9 LINEAR PROGRAMMING

John von Neumann

1903–1957

John von Neumann was born in Budapest, Hungary, where his father was a successful banker. John's genius was recognized at an early age. By the age of ten, his mathematical knowledge was so great that instead of attending regular classes he studied privately under the direction of leading Hungarian mathematicians.

At the age of twenty-one, he acquired two degrees, one in chemical engineering at Zurich, and the other a Ph.D. in mathematics from the University of Budapest. He spent some time teaching at the University of Berlin, and then, in 1930, accepted a visiting professorship at Princeton University. In 1933, John von Neumann and Albert Einstein were among the first full professors to be appointed to the newly organized Institute for Advanced Study at Princeton.

During World War II, von Neumann was a consultant at Los Alamos, and his research helped in the development of the atomic bomb. In 1954, President Eisenhower appointed him to the Atomic Energy Commission.

von Neumann is considered to be the father of modern game theory—a branch of mathematics that deals with strategies and decision making. Much of his results concerning game theory were published in a lengthy paper in 1944 titled *Theory of Games and Economic Behavior*, written with Oskar Morgenstern.

In 1955, John von Neumann was diagnosed with cancer—he died in 1957 at the age of 53. Many stories are told of his mental abilities. Even during the final months of his life, as his brother read to him in German from Goethe's *Faust*, each time a page was turned John would recite from memory the continuation of the passage on the following page.

9.1 SYSTEMS OF LINEAR INEQUALITIES

The following statements are inequalities in two variables.

$$3x - 2y < 6 \quad \text{and} \quad x + y \geq 6$$

An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y, respectively. For instance, $(1, 1)$ is a solution of the inequality $3x - 2y < 6$ because $3(1) - 2(1) = 1 < 6$. The **graph** of an inequality is the collection of all solutions of the inequality. To sketch the graph of an inequality such as

$$3x - 2y < 6$$

we begin by sketching the graph of the *corresponding equation*

$$3x - 2y = 6.$$

The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true. (1) All points in the region are solutions of the inequality. (2) No points in the region are solutions of the inequality. Thus, we can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign, and sketch the graph of the resulting equation. (We use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

In this section, we will work with **linear inequalities** of the form

$$ax + by < c \qquad ax + by \leq c$$

$$ax + by > c \qquad ax + by \geq c.$$

The graph of each of these linear inequalities is a half-plane lying on one side of the line $ax + by = c$. The simplest linear inequalities are those corresponding to horizontal or vertical lines, as shown in Example 1.

EXAMPLE 1 ***Sketching the Graph of a Linear Inequality***

Sketch the graphs of (a) $x > -2$ and (b) $y \leq 3$.

Solution (a) The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 9.1.

(b) The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 9.2.

Figure 9.1

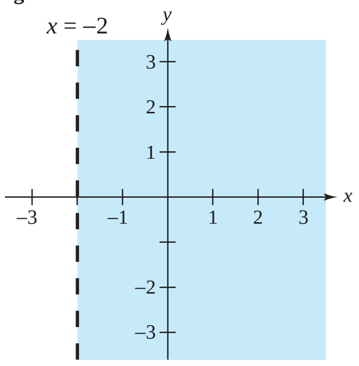
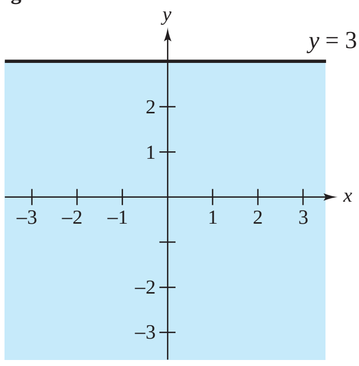


Figure 9.2

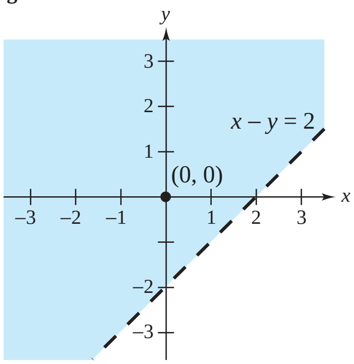


EXAMPLE 2 ***Sketching the Graph of a Linear Inequality***

Sketch the graph of $x - y < 2$.

Solution The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 9.3. Since the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point you choose, you will see that it does not satisfy the inequality.)

Figure 9.3



For a linear inequality in two variables, we can sometimes simplify the graphing procedure by writing the inequality in *slope-intercept* form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

we can see that the solution points lie *above* the line $y = x - 2$, as shown in Figure 9.3. Similarly, by writing the inequality $3x - 2y > 5$ in the form

$$y < \frac{3}{2}x - \frac{5}{2}$$

we see that the solutions lie *below* the line $y = \frac{3}{2}x - \frac{5}{2}$.

Systems of Inequalities

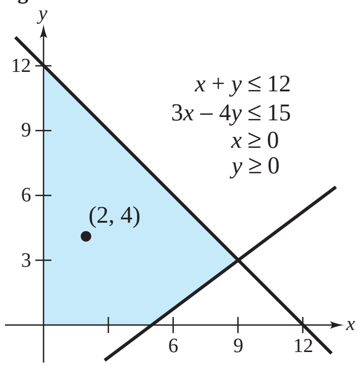
Many practical problems in business, science, and engineering involve systems of linear inequalities. Here is an example of such a system.

$$\begin{aligned} x + y &\leq 12 \\ 3x - 4y &\leq 15 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

A **solution** of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system. For instance, $(2, 4)$ is a solution of this system because $x = 2$ and $y = 4$ satisfy each of the four inequalities in the system. The **graph** of a system of inequalities in two variables is the collection of all points that are solutions of the system. For instance, the graph of the system above is the region shown in Figure 9.4. Note that the point $(2, 4)$ lies in the region because it is a solution of the system of inequalities.

