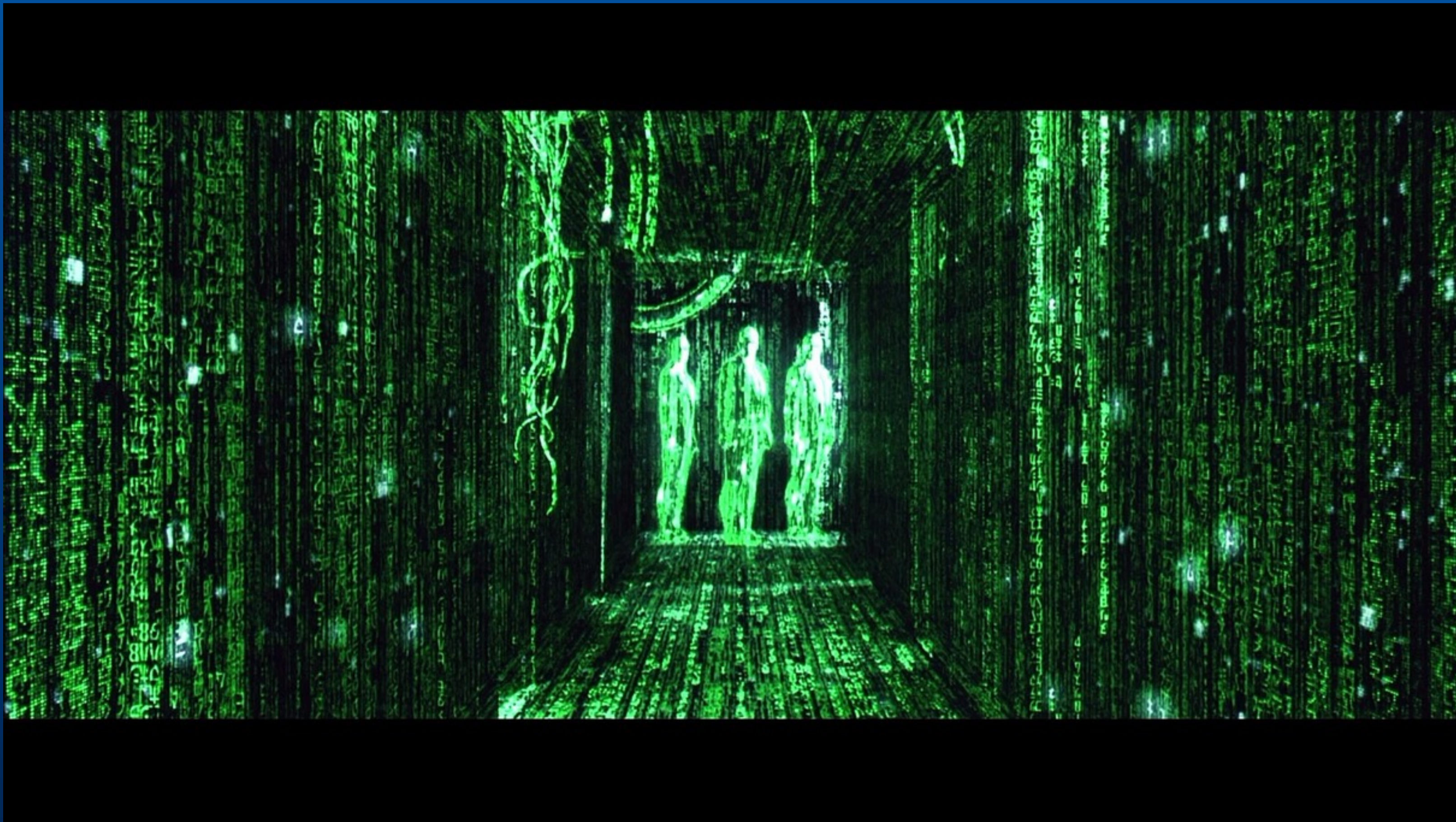


Matrix transformations.

