 4)$.

39. Consider the line segment joining $A(-1, 2)$ and $B(3, 4)$.

(a) Find an equation that expresses the fact that a point $P(x, y)$ is equidistant from A and from B.

(b) Describe geometrically the set of points described by the equation in part (a).

40. Use the distance formula to determine whether the points $A(-1, -5)$, $B(2, 4)$, and $C(4, 10)$ lie on a straight line.

41. Find all points each with x-coordinate 6 such that the distance from each point to $(-1, 2)$ is $\sqrt{85}$.

42. Which point, $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ or $(0.25, 0.97)$, is closer to the origin?

In Problems 43–48, find the midpoint M of the line segment joining the points A and B.

43. $A(4, 1)$, $B(-2, 4)$

44. $A\left(\frac{2}{3}, 1\right)$, $B\left(\frac{7}{3}, -3\right)$

45. $A(-1, 0)$, $B(-8, 5)$

46. $A\left(\frac{1}{2}, -\frac{3}{2}\right)$, $B\left(-\frac{5}{2}, 1\right)$

47. $A(2a, 3b)$, $B(4a, -6b)$

48. $A(x, x)$, $B(-x, x + 2)$

In Problems 49–52, find the point B if M is the midpoint of the line segment joining points A and B.

49. $A(-2, 1)$, $M\left(\frac{3}{2}, 0\right)$

50. $A\left(4, \frac{1}{2}\right)$, $M\left(7, -\frac{5}{2}\right)$

51. $A(5, 8)$, $M(-1, -1)$

52. $A(-10, 2)$, $M(5, 1)$

53. Find the distance from the midpoint M_1 of the line segment joining $A(-1, 3)$ and $B(3, 5)$ to the midpoint M_2 of the line segment joining $C(4, 6)$ and $D(-2, -10)$.

54. Find all points on the x-axis that are 3 units from the midpoint of the line segment joining $(5, 2)$ and $(-5, -6)$.

55. The x-axis is the perpendicular bisector of the line segment through $A(2, 5)$ and $B(x, y)$. Find x and y.

56. Consider the line segment joining the points $A(0, 0)$ and $B(6, 0)$. Find a point $C(x, y)$ in the first quadrant such that A, B, and C are vertices of an equilateral triangle.

1.4 Circles and Graphs

DEFINITION 1.4.1 Circle

A **circle** is the set of all points $P(x, y)$ in the coordinate plane that are a given fixed distance r, called the **radius**, from a given fixed point C, called the **center**.

If the center has coordinates $C(h, k)$, then from the preceding definition a point $P(x, y)$ lies on a circle of radius r if and only if

$$d(P, C) = r \qquad \text{or} \qquad \sqrt{(x-h)^2 + (y-k)^2} = r.$$

Since $(x-h)^2 + (y-k)^2$ is always nonnegative, we obtain an equivalent equation when both sides are squared. We conclude that a circle of radius r and center $C(h, k)$ has the equation

$$(x-h)^2 + (y-k)^2 = r^2. \tag{2}$$

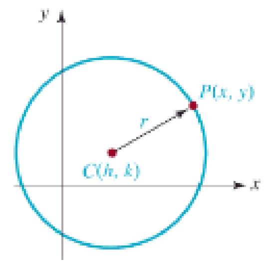


FIGURE 1.4.2 Circle with radius r and center (h, k)

In FIGURE 1.4.2 we have sketched a typical graph of an equation of the form given in (2). Equation (2) is called the **standard form** of the equation of a circle. We note that the symbols h and k in (2) represent real numbers and as such can be positive, zero, or negative. When $h = 0$ and $k = 0$, we see that the standard form of the equation of a circle with center at the origin is

$$x^2 + y^2 = r^2. \tag{3}$$

See FIGURE 1.4.3. When $r = 1$ we say that (2) is an equation of a **unit circle**. For example, $x^2 + y^2 = 1$ is an equation of a unit circle centered at the origin.