To sketch the graph of a system of inequalities in two variables, we first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. For systems of linear inequalities, it is helpful to find the *vertices* of the solution region, as shown in the following example.

Figure 9.4



(2, 4) is a solution because it satisfies the system of inequalities.

EXAMPLE 3 *Solving a System of Inequalities*

Sketch the graph (and label the vertices) of the solution set of the following system.

$$\begin{aligned} x - y &< 2 \\ x &> -2 \\ y &\leq 3 \end{aligned}$$

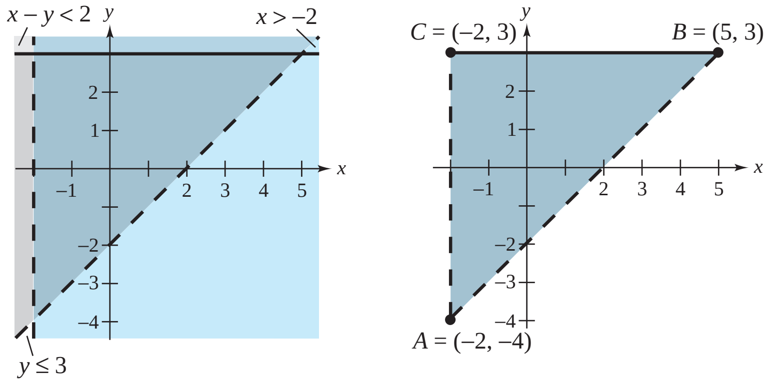
Solution We have already sketched the graph of each inequality in Examples 1 and 2. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown in Figure 9.5. To find the vertices of the region, we find the points of intersection of the boundaries of the region.

Vertex A: $(-2, -4)$
Obtained by finding the point of intersection of
$$\begin{aligned} x - y &= 2 \\ x &= -2. \end{aligned}$$

Vertex B: $(5, 3)$
Obtained by finding the point of intersection of
$$\begin{aligned} x - y &= 2 \\ y &= 3. \end{aligned}$$

Vertex C: $(-2, 3)$
Obtained by finding the point of intersection of
$$\begin{aligned} x &= -2 \\ y &= 3. \end{aligned}$$

Figure 9.5



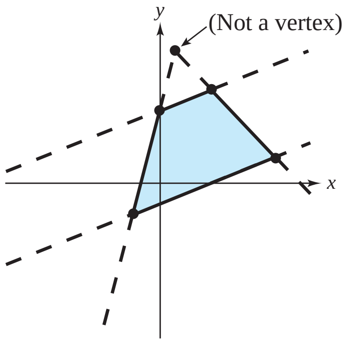
For the triangular region shown in Figure 9.5, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 9.6. In order to keep track of which points of intersection are actually vertices of the region, we suggest that you make a careful sketch of the region and refer to your sketch as you find each point of intersection.

When solving a system of inequalities, you should be aware that the system might have no solution. For instance, the system

$$x + y > 3$$
$$x + y < -1$$

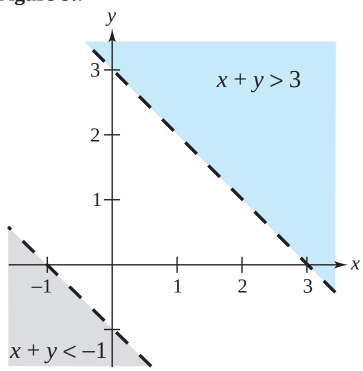
has no solution points because the quantity $(x + y)$ cannot be both less than -1 and greater than 3, as shown in Figure 9.7.

Figure 9.6



Border lines can intersect at a point that is not a vertex.

Figure 9.7



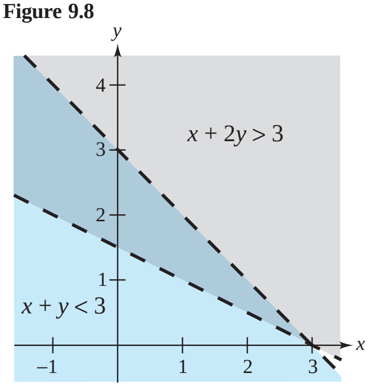
No Solution

Another possibility is that the solution set of a system of inequalities can be unbounded. For instance, the solution set of

$$x + y < 3$$
$$x + 2y > 3$$

forms an *infinite wedge*, as shown in Figure 9.8.

Figure 9.8



Unbounded Region

Applications

Our last example in this section shows how a system of linear inequalities can arise in an applied problem.

EXAMPLE 4 ***An Application of a System of Inequalities***

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes the minimum daily requirements for calories and vitamins.

Solution We let

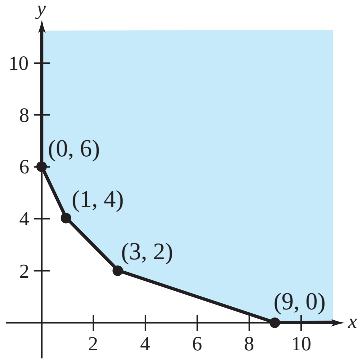
x = number of cups of dietary drink X
y = number of cups of dietary drink Y.

Then, to meet the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{aligned} \text{For calories:} \quad & 60x + 60y \geq 300 \\ \text{For vitamin A:} \quad & 12x + 6y \geq 36 \\ \text{For vitamin C:} \quad & 10x + 30y \geq 90 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of linear inequalities is shown in Figure 9.9.

Figure 9.9



Any point inside the region shown in Figure 9.9 (or on its boundary) meets the minimum daily requirements for calories and vitamins. For instance, 3 cups of dietary drink X and 2 cups of dietary drink Y supply 300 calories, 48 units of vitamin A, and 90 units of vitamin C.