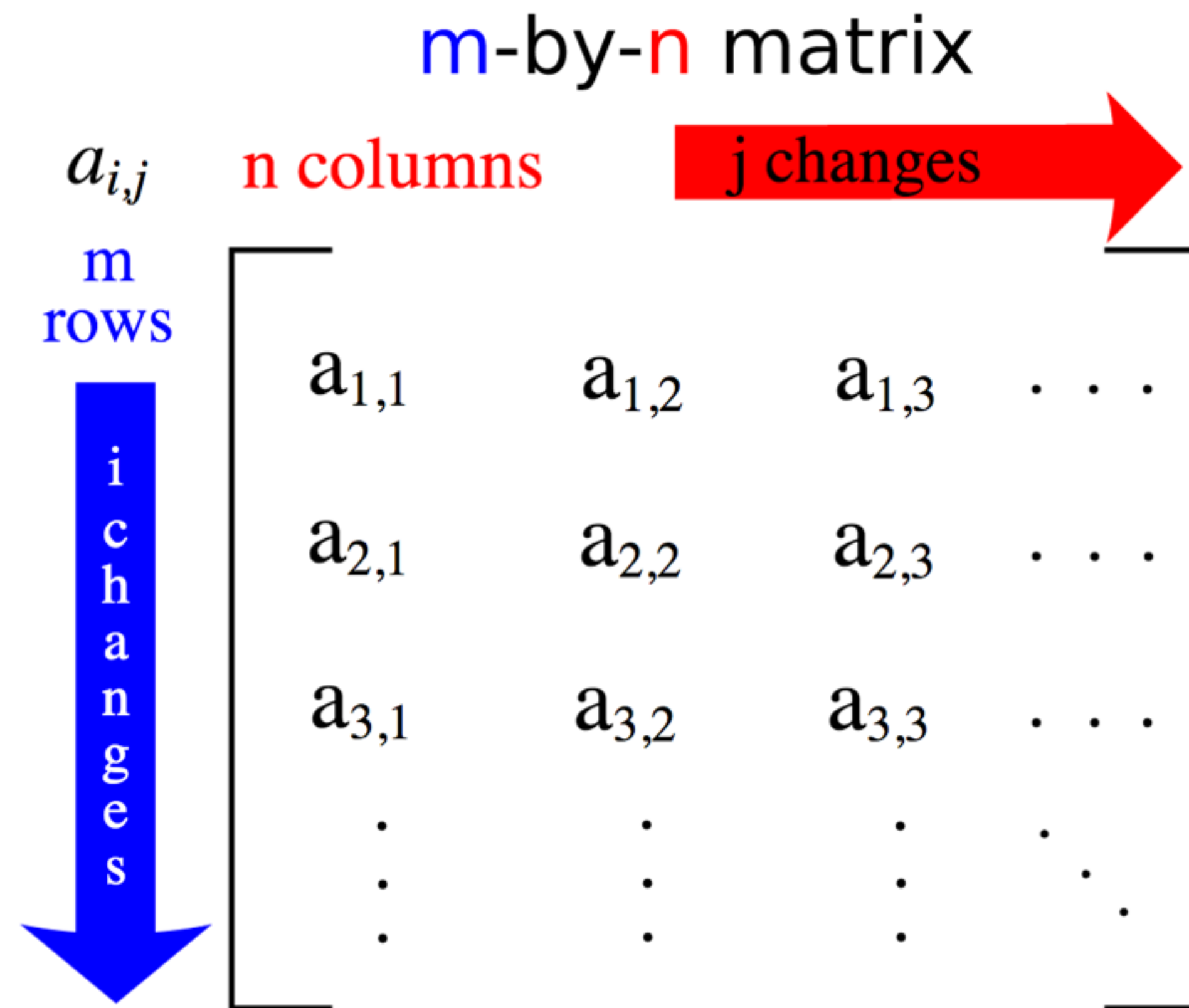
Part 1



Matrix math.

Matrices.

A matrix.



A 2x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

A 3x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

Matrix operations.

Matrix addition.

To add two matrices, add their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} + \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A+J & B+K & C+L \\ D+M & E+N & F+O \\ G+P & H+Q & I+R \end{bmatrix}$$

Matrix subtraction.

To subtract two matrices, subtract their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} - \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A-J & B-K & C-L \\ D-M & E-N & F-O \\ G-P & H-Q & I-R \end{bmatrix}$$

Matrix addition and subtraction can only
happen
with matrices that are the same size!

Transpose of a matrix.

Transpose of a matrix is a matrix whose columns are the rows of the original matrix and its rows are the columns.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$$

M

$$\begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

M^T

Matrix/scalar multiplication.

