

Student: Jeferson Morales Mariciano <jmorale@ethz.ch>

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## Assignment 3

Due date: Thursday, 10 October 2024, 23:59

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### Exercise 3.2, From Natural Language to a Formula (★) (4 Points)

Consider the universe  $U = \mathbb{N} \setminus \{0\}$ . Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are  $divides(x, y)$ ,  $equals(x, y)$  and  $prime(x)$  (instead of  $divides(x, y)$  and  $equals(x, y)$  you can write  $x|y$  and  $x = y$  accordingly). You can also use the symbols  $+$  and  $\cdot$  to denote the addition and multiplication functions, and you can use constants (e.g., 0, 1, . . .). You can also use  $\leftrightarrow$ . No justification is required.

- (i) (★) If a number divides two numbers, then it also divides their sum.
- (ii) (★) The only divisors of a prime number are 1 and the number itself.
- (iii) (★) 1 is the only natural number which has an inverse.
- (iv) (★) A prime number divides the product of two natural numbers if and only if it divides at least one of them.

i)

$$\forall x \forall y \forall z ((x|y) \wedge (x|z)) \rightarrow (x|(y+z)) \quad (1)$$

ii)

$$\forall x \forall y ((prime(x) \wedge (y|x)) \rightarrow ((y=1) \vee (y=x))) \quad (2)$$

iii)

$$\forall x \forall y ((x \cdot y = 1) \rightarrow ((x=1) \wedge (y=1))) \quad (3)$$

iv)

$$\forall x (prime(x) \rightarrow \forall y \forall z ((x|(y \cdot z)) \leftrightarrow ((x|y) \vee (x|z)))) \quad (4)$$

### Exercise 3.8, Proof by Contradiction (★) (4 Points)

Let  $n, m \in \mathbb{N}$  be arbitrary. We say “ $n$  divides  $m$ ” and write  $n|m$  if there exists a  $k \in \mathbb{N}$  such that  $k \cdot n = m$ . Prove that the following statement is true, using a proof by contradiction:

$$n|m \text{ and } n|(m+1) \implies n=1.$$

You are allowed to invoke the statement 3.2 iii) from above to justify one step. You must use the same notation as in the lecture notes, i.e. precisely state what your statements  $S$  and  $T$  are, and justify each of your proof steps.

*Proof.* From Definition 2.17, a proof by contradiction of a statement  $S$  proceeds in 3 steps:

1. Find a suitable mathematical statement  $T$ .

2. Prove that  $T$  is false.

3. Assume that  $S$  is false and prove (from this assumption) that  $T$  is true (a contradiction).

The general informal overview of the proof is to use the statement 3.2 iii) 3 i.e. the only natural number which has an inverse is 1. It will be used to prove that statement  $T$  is false, namely that there exist some pair(s) of number belonging to  $\mathbb{N}$  such that their product is 1 and at least one of them is not 1, meaning that 1 is not the only natural number with an inverse. Then, from assuming that  $\neg S$  is true, we will prove that  $T$  is true through composition of implications, which will lead to a contradiction. The statement provided to be proved will be our statement  $S$ , namely that if a number  $n$  divides two other consecutive numbers, then it must be equal to 1.

The interpretation is fixed by remarking that the universe is  $U = \mathbb{N}$  of natural numbers, the predicates will be *divides*( $x, y$ ), *equals*( $x, y$ ), *sum*( $x, y$ ), *multiplication*( $x, y$ ). Finally, the function symbols used will be  $|, =, +, \cdot$  for divisibility, equality, addition and multiplication, respectively. Nevertheless, we stick to make mathematical statements about formulas.