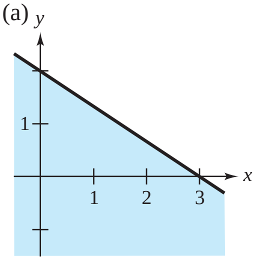
SECTION 9.1 EXERCISES

In Exercises 1–6, match the linear inequality with its graph. [The graphs are labeled (a)–(f).]

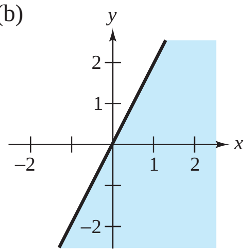
1. $x > 3$ **2.** $y \leq 2$ **3.** $2x + 3y \leq 6$

4. $2x - y \geq -2$ **5.** $x \geq \frac{y}{2}$ **6.** $y > 3x$

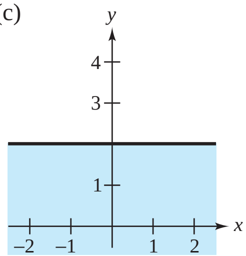
(a)



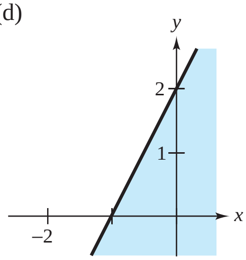
(b)



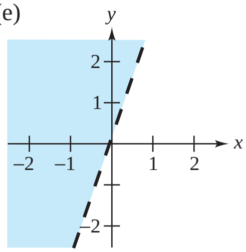
(c)



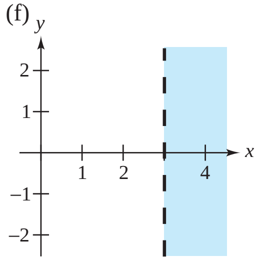
(d)



(e)



(f)



In Exercises 7–22, sketch the graph of the given linear inequality.

7. $x \geq 2$ **8.** $x \leq 4$ **9.** $y \geq -1$

10. $y \leq 3$ **11.** $y < 2 - x$ **12.** $y > 2x - 4$

13. $2y - x \geq 4$ **14.** $5x + 3y \geq -15$

15. $y \leq x$ **16.** $3x > y$ **17.** $y \geq 4 - 2x$

18. $y \leq 3 + x$ **19.** $3y + 4 \geq x$ **20.** $6 - 2y < x$

21. $4x - 2y \leq 12$ **22.** $y + 3x > 6$

In Exercises 23–32, sketch the graph of the solution of the given system of linear inequalities.

23. $\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 2 \\ y &\leq 4 \end{aligned}$

24. $\begin{aligned} x &\geq -1 \\ y &\geq -1 \\ x &\leq 1 \\ y &\leq 2 \end{aligned}$

25. $\begin{aligned} x + y &\leq 1 \\ -x + y &\leq 1 \\ y &\geq 0 \end{aligned}$

26. $\begin{aligned} 3x + 2y &< 6 \\ x \quad &> 0 \\ y &> 0 \end{aligned}$

27. $\begin{aligned} x + y &\le 5 \\ x \quad &\ge 2 \\ y &\ge 0 \end{aligned}$

28. $\begin{aligned} 2x + y &\ge 2 \\ x \quad &\le 2 \\ y &\le 1 \end{aligned}$

29. $\begin{aligned} -3x + 2y &< 6 \\ x + 4y &> -2 \\ 2x + y &< 3 \end{aligned}$

30. $\begin{aligned} x - 7y &> -36 \\ 5x + 2y &> 5 \\ 6x - 5y &> 6 \end{aligned}$

31. $\begin{aligned} x \quad &\ge 1 \\ x - 2y &\le 3 \\ 3x + 2y &\ge 9 \\ x + y &\le 6 \end{aligned}$

32. $\begin{aligned} x + y &< 10 \\ 2x + y &> 10 \\ x - y &< 2 \end{aligned}$

In Exercises 33–36, derive a set of inequalities to describe the given region.

33. Rectangular region with vertices at $(2, 1)$, $(5, 1)$, $(5, 7)$, and $(2, 7)$.

34. Parallelogram with vertices at $(0, 0)$, $(4, 0)$, $(1, 4)$, and $(5, 4)$.

35. Triangular region with vertices at $(0, 0)$, $(5, 0)$, and $(2, 3)$.

36. Triangular region with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

37. A furniture company can sell all the tables and chairs it produces. Each table requires 1 hour in the assembly center and $1\frac{1}{3}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. The company's assembly center is available 12 hours per day, and its finishing center is available 15 hours per day. If x is the number of tables produced per day and y is the number of chairs, find a system of inequalities describing all possible production levels. Sketch the graph of the system.

38. A store sells two models of a certain brand of computer. Because of the demand, it is necessary to stock at least twice as many units of model A as units of model B. The cost to the store for the two models is \$800 and \$1200, respectively. The management does not want more than \$20,000 in computer inventory at any one time, and it wants at least four model A computers and two model B computers in inventory at all times. Devise a system of inequalities describing all possible inventory levels, and sketch the graph of the system.

39. A person plans to invest no more than \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000. Moreover, one account should have at least twice the amount that is in the other account. Find a system of inequalities to describe the various amounts that can be deposited in each account, and sketch the graph of the system.

40. Two types of tickets are to be sold for a concert. One type costs \$15 per ticket and the other type costs \$25 per ticket. The promoter of the concert must sell at least 15,000 tickets including 8000 of the \$15 tickets and 4000 of the \$25 tickets. Moreover, the gross receipts must total at least \$275,000 in order for the concert to be held. Find a system of inequalities describing the different numbers of tickets that can be sold, and sketch the graph of the system.

41. A dietitian is asked to design a special diet supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements in the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B. Find a system of inequalities describing the different amounts of food X and food Y that can be used in the diet, and sketch the graph of the system.

42. Rework Exercise 41 using minimum daily requirements of 280 units of calcium, 160 units of iron, and 180 units of vitamin B.