Multiply each entry of the matrix by the scalar

$$S \times \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} S \times A & S \times B & S \times C \\ S \times D & S \times E & S \times F \\ S \times G & S \times H & S \times I \end{bmatrix}$$

Matrix/matrix multiplication.

You can only multiply two matrices
if the number of columns of the first matrix
equals the number of rows of the second.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = ?$$

It results in a matrix that is number of rows of first matrix by number of columns of second matrix.

$$\begin{bmatrix} \text{A} & \text{B} & \text{C} \\ \text{D} & \text{E} & \text{F} \end{bmatrix} \begin{bmatrix} \text{J} & \text{K} \\ \text{M} & \text{N} \\ \text{P} & \text{Q} \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For each row, find dot product with each column.

The diagram illustrates the dot product of the first row of matrix A with the first column of matrix B. Matrix A is a 2x3 matrix with elements A, B, C in the first row and D, E, F in the second row. Matrix B is a 3x3 matrix with elements J, M, P in the first column and K, N, Q in the second column. A red arrow points from the first row of A to the first column of B, and another red arrow points from the first column of B to the first element of the result vector.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J \\ M \\ P \\ K \\ N \\ Q \end{bmatrix} =$$

$$\begin{bmatrix} AxJ + BxM + CxP \\ \end{bmatrix}$$

For each row, find dot product with each column.

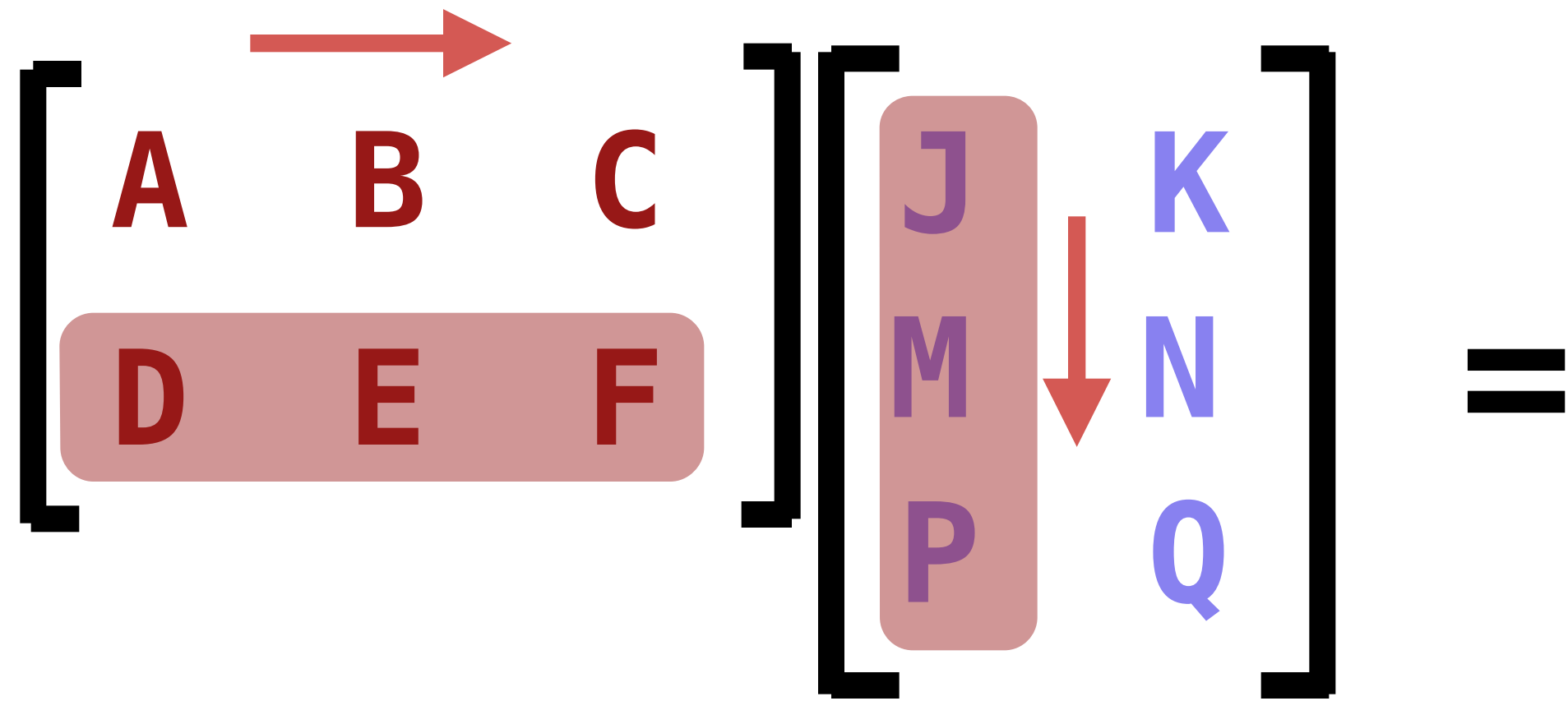
The diagram illustrates the dot product of the first row of matrix A with the second column of matrix B. Matrix A is represented as a 2x3 grid with elements A, B, C in the first row and D, E, F in the second row. A red arrow points from the first row to a red box containing A, B, and C. Matrix B is represented as a 3x2 grid with elements J, K in the first column and M, N, Q in the second column. A red arrow points from the second column to a red box containing K, N, and Q. An equals sign follows the matrices.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

