

Assignment 2

Due date: Thursday, 3 October 2024, 23:59

Exercise 2.3, Simplifying a Formula (★)

(8 Points)

Consider the propositional formula

$$F = ((B \vee C) \rightarrow ((A \vee \neg B) \wedge C)) \vee (A \wedge \neg C)$$

Give a formula G that is equivalent to F , but in which each atomic formula A , B , and C appears at most once. Prove that $F \equiv G$ by providing a sequence of equivalence transformations with *at most* 12 steps.

Expectation. Your proof should be in the form of a sequence of steps, where each step consists of applying the definition of \rightarrow (that is $F \rightarrow G \equiv \neg F \vee G$), one of the rules given in Lemma 2.1 of the lecture notes ¹, or one of the following rules: $F \wedge \neg F \equiv \perp$, $F \wedge \perp \equiv \perp$, $F \vee \perp \equiv F$, $F \vee \neg F \equiv \top$, $F \wedge \top \equiv F$, and $F \vee \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1.3. Each step of your proof should apply a *single* rule *once* and state *which* rule was applied.

The formula G equivalent to F is given by $G = A \vee \neg B$, where each propositional symbol appears at most once with C not comparing at all. The proof is presented below: it is 12-steps long as requested and follows a sequence of equivalence transformations.

$$\begin{aligned}
 F &= ((B \vee C) \rightarrow ((A \vee \neg B) \wedge C)) \vee (A \wedge \neg C) \\
 &\equiv (\neg(B \vee C) \vee ((A \vee \neg B) \wedge C)) \vee (A \wedge \neg C) && \text{def. of implication } \rightarrow \\
 &\equiv ((\neg B \wedge \neg C) \vee ((A \vee \neg B) \wedge C)) \vee (A \wedge \neg C) && \text{def. De Morgan rule} \\
 &\equiv (((A \vee \neg B) \wedge C) \vee (\neg B \wedge \neg C)) \vee (A \wedge \neg C) && \text{def. commutativity of } \vee \\
 &\equiv ((A \vee \neg B) \wedge C) \vee ((\neg B \wedge \neg C) \vee (A \wedge \neg C)) && \text{def. associativity of } \vee \\
 &\equiv ((A \vee \neg B) \wedge C) \vee ((\neg B \wedge \neg C) \vee (\neg C \wedge A)) && \text{def. commutativity of } \wedge \\
 &\equiv ((A \vee \neg B) \wedge C) \vee ((\neg C \wedge \neg B) \vee (\neg C \wedge A)) && \text{def. commutativity of } \wedge \quad (1) \\
 &\equiv ((A \vee \neg B) \wedge C) \vee (\neg C \wedge (\neg B \vee A)) && \text{def. 1st distributivity law} \\
 &\equiv ((A \vee \neg B) \wedge C) \vee (\neg C \wedge (A \vee \neg B)) && \text{def. commutativity of } \vee \\
 &\equiv ((A \vee \neg B) \wedge C) \vee ((A \vee \neg B) \wedge \neg C) && \text{def. commutativity of } \wedge \\
 &\equiv (A \vee \neg B) \wedge (C \vee \neg C) && \text{def. 1st distributivity law} \\
 &\equiv (A \vee \neg B) \wedge \top && F \vee \neg F \equiv \top \\
 &\equiv A \vee \neg B && F \wedge \top \equiv F
 \end{aligned}$$

¹Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).