9.2 LINEAR PROGRAMMING INVOLVING TWO VARIABLES

Many applications in business and economics involve a process called **optimization,** in which we are required to find the minimum cost, the maximum profit, or the minimum use of resources. In this section we discuss one type of optimization problem called **linear programming.**

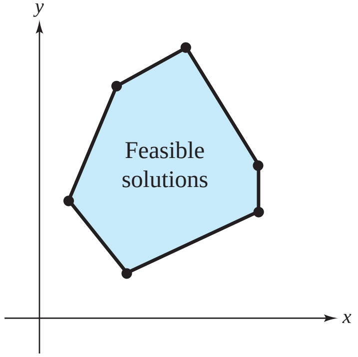
A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints.** The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions.**

For example, consider a linear programming problem in which we are asked to maximize the value of

$$z = ax + by \qquad \text{Objective function}$$

subject to a set of constraints that determine the region indicated in Figure 9.10. Because every point in the region satisfies each constraint, it is not clear how we should go about finding the point that yields a maximum value of z. Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices of the region. In other words, *we can find the maximum value by testing z at each of the vertices*, as illustrated in Example 1.

Figure 9.10



The objective function has its optimal value at one of the vertices of the region determined by the constraints.

Theorem 9.1
Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions. If the problem has more than one solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

EXAMPLE 1 *Solving a Linear Programming Problem*

Find the maximum value of

$$z = 3x + 2y \qquad \text{Objective function}$$

subject to the following constraints.

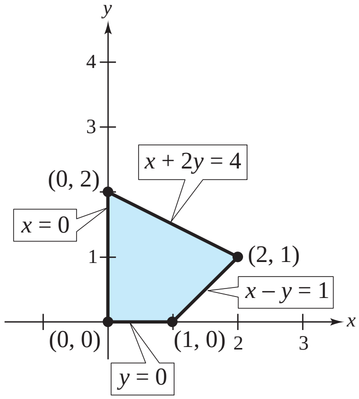
$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\} \quad \text{Constraints}$$

Solution The constraints form the region shown in Figure 9.11. At the four vertices of this region, the objective function has the following values.

$$\text{At } (0, 0): z = 3(0) + 2(0) = 0$$
$$\text{At } (1, 0): z = 3(1) + 2(0) = 3$$
$$\text{At } (2, 1): z = 3(2) + 2(1) = 8 \qquad \textbf{(Maximum value of } z\textbf{)}$$
$$\text{At } (0, 2): z = 3(0) + 2(2) = 4$$

Thus, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

Figure 9.11



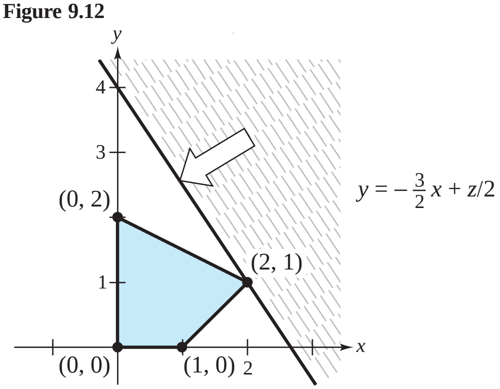
REMARK: In Example 1, try testing some of the interior points in the region. You will see that the corresponding values of z are less than 8.

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in the form

$$y = -\frac{3}{2}x + \frac{z}{2}.$$

This equation represents a family of lines, each of slope $-3/2$. Of these infinitely many lines, we want the one that has the largest z-value, while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-3/2$, we want the one that has the largest y-intercept *and* intersects the given region, as shown in Figure 9.12. It should be clear that such a line will pass through one (or more) of the vertices of the region.

Figure 9.12



We outline the graphical method for solving a linear programming problem as follows.

Graphical Method of Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the following steps.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are called *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

These guidelines will work regardless of whether the objective function is to be maximized *or* minimized. For instance, in Example 1 the same test used to find the maximum value of z can be used to conclude that the minimum value of z is 0, and this occurs at the vertex (0, 0).

EXAMPLE 2 ***Solving a Linear Programming Problem***

Find the maximum value of the objective function

$$z = 4x + 6y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left.\begin{aligned} -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90. \end{aligned}\right\} \quad \text{Constraints}$$

Solution The region bounded by the constraints is shown in Figure 9.13. By testing the objective function at each vertex, we obtain the following.

$$\begin{aligned}
&\text{At } (0, 0)\text{: } z = 4(0) + 6(0) = 0 \\
&\text{At } (0, 11)\text{: } z = 4(0) + 6(11) = 66 \\
&\text{At } (5, 16)\text{: } z = 4(5) + 6(16) = 116 \\
&\text{At } (15, 12)\text{: } z = 4(15) + 6(12) = 132 \qquad \text{(Maximum value of } z\text{)} \\
&\text{At } (27, 0)\text{: } z = 4(27) + 6(0) = 108
\end{aligned}$$

Thus, the maximum value of z is 132, and this occurs when $x = 15$ and $y = 12$.

In the next example, we show that the same basic procedure can be used to solve a linear programming problem in which the objective function is to be *minimized*.

Figure 9.13

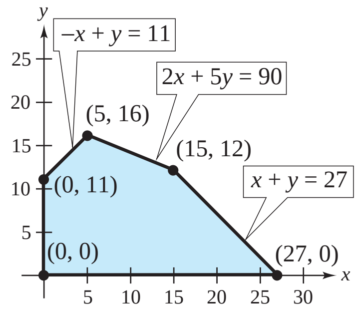
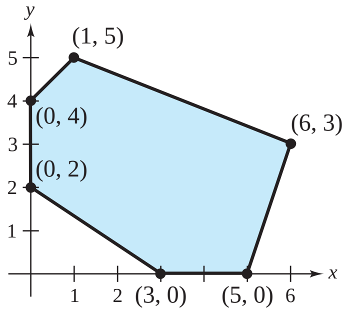


Figure 9.14



EXAMPLE 3 *Minimizing an Objective Function*

Find the minimum value of the objective function

$$z = 5x + 7y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left.\begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27. \end{aligned}\right\} \text{Constraints}$$

Solution The region bounded by the constraints is shown in Figure 9.14. By testing the objective function at each vertex, we obtain the following.

