Lines

One of the basic concepts in geometry is that of a *line*. In this section we will restrict our discussion to lines that lie in a coordinate plane. This will allow us to use algebraic methods to study their properties. Two of our principal objectives may be stated as follows:

(1) Given a line l in a coordinate plane, find an equation whose graph corresponds to l.

(2) Given an equation of a line l in a coordinate plane, sketch the graph of the equation.

The following concept is fundamental to the study of lines.

Definition of Slope of a Line

Let l be a line that is not parallel to the y-axis, and let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points on l. The **slope m** of l is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

If l is parallel to the y-axis, then the slope of l is not defined.

The Greek letter Δ (delta) is used in mathematics to denote "change in." Thus, we can think of the slope m as

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}.$$

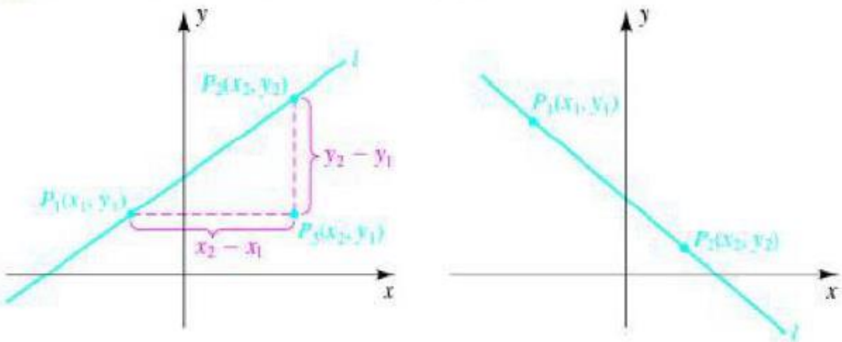
Typical points P_1 and P_2 on a line l are shown in Figure 1. The numerator $y_2 - y_1$ in the formula for m is the vertical change in direction from P_1 to P_2 and may be positive, negative, or zero. The denominator $x_2 - x_1$ is the horizontal change from P_1 to P_2, and it may be positive or negative, but never zero, because l is not parallel to the y-axis if a slope exists. In Figure 1(a) the slope is positive, and we say that the line *rises*. In Figure 1(b) the slope is negative, and the line *falls*.

In finding the slope of a line it is immaterial which point we label as P_1 and which as P_2, since

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{(-1)}{(-1)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

If the points are labeled so that $x_1 < x_2$, as in Figure 1, then $x_2 - x_1 > 0$, and hence the slope is positive, negative, or zero, depending on whether $y_2 > y_1$, $y_2 < y_1$, or $y_2 = y_1$, respectively.

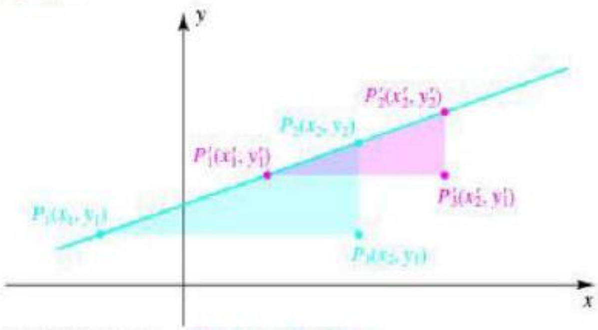
Figure 1



The definition of slope is independent of the two points that are chosen on l. If other points $P_1'(x_1', y_1')$ and $P_2'(x_2', y_2')$ are used, then, as in Figure 2, the triangle with vertices P_1', P_2', and $P_3'(x_2', y_1')$ is similar to the triangle with vertices P_1, P_2, and $P_3(x_2, y_1)$. Since the ratios of corresponding sides of similar triangles are equal,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}.$$

Figure 2



EXAMPLE 1 Finding slopes

Sketch the line through each pair of points, and find its slope m:

