ETH zürich

Discrete Mathematics **2024**

Student: Jeferson Morales Mariciano <jmorale@ethz.ch>

Assignment 2 **Due date:** Thursday, 3 October 2024, 23:59

Exercise 2.3, Simplifying a Formula ($\star$) (8 Points)

Consider the propositional formula

$$F = ((B \lor C) \to ((A \lor \neg B) \land C)) \lor (A \land \neg C)$$

Give a formula G that is equivalent to F, but in which each atomic formula A, B, and C appears at most once. Prove that $F \equiv G$ by providing a sequence of equivalence transformations with *at most* 12 steps.

Expectation. Your proof should be in the form of a sequence of steps, where each step consists of applying the definition of $\to$ (that is $F \to G \equiv \neg F \lor G$), one of the rules given in Lemma 2.1 of the lecture notes [1], or one of the following rules: $F \land \neg F \equiv \bot$, $F \land \bot \equiv \bot$, $F \lor \bot \equiv F$, $F \lor \neg F \equiv \top$, $F \land \top \equiv F$, and $F \lor \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1.3. Each step of your proof should apply a *single* rule *once* and state *which* rule was applied.

The formula G equivalent to F is given by $G = A \lor \neg B$, where each propositional symbol appears at most once with C not comparing at all. The proof is presented below: it is 12-steps long as requested and follows a sequence of equivalence transformations.

$$\begin{aligned}
F &= ((B \lor C) \to ((A \lor \neg B) \land C)) \lor (A \land \neg C) && \\
&\equiv \Big(\neg (B \lor C) \lor ((A \lor \neg B) \land C)\Big) \lor (A \land \neg C) && \text{def. of implication } \to \\
&\equiv \Big(\Big(\neg B \land \neg C\Big) \lor ((A \lor \neg B) \land C)\Big) \lor (A \land \neg C) && \text{def. De Morgan rule} \\
&\equiv \Big(((A \lor \neg B) \land C) \lor (\neg B \land \neg C)\Big) \lor (A \land \neg C) && \text{def. commutativity of } \lor \\
&\equiv ((A \lor \neg B) \land C) \lor \Big((\neg B \land \neg C) \lor (A \land \neg C)\Big) && \text{def. associativity of } \lor \\
&\equiv ((A \lor \neg B) \land C) \lor \Big((\neg B \land \neg C) \lor \Big(\neg C \land A\Big)\Big) && \text{def. commutativity of } \land \\
&\equiv ((A \lor \neg B) \land C) \lor \Big(\Big(\neg C \land \neg B\Big) \lor (\neg C \land A)\Big) && \text{def. commutativity of } \land \\
&\equiv ((A \lor \neg B) \land C) \lor \Big(\neg C \land (\neg B \lor A)\Big) && \text{def. 1st distributivity law} \\
&\equiv ((A \lor \neg B) \land C) \lor \Big(\neg C \land \Big(A \lor \neg B\Big)\Big) && \text{def. commutativity of } \lor \\
&\equiv ((A \lor \neg B) \land C) \lor \Big((A \lor \neg B) \land \neg C\Big) && \text{def. commutativity of } \land \\
&\equiv (A \lor \neg B) \land (C \lor \neg C) && \text{def. 1st distributivity law} \\
&\equiv (A \lor \neg B) \land \top && F \lor \neg F \equiv \top \\
&\equiv A \lor \neg B && F \land \top \equiv F
\end{aligned} \tag{1}$$

[1] Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).