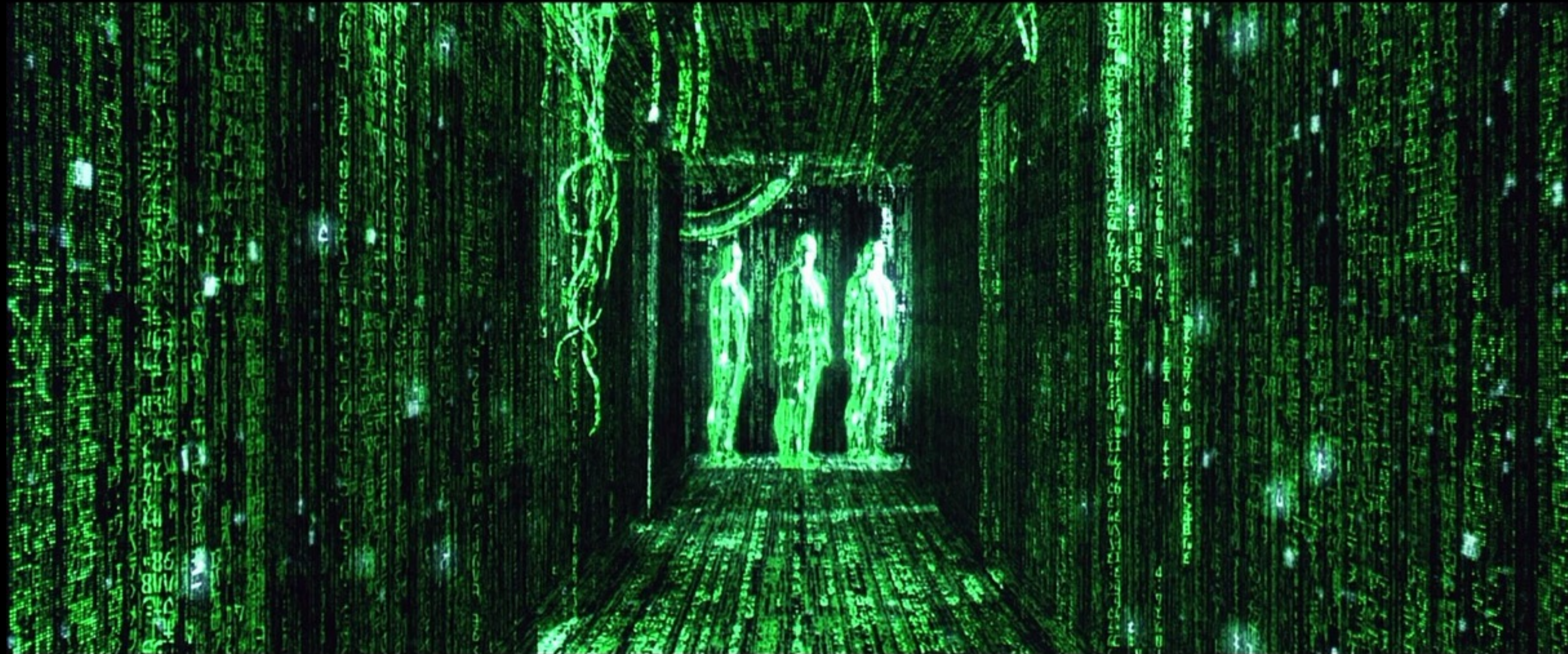


# Matrix transformations.

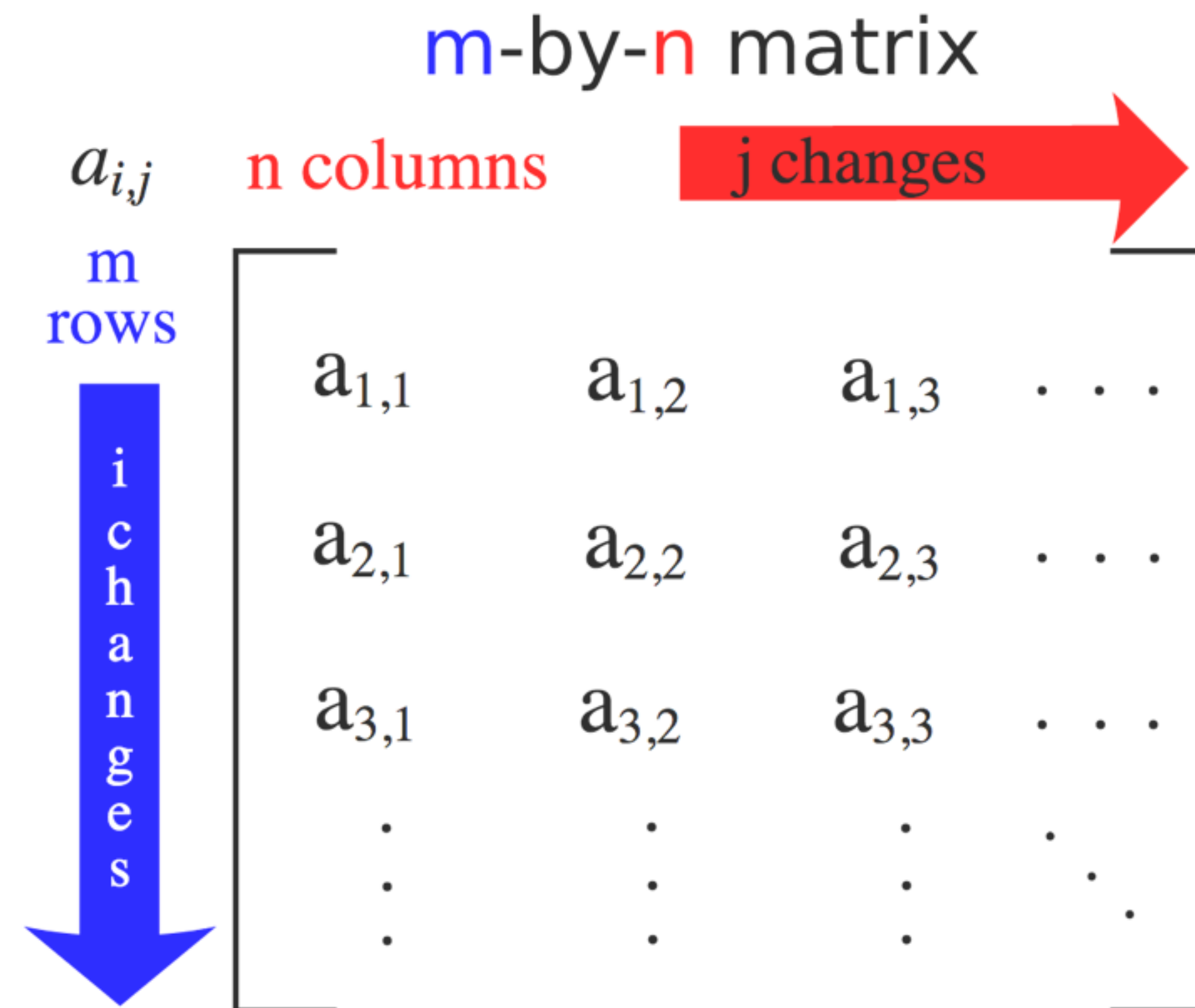
## Part 1





Matrix **math**.

# A matrix.



A **2x3** matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

A **3x3** matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

**Matrix operations.**

Matrix **addition**.

To **add** two matrices, **add** their **corresponding** entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} + \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A+J & B+K & C+L \\ D+M & E+N & F+O \\ G+P & H+Q & I+R \end{bmatrix}$$



Matrix **subtraction**.

To **subtract** two matrices, **subtract** their **corresponding** entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} - \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A-J & B-K & C-L \\ D-M & E-N & F-O \\ G-P & H-Q & I-R \end{bmatrix}$$

Matrix addition and subtraction can only happen  
with **matrices that are the same size!**

**Transpose** of a matrix.



**Transpose** of a matrix is a matrix whose **columns are the rows** of the original matrix (and its **rows are the columns**).

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$$

**M**

$$\begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

**M<sup>T</sup>**

**Matrix/scalar multiplication.**

**Multiply** each entry of the matrix **by the scalar**.

$$S \times \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} S \times A & S \times B & S \times C \\ S \times D & S \times E & S \times F \\ S \times G & S \times H & S \times I \end{bmatrix}$$

Matrix/**matrix** multiplication.