(a) $A(-1, 4)$ and $B(3, 2)$ **(b)** $A(2, 5)$ and $B(-2, -1)$

(c) $A(4, 3)$ and $B(-2, 3)$ **(d)** $A(4, -1)$ and $B(4, 4)$

SOLUTION

EXAMPLE 2 Sketching a line with a given slope

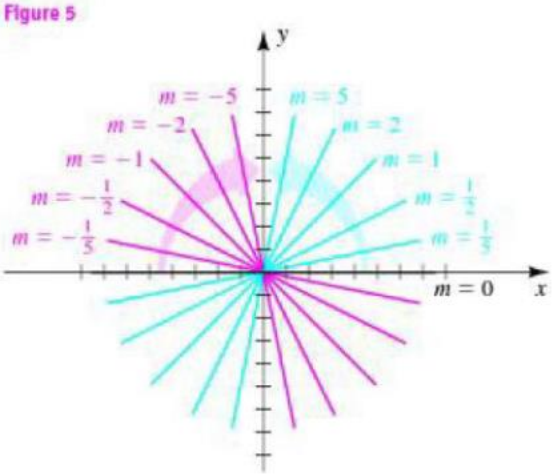
Sketch a line through $P(2, 1)$ that has

(a) slope $\frac{5}{3}$ **(b)** slope $-\frac{5}{3}$

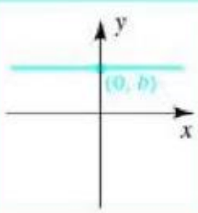
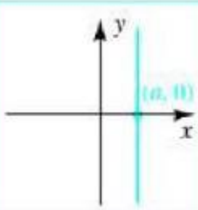
SOLUTION

The diagram in Figure 5 indicates the slopes of several lines through the origin. The line that lies on the x-axis has slope $m = 0$. If this line is rotated about O in the *counterclockwise* direction (as indicated by the blue arrow), the slope is positive and increases, reaching the value 1 when the line bisects the first quadrant and continuing to increase as the line gets closer to the y-axis. If we rotate the line of slope $m = 0$ in the *clockwise* direction (as indicated by the red arrow), the slope is negative, reaching the value -1 when the line bisects the second quadrant and becoming large and negative as the line gets closer to the y-axis.

Figure 5



Lines that are horizontal or vertical have simple equations, as indicated in the following chart.

Terminology	Definition	Graph	Equation	Slope
Horizontal line	A line parallel to the x-axis		$y = b$ y-intercept is b	Slope is 0
Vertical line	A line parallel to the y-axis		$x = a$ x-intercept is a	Slope is undefined

A common error is to regard the graph of $y = b$ as consisting of only the one point $(0, b)$. If we express the equation in the form $0 \cdot x + y = b$, we see that the value of x is immaterial; thus, the graph of $y = b$ consists of the points (x, b) for *every* x and hence is a horizontal line. Similarly, the graph of $x = a$ is the vertical line consisting of all points (a, y), where y is a real number.

EXAMPLE 3 Finding equations of horizontal and vertical lines

Find an equation of the line through $A(-3, 4)$ that is parallel to

(a) the x-axis **(b)** the y-axis

SOLUTION

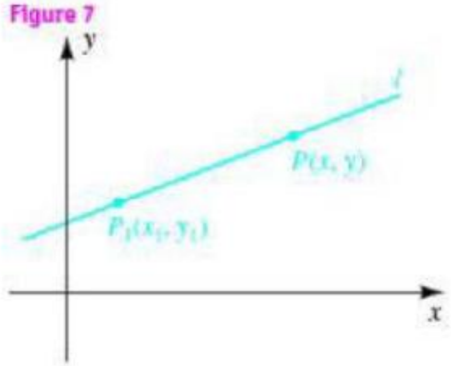
Let us next find an equation of a line l through a point $P_1(x_1, y_1)$ with slope m. If $P(x, y)$ is any point with $x \neq x_1$ (see Figure 7), then P is on l if and only if the slope of the line through P_1 and P is m—that is, if

$$\frac{y - y_1}{x - x_1} = m.$$

This equation may be written in the form

$$y - y_1 = m(x - x_1).$$

Figure 7



Note that (x_1, y_1) is a solution of the last equation, and hence the points on l are precisely the points that correspond to the solutions. This equation for l is referred to as the **point-slope form.**

