Dr. Maria Giuseppina Nestola	Assignment 1 Optimization Methods – SP24	March 15, 2024 Deadline: March 17, 2024

Please refer to the Assignment rules document.

Exercise 1

Consider the following vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x_1, x_2) = 200(x_2 - x_1^2)^2 + (1 - x_1)^2 \tag{1}$$

1. Compute the gradient $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$ and the Hessian $H_f : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, which are respectively defined as:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]^T \qquad H_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

2. Write the Taylor's expansion of f up to the second order around the point $(x_1, x_2) = (0, 0)$.

Exercise 2

Consider the quadratic minimization problem

$$\min_{x \in \mathbb{R}^n} J(x) = \frac{1}{2} x^T A x - b^T x \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $x, b \in \mathbb{R}^n$.

1. Compute gradient and Hessian of functional J.
2. Write down the first order necessary condition for (2).
3. Write down the second order necessary and sufficient conditions for (2).

Exercise 3

Consider the following function:

$$f(x, y) = x^2 + \mu y^2 \tag{3}$$

1. Write it down in quadratic form, i.e. $\frac{1}{2} x^T A x - b^T x$, where $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$.
2. Plot the surface of the functions (Matlab function: `surf`) and the corresponding contour plot (Matlab function: `contour`) for values $\mu = 1$ and $\mu = 10$. In both cases use the square $[-10, 10] \times [-10, 10]$. Comment on the behaviour of the isolines.
3. Considering that A is a symmetric positive-definite matrix, find the exact optimal step-length α. Show your computations.
4. Write a Matlab code for the gradient method with maximum number of iterations $N = 100$ and a tolerance $\text{tol} = 10^{-8}$. Minimize f for $\mu = (1, 10)$ and starting points: $(x_0, y_0) = (10, 0), (0, 10), (10, 10)$.
5. For each case plot the iterations on the energy landscape in 2D (the plot of the objective function), the log10 of the norm of the gradient and the value of the energy function (objective function) as functions of the iterations. Comment the results.