You can only multiply **two matrices**  
if the **number of columns of the first matrix** equals the  
**number of rows of the second.**

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = ?$$

It results in a matrix that is **number of rows of first matrix** by **number of columns of second matrix**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For **each row**, find **dot product with each column**.

The diagram illustrates the dot product of a row from a 2x3 matrix and a column from a 3x3 matrix. The first matrix has elements A, B, C in the first row and D, E, F in the second row. A red arrow points to the first row. The second matrix has elements J, M, P in the first column and K, N, Q in the second column. A red arrow points to the first column. The two matrices are separated by an equals sign.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

The diagram shows the calculation of the dot product for the first row and first column. The expression is enclosed in large square brackets. The expression is: A x J + B x M + C x P. The elements A, B, and C are red, and J, M, and P are blue.

$$\left[ A \times J + B \times M + C \times P \right]$$

For **each row**, find **dot product with each column**.

The diagram illustrates the dot product of a row and a column from two matrices. The first matrix is  $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$ . A red arrow points to the first row  $[A \ B \ C]$ , which is highlighted in a light red box. The second matrix is  $\begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix}$ . A red arrow points to the second column  $\begin{bmatrix} K \\ N \\ Q \end{bmatrix}$ , which is highlighted in a light red box. An equals sign follows the matrices.

The diagram shows the resulting dot products for the first row of the first matrix. The first dot product is  $A \times J + B \times M + C \times P$ , and the second dot product is  $A \times K + B \times N + C \times Q$ . These are enclosed in large square brackets.



For **each row**, find **dot product with each column**.

The diagram illustrates the dot product of a row from matrix A with a column from matrix B. Matrix A is represented as a 2x3 grid with elements A, B, C in the first row and D, E, F in the second row. A red arrow points from the second row of A to the first row of the result matrix. Matrix B is represented as a 3x2 grid with elements J, K in the first column and M, N in the second column. A red arrow points from the first column of B to the first column of the result matrix. The result matrix is shown as a 2x2 grid with elements J, K in the first column and M, N in the second column. The elements J, K, M, N are highlighted in a light blue box.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

The diagram illustrates the dot product of a row from matrix A with a column from matrix B. Matrix A is represented as a 2x3 grid with elements A, B, C in the first row and D, E, F in the second row. A red arrow points from the second row of A to the first row of the result matrix. Matrix B is represented as a 3x2 grid with elements J, K in the first column and M, N in the second column. A red arrow points from the first column of B to the first column of the result matrix. The result matrix is shown as a 2x2 grid with elements J, K in the first column and M, N in the second column. The elements J, K, M, N are highlighted in a light blue box.

