Operations on Functions

In general, if f and g are *any* functions, we use the terminology and notation given in the following chart.

Sum, Difference, Product, and Quotient of Functions

Terminology	Function value
sum $f + g$	$(f+g)(x) = f(x) + g(x)$
difference $f - g$	$(f-g)(x) = f(x) - g(x)$
product fg	$(fg)(x) = f(x)g(x)$
quotient $\dfrac{f}{g}$	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)},\ g(x) \neq 0$

The domains of $f + g$, $f - g$, and fg are the intersection I of the domains of f and g—that is, the numbers that are *common* to both domains. The domain of f/g is the subset of I consisting of all x in I such that $g(x) \neq 0$.

EXAMPLE 1 Finding function values of $f+g$, $f-g$, fg, and f/g

If $f(x) = 3x - 2$ and $g(x) = x^3$, find $(f+g)(2)$, $(f-g)(2)$, $(fg)(2)$, and $(f/g)(2)$.

SOLUTION Since $f(2) = 3(2) - 2 = 4$ and $g(2) = 2^3 = 8$, we have

$$(f+g)(2) = f(2) + g(2) = 4 + 8 = 12$$

$$(f-g)(2) = f(2) - g(2) = 4 - 8 = -4$$

$$(fg)(2) = f(2)g(2) = (4)(8) = 32$$

$$\left(\frac{f}{g}\right)(2) = \frac{f(2)}{g(2)} = \frac{4}{8} = \frac{1}{2}.$$

EXAMPLE 2 Finding $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $(f/g)(x)$

If $f(x) = \sqrt{4 - x^2}$ and $g(x) = 3x + 1$, find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $(f/g)(x)$, and state the domains of the respective functions.

SOLUTION The domain of f is the closed interval $[-2, 2]$, and the domain of g is $\mathbb{R}$. The intersection of these domains is $[-2, 2]$, which is the domain of $f+g$, $f-g$, and fg. For the domain of f/g, we exclude each number x in $[-2, 2]$ such that $g(x) = 3x + 1 = 0$ $\left(\text{namely, } x = -\frac{1}{3}\right)$. Thus, we have the following:

$$(f+g)(x)=\sqrt{4-x^2}+(3x+1), \qquad -2\le x\le 2$$
$$(f-g)(x)=\sqrt{4-x^2}-(3x+1), \qquad -2\le x\le 2$$
$$(fg)(x)=\sqrt{4-x^2}\,(3x+1), \qquad -2\le x\le 2$$
$$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{4-x^2}}{3x+1}, \qquad -2\le x\le 2 \text{ and } x\ne -\frac{1}{3}$$

A function f is a **polynomial function** if $f(x)$ is a polynomial—that is, if

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers and the exponents are nonnegative integers.

An **algebraic function** is a function that can be expressed in terms of finite sums, differences, products, quotients, or roots of polynomial functions.

Algebraic Function

- $f(x)=5x^4-2\sqrt[3]{x}+\dfrac{x(x^2+5)}{\sqrt{x^3+\sqrt{x}}}$

Functions that are not algebraic are **transcendental.**

Definition of Composite Function

The **composite function** $f\circ g$ of two functions f and g is defined by

$$(f\circ g)(x)=f(g(x)).$$

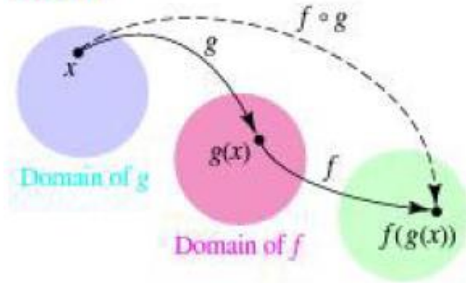
The domain of $f\circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

For the composite function $g\circ f$, we reverse this order, first finding $f(x)$ and second finding $g(f(x))$. The domain of $g\circ f$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

Since the notation $g(x)$ is read "g of x," we sometimes say that g *is a function of* x. For the composite function $f\circ g$, the notation $f(g(x))$ is read "f of g of x," and we could regard f as a function of $g(x)$. In this sense, *a composite function is a function of a function* or, more precisely, a function of another function's values.

*A number x is in the domain of $(f \circ g)(x)$ if and only if **both** $g(x)$ **and** $f(g(x))$ are defined.*

Figure 1



EXAMPLE 3 Finding composite functions

Let $f(x) = x^2 - 1$ and $g(x) = 3x + 5$.

(a) Find $(f \circ g)(x)$ and the domain of $f \circ g$.

(b) Find $(g \circ f)(x)$ and the domain of $g \circ f$.

(c) Find $f(g(2))$ in two different ways: first using the functions f and g separately and second using the composite function $f \circ g$.

SOLUTION

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g\\ &= f(3x + 5) && \text{definition of } g\\ &= (3x + 5)^2 - 1 && \text{definition of } f\\ &= 9x^2 + 30x + 24 && \text{simplify}\end{aligned}$$

The domain of both f and g is $\mathbb{R}$. Since for each x in $\mathbb{R}$ (the domain of g), the function value $g(x)$ is in $\mathbb{R}$ (the domain of f), the domain of $f \circ g$ is also $\mathbb{R}$. Note that *both* $g(x)$ *and* $f(g(x))$ are defined for all real numbers.

(b)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{definition of } g \circ f\\ &= g(x^2 - 1) && \text{definition of } f\\ &= 3(x^2 - 1) + 5 && \text{definition of } g\\ &= 3x^2 + 2 && \text{simplify}\end{aligned}$$

Since for each x in $\mathbb{R}$ (the domain of f), the function value $f(x)$ is in $\mathbb{R}$ (the domain of g), the domain of $g \circ f$ is $\mathbb{R}$. Note that *both* $f(x)$ *and* $g(f(x))$ are defined for all real numbers.

(c) To find $f(g(2))$ using $f(x) = x^2 - 1$ and $g(x) = 3x + 5$ separately, we may proceed as follows:

$$g(2) = 3(2) + 5 = 11$$
$$f(g(2)) = f(11) = 11^2 - 1 = 120$$

To find $f(g(2))$ using $f \circ g$, we refer to part (a), where we found

$$(f \circ g)(x) = f(g(x)) = 9x^2 + 30x + 24.$$

Hence,

$$\begin{aligned} f(g(2)) &= 9(2)^2 + 30(2) + 24 \\ &= 36 + 60 + 24 = 120. \end{aligned}$$

Note that in Example 3, $f(g(x))$ and $g(f(x))$ are not always the same; that is, $f \circ g \neq g \circ f$.

EXAMPLE 4 **Finding composite functions**

Let $f(x) = x^2 - 16$ and $g(x) = \sqrt{x}$.

(a) Find $(f \circ g)(x)$ and the domain of $f \circ g$.

(b) Find $(g \circ f)(x)$ and the domain of $g \circ f$.

SOLUTION We first note that the domain of f is $\mathbb{R}$ and the domain of g is the set of all nonnegative real numbers—that is, the interval $[0, \infty)$. We may proceed as follows.

(a)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g \\ &= f\left(\sqrt{x}\right) && \text{definition of } g \\ &= \left(\sqrt{x}\right)^2 - 16 && \text{definition of } f \\ &= x - 16 && \text{simplify} \end{aligned}$$

If we consider only the final expression, $x - 16$, we might be led to believe that the domain of $f \circ g$ is $\mathbb{R}$, since $x - 16$ is defined for every real number x. However, this is not the case. By definition, the domain of $f \circ g$ is the set of all x in $[0, \infty)$ (the domain of g) such that $g(x)$ is in $\mathbb{R}$ (the domain of f). Since $g(x) = \sqrt{x}$ is in $\mathbb{R}$ for every x in $[0, \infty)$, it follows that the domain of $f \circ g$ is $[0, \infty)$. Note that *both* $g(x)$ *and* $f(g(x))$ are defined for x in $[0, \infty)$.

(b)

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{definition of } g \circ f \\ &= g(x^2 - 16) && \text{definition of } f \\ &= \sqrt{x^2 - 16} && \text{definition of } g \end{aligned}$$

By definition, the domain of $g \circ f$ is the set of all x in $\mathbb{R}$ (the domain of f) such that $f(x) = x^2 - 16$ is in $[0, \infty)$ (the domain of g). The statement "$x^2 - 16$ is in $[0, \infty)$" is equivalent to each of the inequalities

$$x^2 - 16 \geq 0, \qquad x^2 \geq 16, \qquad |x| \geq 4.$$

Thus, the domain of $g \circ f$ is the union $(-\infty, -4] \cup [4, \infty)$. Note that *both* $f(x)$ *and* $g(f(x))$ are defined for x in $(-\infty, -4] \cup [4, \infty)$. Also note that this domain is different from the domains of both f and g.

EXAMPLE 5 **Finding composite function values from tables**

Several values of two functions f and g are listed in the following tables.

x	1	2	3	4
$f(x)$	3	4	2	1

x	1	2	3	4
$g(x)$	4	1	3	2

Find $(f \circ g)(2)$, $(g \circ f)(2)$, $(f \circ f)(2)$, and $(g \circ g)(2)$.

SOLUTION Using the definition of composite function and referring to the tables above, we obtain

$$(f \circ g)(2) = f(g(2)) = f(1) = 3$$
$$(g \circ f)(2) = g(f(2)) = g(4) = 2$$
$$(f \circ f)(2) = f(f(2)) = f(4) = 1$$
$$(g \circ g)(2) = g(g(2)) = g(1) = 4.$$

EXAMPLE 7 **Finding a composite function form**

Express $y = (2x + 5)^8$ as a composite function form.

SOLUTION Suppose, for a real number x, we wanted to evaluate the expression $(2x + 5)^8$ by using a calculator. We would first calculate the value of $2x + 5$ and then raise the result to the eighth power. This suggests that we let

$$u = 2x + 5 \quad \text{and} \quad y = u^8,$$

which is a composite function form for $y = (2x + 5)^8$.