The diagram shows the resulting row vector from the dot products. It is a 1x2 grid with two entries. The first entry is the dot product of the first row of A with the first column of B, calculated as A times J plus B times M plus C times P. The second entry is the dot product of the first row of A with the second column of B, calculated as A times K plus B times N plus C times Q. The entire row vector is enclosed in large square brackets.

$$\begin{bmatrix} AxJ + BxM + CxP & AxK + BxN + CxQ \end{bmatrix}$$

For each row, find dot product with each column.



The diagram illustrates the process of finding the dot product for each row of a matrix. The first matrix is $\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$. A red arrow points from the top row to the second row, indicating the current row being processed. The second matrix is $\begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix}$. A red arrow points from the first column to the second column, indicating the current column being processed. The two matrices are separated by an equals sign, suggesting the result of the dot product operation.

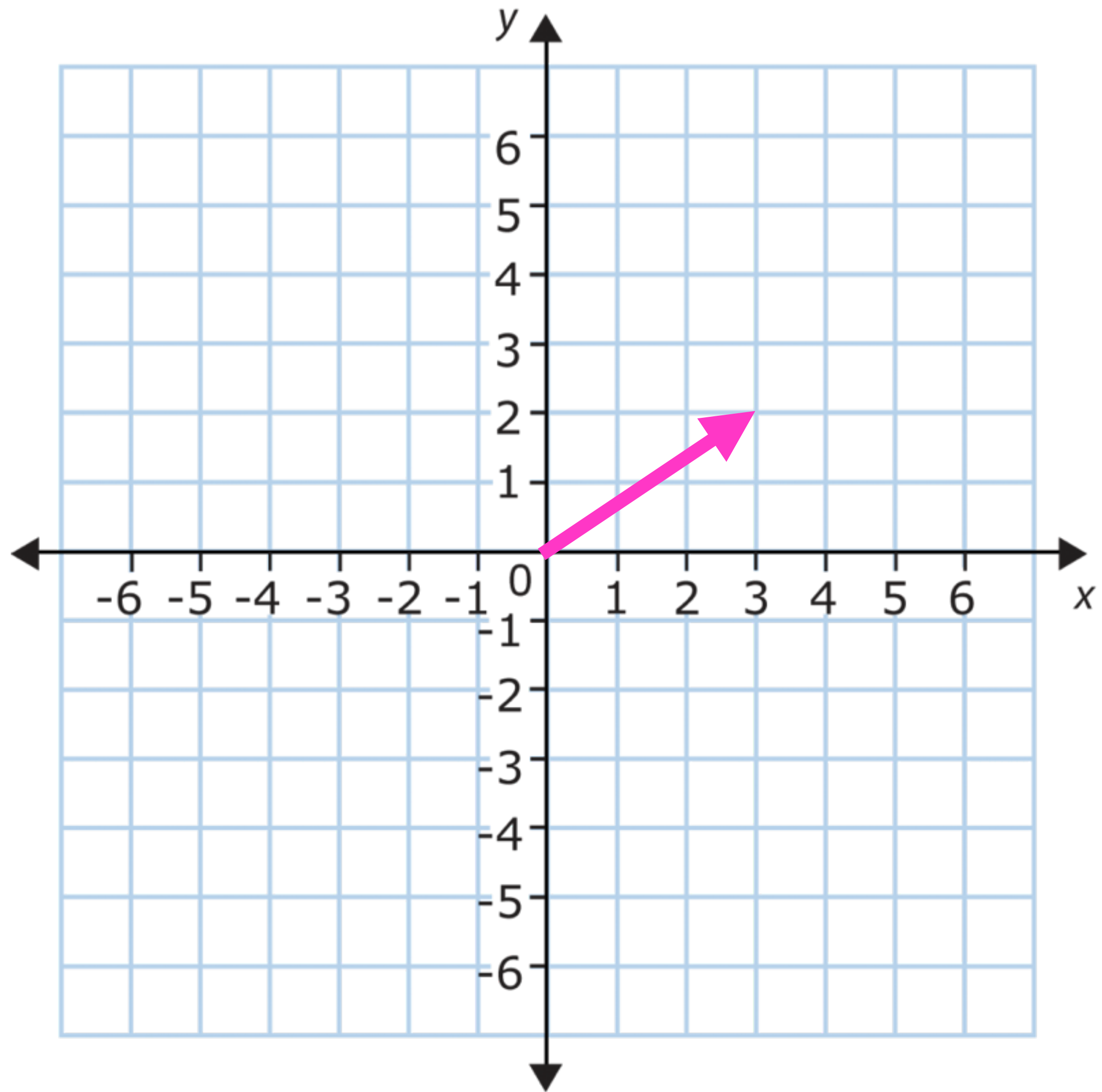
$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P \end{bmatrix}$$

