Discrete Mathematics **2024**

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Assignment 4 — **Due date:** Thursday, 17 October 2024, 23:59

Exercise 4.5, Proving/Disproving Set Properties ($\star\star$) (8 Points)

Prove or disprove the following statements.

a) For any sets A, B, C it holds $(A \cup (B \setminus C)) \cap (B \cap C) = \varnothing$

b) For any sets A, B, C it holds $A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C)$

c) For any sets A, B it holds $|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| \geq 2$

Expectation: Argue using the definitions of $\subseteq, \cup, \cap, |\cdot|, \mathcal{P}(\cdot), \setminus, \times$ from the lecture notes. You are allowed to use any results you have already seen in the lecture, including facts from Chapter 2 (e.g. the rules of Lemma 2.1), as well as $F \vee \bot \equiv F$ and $F \wedge \top \equiv F$. You can apply several rules/results in one step, but have to clearly state which rules/results you apply. To disprove a statement, provide a concrete counterexample.

a)

$$(A \cup (B \setminus C)) \cap (B \cap C) = \varnothing$$

This is false and a counterexample follows:

$$(A \cup (B \setminus C)) \cap (B \setminus C) \quad \text{for} \quad A = \{2\}, \quad B = \{2\}, \quad C = \{3\}$$
$$B \setminus C = \{2\} \implies A \cup (B \setminus C) = \{2\} \implies (A \cup (B \setminus C)) \cap (B \setminus C) = \{2\} \neq \varnothing$$

it can be proven as follows:

$$
\begin{aligned}
& (A \cup (B \setminus C)) \cap (B \setminus C) \\
\implies\ & x \in (A \cup (B \setminus C)) \wedge x \in (B \setminus C) && \text{(def. intersection)} \\
\implies\ & (x \in A \vee x \in (B \setminus C)) \wedge x \in (B \setminus C) && \text{(def. union)} \\
\implies\ & (x \in A \vee (x \in B \wedge \neg(x \in C))) \wedge (x \in B \wedge \neg(x \in C)) && \text{(def. difference of sets)} \\
\implies\ & (x \in B \wedge \neg(x \in C)) \wedge (x \in A \vee (x \in B \wedge \neg(x \in C))) && \text{(def. commutativity of } \wedge) \\
\implies\ & ((x \in B \wedge \neg(x \in C)) \wedge x \in A) \vee \\
& \quad ((x \in B \wedge \neg(x \in C)) \wedge (x \in B \wedge \neg(x \in C))) && \text{(def. 1st distributivity law)} \\
\implies\ & ((x \in B \wedge \neg(x \in C)) \wedge x \in A) \vee (x \in B \wedge \neg(x \in C)) && \text{(def. idempotence)} \\
\implies\ & (x \in B \wedge \neg(x \in C)) \vee ((x \in B \wedge \neg(x \in C)) \wedge x \in A) && \text{(def. commutativity of } \vee) \\
\implies\ & (x \in B \wedge \neg(x \in C)) && \text{(def. absorption)} \\
\implies\ & B \setminus C && \text{(def. difference of sets)} \\
\implies\ & B \setminus C \neq \varnothing && \square
\end{aligned}
$$

b)

$$A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C)$$

This is true and it can be proven as follows by working on the right hand side (RHS):

