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## Assignment 12

Due date: Thursday, 12 December 2024, 23:59

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### Exercise 12.6, Semantics of Predicate Logic

(8 Points)

Prove the following statements using the semantics of predicate logic (Definition 6.36). Do not use other results from the lecture notes.

a)  $\exists x \forall y P(x, y) \models \exists x P(x, f(x))$ .

b)  $\neg(\forall x P(x)) \models \exists x \neg P(x)$ .

a)

Let  $\mathcal{A} = (U, \phi, \psi, \xi)$  be an arbitrary suitable structure for the formulas. Assume  $\mathcal{A}$  is a model for the first formula, i.e.  $\mathcal{A} \models \exists x \forall y P(x, y)$ . There are no free variables in the formula since all are bound, hence it is a closed formula. The  $P^{\mathcal{A}}$  is omitted for simplicity, as it remains a free predicate symbol not involved in the proof. By Definition 6.9 of suitable interpretation for a formula:

$$\mathcal{A} \models \exists x \forall y P(x, y) \implies \mathcal{A}(\exists x \forall y P(x, y)) = 1$$

Recall the semantics of  $\exists$ ,  $\mathcal{A}(\exists x G) = 1$  if  $\mathcal{A}_{[x \rightarrow u]}(G) = 1$  for some  $u \in U$  (Definition 6.36). Since the formula  $\forall y P(u, y)$  is true for some  $u \in U^{\mathcal{A}}$ , applying the semantics results in overriding  $\xi(x) = u$  for:

$$\mathcal{A}_{[x \rightarrow u]}(\forall y P(x, y)) = 1 \quad \text{for some } u \in U^{\mathcal{A}}$$

Recall the semantics of  $\forall$ ,  $\mathcal{A}(\forall x G) = 1$  if  $\mathcal{A}_{[x \rightarrow u]}(G) = 1$  for all  $u \in U$  (Definition 6.36). Since the formula  $P(u, w)$  is true (for some  $u \in U^{\mathcal{A}}$  and) for all  $w \in U^{\mathcal{A}}$ , applying the semantics results in overriding also  $\xi(y) = w$  for:

$$\mathcal{A}_{[x \rightarrow u][y \rightarrow w]}(P(x, y)) = 1 \quad \text{for some } u \in U^{\mathcal{A}} \text{ and for all } w \in U^{\mathcal{A}}$$

Notice that the constraint on  $w$  is very loose, and most importantly, that it is satisfied by definition of  $\phi : U^k \rightarrow U$  with  $k = 1$  for any arbitrary unary function symbol  $f$  in the structure (Definition 6.36). Specifically,  $\phi : U \rightarrow U$  where  $w = \phi(f)(\mathcal{A}(x)) = \phi(f)(\xi(x)) = \phi(f)(u)$  with  $u \in U^{\mathcal{A}}$  and  $f$  arbitrary. Instead of  $\phi(f)$ , the notation  $f^{\mathcal{A}}$  is used for simplicity. Thus:

$$\mathcal{A}_{[x \rightarrow u][y \rightarrow f^{\mathcal{A}}(x)]}(P(x, y)) = 1 \quad \text{for some } u \in U^{\mathcal{A}} \text{ and arbitrary } f^{\mathcal{A}}$$

By using the recursivity of the semantics of Predicate Logic, the unary function symbols is embedded as a term in the formula (Definition 6.36). The superscript  $\mathcal{A}$  is omitted for simplicity on unary function symbol  $f$  of arbitrary suitable structure  $\mathcal{A}$ .

$$\mathcal{A}_{[x \rightarrow u]}(P(x, f(x))) = 1 \quad \text{for some } u \in U^{\mathcal{A}}$$

Thus, by applying the semantics of  $\exists$  again, since the formula  $P(u, f(u))$  is true for some  $u \in U^{\mathcal{A}}$ , the proof to show that the structure  $\mathcal{A}$  is a model for the second formula is completed:

$$\mathcal{A}(\exists x P(x, f(x))) = 1$$

Finally,

$$\exists x \forall y P(x, y) \models \exists x P(x, f(x))$$

**b)**

Let  $\mathcal{A} = (U, \phi, \psi, \xi)$  be an arbitrary suitable structure for the formulas. Assume  $\mathcal{A}$  is a model for the first formula, i.e.  $\mathcal{A} \models \neg(\forall x P(x))$ . There are no free variables in the formula since all are bound, hence it is a closed formula. The  $P^{\mathcal{A}}$  is omitted for simplicity, as it remains a free predicate symbol not involved in the proof. By Definition 6.9 of suitable interpretation for a formula:

$$\mathcal{A} \models \neg(\forall x P(x)) \implies \mathcal{A}(\neg(\forall x P(x))) = 1$$