For each row, find dot product with each column.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & D \times K + E \times N + F \times Q \end{bmatrix}$$

Vectors.



A 2 dimensional vector can be represented
as a 2x1 matrix.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A 3 dimensional vector can be represented
as a 3x1 matrix.

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Matrix vector multiplication.

Multiplying a matrix and a vector is basically just multiplying two matrices.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Row by row, multiply each column value with the each row of the vector and add them together.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \end{bmatrix}$$

Row by row, multiply each column value with the each row of the vector and add them together.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ \end{bmatrix}$$

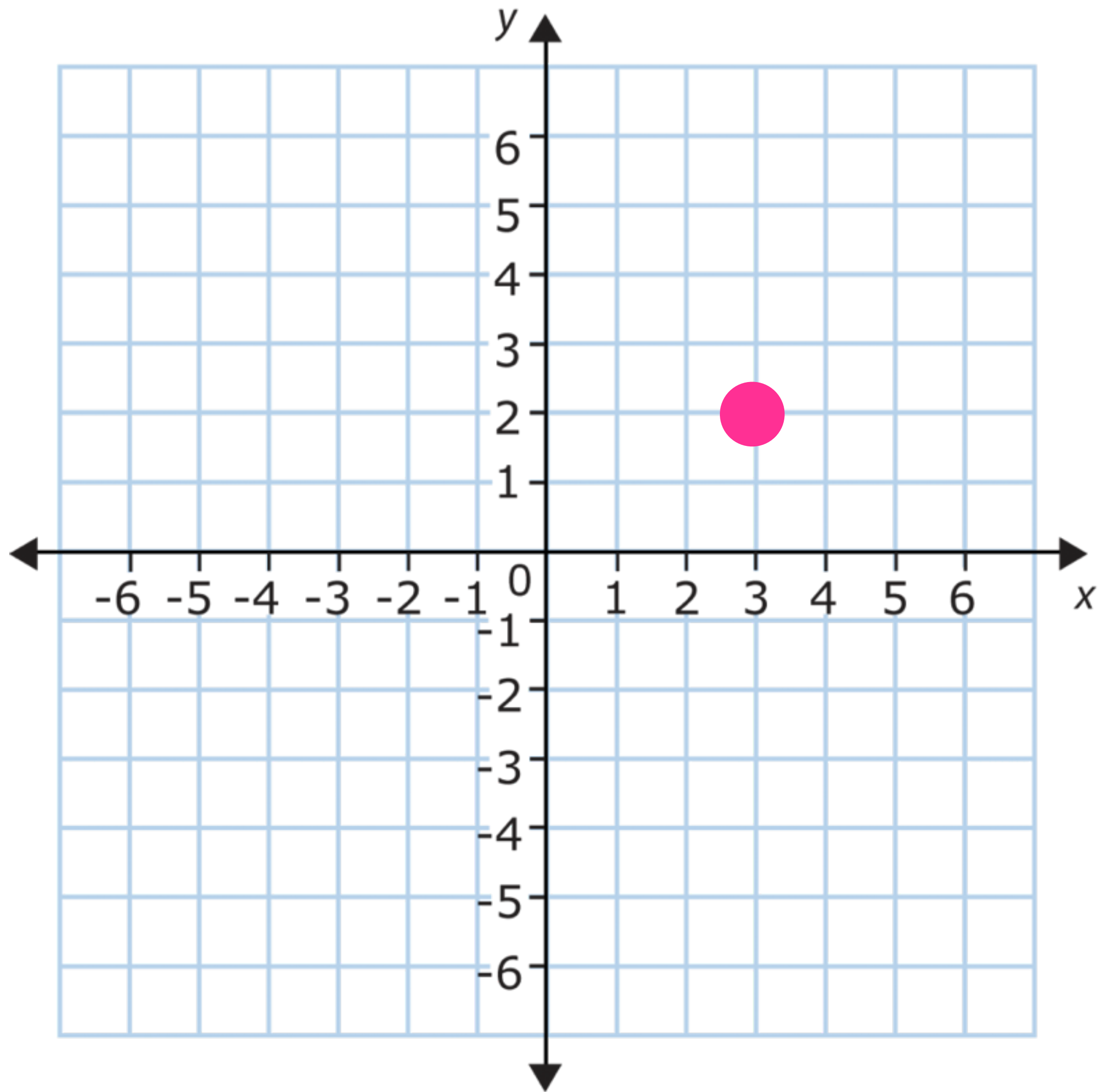
Row by row, multiply each column value with the each row of the vector and add them together.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

Why are we doing all this?

Transformation matrices.

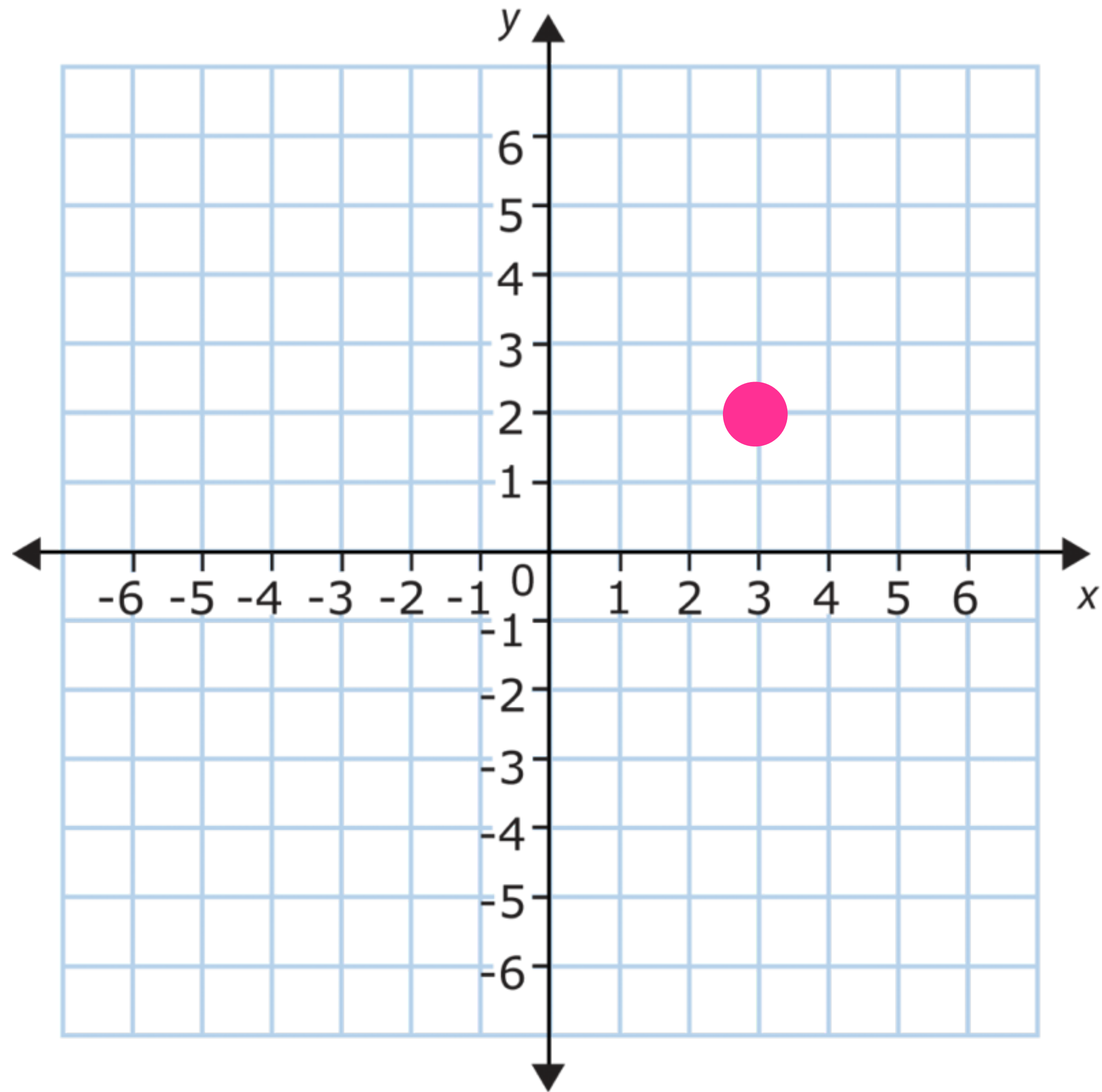
Linear transformations stored
as matrices.