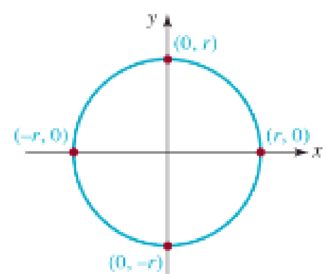


FIGURE 1.4.3 Circle with radius r and center $(0, 0)$

EXAMPLE 1 Center and Radius

Find the center and radius of the circle whose equation is

$$(x-8)^2 + (y+2)^2 = 49. \tag{4}$$

Solution To obtain the standard form of the equation, we rewrite (4) as

$$(x-8)^2 + (y-(-2))^2 = 7^2.$$

From this last form we identify $h = 8$, $k = -2$, and $r = 7$. Thus the circle is centered at $(8, -2)$ and has radius 7. ≡

EXAMPLE 2 Equation of a Circle

Find an equation of the circle with center $C(-5, 4)$ with radius $\sqrt{2}$.

Solution Substituting $h = -5$, $k = 4$, and $r = \sqrt{2}$ in (2), we obtain

$$(x-(-5))^2 + (y-4)^2 = (\sqrt{2})^2 \qquad \text{or} \qquad (x+5)^2 + (y-4)^2 = 2. \quad ≡$$

EXAMPLE 3 Equation of a Circle

Find an equation of the circle with center $C(4, 3)$ and passing through $P(1, 4)$.

Solution With $h = 4$ and $k = 3$, we have from (2)

$$(x-4)^2 + (y-3)^2 = r^2. \tag{5}$$

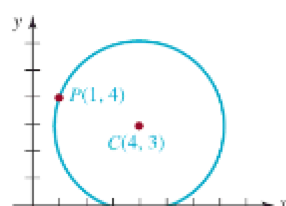


FIGURE 1.4.4 Circle in Example 3

Since the point $P(1, 4)$ lies on the circle as shown in FIGURE 1.4.4, its coordinates must satisfy equation (5). That is,

$$(1-4)^2+(4-3)^2=r^2 \qquad \text{or} \qquad 10=r^2.$$

Thus the required equation in standard form is $(x-4)^2+(y-3)^2=10$. ≡

☐ Completing the Square

If the terms $(x-h)^2$ and $(y-k)^2$ are expanded and the like terms grouped together, an equation of a circle in standard form can be written as

$$x^2+y^2+ax+by+c=0. \tag{6}$$

Of course in this last form the center and radius are not apparent. To reverse the process—in other words, to go from (6) to the standard form (2)—we must **complete the square** in both x and y. Recall from algebra that adding $(a/2)^2$ to an expression such as x^2+ax yields $x^2+ax+(a/2)^2$, which is the perfect square $(x+a/2)^2$. By rearranging the terms in (6),

$$(x^2+ax \qquad)+(y^2+by \qquad)=-c,$$

and then adding $(a/2)^2$ and $(b/2)^2$ to *both* sides of the last equation,

$$\left(x^2+ax+\left(\frac{a}{2}\right)^2\right)+\left(y^2+by+\left(\frac{b}{2}\right)^2\right)=\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2-c,$$

we obtain the standard form of the equation of a circle:

$$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2=\frac{1}{4}(a^2+b^2-4c).$$

You should *not* memorize the last equation; we strongly recommend that you work through the process of completing the square each time.

EXAMPLE 4 Completing the Square

Find the center and radius of the circle whose equation is

$$x^2+y^2+10x-2y+17=0. \tag{7}$$

Solution To find the center and radius we rewrite equation (7) in the standard form (2). First, we rearrange the terms,

$$(x^2+10x \qquad)+(y^2-2y \qquad)=-17.$$

Then, we complete the square in x and y by adding, in turn, $(10/2)^2$ in the first set of parentheses and $(-2/2)^2$ in the second set of parentheses. Proceed carefully here because we must add these numbers to both sides of the equation:

$$\begin{aligned}\left[x^2+10x+\left(\tfrac{10}{2}\right)^2\right]+\left[y^2-2y+\left(\tfrac{-2}{2}\right)^2\right]&=-17+\left(\tfrac{10}{2}\right)^2+\left(\tfrac{-2}{2}\right)^2\\ (x^2+10x+25)+(y^2-2y+1)&=9\\ (x+5)^2+(y-1)^2&=3^2.\end{aligned}$$

