Assignment 3 Jeferson Morales Mariciano

Exercise 1

Use field and order axioms to show that $0 \leq y-x$ $\forall x,y \in \mathbb{Q}$ with $x \leq y$

$0 \leq y-x$ $\boxed{O_4}$ $0+x \leq y-x+x$ $\boxed{F_3}$ $x \leq y-x+x$ $\boxed{F_4}$ $x \leq y$

Exercise 2

Use field and order axioms to show that

$$x \leq \frac{x+y}{2} \leq y \quad \forall x,y \in \mathbb{Q} \quad \text{with} \quad x \leq y$$

$x \leq \frac{x+y}{2}$ $\boxed{O5}$ $2 \cdot x \leq \frac{x+y}{\cancel{2}} \cancel{2}$ $\Rightarrow$ $2x \leq x+y$ $\boxed{O4}$ $2x+(-x) \leq x+y+(-x)$

$\boxed{F_4}$ $x \leq y$ ✓

$\frac{x+y}{2} \leq y$ $\boxed{O5}$ $\cancel{2} \cdot \frac{x+y}{\cancel{2}} \leq y \cdot 2$ $\Rightarrow$ $x+y \leq 2y$ $\boxed{O4}$ $x+y+(-y) \leq 2y+(-y)$

$\boxed{F_4}$ $x \leq y$ ✓

$x \leq \frac{x+y}{2} \leq y$ with $z = \frac{x+y}{2}$, then $x \leq z \leq y$ respects transitivity O_3 axiom.

Exercise 3

Prove by induction that $|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$ $\forall n \in \mathbb{N}$

and $\forall x_1, \dots, x_n \in \mathbb{Q}$

1) base case: $n=1$ $|x_1| \leq |x_1|$ $1 \leq 1$ ✓

2) inductive step: $n = n+1$

$\left|\sum_{k=1}^{n} x_k\right| = |x_1 + x_2 + \dots + x_n|$ $\Rightarrow$ $\left|\sum_{k=1}^{n+1} x_k\right| = \left|\sum_{k=1}^{n} x_k + x_{n+1}\right|$

$\sum_{k=1}^{n} |x_k| = |x_1| + |x_2| + \dots + |x_n|$ $\Rightarrow$ $\sum_{k=1}^{n+1} |x_k| = \sum_{k=1}^{n} |x_k| + |x_{n+1}|$

I.H. $\left|\underbrace{\sum_{k=1}^{n} x_k}_{x} + \underbrace{x_{n+1}}_{y}\right| \leq \underbrace{\sum_{k=1}^{n} |x_k|}_{x} + \underbrace{|x_{n+1}|}_{y}$

$|x+y| \leq |x| + |y|$ triangle inequality $\forall x,y \in \mathbb{Q}$, thus for any $x_1, \dots, x_n \in \mathbb{Q}$

Bonus Exercise

$F_n = F_{n-2} + F_{n-1}$ $F_0 = 0$ $F_1 = 1$ $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ $\forall n \in \mathbb{N}$

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$

Prove by induction that the numbers F_{3n} are even $\forall n \in \mathbb{N}$ $\Rightarrow F_{3n} = 2K$

1) base case: $(n=1)$ $F_{3 \cdot 1} = 2$ $2K = 2$ True with $K = 1$

2) inductive step: $(n = n+1)$ $F_{3(n+1)} \Rightarrow F_{3n+3}$

if $F_n = F_{n-2} + F_{n-1}$
Then, $F_{3n+3} = F_{3n+1} + F_{3n+2}$
then, $F_{3n+2} = F_{3n} + F_{3n+1}$

I.H. $F_{3n+3} = 2K$

$F_{3n+1} + F_{3n+2}$

$F_{3n+1} + F_{3n} + F_{3n+1} \Rightarrow \underbrace{2(F_{3n+1})}_{2K} + F_{3n}$

true with $K = F_{3n+1}$ and $F_{3n} = 2K$ by I.H

So, always even number $\forall n \in \mathbb{N}$