Point-Slope Form for the Equation of a Line

An equation for the line through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1).$$

The point-slope form is only one possibility for an equation of a line. There are many equivalent equations. We sometimes simplify the equation obtained using the point-slope form to either

$$ax + by = c \qquad \text{or} \qquad ax + by + d = 0,$$

where a, b, and c are integers with no common factor, $a > 0$, and $d = -c$.

EXAMPLE 4 **Finding an equation of a line through two points**

Find an equation of the line through $A(1, 7)$ and $B(-3, 2)$.

SOLUTION

The point-slope form for the equation of a line may be rewritten as $y = mx - mx_1 + y_1$, which is of the form

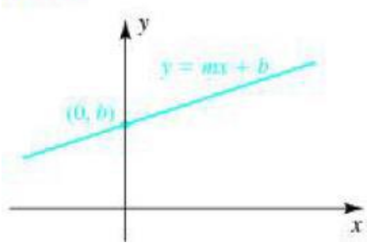
$$y = mx + b$$

with $b = -mx_1 + y_1$. The real number b is the y-intercept of the graph, as indicated in Figure 9. Since the equation $y = mx + b$ displays the slope m and y-intercept b of l, it is called the **slope-intercept form** for the equation of a line. Conversely, if we start with $y = mx + b$, we may write

$$y - b = m(x - 0).$$

Comparing this equation with the point-slope form, we see that the graph is a line with slope m and passing through the point $(0, b)$. We have proved the following result.

Figure 9



Slope-Intercept Form for the Equation of a Line

The graph of $y = mx + b$ is a line having slope m and y-intercept b.

EXAMPLE 5 **Expressing an equation in slope-intercept form**

Express the equation $2x - 5y = 8$ in slope-intercept form.

SOLUTION

It follows from the point-slope form that every line is a graph of an equation

$$ax + by = c,$$

where a, b, and c are real numbers and a and b are not both zero. We call such an equation a **linear equation** in x and y. Let us show, conversely, that the graph of $ax + by = c$, with a and b not both zero, is always a line. If $b \neq 0$, we may solve for y, obtaining

$$y = \left(-\frac{a}{b}\right)x + \frac{c}{b},$$

which, by the slope-intercept form, is an equation of a line with slope $-a/b$ and y-intercept c/b. If $b = 0$ but $a \neq 0$, we may solve for x, obtaining $x = c/a$, which is the equation of a vertical line with x-intercept c/a. This discussion establishes the following result.

General Form for the Equation of a Line

The graph of a linear equation $ax + by = c$ is a line, and conversely, every line is the graph of a linear equation.

For simplicity, we use the terminology *the line* $ax + by = c$ rather than *the line with equation* $ax + by = c$.

EXAMPLE 6 Sketching the graph of a linear equation

Sketch the graph of $2x - 5y = 8$.

SOLUTION

The following theorem specifies the relationship between **parallel lines** (lines in a plane that do not intersect) and slope.

Theorem on Slopes of Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.

