ETH zürich

Discrete Mathematics **2024**

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Assignment 5 — **Due date:** Thursday, 24 October 2024, 23:59

Exercise 5.5, Properties of Relations ($\star$) (8 Points)

Prove or disprove the following claims:

a) A relation ρ on a set A is symmetric on A if and only if ρ^2 is symmetric on A.

b) If ρ is a relation on a set A that is symmetric and antisymmetric, then it must hold $\rho = \mathsf{id}_A$.

c) Define the relations ρ_1 and ρ_2 on $\mathbb{Z}$ as

$$a\,\rho_1\,b \iff b = a + 1, \qquad a\,\rho_2\,b \iff b \equiv_2 a.$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$.

a)

The claim is false, a counterexample follows:

$$\rho = \{(a,c),(b,d),(c,b),(d,a)\}$$
$$\rho^2 = \{(a,b),(b,a),(c,d),(d,c)\}$$

The relation ρ^2 is symmetric, but ρ is not, which disprove the claim if and only if ($\iff$) from the right to left part ($\Longleftarrow$).

The intuition behind the counterexample is given by expanding the right hand side ($\Longleftarrow$) of the claim into a composition of implications as follows. Note the universe $\mathcal{U}$ is an arbitrary set A of elements.

ρ^2 symmetric on A

$\overset{.}{\Longrightarrow}$ $\forall a \forall b \left(a\,\rho^2\,b \iff b\,\rho^2\,a\right)$ (def 3.15 symmetry)

$\overset{.}{\Longrightarrow}$ $\forall a \forall b \left((a,b) \in \rho^2 \land (b,a) \in \rho^2\right)$ (def 3.9 relation)

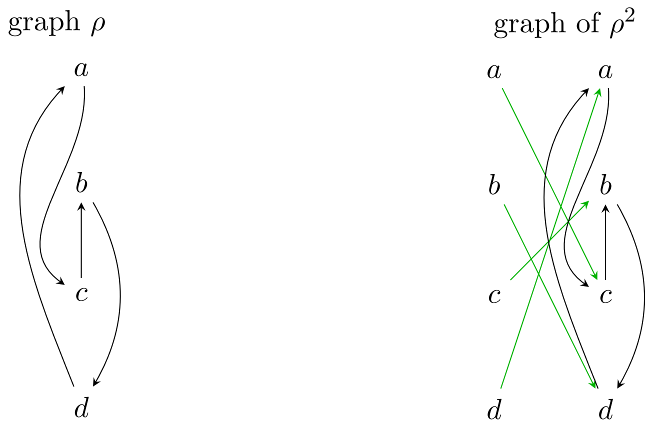
$\overset{.}{\Longrightarrow}$ $\forall a \forall b \left(\exists c\,((a,c) \in \rho \land (c,b) \in \rho) \land \exists d\,((b,d) \in \rho \land (d,a) \in \rho)\right)$ (def 3.12 composition)

So it is sufficient to find a relation ρ that is not symmetric, i.e. which have $c \neq d$, satisfying the constraint of the composition of implications above. A visualization is provided in the following matrix:

$$M^{\rho^2} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = M^{\rho} \cdot M^{\rho}\,, \quad \text{where} \quad M^{\rho} = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The matrices resemble the counterexample, where ρ^2 is symmetric, while ρ is clearly not, i.e. $M^\rho \neq (M^\rho)^\intercal \iff \rho \neq \hat{\rho}$.

A graph representation is provided for the sake of visualization, where the composition of the same structure choosen by the matrices is visualize. The elements of ρ^2 are given by choosing a path with length 2, meaning a green edge and the subsequent black one on the right. Doing so for every vertex gives the ρ^2 relation.



b)

The claim is false, a counterexample follows:

$$\rho = \emptyset \neq \mathsf{id}_A$$

The relation ρ is both symmetric and antisymmetric. However, ρ is not the identity relation as the claim states.

The satisfiability conditions for the relation being both symmetric and antisymmetric are explained as follows. The matrix for the counterexample is the all-zeros matrix $\mathbf{0}$ given by:

$$M^\rho = \mathbf{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

It is clear that the relation is symmetric by Definition 3.15, $\forall a \forall b \in A \;\; a\,\rho\,b \iff b\,\rho\,a$, i.e. the matrix is symmetric $M^\rho = (M^\rho)^\intercal \iff \rho = \hat{\rho}$.