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A transformation matrix is a matrix that we can multiply with a vector to transform the vector.

Example: scale



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX + BY \\ CX + DY \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} sxX + 0Y \\ 0X + syY \end{bmatrix}$$

Example: translate??

Affine transformations
and homogenous
coordinates.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Translate

Translate

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

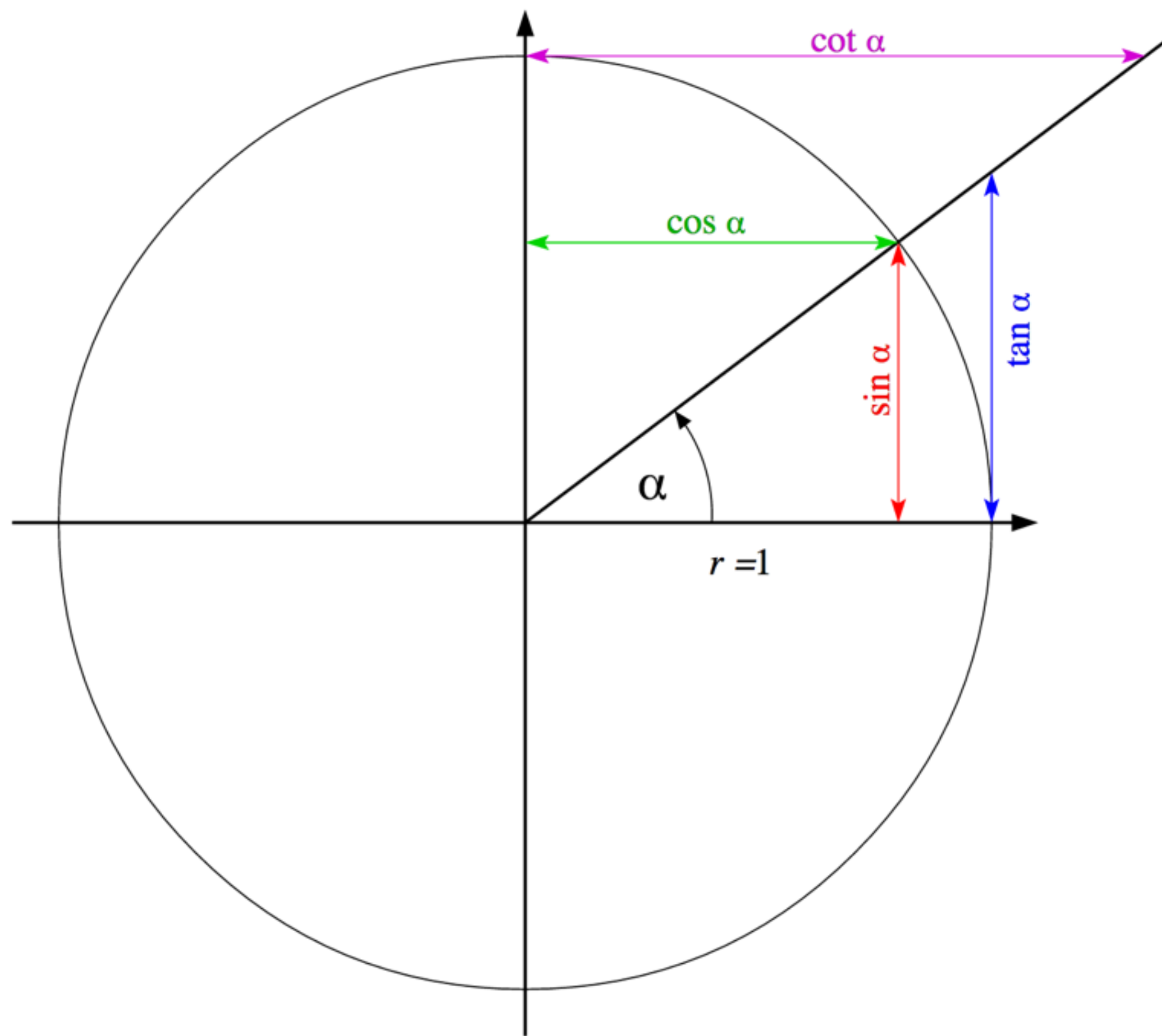
$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1T_x \\ 0X + 1Y + 1T_y \\ 0X + 0Y + 1 \times 1 \end{bmatrix}$$