From the statement 3.2 iii) 3:

$$\forall x \forall y ((x \cdot y = 1) \rightarrow ((x = 1) \wedge (y = 1)))$$

its negation leads to state that there exists some pair of numbers which product is 1 and at least one of these numbers is not 1, meaning 1 is not the only natural number with an inverse as shown below:

$$\begin{aligned} & \neg \forall x \forall y ((x \cdot y = 1) \rightarrow ((x = 1) \wedge (y = 1))) \\ \implies & \exists x \neg \forall y ((x \cdot y = 1) \rightarrow ((x = 1) \wedge (y = 1))) \\ \implies & \exists x \exists y \neg ((x \cdot y = 1) \rightarrow ((x = 1) \wedge (y = 1))) \\ \implies & \exists x \exists y ((x \cdot y = 1) \wedge \neg ((x = 1) \wedge (y = 1))) \\ \implies & \exists x \exists y ((x \cdot y = 1) \wedge (\neg (x = 1) \vee \neg (y = 1))) \end{aligned}$$

Hence, w.r.t. interpretation the clearly false statement  $T$  is "1 is not the only natural number which has an inverse". Let  $n, k \in U$

$$T \implies \exists n \exists k ((n \cdot k = 1) \wedge (\neg (n = 1) \vee \neg (k = 1)))$$

Now, the statement to be proved  $S$  is "given  $n$  dividing two consecutive numbers, then it must be equal to 1":

$$S \implies \forall n \forall m (((n | m) \wedge (n | (m + 1))) \longrightarrow n = 1)$$

Assuming that the statement  $S$  is false, meaning that there exist some  $n, m$  such that  $n$  divides  $m$  and  $n$  divides  $m + 1$ , and  $n$  is not equal to 1:

$$\begin{aligned} \neg S \implies & \neg \forall n \forall m (((n | m) \wedge (n | (m + 1))) \longrightarrow n = 1) \\ \implies & \exists n \neg \forall m (((n | m) \wedge (n | (m + 1))) \longrightarrow n = 1) \\ \implies & \exists n \exists m \neg (((n | m) \wedge (n | (m + 1))) \longrightarrow n = 1) \\ \implies & \exists n \exists m \neg (\neg ((n | m) \wedge (n | (m + 1))) \vee (n = 1)) \\ \implies & \exists n \exists m (\neg \neg ((n | m) \wedge (n | (m + 1))) \wedge \neg (n = 1)) \\ \implies & \exists n \exists m (((n | m) \wedge (n | (m + 1))) \wedge \neg (n = 1)) \end{aligned}$$

from definition of divisibility we have that:

$$n | m \stackrel{\text{def}}{\iff} \exists k (k \cdot n = m)$$

so we can rewrite the statements  $n | m$  and  $n | (m + 1)$  as:

$$\begin{aligned} n | m & \implies k_1 \cdot n = m \\ n | (m + 1) & \implies k_2 \cdot n = m + 1 \implies (k_2 \cdot n) - 1 = m \end{aligned}$$

since  $m = m$ , we can inject the first equation into the second one:

$$\begin{aligned}
& k_1 \cdot n = (k_2 \cdot n) - 1 \\
\implies & (k_1 \cdot n) - (k_2 \cdot n) = -1 \\
\implies & n \cdot (k_1 - k_2) = -1 \\
\implies & n \cdot (k_2 - k_1) = 1 \\
\implies & n \cdot k = 1
\end{aligned}$$

By plugging all back into the formula derived from assumption of  $\neg S$ , using composition of implications, we get a contradiction that  $n$  time  $k$  gives 1, meaning it has an inverse, but  $n$  is not equal to 1, meaning that there exists a number that has an inverse and is not 1. It corresponds to the form of the statement  $T$  already proved to be false, as shown below:

$$\exists n \exists m \exists k \left( ((n \mid m) \wedge (n \mid (m + 1))) \wedge \underbrace{\neg(n = 1) \wedge ((k \cdot n) = 1)}_{T \text{ false for arbitrary } k \text{ and } n} \right)$$

Therefore, the statement  $S$  must be true, it is satisfiable and holds for our interpretation, concluding the proof.  $\square$