Moreover, the relation is antisymmetric by Definition 3.16, $\forall a \forall b \in A \;\; a\,\rho\,b \wedge b\,\rho\,a \implies a = b$, i.e. the intersection between the relation and its inverse is a subset of the indentity relation, $\rho \cap \hat{\rho} \subseteq \mathsf{id}_A$. Notice that for the counterexample, the intersection is the empty set, which is a subset of every set by Lemma 3.6, including the identity relation.

$$\rho = \emptyset \subseteq (A \times A), \quad \rho \cap \hat{\rho} = \emptyset \subseteq \mathsf{id}_A \quad \because \quad \emptyset \in \mathcal{P}(\mathsf{id}_A)$$

In general symmetry and antisymmetry are independent properties of a relation.

c)

The claim is true, and it can be proven as follows:
recall that the universe $\mathcal{U} = \mathbb{Z}$, so relations will be a subset of the cartesian product $\mathcal{U} \times \mathcal{U}$, i.e. $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$.
By analysing both relations:

- $\rho_1 = \{(a, b) \mid b = a + 1\}$, meaning that the relation denotes the successor b of a, as usually denoted by $s(\cdot)$. It's matrix representation is scratched hereby:

$$
M^{\rho_1} = \begin{array}{c} \\ \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} \cdots & -1 & 0 & 1 & 2 & \cdots \end{array} \\
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{array}
$$

- $\rho_2 = \{(a,b) \mid b \equiv_2 a\}$, meaning that the relation denotes all elements with same parity, including the same element, thus reflexive since $\mathsf{id}_{\mathbb{Z}} \subseteq \rho_2$. It is symmetric on $\mathbb{Z}$ from Example 3.25; and it is also transitive from same parity to same parity as stated in Example 3.28. The definition of the $\equiv_m$ relation on $\mathbb{Z}$ from Example 3.11 states $a \equiv_m b \overset{def}{\iff} \exists k\,(a - b = km)$. It's matrix representation is scratched hereby:

$$
M^{\rho_2} = \begin{array}{c} \\ \vdots \\ \textit{even} \\ \textit{odd} \\ \textit{even} \\ \textit{odd} \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} \cdots & \textit{even} & \textit{odd} & \textit{even} & \textit{odd} & \cdots \end{array} \\
\begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 1 & 0 & 1 & 0 & \ddots \\
\ddots & 0 & 1 & 0 & 1 & \ddots \\
\ddots & 1 & 0 & 1 & 0 & \ddots \\
\ddots & 0 & 1 & 0 & 1 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\end{array}
$$

The informal overview of the proof is: decomposing the claim through composition of implications we will try a direct proof which will lead to a clear satisfiability statement. Then, proving by case distinction that the statement is always satisfiable w.r.t. fixed interpretation of universe, meaning of predicates and relation symbols.

Composition of Implications

from the statement S about **definitions of ρ_1, ρ_2 with $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$** . Assuming S is true, then

$$
\begin{aligned}
\rho^2 &\overset{\cdot}{\implies} \forall a\, \forall b\ \big((a,b) \in \rho^2\big) && \text{(def 3.9 relations)} \\
&\overset{\cdot}{\implies} \forall a\, \forall b\ ((a,b) \in (\rho \circ \rho)) && \text{(def 3.12 composition)} \\
&\overset{\cdot}{\implies} \forall a\, \forall b\, \exists c\ ((a,c) \in \rho \wedge (c,b) \in \rho) && \text{(def 3.12 composition)} \\
&\overset{\cdot}{\implies} \forall a\, \forall b\, \exists c\ ((a,c) \in (\rho_1 \cup \rho_2) \wedge (c,b) \in (\rho_1 \cup \rho_2)) && \text{(def of } \rho\text{)} \\
&\overset{\cdot}{\implies} \forall a\, \forall b\, \exists c\ (((a,c) \in \rho_1 \vee (a,c) \in \rho_2) \wedge ((c,b) \in \rho_1 \vee (c,b) \in \rho_2)) && \text{(def of } \cup\text{)} \\
&\overset{\cdot}{\implies} \forall a\, \forall b\, \exists c\ (((c = a+1) \vee (c \equiv_2 a)) \wedge ((b = c+1) \vee (b \equiv_2 c))) && \text{(def of } \cup\text{)}
\end{aligned}
$$

Check that the statement is always satisfiable, i.e. that there always exists a suitable $\exists c \in \mathbb{Z}$ to satisfy the claim.

Case Distinction

For all a, b in $\mathbb{Z}$, a can be either even or odd, and same for b. Recall that the sum of 2 even numbers is even, and also the sum of 2 odd numbers is even in $\mathbb{Z}$. The possible cases are then 4, and are represented and solved by a suitable c by the following matrix:

a	b	c
even	*even*	*even*
even	*odd*	$s(a)$
odd	*even*	$s(a)$
odd	*odd*	*odd*

where $s(x)$ is the successor of x in the universe.
Explanation:

- for a even and b even, c is even: $(a, c \text{ even}) \circ (c \text{ even}, b)$
- for a even and b odd, c is successor of a: $(a, a+1) \circ (a+1, b)$
- for a odd and b even, c is successor of a: $(a, a+1) \circ (a+1, b)$
- for a odd and b odd, c is odd: $(a, c \text{ odd}) \circ (c \text{ odd}, b)$

Finally, it follows that the claim is always satisfiable because there always exists a suitable c for the statement, meaning is true for all cases of $a, b \in \mathcal{U} = \mathbb{Z} \times \mathbb{Z}$. Concluding, $\rho^2 = \mathbb{1} = \mathbb{Z} \times \mathbb{Z}$.