At $(0, 2)$: $z = 5(0) + 7(2) = 14$ **(Minimum value of z)**

At $(0, 4)$: $z = 5(0) + 7(4) = 28$

At $(1, 5)$: $z = 5(1) + 7(5) = 40$

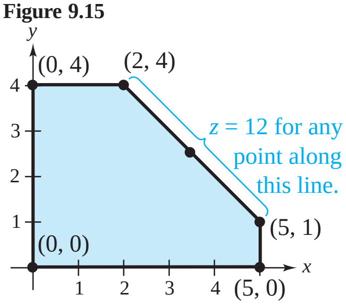
At $(6, 3)$: $z = 5(6) + 7(3) = 51$

At $(5, 0)$: $z = 5(5) + 7(0) = 25$

At $(3, 0)$: $z = 5(3) + 7(0) = 15$

Thus, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$.

Figure 9.15



REMARK: In Example 3, note that the steps used to find the minimum value are precisely the same ones we would use to find the maximum value. In other words, once we have evaluated the objective function at the vertices of the feasible region, we simply choose the largest value as the maximum and the smallest value as the minimum.

When solving a linear programming problem, it is possible that the maximum (or minimum) value occurs at *two* different vertices. For instance, at the vertices of the region shown in Figure 9.15, the objective function

$$z = 2x + 2y \qquad \text{Objective function}$$

has the following values.

$$\text{At } (0, 0)\colon z = 2(0) + 2(0) = 0$$
$$\text{At } (0, 4)\colon z = 2(0) + 2(4) = 8$$
$$\text{At } (2, 4)\colon z = 2(2) + 2(4) = 12 \qquad \text{(Maximum value of } z\text{)}$$
$$\text{At } (5, 1)\colon z = 2(5) + 2(1) = 12 \qquad \text{(Maximum value of } z\text{)}$$
$$\text{At } (5, 0)\colon z = 2(5) + 2(0) = 10$$

In this case, we can conclude that the objective function has a maximum value not only at the vertices (2, 4) and (5, 1), it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices*.

Some linear programming problems have no optimal solution. This can occur if the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

EXAMPLE 4 *An Unbounded Region*

Find the maximum value of

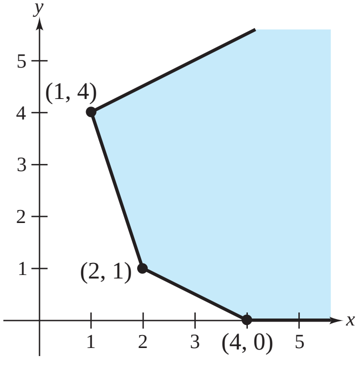
$$z = 4x + 2y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left.\begin{aligned} x + 2y &\geq 4 \\ 3x + y &\geq 7 \\ -x + 2y &\leq 7. \end{aligned}\right\} \quad \text{Constraints}$$

Solution The region determined by the constraints is shown in Figure 9.16. For this unbounded region, there is no maximum value of z. To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing x to be large, we can obtain values of $z = 4(x) + 2(0) = 4x$ that are as large as we want. Thus, there is no maximum value of z.

Figure 9.16



Applications

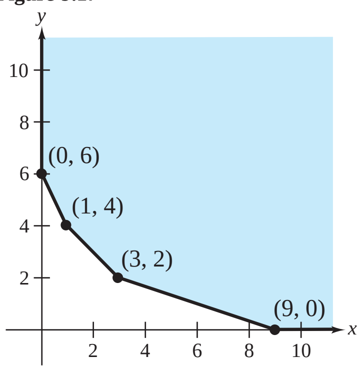
EXAMPLE 5 ***An Application: Minimum Cost***

In Example 4 in Section 9.1, we set up a system of linear equations for the following problem. The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, suppose that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimize the cost and still meet the stated daily requirements?

Solution We begin by letting x be the number of cups of dietary drink X and y be the number of cups of dietary drink Y. Moreover, to meet the minimum daily requirements, the following inequalities must be satisfied.

$$\left.\begin{array}{lrl} \text{For calories:} & 60x + 60y \geq & 300 \\ \text{For vitamin A:} & 12x + 6y \geq & 36 \\ \text{For vitamin C:} & 10x + 30y \geq & 90 \\ & x \geq & 0 \\ & y \geq & 0 \end{array}\right\} \quad \text{Constraints}$$

Figure 9.17



The cost C is given by

$$C = 0.12x + 0.15y. \qquad \text{Objective function}$$

The graph of the region corresponding to the constraints is shown in Figure 9.17. To determine the minimum cost, we test C at each vertex of the region as follows.

At $(0, 6)$: $C = 0.12(0) + 0.15(6) = 0.90$

At $(1, 4)$: $C = 0.12(1) + 0.15(4) = 0.72$

At $(3, 2)$: $C = 0.12(3) + 0.15(2) = 0.66$ **(Minimum value of C)**

At $(9, 0)$: $C = 0.12(9) + 0.15(0) = 1.08$

Thus, the minimum cost is \$0.66 per day, and this occurs when three cups of drink X and two cups of drink Y are consumed each day.

SECTION 9.2 ❑ EXERCISES

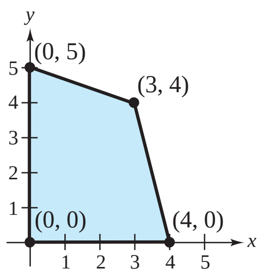
In Exercises 1–16, find the minimum and maximum values of the given objective function, subject to the indicated constraints.

