Assignment 4 Jeferson Morales Mariciano

Exercise 1

a) $\{2, 3, 5, 7, 11, 13\}$ this set S is bounded.

Is bounded from above by the upper bounds $\{13, 14\}$, because $x \leq \{13, 14\}$ $\forall x \in S$
Is bounded from below by the lower bounds $\{2, 1\}$, because $\{2, 1\} \leq x$ $\forall x \in S$

b) $\{x \in \mathbb{R} : x \geq 0\}$ This set S is bounded from below. Upper bound of $\mathbb{R}$ is $+\infty$.

Is bounded from below by the lower bounds $\{0, -1\}$, because $\{0, -1\} \leq x$ $\forall x \in S$

c) $\{2 + \frac{1}{2^n} : n \in \mathbb{N}\}$ This set S is bounded. $\{2.5, 2.25, 2.125, \ldots\}$
Is bounded from above by the upper bounds $\{3, 10\}$, because $\{3, 10\} \leq x$ $\forall x \in S$
Is bounded from below by the lower bounds $\{2, 1\}$, because $\{2, 1\} \leq x$ $\forall x \in S$

d) $\{x^3 - 2 : x \in \mathbb{R}\}$ this set S is unbounded.
The set $\mathbb{R}$ has no bounds $(-\infty, +\infty)$

e) $(0, 1) \cup (1, 2]$ this set S is bounded.

Is bounded from above by the upper bounds $\{2, 3\}$, because $x \leq \{2, 3\}$ $\forall x \in S$
Is bounded from below by the lower bounds $\{0, -3\}$, because $\{0, -3\} \leq x$ $\forall x \in S$

Exercise 2

a) $\{2, 3, 5, 7, 11, 13\}$ this set S has a maximum and an infimum

- Maximum because, let $b = 13$ and S a non-empty subset of $\mathbb{R}$. Then, $b \in \mathbb{R}$ is the maximum of S, denoted by $b = \max S$ and $x \leq b$ $\forall x \in S$
- Infimum because, let $a = 2$ and S a non-empty subset of $\mathbb{R}$. Then, the lower bound a of S is the greatest lower bound or infimum of S, denoted by $a = \inf S$, if $a \geq a'$ $\forall a' \in S$

b) $\{x \in \mathbb{R} : x \geq 0\}$ this set S has an infimum but no maximum
- No maximum because set $\mathbb{R}$ has no maximum.
- Infimum because, let $a = 0$ and S a non-empty subset of $\mathbb{R}$. Then, the lower bound a of S is the greatest lower bound or infimum of S, denoted by $a = \inf S$, if $a \geq a'$ $\forall a' \in S$

c) $\{2 + \frac{1}{2^n} : n \in \mathbb{N}\}$ this set S has a maximum and an infimum.
- Maximum because, let $b = 2.5$ ($\emptyset$ excluded) and S a non-empty subset of $\mathbb{R}$. Then, $b \in \mathbb{R}$ is the maximum of S, denoted by $b = \max S$ and $x \leq b$ $\forall x \in S$
- Infimum because, let $a = 2$ and S a non-empty subset of $\mathbb{R}$. Then, the lower bound a of S is the greatest lower bound or infimum of S, denoted by $a = \inf S$, if $a \geq a'$ $\forall a' \in S$

d) $\{x^3 - 2 : x \in \mathbb{R}\}$ this set S has neither maximum nor an infimum because set $\mathbb{R}$ is unbounded $(-\infty, +\infty)$

e) $(0, 1) \cup (1, 2]$ this set S has a maximum and an infimum.

- Maximum because, let $b=2$ and S a non-empty subset of $\mathbb{R}$. Then, $b \in \mathbb{R}$ is the maximum of S, denoted by $b = \max S$ and $x \leq b \ \forall x \in S$
- Infimum because, let $a = 0$ and S a non-empty subset of $\mathbb{R}$. Then, the lower bound a of S is the greatest lower bound or infimum of S, denoted by $a = \inf S$, if $a \geq a' \ \forall a' \in S$

Exercise 3

Let S be a non empty set of real numbers. Prove $\inf S \leq \sup S$. Can it happen that $\inf S = \sup S$?

- an upper bound b of S is the supremum of S, denoted by $b = \sup S$, if $b \leq b' \ \forall b' \in S$
- an lower bound a of S is the infimum of S, denoted by $a = \inf S$, if $a \geq a' \ \forall a' \in S$

In any set of numbers is possible to order it numerically and by definition the greatest lower bound is smaller than the least upper bound, thus $\ldots, a'', a', a < b, b', b'', \ldots \Rightarrow a < b$

assume S is a set of $\mathbb{R}$ with one element $\Rightarrow S = \{x\}$, then, the interval is $[x, x]$ or $[x]$, so x is at the same time maximum and minimum. Maximum and minimum correspond to supremum and infimum of a set, so $a = b \Rightarrow \inf S = \sup S$

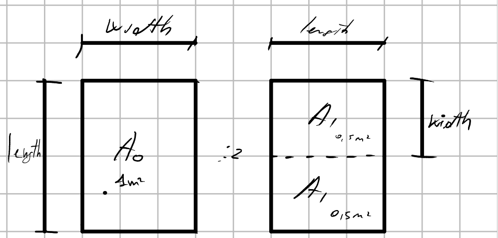
So we can state $a \leq b$ $\quad \inf S \leq \sup S$

Bonus exercise

Length = long side ; width = short side

i) the area of A_0 is $1m^2$

ii) if we cut A_0 in two identical pieces A_1, the length/width ratio of A_1 is the same as that of A_0