From the last equation we see that the circle is centered at $(-5, 1)$ and has radius 3. See FIGURE 1.4.5. ≡

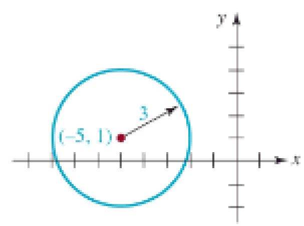


FIGURE 1.4.5 Circle in Example 4

It is possible that an expression for which we must complete the square has a leading coefficient other than 1. For example,

Note:

$$3x^2+3y^2-18x+6y+2=0$$

is an equation of circle. As in Example 4, we start by rearranging the equation:

$$(3x^2-18x \qquad)+(3y^2+6y \qquad)=-2.$$

Now, however, we must do one extra step before attempting completion of the square, that is, we must divide both sides of the equation by 3 so that the coefficients of x^2 and y^2 are each 1:

$$(x^2-6x \qquad)+(y^2+2y \qquad)=-\tfrac{2}{3}.$$

At this point we can now add the appropriate numbers within each set of parentheses *and* to the right-hand side of the equality. You should verify that the resulting standard form is $(x - 3)^2 + (y + 1)^2 = \frac{28}{3}$.

Semicircles If we solve (3) for y we get $y^2 = r^2 - x^2$ or $y = \pm\sqrt{r^2 - x^2}$. This last expression is equivalent to two equations, $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. In like manner if we solve (3) for x we obtain $x = \sqrt{r^2 - y^2}$ and $x = -\sqrt{r^2 - y^2}$.

By convention, the symbol $\sqrt{\ }$ denotes a nonnegative quantity, thus the y-values defined by an equation such as $y = \sqrt{r^2 - x^2}$ are nonnegative. The graphs of the four equations highlighted in color are, in turn, the upper half, lower half, right half, and left half of the circle shown in Figure 1.4.3. Each graph in FIGURE 1.4.6 is called a **semicircle**.

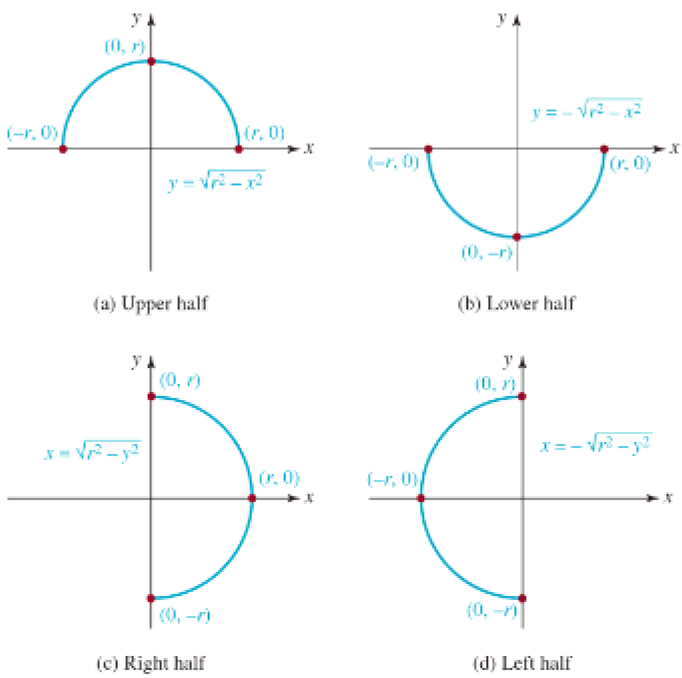


FIGURE 1.4.6 Semicircles