1. Objective function:

$z = 3x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 3y &\leq 15 \\ 4x + y &\leq 16 \end{aligned}$$

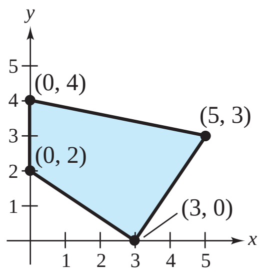


2. Objective function:

$z = 4x + 3y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\geq 6 \\ 3x - 2y &\leq 9 \\ x + 5y &\leq 20 \end{aligned}$$



3. Objective function:

$z = 5x + 0.5y$

Constraints:

(See Exercise 1.)

4. Objective function:

$z = x + 6y$

Constraints:

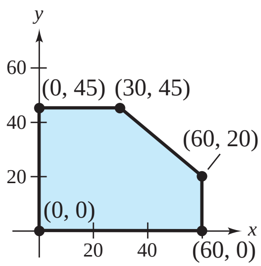
(See Exercise 2.)

5. Objective function:

$z = 10x + 7y$

Constraints:

$$\begin{aligned} 0 \leq x &\leq 60 \\ 0 \leq y &\leq 45 \\ 5x + 6y &\leq 420 \end{aligned}$$

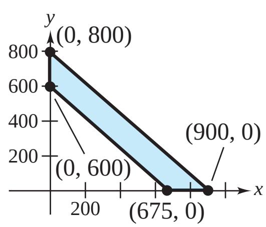


6. Objective function:

$z = 50x + 35y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 8x + 9y &\leq 7200 \\ 8x + 9y &\geq 5400 \end{aligned}$$



7. Objective function:

$z = 25x + 30y$

Constraints:

(See Exercise 5.)

8. Objective function:

$z = 16x + 18y$

Constraints:

(See Exercise 6.)

9. Objective function:

$z = 4x + 5y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 4x + 3y &\geq 27 \\ x + y &\geq 8 \\ 3x + 5y &\geq 30 \end{aligned}$$

10. Objective function:

$z = 4x + 5y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 2y &\leq 10 \\ x + 2y &\leq 6 \end{aligned}$$

11. Objective function:

$z = 2x + 7y$

Constraints:

(See Exercise 9.)

12. Objective function:

$z = 2x - y$

Constraints:

(See Exercise 10.)

13. Objective function:

$z = 4x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 40 \\ x + y &\geq 30 \\ 2x + 3y &\geq 72 \end{aligned}$$

14. Objective function:

$z = x$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\leq 60 \\ 2x + y &\leq 28 \\ 4x + y &\leq 48 \end{aligned}$$

15. Objective function:

$z = x + 4y$

Constraints:

(See Exercise 13.)

16. Objective function:

$z = y$

Constraints:

(See Exercise 14.)

In Exercises 17–20, maximize the given objective function subject to the constraints $3x_1 + x_2 \leq 15$ and $4x_1 + 3x_2 \leq 30$ where $x_1, x_2 \geq 0$.

17. $z = 2x_1 + x_2$

18. $z = 5x_1 + x_2$

19. $z = x_1 + x_2$

20. $z = 3x_1 + x_2$

21. A merchant plans to sell two models of home computers at costs of \$250 and \$400, respectively. The \$250 model yields a profit of \$45 and the \$400 model yields a profit of \$50. The merchant estimates that the total monthly demand will not exceed 250 units. Find the number of units of each model that should be stocked in order to maximize profit. Assume that the merchant does not want to invest more than \$70,000 in computer inventory.

22. A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking. Find the number of acres of each fruit that should be planted to maximize profit, assuming that the profit is \$140 per acre for crop A and \$235 per acre for crop B.

23. A farming cooperative mixes two brands of cattle feed. Brand X costs \$25 per bag and contains 2 units of nutritional element A, 2 units of element B, and 2 units of element C. Brand Y costs \$20 per bag and contains 1 unit of nutritional element A, 9 units of element B, and 3 units of element C. Find the number of bags of each brand that should be mixed to produce a mixture having a minimum cost per bag. The minimum requirements of nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.

24. Two gasolines, type A and type B, have octane ratings of 80 and 92, respectively. Type A costs \$0.83 per liter and type B costs \$0.98 per liter. Determine the blend of minimum cost with an octane rating of at least 90. [Hint: Let x be the fraction of each liter that is type A and y be the fraction that is type B.]

In Exercises 25–30, the given linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. (In each problem, the objective function is to be maximized.)

25. Objective function:

$z = 2.5x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 3x + 5y &\leq 15 \\ 5x + 2y &\leq 10 \end{aligned}$$

26. Objective function:

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -x + 2y &\leq 4 \end{aligned}$$

27. Objective function:

$z = -x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 10 \\ x + y &\leq 7 \end{aligned}$$

28. Objective function:

$z = x + y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 0 \\ -3x + y &\geq 3 \end{aligned}$$

29. Objective function:

$z = 3x + 4y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 1 \\ 2x + y &\geq 4 \end{aligned}$$

30. Objective function:

$z = x + 2y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ 2x + y &\leq 4 \end{aligned}$$

In Exercises 31 and 32, determine t-values such that the given objective function has a maximum value at the indicated vertex.

31. Objective function:

$z = x + ty$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 1 \\ y &\leq 1 \end{aligned}$$

(a) (0, 0) (b) (1, 0)

(c) (1, 1) (d) (0, 1)

32. Objective function:

$z = 3x + ty$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}$$

(a) (0, 0) (b) (1, 0)

(c) (2, 1) (d) (0, 2)

[Note: For $t = 2$, this problem is equivalent to that given in Example 1.]

9.3 THE SIMPLEX METHOD: MAXIMIZATION

For linear programming problems involving two variables, the graphical solution method introduced in Section 9.2 is convenient. However, for problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers. One such method is called the **simplex method,** developed by George Dantzig in 1946. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function. We introduce this method with an example.

