Discrete Mathematics **2024**

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Assignment 12

Due date: Thursday, 12 December 2024, 23:59

Exercise 12.6, Semantics of Predicate Logic (8 Points)

Prove the following statements using the semantics of predicate logic (Definition 6.36). Do not use other results from the lecture notes.

a) $\exists x\,\forall y\,P(x,y) \models \exists x\,P(x,f(x))$.

b) $\neg(\forall x\;P(x)) \models \exists x\,\neg P(x)$.

a)

Let $\mathcal{A} = (U,\phi,\psi,\xi)$ be an arbitrary suitable structure for the formulas. Assume $\mathcal{A}$ is a model for the first formula, i.e. $\mathcal{A} \models \exists x\,\forall y\,P(x,y)$. There are no free variables in the formula since all are bound, hence it is a closed formula. The $P^{\mathcal{A}}$ is omitted for simplicity, as it remains a free predicate symbol not involved in the proof. By Definition 6.9 of suitable interpretation for a formula:

$$\mathcal{A} \models \exists x\,\forall y\,P(x,y) \implies \mathcal{A}\left(\exists x\,\forall y\,P(x,y)\right) = 1$$

Recall the semantics of $\exists$, $\mathcal{A}(\exists x\,G) = 1$ if $\mathcal{A}_{[x\to u]}(G) = 1$ for some $u \in U$ (Definition 6.36). Since the formula $\forall y\,P(u,y)$ is true for some $u \in U^{\mathcal{A}}$, applying the semantics results in overriding $\xi(x) = u$ for:

$$\mathcal{A}_{[x\to u]}\left(\forall y\,P(x,y)\right) = 1 \quad \text{for some } u \in U^{\mathcal{A}}$$

Recall the semantics of $\forall$, $\mathcal{A}(\forall x\,G) = 1$ if $\mathcal{A}_{[x\to u]}(G) = 1$ for all $u \in U$ (Definition 6.36). Since the formula $P(u,w)$ is true (for some $u \in U^{\mathcal{A}}$ and) for all $w \in U^{\mathcal{A}}$, applying the semantics results in overriding also $\xi(y) = w$ for:

$$\mathcal{A}_{[x\to u][y\to w]}\left(P(x,y)\right) = 1 \quad \text{for some } u \in U^{\mathcal{A}} \text{ and for all } w \in U^{\mathcal{A}}$$

Notice that the contraint on w is very loose, and most importantly, that it is satisfied by definition of $\phi : U^k \to U$ with $k = 1$ for any arbitrary unary function symbol f in the structure (Definition 6.36). Specifically, $\phi : U \to U$ where $w = \phi(f)(\mathcal{A}(x)) = \phi(f)(\xi(x)) = \phi(f)(u)$ with $u \in U^{\mathcal{A}}$ and f arbitrary. Instead of $\phi(f)$, the notation $f^{\mathcal{A}}$ is used for simplicity. Thus:

$$\mathcal{A}_{[x\to u][y\to f^{\mathcal{A}}(x)]}\left(P(x,y)\right) = 1 \quad \text{for some } u \in U^{\mathcal{A}} \text{ and arbitrary } f^{\mathcal{A}}$$

By using the recursivity of the semantics of Predicate Logic, the unary function symbols is embedded as a term in the formula (Definition 6.36). The superscript $\mathcal{A}$ is omitted for simplicity on unary function symbol f of arbitrary suitable structure $\mathcal{A}$.

$$\mathcal{A}_{[x\to u]}\left(P(x,f(x))\right) = 1 \quad \text{for some } u \in U^{\mathcal{A}}$$

Thus, by applying the semantics of $\exists$ again, since the formula $P(u, f(u))$ is true for some $u \in U^{\mathcal{A}}$, the proof to show that the structure $\mathcal{A}$ is a model for the second formula is completed:

$$\mathcal{A}(\exists x\, P(x, f(x))) = 1$$

Finally,

$$\exists x\, \forall y\, P(x, y) \models \exists x\, P(x, f(x))$$

b)

Let $\mathcal{A} = (U, \phi, \psi, \xi)$ be an arbitrary suitable structure for the formulas. Assume $\mathcal{A}$ is a model for the first formula, i.e. $\mathcal{A} \models \neg(\forall x\, P(x))$. There are no free variables in the formula since all are bound, hence it is a closed formula. The $P^{\mathcal{A}}$ is omitted for simplicity, as it remains a free predicate symbol not involved in the proof. By Definition 6.9 of suitable interpretation for a formula:

$$\mathcal{A} \models \neg(\forall x\, P(x)) \implies \mathcal{A}\,(\neg(\forall x\, P(x))) = 1$$

Recall the semantics of $\neg$ for a formula in predicate logic (Definition 6.16, 6.24). Since $\forall x\, P(x)$ is a formula in predicate logic (Definition 6.36), then also its negation is (Definition 6.31). Since $\mathcal{A}(\neg F) = 1 \iff \mathcal{A}(F) = 0$, recursively decoupling the formula yields:

$$\mathcal{A}\,(\forall x\, P(x)) = 0$$

Recall the semantics of $\forall$, $\mathcal{A}(\forall x\, G) = 1$ if $\mathcal{A}_{[x \to u]}(G) = 1$ for all $u \in U$, otherwise $\mathcal{A}(\forall x\, G) = 0$ (Definition 6.36). Since the formula $P(u)$ is **not** true for all $u \in U^{\mathcal{A}}$, applying the semantics results in overriding $\xi(x) = u$ for:

$$\mathcal{A}_{[x \to u]}\,(P(x)) = 0 \quad \text{for } \textbf{not} \text{ all } u \in U^{\mathcal{A}}$$

Reasoning about the formula $P(u)$ not being true for all $u \in U^{\mathcal{A}}$, means that there is some, at least one, u in the Universe for which $P(u)$ is false. Rephrasing the statement, $P(u)$ is false for some $u \in U^{\mathcal{A}}$, or equivalently, $\neg P(u)$ is true for some $u \in U^{\mathcal{A}}$ by semantics of $\neg$ (Definition 6.16). Thus:

$$\mathcal{A}_{[x \to u]}\,(\neg P(x)) = 1 \quad \text{for some } u \in U^{\mathcal{A}}$$

Recall semantics of $\exists$, $\mathcal{A}(\exists x\, G) = 1$ if $\mathcal{A}_{[x \to u]}(G) = 1$ for some $u \in U$ (Definition 6.36). Since the formula $\neg P(u)$ is true for some $u \in U^{\mathcal{A}}$, applying the semantics results in:

$$\mathcal{A}\,(\exists x\, \neg P(x)) = 1$$

Finally, this complete the proof that for any arbitrary structure for the first formula, it is also a model for the second formula:

$$\neg\,(\forall x\, P(x)) \models \exists x\, \neg P(x)$$