Determine length and width of the rectangle A_0

by def. the area of a rectangle $A = b \cdot h$. Ratio $r = \frac{\text{length}}{\text{width}}$

$A_0 \ r = \frac{L}{W}$ $\quad A_1 \ r = \frac{W}{L/2}$ $\quad \frac{L}{W} = \frac{W}{L/2}$ $\quad L = \frac{W^2}{1} \cdot \frac{2}{L}$ $\quad L = \frac{2W^2}{L}$ $\quad L^2 = 2W^2$

$L = \sqrt{2W^2}$

$L = W\sqrt{2}$

$W\sqrt{2} = L$ $\quad W = \frac{L}{\sqrt{2}} = \frac{L\sqrt{2}}{2}$

$L \cdot W = A_0$

$W\sqrt{2} \cdot W = 1$

$W^2 \sqrt{2} = 1$

$W^2 = \frac{1}{\sqrt{2}}$ $\quad W = \frac{1}{\sqrt[4]{2}} \cdot \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}} = \frac{\sqrt[4]{2^3}}{2}$

$L = \frac{\sqrt[4]{2^3}}{2^1} \cdot \frac{\sqrt[2]{2^1}}{1} = \frac{\sqrt[4]{2^3} \cdot \sqrt[2]{2^1}}{2^1}$

$A_0 = L \cdot W = \frac{\sqrt[4]{2^3} \cdot \sqrt[2]{2^1}}{2^1} \cdot \frac{\sqrt[4]{2^3}}{2^1} = \frac{\sqrt[4]{2^6} \cdot \sqrt{2}}{2^2} = \frac{\sqrt[2]{2^3} \cdot \sqrt[2]{2}}{2^2} = \frac{\sqrt[2]{2^2 \cdot 2^1} \cdot \sqrt{2}}{2^2} = \frac{2 \cdot \sqrt[2]{2^2}}{2^2} = 1$

$A_0 = 1$ $\quad L_0 = \frac{\sqrt[4]{2^3} \cdot \sqrt[2]{2^1}}{2^1}$ $\quad W_0 = \frac{\sqrt[4]{2^3}}{2}$

Sizes of A_4? (A_4, L_4, W_4)

$W_1 = L_0/2 = \frac{\sqrt[4]{2^3} \cdot \sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt[4]{2^3} \cdot \sqrt{2}}{2^2}$ $\quad L_1 = W_0 = \frac{\sqrt[4]{2^3}}{2}$

$$A_1 = L_1 \cdot W_1 = \frac{\sqrt[4]{2^3}}{2} \cdot \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2} = \frac{\sqrt[4]{2^6}\cdot\sqrt{2}}{2^3} = \frac{\sqrt[2]{2^2\cdot 2}\cdot\sqrt{2}}{2^3} = \frac{2^2}{2^3} = \frac{4}{8} = \frac{1}{2} = 0.5$$

$$A_1 = \frac{1}{2} \qquad L_1 = \frac{\sqrt[4]{2^3}}{2} \qquad W_1 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2}$$

$$W_2 = L_1/2 = \frac{\sqrt[4]{2^3}}{2}\cdot\frac{1}{2} = \frac{\sqrt[4]{2^3}}{2^2} \qquad L_2 = W_1 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2}$$

$$A_2 = L_2 \cdot W_2 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2}\cdot\frac{\sqrt[4]{2^3}}{2^2} = \frac{\sqrt[4]{2^6}\cdot\sqrt{2}}{2^4} = \frac{2^2}{2^4} = \frac{4}{16} = \frac{1}{4} = 0.25$$

$$A_2 = \frac{1}{4} \qquad L_2 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2} \qquad W_2 = \frac{\sqrt[4]{2^3}}{2^2}$$

$$W_3 = L_2/2 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^2}\cdot\frac{1}{2} = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3} \qquad L_3 = W_2 = \frac{\sqrt[4]{2^3}}{2^2}$$

$$A_3 = L_3 \cdot W_3 = \frac{\sqrt[4]{2^3}}{2^2}\cdot\frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3} = \frac{\sqrt[4]{2^6}\cdot\sqrt{2}}{2^5} = \frac{2^2}{2^5} = \frac{4}{32} = \frac{1}{8} = 0.125$$

$$A_3 = \frac{1}{8} \qquad L_3 = \frac{\sqrt[4]{2^3}}{2^2} \qquad W_3 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3}$$

$$W_4 = L_3/2 = \frac{\sqrt[4]{2^3}}{2^2}\cdot\frac{1}{2} = \frac{\sqrt[4]{2^3}}{2^3} \qquad L_4 = W_3 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3}$$

$$A_4 = L_4 \cdot W_4 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3}\cdot\frac{\sqrt[4]{2^3}}{2^3} = \frac{\sqrt[4]{2^6}\cdot\sqrt{2}}{2^6} = \frac{2^2}{2^6} = \frac{4}{64} = \frac{1}{16} = 0.0625$$

$$A_4 = \frac{1}{16} \qquad L_4 = \frac{\sqrt[4]{2^3}\cdot\sqrt{2}}{2^3} \qquad W_4 = \frac{\sqrt[4]{2^3}}{2^3}$$

IT looks familiar indeed! the sequence we got from areas $A_0, A_1, A_2 \ldots$ $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$ from the $\frac{\text{length}}{\text{width}}$ ratio of $\sqrt{2}$ reminds me about the infinite series