Suppose we want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

Since the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned} -x_1 + x_2 + s_1 &= 11 \\ x_1 + x_2 + s_2 &= 27 \\ 2x_1 + 5x_2 + s_3 &= 90 \end{aligned}$$

The numbers s_1, s_2 and s_3 are called **slack variables** because they take up the "slack" in each inequality.

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** if it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &= b_2 \\ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

where $s_i \geq 0$.

REMARK: Note that for a linear programming problem in standard form, the objective function is to be maximized, not minimized. (Minimization problems will be discussed in Sections 9.4 and 9.5.)

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ of the constraint equations in which *at most m* variables are nonzero—the variables that are nonzero are called **basic variables.** A basic solution for which all variables are nonnegative is called a **basic feasible solution.**

The Simplex Tableau

The simplex method is carried out by performing elementary row operations on a matrix that we call the **simplex tableau.** This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + (0)s_1 + (0)s_2 + \cdots + (0)\,s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of z. For instance, the simplex tableau for the linear programming problem

$$z = 4x_1 + 6x_2 \qquad \text{Objective function}$$

$$\left.\begin{aligned} -x_1 + x_2 + s_1 \qquad\qquad &= 11 \\ x_1 + x_2 \qquad + s_2 \qquad &= 27 \\ 2x_1 + 5x_2 \qquad\qquad + s_3 &= 90 \end{aligned}\right\} \qquad \text{Constraints}$$

is as follows.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

↑ *Current z–value*

For this **initial simplex tableau,** the **basic variables** are s_1, s_2, and s_3, and the **nonbasic variables** (which have a value of zero) are x_1 and x_2. Hence, from the two columns that are farthest to the right, we see that the current solution is

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 11, \quad s_2 = 27, \quad \text{and} \quad s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower–right corner of the simplex tableau is the current value of z. Note that the bottom–row entries under x_1 and x_2 are the negatives of the coefficients of x_1 and x_2 in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, we look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

Pivoting

Once we have set up the initial simplex tableau for a linear programming problem, the simplex method consists of checking for optimality and then, if the current solution is not optimal, improving the current solution. (An improved solution is one that has a larger z-value than the current solution.) To improve the current solution, we bring a new basic variable into the solution—we call this variable the **entering variable.** This implies that one of the current basic variables must leave, otherwise we would have too many variables for a basic solution—we call this variable the **departing variable.** We choose the entering and departing variables as follows.

1. The **entering variable** corresponds to the smallest (the most negative) entry in the bottom row of the tableau.
2. The **departing variable** corresponds to the smallest nonnegative ratio of b_i/a_{ij}, in the column determined by the entering variable.
3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is called the **pivot.**

Finally, to form the improved solution, we apply Gauss-Jordan elimination to the column that contains the pivot, as illustrated in the following example. (This process is called **pivoting.**)

EXAMPLE 1 *Pivoting to Find an Improved Solution*

Use the simplex method to find an improved solution for the linear programming problem represented by the following tableau.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

The objective function for this problem is $z = 4x_1 + 6x_2$.

Solution Note that the current solution $(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$ corresponds to a z–value of 0. To improve this solution, we determine that x_2 is the entering variable, because -6 is the smallest entry in the bottom row.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	
	↑ *Entering*					

To see *why* we choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. Hence, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

To find the departing variable, we locate the b_i's that have corresponding positive elements in the entering variables column and form the following ratios.

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \frac{90}{5} = 18$$

Here the smallest positive ratio is 11, so we choose s_1 as the departing variable.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*	
-1	1	1	0	0	11	s_1	← *Departing*
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
-4	-6	0	0	0	0		
	↑ *Entering*						

Note that the pivot is the entry in the first row and second column. Now, we use Gauss-Jordan elimination to obtain the following improved solution.

Before Pivoting *After Pivoting*

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

The new tableau now appears as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basis column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

In Example 1 the improved solution is not yet optimal since the bottom row still has a negative entry. Thus, we can apply another iteration of the simplex method to further improve our solution as follows. We choose x_1 as the entering variable. Moreover, the smallest nonnegative ratio of $11/(-1)$, $16/2 = 8$, and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the following.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

Thus, the new simplex tableau is as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. Thus, we choose s_1 as the entering variable and s_2 as the departing variable, as shown in the following tableau.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*	
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2	
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2	← *Departing*
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1	
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116		
		↑ *Entering*					

By performing one more iteration of the simplex method, we obtain the following tableau. (Try checking this.)

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*	
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2	
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1	
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1	
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132		← *Maximum z-value*

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

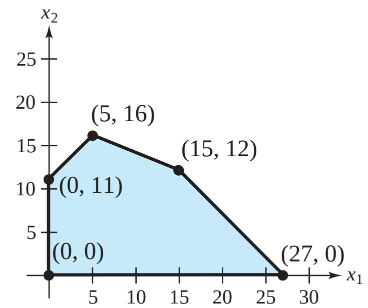
with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

REMARK: Ties may occur in choosing entering and/or departing variables. Should this happen, any choice among the tied variables may be made.

Because the linear programming problem in Example 1 involved only two decision variables, we could have used a graphical solution technique, as we did in Example 2, Section 9.2. Notice in Figure 9.18 that each iteration in the simplex method corresponds to moving from a given vertex to an adjacent vertex with an improved z-value.

Figure 9.18



$$\begin{array}{ccccccc} (0, 0) & \rightarrow & (0, 11) & \rightarrow & (5, 16) & \rightarrow & (15, 12) \\ z = 0 & & z = 66 & & z = 116 & & z = 132 \end{array}$$

The Simplex Method

We summarize the steps involved in the simplex method as follows.

