Discrete Mathematics **2024**

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Assignment 5 **Due date:** Thursday, 24 October 2024, 23:59

Exercise 5.5, Properties of Relations ($\star$) (8 Points)

Prove or disprove the following claims:

a) A relation ρ on a set A is symmetric on A if and only if ρ^2 is symmetric on A.

b) If ρ is a relation on a set A that is symmetric and antisymmetric, then it musta hold $\rho = \mathsf{id}_A$.

c) Define the relations ρ_1 and rho_2 on $\mathbb{Z}$ as

$$a \, \rho_1 \, b \iff b = a + 1, \qquad a \, \rho_2 \, b \iff b \equiv_2 a.$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$.

a)

The claim is false, a counterexample follows:

$$\rho = \{(a, c), (b, d), (c, b), (d, a)\}$$
$$\rho^2 = \{(a, b), (b, a), (c, d), (d, c)\}$$

The relation ρ^2 is symmetric, but ρ is not, which disprove the claim if and only if ($\iff$) from the right to left part ($\Longleftarrow$).

The intuition behind the counterexample is given by expanding the right hand side ($\Longleftarrow$) of the claim into a composition of implications as follows. Note the universe $\mathcal{U}$ is an arbitrary set A of elements.

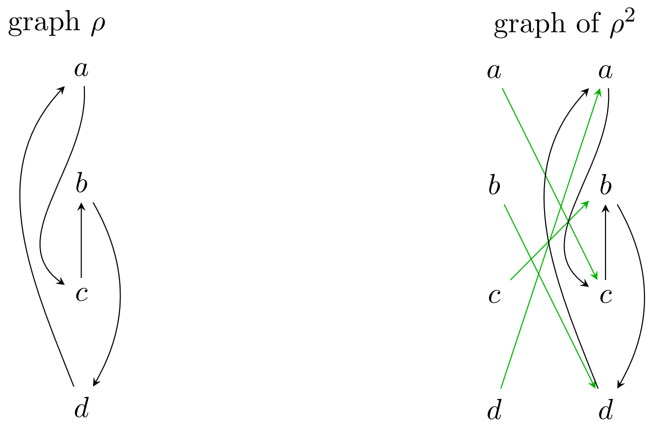
$$\begin{aligned}
&\rho^2 \text{ symmetric on } A \\
&\overset{\cdot}{\Longrightarrow} \forall a \forall b \left(a \, \rho^2 \, b \iff b \, \rho^2 \, a \right) && \text{(def 3.15 symmetry)} \\
&\overset{\cdot}{\Longrightarrow} \forall a \forall b \left((a, b) \in \rho^2 \land (b, a) \in \rho^2 \right) && \text{(def 3.9 relation)} \\
&\overset{\cdot}{\Longrightarrow} \forall a \forall b \left(\exists c \left((a, c) \in \rho \land (c, b) \in \rho \right) \land \exists d \left((b, d) \in \rho \land (d, a) \in \rho \right) \right) && \text{(def 3.12 composition)}
\end{aligned}$$

So it is sufficient to find a relation ρ that is not symmetric, i.e. which have $c \neq d$, satisfying the constraint of the composition of implications above. A visualization is provided in the following matrix:

$$M^{\rho^2} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = M^{\rho} \cdot M^{\rho}, \quad \text{where} \quad M^{\rho} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The matrices resemble the counterexample, where ρ^2 is symmetric, while ρ is clearly not, i.e. $M^\rho \neq (M^\rho)^\intercal \iff \rho \neq \hat{\rho}$.

A graph representation is provided for the sake of visualization, where the composition of the same structure choosen by the matrices is visualize. The elements of ρ^2 are given by choosing a path with length 2, meaning a green edge and the subsequent black one on the right. Doing so for every vertex gives the ρ^2 relation.



b)

The claim is false, a counterexample follows:

$$\rho = \emptyset \neq \mathsf{id}_A$$

The relation ρ is both symmetric and antisymmetric. However, ρ is not the identity relation as the claim states.

c)

The claim is true, and it can be proven as follows:

we will decompose the claim through composition of implications from which will lead to a clear satisfiability problem that show is satisfiable using a proof by case distinction.

Composition of Implications

Case Distinction

Check that the statement is always satisfiable for all cases of a, b, i.e. that there always exist some suitable c in the universe to satisfy the claim.

The cases are represented by the following matrix:

a	b	c
0	0	0
0	1	$s(a)$
1	0	$s(a)$
1	1	1

where $s(x)$ is the successor of x in the universe. Finally, it follows that the claim is always satisfiable, meaning is true for all cases of $a, b \in \mathcal{U} = \mathbb{Z} \times \mathbb{Z}$. Concluding, $\rho^2 = \mathbb{1} = \mathbb{Z} \times \mathbb{Z}$.