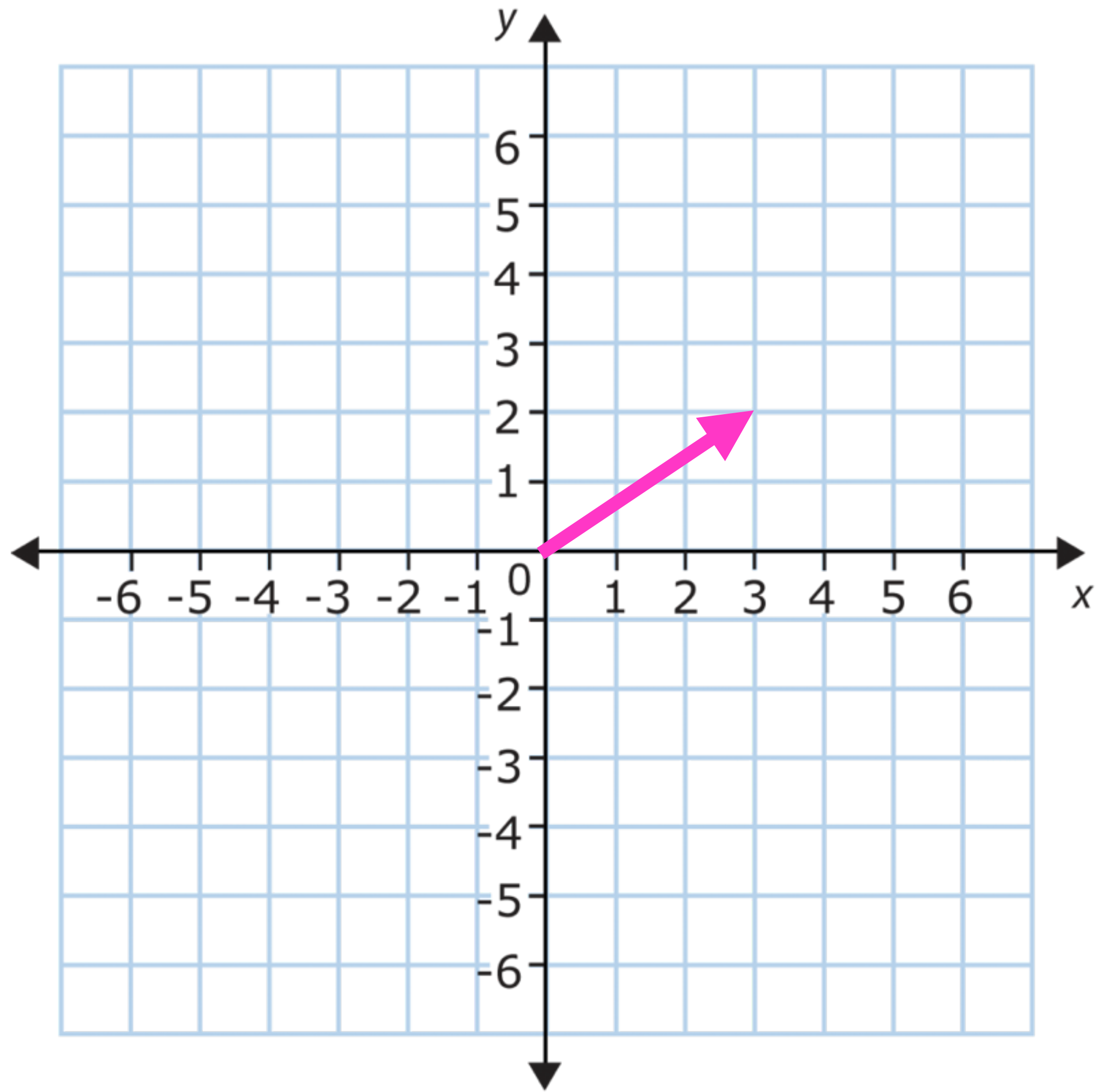
$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & \end{bmatrix}$$

For **each row**, find **dot product with each column**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & D \times K + E \times N + F \times Q \end{bmatrix}$$

Vectors.





A **2 dimensional** vector can be represented  
as a **2x1 matrix**.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A **3 dimensional** vector can be represented  
as a **3x1 matrix**.

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

**Matrix vector** multiplication.

Multiplying a **matrix** and a **vector** is basically just  
**multiplying two matrices.**