The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the following steps.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row. The column for this entry is called the **entering column.** (If ties occur, any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "b-column" with their corresponding positive entries in the entering column. The **departing row** corresponds to the smallest nonnegative ratio b_i/a_{ij}. (If all entries in the entering column are 0 or negative, then there is no maximum solution. For ties, choose either entry.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations so that the pivot is 1, and all other entries in the entering column are 0. This process is called **pivoting.**
6. If all entries in the bottom row are zero or positive, this is the final tableau. If not, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m).$$

This solution is basic because at most m variables are nonzero (namely the slack variables). It is feasible because each variable is nonnegative.

In the next two examples, we illustrate the use of the simplex method to solve a problem involving three decision variables.

EXAMPLE 2 ***The Simplex Method with Three Decision Variables***

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \phantom{{}-2x_3} &\le 10 \\ x_1 + 2x_2 - 2x_3 &\le 20 \\ x_2 + 2x_3 &\le 5 \end{aligned}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Solution Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial simplex tableau for this problem is as follows. (Try checking these computations, and note the "tie" that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	
1	2	-2	0	1	0	20	s_2	
0	1	2	0	0	1	5	s_3	← *Departing*
-2	1	-2	0	0	0	0		
		↑ *Entering*						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	← *Departing*
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
-2	2	0	0	0	1	5		
↑ *Entering*								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \tfrac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

Occasionally, the constraints in a linear programming problem will include an equation. In such cases, we still add a "slack variable" called an **artificial variable** to form the initial simplex tableau. Technically, this new variable is not a slack variable (because there is no slack to be taken). Once you have determined an optimal solution in such a problem, you should check to see that any equations given in the original constraints are satisfied. Example 3 illustrates such a case.

EXAMPLE 3 *The Simplex Method with Three Decision Variables*

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \qquad \textbf{Objective function}$$

subject to the constraints

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 30 \\ 2x_1 + 3x_2 + x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 40 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Solution Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$$

the initial simplex tableau for this problem is as follows. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
4	1	1	1	0	0	30	s_1	← Departing
2	3	1	0	1	0	60	s_2	
1	2	3	0	0	1	40	s_3	
−3	−2	−1	0	0	0	0		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1	
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	s_2	← Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3	
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$		
	↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$$

and the maximum value of z is 45. (This solution satisfies the equation given in the constraints because $4(3) + 1(18) + 1(0) = 30$.)

Applications

EXAMPLE 4 ***A Business Application: Maximum Profit***

A manufacturer produces three types of plastic fixtures. The time required for molding, trimming, and packaging is given in Table 9.1. (Times are given in hours per dozen fixtures.)

TABLE 9.1

Process	*Type A*	*Type B*	*Type C*	*Total time available*
Molding	1	2	$\frac{3}{2}$	12,000
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1	4,600
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	2,400
Profit	\$11	\$16	\$15	—

How many dozen of each type of fixture should be produced to obtain a maximum profit?

Solution Letting x_1, x_2, and x_3 represent the number of dozen units of Types A, B, and C, respectively, the objective function is given by

$$\text{Profit} = P = 11x_1 + 16x_2 + 15x_3.$$

Moreover, using the information in the table, we construct the following constraints.

$$\begin{aligned} x_1 + 2x_2 + \tfrac{3}{2}x_3 &\le 12{,}000 \\ \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 &\le 4{,}600 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &\le 2{,}400 \end{aligned}$$

(We also assume that $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.) Now, applying the simplex method with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4{,}600, 2{,}400)$$

we obtain the following tableaus.

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
1	2	$\frac{3}{2}$	1	0	0	12,000	s_1	← *Departing*
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	4,600	s_2	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	2,400	s_3	
−11	−16	−15	0	0	0	0		
	↑ *Entering*							

$$
\begin{array}{rrrrrrr|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
\frac{1}{2} & 1 & \frac{3}{4} & \frac{1}{2} & 0 & 0 & 6{,}000 & x_2 \\
\frac{1}{3} & 0 & \frac{1}{2} & -\frac{1}{3} & 1 & 0 & 600 & s_2 \\
\frac{1}{3} & 0 & \frac{1}{4} & -\frac{1}{6} & 0 & 1 & 400 & s_3 \leftarrow \text{Departing} \\
\hline
-3 & 0 & -3 & 8 & 0 & 0 & 96{,}000 & \\
\uparrow & & & & & & & \\
\text{Entering} & & & & & & &
\end{array}
$$

$$
\begin{array}{rrrrrrr|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
0 & 1 & \frac{3}{8} & \frac{3}{4} & 0 & -\frac{3}{2} & 5{,}400 & x_2 \\
0 & 0 & \frac{1}{4} & -\frac{1}{6} & 1 & -1 & 200 & s_2 \leftarrow \text{Departing} \\
1 & 0 & \frac{3}{4} & -\frac{1}{2} & 0 & 3 & 1{,}200 & x_1 \\
\hline
0 & 0 & -\frac{3}{4} & \frac{13}{2} & 0 & 9 & 99{,}600 & \\
 & & \uparrow & & & & & \\
 & & \text{Entering} & & & & &
\end{array}
$$

$$
\begin{array}{rrrrrrr|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
0 & 1 & 0 & 1 & -\frac{3}{2} & 0 & 5{,}100 & x_2 \\
0 & 0 & 1 & -\frac{2}{3} & 4 & -4 & 800 & x_3 \\
1 & 0 & 0 & 0 & -3 & 6 & 600 & x_1 \\
\hline
0 & 0 & 0 & 6 & 3 & 6 & 100{,}200 &
\end{array}
$$

From this final simplex tableau, we see that the maximum profit is $100,200, and this is obtained by the following production levels.

Type A:	600 dozen units
Type B:	5,100 dozen units
Type C:	800 dozen units

REMARK: In Example 4, note that the second simplex tableau contains a “tie” for the minimum entry in the bottom row. (Both the first and third entries in the bottom row are -3.) