Discrete Mathematics **2024**

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Assignment 6

Due date: Thursday, 31 October 2024, 23:59

Exercise 6.5, Countability (8 Points)

Prove that for all $l \in \mathbb{N}$ with $l \geq 1$ the set

$$A_l := \left\{ f : \mathbb{N} \to \{0,1\} \middle| \sum_{i=0}^{k} f(i) \leq \frac{k}{l} + 1 \quad \forall k \in \mathbb{N} \right\}.$$

is uncountable.
Hint: For all $l \geq 1$, explicitly write an injection from a known uncountable set into A_l.

Following the hint, an injection from an uncountable set to A_l is going to be built. Notice that to belong to A_l $\forall l \geq 1$, it is sufficient that the function f satisfies the following conditions: yielding either always 0 for any input or a limited number of 1s following the bound pattern.

$$\begin{array}{cccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\ f(n) & 0 & 1 & 0 & 0 & 1 & 0 & \ldots \end{array}$$

$$\underbrace{\qquad\qquad}_{l}\underbrace{\qquad\qquad}_{l}$$

So, imagining the string presented above divided in chuncks of lenght l, every l characters there can be a $f(n) = 1$ for some n indexed within the chunk. Since this holds for any size of k, the condition of A_l stating $\sum_{i=0}^{k} f(i) \leq \frac{k}{l} + 1 \quad \forall k \in \mathbb{N}$ is satisfied. This means that **at most** 1 of $(f(0), f(1), \ldots, f(l))$ can be 1, for every l sized chunk in order for all such function to belong to A_l.

The construction of binary sequences denoting results of $f(i)$ $\forall i \in \mathbb{N}$ satisfying the above condition are a valid candidate for the injection construction into A_l: it allows for the function to yield $1s$ for the i-th positions with $i \equiv_l 0$. Such binary construction can be graphically denoted as:

$$\begin{array}{ccccccccc} 0 & 1 & \ldots & l-1 & l & l+1 & \ldots & 2 \cdot l - 1 & \\ \alpha_0 & 0 & \ldots & 0 & \alpha_1 & 0 & \ldots & 0 & \ldots \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{l}\underbrace{\qquad\qquad\qquad\qquad}_{l}$$

$$(\alpha_i 0^{l-1})^m \quad \alpha_i \in \{0,1\},\ \forall i \in \mathbb{N},\ \forall m \in \mathbb{N}$$

Thus, encoding the possible image of a valid function $f \in A_l$ along arbitrary $m \cdot l$ sized bit string.

From Theorem 3.23, the set $\{0,1\}^\infty$ is uncountable. Building an injection from $\{0,1\}^\infty$ to such above construction enconding f functions in A_l is proposed. Let's define the function g:

$$g : \{0,1\}^\infty \to A_l, \quad \alpha \in \{0,1\}^\infty, \quad f \in A_l, \quad \alpha_i \in \{0,1\}$$

$$\alpha \mapsto f, \quad \forall j \in \mathbb{N} : \ f(j) = \begin{cases} \alpha_i, & \text{if } j = l \cdot i \quad i \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

So we have α corresponding to a semi infinite binary string, and α_i denoting the i-th bit of the string. Hence, using the mapping defined, **every possible semi infinite binary sequence has a valid function f mapping in A_l**. This injective map denoting $\{0,1\}^\infty \preceq A_l$, i.e. A_l dominates $\{0,1\}^\infty$, implies that A_l **is uncountable**.

To prove the injectivity of the function, let $\alpha, \beta \in \{0,1\}^\infty$ be two different semi infinite binary sequences. Let i be the position where $\alpha_i \neq \beta_i$. Let $g(\alpha) = f_\alpha,\ g(\beta) = f_\beta$.
By construction of $f_\alpha,\ f_\beta:\ f_\alpha(l \cdot i) = \alpha_i \neq \beta_i = f_\beta(l \cdot i) \quad \Longrightarrow \quad f_\alpha \neq f_\beta$