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

**Row by row**, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \end{bmatrix}$$

**Row by row**, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ \end{bmatrix}$$



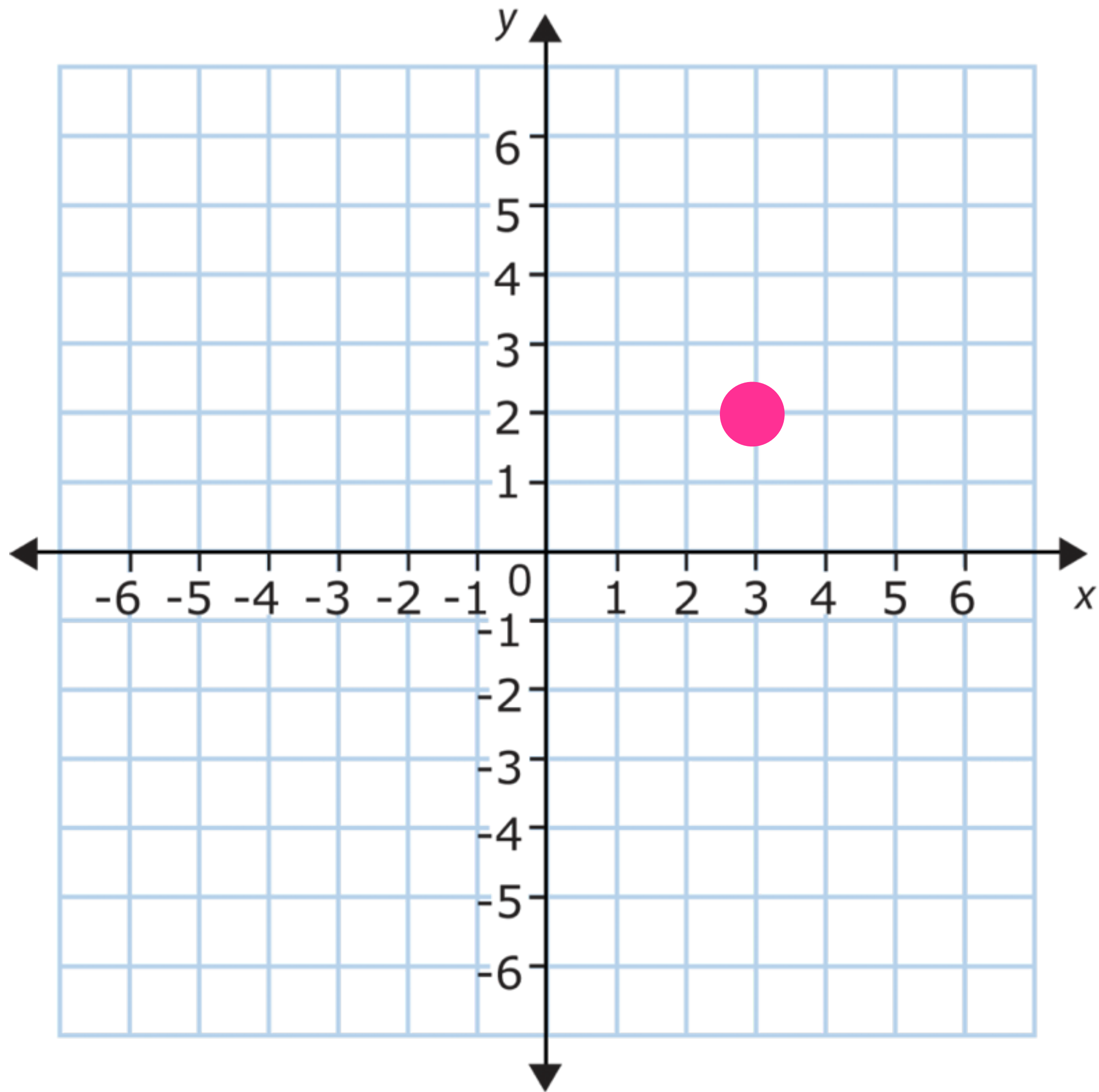
**Row by row**, multiply **each column value** with the **each row of the vector** and **add them together**.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

Why are we doing all this?

