ETH zürich

Discrete Mathematics **2024**

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Assignment 4

Due date: Thursday, 17 October 2024, 23:59

Exercise 4.5, Proving/Disproving Set Properties (⋆⋆) (8 Points)

Prove or disprove the following statements.

a) For any sets A, B, C it holds $(A \cup (B \setminus C)) \cap (B \cap C) = \varnothing$

b) For any sets A, B, C it holds $A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C)$

c) For any sets A, B it holds $|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| \geq 2$

Expectation: Argue using the definitions of $\subseteq, \cup, \cap, |\cdot|, \mathcal{P}(\cdot), \setminus, \times$ from the lecture notes. You are allowed to use any results you have already seen in the lecture, including facts from Chapter 2 (e.g. the rules of Lemma 2.1), as well as $F \vee \bot \equiv F$ and $F \wedge \top \equiv F$. You can apply several rules/results in one step, but have to clearly state which rules/results you apply. To disprove a statement, provide a concrete counterexample.

a)

$$(A \cup (B \setminus C)) \cap (B \cap C) = \varnothing$$

This is false and a counterexample follows:

$$\begin{aligned}
&(A \cup (B \setminus C)) \cap (B \cap C) \quad \text{for} \quad A = \{2\}, \quad B = \{2\}, \quad C = \{2\} \\
&B \setminus C = \emptyset \implies A \cup (B \setminus C) = \{2\} \implies B \cap C = \{2\} \\
&\implies \quad (A \cup (B \setminus C)) \cap (B \cap C) = \{2\} \cap \{2\} = \{2\} \neq \varnothing
\end{aligned}$$

it can be proven as follows:

$$\begin{aligned}
& (A \cup (B \setminus C)) \cap (B \cap C) \\
\implies\; & x \in (A \cup (B \setminus C)) \wedge x \in (B \cap C) && \text{(intersection)} \\
\implies\; & (x \in A \vee x \in (B \setminus C)) \wedge x \in (B \cap C) && \text{(union)} \\
\implies\; & (x \in A \vee (x \in B \wedge \neg(x \in C))) \wedge x \in (B \cap C) && \text{(difference of sets)} \\
\implies\; & (x \in A \vee (x \in B \wedge \neg(x \in C))) \wedge (x \in B \wedge x \in C) && \text{(intersection)} \\
\implies\; & (x \in B \wedge x \in C) \wedge (x \in A \vee (x \in B \wedge \neg(x \in C))) && \text{(commutativity of } \wedge) \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee ((x \in B \wedge x \in C) \wedge (x \in B \wedge \neg(x \in C))) && \text{(1st distributivity law)} \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee ((x \in B \wedge x \in C) \wedge (\neg(x \in C) \wedge x \in B)) && \text{(commutativity of } \wedge) \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee (x \in B \wedge (x \in C \wedge \neg(x \in C)) \wedge x \in B) && \text{(associativity of } \wedge) \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee (x \in B \wedge \bot \wedge x \in B) && F \wedge \neg F \equiv \bot \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee (\bot \wedge x \in B) && F \wedge \bot \equiv \bot \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee (x \in B \wedge \bot) && \text{(commutativity of } \wedge) \\
\implies\; & ((x \in B \wedge x \in C) \wedge x \in A) \vee \bot && F \wedge \bot \equiv \bot
\end{aligned}$$

$$
\begin{aligned}
&\implies (x \in B \land x \in C) \land x \in A && F \lor \bot \equiv F \\
&\implies x \in A \land (x \in B \land x \in C) && \text{(commutativity of } \land) \\
&\implies (x \in A \land x \in B) \land x \in C && \text{(associativity of } \land) \\
&\implies x \in (A \cap B) \land x \in C && \text{(intersection)} \\
&\implies (A \cap B) \cap C && \text{(intersection)}
\end{aligned}
$$

$\square$

Which is different from the empty set. Thus the statement is not always valid, hence false given the counter example provided. I.e. it is not valid for all possible elements of A, B, C.

b)

$$A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C)$$

This is true and it can be proven as follows by working on the right hand side (RHS):