Composite Function Forms

Function value	Choice for $u = g(x)$	Choice for $y = f(u)$
$y = (x^3 - 5x + 1)^4$	$u = x^3 - 5x + 1$	$y = u^4$
$y = \sqrt{x^2 - 4}$	$u = x^2 - 4$	$y = \sqrt{u}$
$y = \dfrac{2}{3x + 7}$	$u = 3x + 7$	$y = \dfrac{2}{u}$

The composite function form is never unique. For example, consider the first expression in the preceding illustration:

$$y = (x^3 - 5x + 1)^4$$

If n is any nonzero integer, we could choose

$$u = (x^3 - 5x + 1)^n \quad \text{and} \quad y = u^{4/n}.$$

Thus, there are an *unlimited* number of composite function forms. Generally, our goal is to choose a form such that the expression for y is simple, as we did in the illustration.

Exercises

Exer. 1–2: Find

(a) $(f+g)(3)$ **(b)** $(f-g)(3)$

(c) $(fg)(3)$ **(d)** $(f/g)(3)$

1 $f(x) = x + 3, \quad g(x) = x^2$

2 $f(x) = -x^2, \quad g(x) = 2x - 1$

Exer. 3–8: Find

(a) $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, **and** $(f/g)(x)$

(b) the domain of $f+g$, $f-g$, **and** fg

(c) the domain of f/g

3 $f(x) = x^2 + 2, \quad g(x) = 2x^2 - 1$

4 $f(x) = x^2 + x, \quad g(x) = x^2 - 3$

5 $f(x) = \sqrt{x+5}, \quad g(x) = \sqrt{x+5}$

6 $f(x) = \sqrt{3-2x}, \quad g(x) = \sqrt{x+4}$

7 $f(x) = \dfrac{2x}{x-4}, \quad g(x) = \dfrac{x}{x+5}$

8 $f(x) = \dfrac{x}{x-2}, \quad g(x) = \dfrac{3x}{x+4}$

Exer. 9–10: Find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

(c) $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

9 $f(x) = 2x - 1, \quad g(x) = -x^2$

10 $f(x) = 3x^2, \quad g(x) = x - 1$

Exer. 11–20: Find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

(c) $f(g(-2))$ **(d)** $g(f(3))$

11 $f(x) = 2x - 5, \quad g(x) = 3x + 7$

12 $f(x) = 5x + 2, \quad g(x) = 6x - 1$

13 $f(x) = 3x^2 + 4, \quad g(x) = 5x$

14 $f(x) = 3x - 1, \quad g(x) = 4x^2$

15 $f(x) = 2x^2 + 3x - 4, \quad g(x) = 2x - 1$

16 $f(x) = 5x - 7, \quad g(x) = 3x^2 - x + 2$

17 $f(x) = 4x, \quad g(x) = 2x^3 - 5x$

18 $f(x) = x^3 + 2x^2, \quad g(x) = 3x$

19 $f(x) = |x|, \quad g(x) = -7$

20 $f(x) = 5, \quad g(x) = x^2$

Exer. 21–34: Find (a) $(f \circ g)(x)$ and the domain of $f \circ g$ and (b) $(g \circ f)(x)$ and the domain of $g \circ f$.

21 $f(x) = x^2 - 3x, \quad g(x) = \sqrt{x + 2}$

22 $f(x) = \sqrt{x - 15}, \quad g(x) = x^2 + 2x$

23 $f(x) = x^2 - 4, \quad g(x) = \sqrt{3x}$

24 $f(x) = -x^2 + 1, \quad g(x) = \sqrt{x}$

25 $f(x) = \sqrt{x - 2}, \quad g(x) = \sqrt{x + 5}$

26 $f(x) = \sqrt{3 - x}, \quad g(x) = \sqrt{x + 2}$

27 $f(x) = \sqrt{3 - x}, \quad g(x) = \sqrt{x^2 - 16}$

28 $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$

29 $f(x) = \dfrac{3x + 5}{2}, \quad g(x) = \dfrac{2x - 5}{3}$

30 $f(x) = \dfrac{1}{x - 1}, \quad g(x) = x - 1$

31 $f(x) = x^2, \quad g(x) = \dfrac{1}{x^3}$

32 $f(x) = \dfrac{x}{x - 2}, \quad g(x) = \dfrac{3}{x}$

33 $f(x) = \dfrac{x-1}{x-2}, \qquad g(x) = \dfrac{x-3}{x-4}$

34 $f(x) = \dfrac{x+2}{x-1}, \qquad g(x) = \dfrac{x-5}{x+4}$

Exer. 35–36: Solve the equation $(f \circ g)(x) = 0$.

35 $f(x) = x^2 - 2, \qquad g(x) = x + 3$

36 $f(x) = x^2 - x - 2, \quad g(x) = 2x - 1$

37 Several values of two functions f and g are listed in the following tables:

x	5	6	7	8	9
$f(x)$	8	7	6	5	4

x	5	6	7	8	9
$g(x)$	7	8	6	5	4

If possible, find

(a) $(f \circ g)(6)$ (b) $(g \circ f)(6)$ (c) $(f \circ f)(6)$

(d) $(g \circ g)(6)$ (e) $(f \circ g)(9)$