Discrete Mathematics **2024**

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Assignment 2 **Due date:** Thursday, 3 October 2024, 23:59

Exercise 2.3, Simplifying a Formula ($\star$) (8 Points)

Consider the propositional formula

$$F = ((B \lor C) \to ((A \lor \neg B) \land C)) \lor (A \land \neg C)$$

Give a formula G that is equivalent to F, but in which each atomic formula A, B, and C appears at most once. Prove that $F \equiv G$ by providing a sequence of equivalence transformations with *at most* 12 steps.

Expectation. Your proof should be in the form of a sequence of steps, where each step consists of applying the definition of $\to$ (that is $F \to G \equiv \neg F \lor G$), one of the rules given in Lemma 2.1 of the lecture notes [1], or one of the following rules: $F \land \neg F \equiv \bot$, $F \land \bot \equiv \bot$, $F \lor \bot \equiv F$, $F \lor \neg F \equiv \top$, $F \land \top \equiv F$, and $F \lor \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1.3. Each step of your proof should apply a *single* rule *once* and state *which* rule was applied.

$$\begin{aligned}
F &= ((B \lor C) \to ((A \lor \neg B) \land C)) \lor (A \land \neg C) \\
&\equiv \Big(\neg (B \lor C) \lor ((A \lor \neg B) \land C)\Big) \lor (A \land \neg C) && \text{def. of implication} \to \\
&\equiv \Big(\big(\neg B \land \neg C\big) \lor ((A \lor \neg B) \land C)\Big) \lor (A \land \neg C) && \text{def. De Morgan rule} \\
&\equiv \Big((\neg B \land \neg C) \lor \big(C \land (A \lor \neg B)\big)\Big) \lor (A \land \neg C) && \text{def. commutative of } \land \\
&\equiv \Big((\neg B \land \neg C) \lor (C \land A) \lor (C \land \neg B)\Big) \lor (A \land \neg C) && \text{def. of 1st distributive law} \\
&\equiv \Big((\neg B \land \neg C) \lor (C \land \neg B) \lor (C \land A)\Big) \lor (A \land \neg C) && \text{def. commutative of } \lor \\
&\equiv \Big(\big(\neg B \land (C \lor \neg C)\big) \lor (C \land A)\Big) \lor (A \land \neg C) && \text{def 1st distributive law} \\
&\equiv \Big(\big(\neg B \land \top\big) \lor (C \land A)\Big) \lor (A \land \neg C) && F \lor \neg F \equiv \top \\
&\equiv \Big(\neg B \lor (C \land A)\Big) \lor (A \land \neg C) && F \land \top \equiv F \\
&\equiv \neg B \lor \Big((C \land A) \lor (A \land \neg C)\Big) && \text{def. associativity of } \lor \\
&\equiv \neg B \lor \Big(A \land (C \lor \neg C)\Big) && \text{def. 1st distributive law} \\
&\equiv \neg B \lor \Big(A \land \top\Big) && F \lor \neg F \equiv \top \\
&\equiv \neg B \lor A && F \land \top \equiv F
\end{aligned}$$

Finally, for the formula G defined as $G = \neg B \lor A$, we have shown that $F \equiv G$ by applying a 12-step sequence of equivalence transformations.

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[1] Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).