$$
\begin{aligned}
&A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C) \\
&(A \cap B) \setminus ((A \cap B) \cap C) \implies x \in (A \cap B) \land \neg (x \in ((A \cap B) \cap C)) && \text{(difference of sets)} \\
&\implies (x \in A \land x \in B) \land \neg (x \in ((A \cap B) \cap C)) && \text{(intersection)} \\
&\implies (x \in A \land x \in B) \land \neg (x \in (A \cap B) \land x \in C) && \text{(intersection)} \\
&\implies (x \in A \land x \in B) \land \neg ((x \in A \land x \in B) \land x \in C) && \text{(intersection)} \\
&\implies (x \in A \land x \in B) \land (\neg (x \in A \land x \in B) \lor \neg (x \in C)) && \text{(de morgan)} \\
&\implies (x \in A \land x \in B) \land ((\neg (x \in A) \lor \neg (x \in B)) \lor \neg (x \in C)) && \text{(de morgan)} \\
&\implies ((x \in A \land x \in B) \land (\neg (x \in A) \lor \neg (x \in B))) \lor \\
&\qquad ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(1st distributivity law)} \\
&\implies (((x \in A \land x \in B) \land \neg (x \in A)) \lor ((x \in A \land x \in B) \land \neg (x \in B))) \lor \\
&\qquad ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(1st distributivity law)} \\
&\implies (((x \in A \land x \in B) \land \neg (x \in A)) \lor (x \in A \land (x \in B \land \neg (x \in B)))) \lor \\
&\qquad ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(associativity)} \\
&\implies (((x \in A \land x \in B) \land \neg (x \in A)) \lor (x \in A \land \bot)) \lor \\
&\qquad ((x \in A \land x \in B) \land \neg (x \in C)) && F \land \neg F \equiv \bot \\
&\implies (((x \in A \land x \in B) \land \neg (x \in A)) \lor \bot) \lor \\
&\qquad ((x \in A \land x \in B) \land \neg (x \in C)) && F \land \neg F \equiv \bot \\
&\implies ((x \in A \land x \in B) \land \neg (x \in A)) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && F \lor \bot \equiv F \\
&\implies (\neg (x \in A) \land (x \in A \land x \in B)) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(commutativity of } \land) \\
&\implies ((\neg (x \in A) \land x \in A) \land x \in B) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(associativity of } \land) \\
&\implies ((x \in A \land \neg (x \in A)) \land x \in B) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(commutativity of } \land) \\
&\implies (\bot \land x \in B) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && F \land \neg F \equiv \bot \\
&\implies (x \in B \land \bot) \lor ((x \in A \land x \in B) \land \neg (x \in C)) && \text{(commutativity of } \land) \\
&\implies \bot \lor ((x \in A \land x \in B) \land \neg (x \in C)) && F \land \bot \equiv \bot \\
&\implies ((x \in A \land x \in B) \land \neg (x \in C)) \lor \bot && \text{(commutativity of } \lor) \\
&\implies (x \in A \land x \in B) \land \neg (x \in C) && F \lor \bot \equiv F \\
&\implies x \in A \land (x \in B \land \neg (x \in C)) && \text{(associativity of } \land) \\
&\implies x \in A \land x \in (B \setminus C) && \text{(difference of sets)} \\
&\implies A \cap (B \setminus C) && \text{(intersection)}
\end{aligned}
$$

$\square$

c)

$$|\mathcal{P}\left(\mathcal{P}\left(A\right)\times\mathcal{P}\left(B\right)\right)|\geq 2$$

This is true and it can be proven as follows:
From Definition 3.7, the power set of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A)\overset{\text{def}}{=}\{S\mid S\subseteq A\}$$

where for a finite set with cardinality $|A|=k$, the power set has cardinality $2^{|A|}=2^k$.
From Definition 3.8, the Cartesian product $A\times B$ of two sets A,B is the set of all ordered pairs with the first component from A and the second component from B:

$$A\times B\overset{\text{def}}{=}\{(a,b)\mid a\in A\land b\in B\}$$

For finite sets, the cardinality of the Cartesian product of some sets is the product of their cardinalities: $|A\times B|=|A|\cdot|B|$.
Then, we want to show:

$$|\mathcal{P}\left(\mathcal{P}\left(A\right)\times\mathcal{P}\left(B\right)\right)|=2^{|\mathcal{P}(A)\times\mathcal{P}(B)|}=2^{2^{|A|}\cdot 2^{|B|}}\geq 2$$

By induction, the base case is $n=|A|=|B|=0\in\mathbb{N}$, where both $A=B=\emptyset$ and $\mathcal{P}(A)=\mathcal{P}(B)=\{\emptyset\}$.
Then,

$$\mathcal{P}\left(\mathcal{P}\left(A\right)\times\mathcal{P}\left(B\right)\right)=\mathcal{P}\left(\mathcal{P}\left(\emptyset\right)\times\mathcal{P}\left(\emptyset\right)\right)=\mathcal{P}\left(\{\emptyset\}\times\{\emptyset\}\right)=\mathcal{P}\left(\{(\emptyset,\emptyset)\}\right)=\{\emptyset,\{(\emptyset,\emptyset)\}\}$$

Which respects the inequality:

$$2^{2^{|A|}\cdot 2^{|B|}}=2^{2^0\cdot 2^0}=2^1=2\geq 2$$

Then, the base cased holds.
For the next inductive step, the Inductive Hypothesis (I.H.) assume the statement holds for $n=|A|=|B|$, we want to show it holds for $|A|=|B|=n+1$. The cardinality of the sets A,B are independent, we do not consider n to be the same number for both. Thus,

$$\mathcal{P}\left(\mathcal{P}\left(A\right)\times\mathcal{P}\left(B\right)\right)=2^{2^{|A|}\cdot 2^{|B|}}=2^{2^{n+1}\cdot 2^{n+1}}=2^{2^n\cdot 2^1\cdot 2^n\cdot 2^1}\overset{\text{(I.H.)}}{=}2^{2^n\cdot 2^1\cdot 2^n\cdot 2^1\geq 1}=2^{2^{2\cdot(n+1)}\geq 1}=2^{2^{2\cdot(n+1)}}\geq 2$$

the inductive step holds and the statement holds for all $n\in\mathbb{N}$.