$$
\begin{aligned}
&A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C) \\
&(A \cap B) \setminus ((A \cap B) \cap C) \quad \overset{\cdot}{\Longrightarrow} \quad x \in (A \cap B) \wedge \neg\,(x \in ((A \cap B) \cap C)) && \text{(def. difference of sets)} \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge \neg\,(x \in ((A \cap B) \cap C)) && \text{(def. intersection)} \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge \neg\,(x \in (A \cap B) \wedge x \in C) && \text{(def. intersection)} \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge \neg\,((x \in A \wedge x \in B) \wedge x \in C) && \text{(def. intersection)} \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge (\neg\,(x \in A \wedge x \in B) \vee \neg\,(x \in C)) && \text{(def. de morgan)} \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge ((\neg\,(x \in A) \vee \neg\,(x \in B)) \vee \neg\,(x \in C)) && \text{(def. de morgan)} \\
&\overset{\cdot}{\Longrightarrow} \quad ((x \in A \wedge x \in B) \wedge (\neg\,(x \in A) \vee \neg\,(x \in B))) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. 1st distributivity law)} \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge x \in B) \wedge \neg\,(x \in A)) \vee ((x \in A \wedge x \in B) \wedge \neg\,(x \in B))) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. 1st distributivity law)} \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge x \in B) \wedge \neg\,(x \in A)) \vee (x \in A \wedge (x \in B \wedge \neg\,(x \in B)))) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. associativity)} \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge x \in B) \wedge \neg\,(x \in A)) \vee (x \in A \wedge \bot)) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && F \wedge \neg F \equiv \bot \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge x \in B) \wedge \neg\,(x \in A)) \vee \bot) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && F \wedge \neg F \equiv \bot \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge x \in B) \wedge \neg\,(x \in A))) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && F \vee \bot \equiv F \\
&\overset{\cdot}{\Longrightarrow} \quad ((\neg\,(x \in A) \wedge (x \in A \wedge x \in B))) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. commutativity of } \wedge) \\
&\overset{\cdot}{\Longrightarrow} \quad (((\neg\,(x \in A) \wedge x \in A) \wedge x \in B)) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. associativity of } \wedge) \\
&\overset{\cdot}{\Longrightarrow} \quad (((x \in A \wedge \neg\,(x \in A)) \wedge x \in B)) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. commutativity of } \wedge) \\
&\overset{\cdot}{\Longrightarrow} \quad ((\bot \wedge x \in B)) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && F \wedge \neg F \equiv \bot \\
&\overset{\cdot}{\Longrightarrow} \quad ((x \in B \wedge \bot)) \vee \\
&\qquad\qquad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && \text{(def. commutativity of } \wedge) \\
&\overset{\cdot}{\Longrightarrow} \quad \bot \vee ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) && F \wedge \bot \equiv \bot \\
&\overset{\cdot}{\Longrightarrow} \quad ((x \in A \wedge x \in B) \wedge \neg\,(x \in C)) \vee \bot && \text{(def. commutativity of } \vee) \\
&\overset{\cdot}{\Longrightarrow} \quad (x \in A \wedge x \in B) \wedge \neg\,(x \in C) && F \vee \bot \equiv F \\
&\overset{\cdot}{\Longrightarrow} \quad x \in A \wedge (x \in B \wedge \neg\,(x \in C)) && \text{(def. associativity of } \wedge) \\
&\overset{\cdot}{\Longrightarrow} \quad x \in A \wedge x \in (B \setminus C) && \text{(def. difference of sets)} \\
&\overset{\cdot}{\Longrightarrow} \quad A \cap (B \setminus C) && \text{(def. intersection)}
\end{aligned}
$$

c)

$$|\mathcal{P}\,(\mathcal{P}\,(A) \times \mathcal{P}\,(B))\,| \geq 2$$

This is true and it can be proven as follows:

From Definition 3.7, the power set of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{S \mid S \subseteq A\}$$

where for a finite set with cardinality $|A| = k$, the power set has cardinality $2^{|A|} = 2^k$.

From Definition 3.8, the Cartesian product $A \times B$ of two sets A, B is the set of all ordered pairs with the first component from A and the second component from B:

$$A \times B \stackrel{\text{def}}{=} \{(a, b) \mid a \in A \wedge b \in B\}$$

For finite sets, the cardinality of the Cartesian product of some sets is the product of their cardinalities: $|A \times B| = |A| \cdot |B|$.

Then, we want to show:

$$|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| = 2^{|\mathcal{P}(A) \times \mathcal{P}(B)|} = 2^{2^{|A|} . 2^{|B|}} \geq 2$$

By induction, the base case is $n = |A| = |B| = 0 \in \mathbb{N}$, where both $A = B = \emptyset$ and $\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset\}$.

Then,

$$\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B)) = \mathcal{P}(\mathcal{P}(\emptyset) \times \mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\} \times \{\emptyset\}) = \mathcal{P}(\{(\emptyset, \emptyset)\}) = \{\emptyset, \{(\emptyset, \emptyset)\}\}$$

Which respects the inequality:

$$2^{2^{|A|} . 2^{|B|}} = 2^{2^0 . 2^0} = 2^1 = 2 \geq 2$$

Then, the base cased holds.

For the next inductive step, the Inductive Hypothesis (I.H.) assume the statement holds for $n = |A| = |B|$, we want to show it holds for $|A| = |B| = n + 1$. The cardinality of the sets A, B are independent, we do not consider n to be the same number for both. Thus,

$$\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B)) = 2^{2^{|A|} . 2^{|B|}} = 2^{2^{n+1} . 2^{n+1}} = 2^{2^n . 2^1 . 2^n . 2^1} \stackrel{I.H.}{=} 2^{2^n . 2^1 . 2^n . 2^1 \geq 1} = 2^{2^{2 \cdot (n+1)} \geq 1} = 2^{2^{2 \cdot (n+1)}} \geq 2$$

the inductive step holds and the statement holds for all $n \in \mathbb{N}$.