PROOF Let l_1 and l_2 be distinct lines of slopes m_1 and m_2, respectively. If the y-intercepts are b_1 and b_2 (see Figure 11), then, by the slope-intercept form, the lines have equations

$$y = m_1x + b_1 \qquad \text{and} \qquad y = m_2x + b_2.$$

The lines intersect at some point (x, y) if and only if the values of y are equal for some x—that is, if

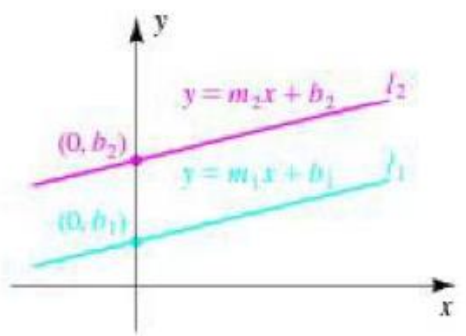
$$m_1x + b_1 = m_2x + b_2,$$

or

$$(m_1 - m_2)x = b_2 - b_1.$$

The last equation can be solved for x if and only if $m_1 - m_2 \neq 0$. We have shown that the lines l_1 and l_2 intersect if and only if $m_1 \neq m_2$. Hence, they do *not* intersect (are parallel) if and only if $m_1 = m_2$.

Figure 11



EXAMPLE 7 Finding an equation of a line parallel to a given line

Find an equation of the line through $P(5, -7)$ that is parallel to the line $6x + 3y = 4$.

SOLUTION

If the slopes of two nonvertical lines are not the same, then the lines are not parallel and intersect at exactly one point.

The next theorem gives us information about **perpendicular lines** (lines that intersect at a right angle).

Theorem on Slopes of Perpendicular Lines

Two lines with slope m_1 and m_2 are perpendicular if and only if

$$m_1m_2 = -1.$$

PROOF For simplicity, let us consider the special case of two lines that intersect at the origin O, as illustrated in Figure 13. Equations of these lines are $y = m_1x$ and $y = m_2x$. If, as in the figure, we choose points $A(x_1, m_1x_1)$ and

$B(x_2, m_2x_2)$ different from O on the lines, then the lines are perpendicular if and only if angle AOB is a right angle. Applying the Pythagorean theorem, we know that angle AOB is a right angle if and only if

$$[d(A, B)]^2 = [d(O, B)]^2 + [d(O, A)]^2$$

or, by the distance formula,

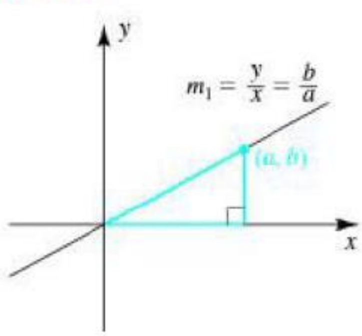
$$(x_2 - x_1)^2 + (m_2x_2 - m_1x_1)^2 = x_2^2 + (m_2x_2)^2 + x_1^2 + (m_1x_1)^2.$$

Squaring terms, simplifying, and factoring gives us

$$-2m_1m_2x_1x_2 - 2x_1x_2 = 0$$
$$-2x_1x_2(m_1m_2 + 1) = 0.$$

Since both x_1 and x_2 are not zero, we may divide both sides by $-2x_1x_2$, obtaining $m_1m_2 + 1 = 0$. Thus, the lines are perpendicular if and only if $m_1m_2 = -1$.

Figure 14



A convenient way to remember the conditions on slopes of perpendicular lines is to note that m_1 and m_2 must be *negative reciprocals* of each other—that is, $m_1 = -1/m_2$ and $m_2 = -1/m_1$.

We can visualize the result of the last theorem as follows. Draw a triangle as in Figure 14; the line containing its hypotenuse has slope $m_1 = b/a$. Now rotate the triangle 90° as in Figure 15. The line now has slope $m_2 = a/(-b)$, the negative reciprocal of m_1.

Figure 14

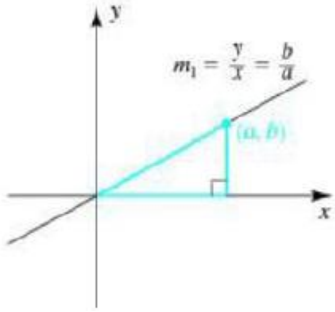


Figure 15