**Transformation matrices.**

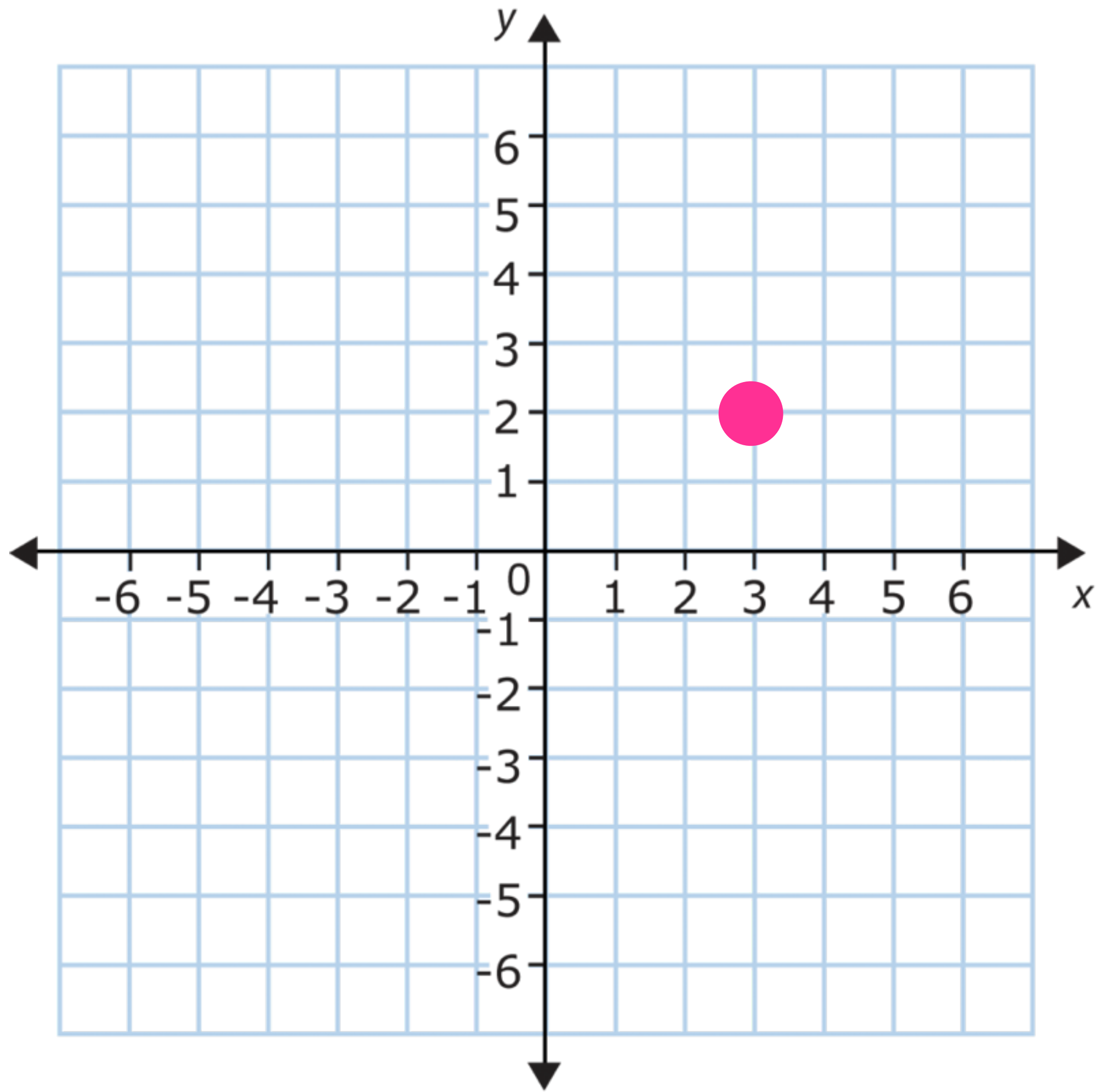
**Affine linear transformations stored  
as matrices.**



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

**A transformation matrix is a matrix that  
we can multiply with a vector to  
transform the vector.**

Example: **scale**



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



# Scale

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

# Scale

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX + BY \\ CX + DY \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} sxX + 0Y \\ 0X + syY \end{bmatrix}$$

