Definition. To specify a ***function*** f you must

(1) give a ***rule*** which tells you how to compute the value $f(x)$ of the function for a given real number x, and:
(2) say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its ***domain***. The set of all possible numbers $f(x)$ as x runs over the domain is called the ***range*** of the function. The rule must be ***unambiguous:*** the same $x$must always lead to the same $f(x)$.

For instance, one can define a function f by putting $f(x) = \sqrt{x}$ for all $x \geq 0$. Here the rule defining f is "take the square root of whatever number you're given", and the function f will accept all nonnegative real numbers.

The rule which specifies a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases} \qquad \text{domain of } g = \text{all real numbers.}$$

Linear functions. A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a ***linear function***. Its graph is a straight line. The constants m and n are the ***slope*** and ***y-intercept*** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y-axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

Example – find the domain and range of $f(x) = 1/x^2$. The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\text{"all real numbers except 0"} = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask "for which y can we solve the equation $y = f(x)$ for x," i.e. we for which y can you solve $y = 1/x^2$ for x?

If $y = 1/x^2$ then we must have $x^2 = 1/y$, so first of all, since we have to divide by y, y can't be zero. Furthermore, $1/y = x^2$ says that y must be positive. On the other hand, if $y > 0$ then $y = 1/x^2$ has a solution (in fact two solutions), namely $x = \pm 1/\sqrt{y}$. This shows that the range of f is

$$\text{"all positive real numbers"} = \{x \mid x > 0\} = (0, \infty).$$

Maxima and Minima

A function has a ***global maximum*** at some a in its domain if $f(x) \leq f(a)$ for all other x in the domain of f. Global maxima are sometimes also called "absolute maxima."

A function has a ***local maximum*** at some a in its domain if there is a small $\delta > 0$ such that $f(x) \leq f(a)$ for all x with $a - \delta < x < a + \delta$ which lie in the domain of f.

Every global maximum is a local maximum, but a local maximum doesn't have to be a global maximum.

7.1. Where to find local maxima and minima. Any x value for which $f'(x) = 0$ is called a ***stationary point*** for the function f.

Derivatives Defined

Definition. *Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.*

*The **derivative of the function** f **at** a is the value of the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

*f is said to be **differentiable at** a if this limit exists.*

*f is called **differentiable on the interval** (c, d) if it is differentiable at every point a in (c, d).*

The slope of the tangent the tangent to the graph of f at the point $(a, f(a))$ is

$$m = f'(a)$$

and hence the equation for the tangent is

$$y = f(a) + f'(a)(x - a).$$