Rotation

Rotation

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta X + \sin\theta Y + 1 \times 0 \\ -\sin\theta X + \cos\theta Y + 1 \times 0 \\ 0X + 0Y + 1 \times 1 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta X + \sin\theta Y + 1x0 \\ -\sin\theta X + \cos\theta Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Identity

Identity

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1 \times 0 \\ 0X + 1Y + 1 \times 0 \\ 0X + 0Y + 1 \times 1 \end{bmatrix}$$

Multiplying affine matrices.

You can only multiply two matrices
if the number of columns of the first matrix equals the number of rows of
the second.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \text{X}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{✓}$$

```
glLoadIdentity();
```

```
glScalef(2.0, 4.0, 1.0);
```

```
glTranslatef(5.0, 4.0, 0.0);
```

```
// draw vertex at 3,2
```

`glLoadIdentity();`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
glLoadIdentity();    glScalef(2.0, 4.0, 1.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

`glLoadIdentity();`

`glScalef(2.0f, 4.0f, 1.0f);`

`glTranslatef(5.0f, 4.0f, 0.0f);`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & 4 & 16 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \\ 1 \end{bmatrix}$$

```
glLoadIdentity();
```

```
glTranslatef(5.0, 4.0, 0.0);
```

```
glScalef(2.0, 4.0, 1.0);
```

```
// draw vertex at 3,2
```


`glLoadIdentity();`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
glLoadIdentity();
```

```
glTranslatef(5.0, 4.0, 0.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

`glLoadIdentity();`

`glTranslatef(5.0, 4.0, 0.0);`

`glScalef(2.0, 4.0, 1.0);`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 1 \end{bmatrix}$$

Moving into 3D

3D identity matrix and 3d position in homogenous coordinates.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

All transformations in 3D

$$\begin{array}{lll} \text{X-Rotation in 3D} & \text{Z-Rotation in 3D} & \text{Scale in 3D} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \\ \text{Y-Rotation in 3D} & \text{Translation in 3D} & \\ \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} & \end{array}$$

Projection matrices are the same.

glOrtho(l, r, t, b, n, f);

$$\mathbf{P}_o = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{1}{f-n} & -\frac{n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


```
glOrtho(-1.33, 1.33, 1.0, -1.0, -1.0, 1.0);
```

$$\begin{bmatrix} 0.751 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.33 \\ 0.0 \\ 0.0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.999 \\ 0.0 \\ 0.5 \\ 1 \end{bmatrix}$$