Example: **translate?**

**Homogenous** coordinates.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

**Translate**

# Translate

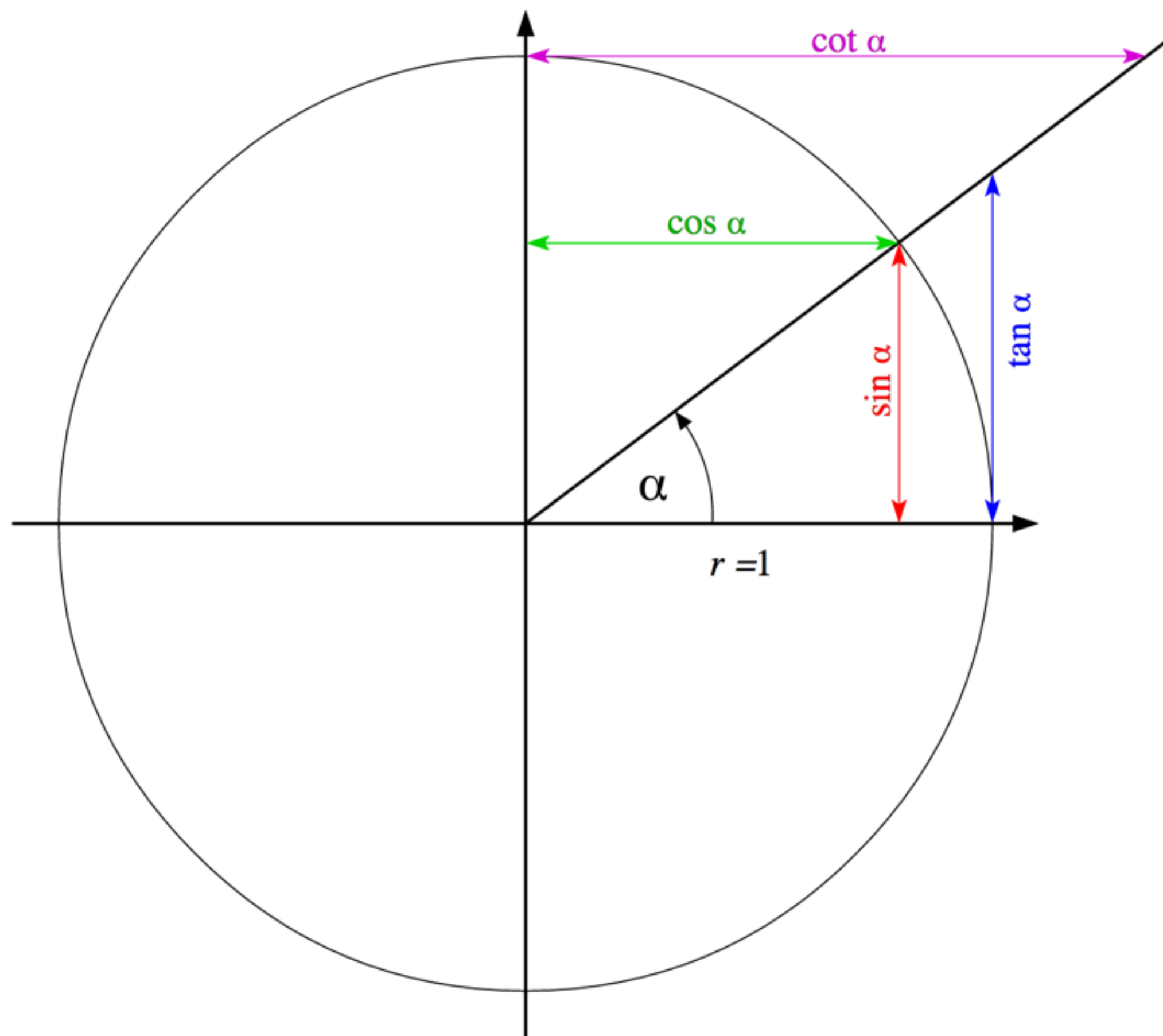
$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1T_x \\ 0X + 1Y + 1T_y \\ 0X + 0Y + 1 \times 1 \end{bmatrix}$$

# Rotation



# Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta X + -\sin\theta Y + 1x0 \\ \sin\theta X + \cos\theta Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$



$$\begin{bmatrix} \cos \theta X + -\sin \theta Y + 1x0 \\ \sin \theta X + \cos \theta Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$

# Identity

# Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1 \times 0 \\ 0X + 1Y + 1 \times 0 \\ 0X + 0Y + 1 \times 1 \end{bmatrix}$$

**Multiplying** affine transformation **matrices**.

You can only multiply **two matrices**  
if the **number of columns of the first matrix** equals the  
**number of rows of the second.**

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = ?$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \checkmark$$

```
matrix.identity();  
matrix.Translate(5.0, 4.0, 0.0);  
matrix.Scale(2.0, 4.0, 1.0);
```

```
// draw vertex at 3,2
```



```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity(); matrix.Translate(5.0, 4.0, 0.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

```
matrix.identity();    matrix.Translate(5.0, 4.0, 0.0);    matrix.Scale(2.0, 4.0, 1.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 1 \end{bmatrix}$$

```
matrix.identity();  
matrix.Scale(2.0, 4.0, 1.0);  
matrix.Translate(5.0, 4.0, 0.0);
```

```
// draw vertex at 3,2
```

```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity();    matrix.Scale(2.0, 4.0, 1.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

```
matrix.identity();
```

```
matrix.Scale(2.0f, 4.0f, 1.0f);
```

```
matrix.Translate(5.0f, 4.0f, 0.0f);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & 4 & 16 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \\ 1 \end{bmatrix}$$

Moving into **3D**



**3D identity matrix and 3d position in homogenous coordinates.**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# All transformations in 3D

<p>X-Rotation in 3D</p> $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	<p>Z-Rotation in 3D</p> $\begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	<p>Scale in 3D</p> $\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
<p>Y-Rotation in 3D</p> $\begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	<p>Translation in 3D</p> $\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

**Projection** matrices are the same.

`matrix.setOrthoProjection(l, r, b, t, n, f);`

$$\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{(r+l)}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{(t+b)}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{(f+n)}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$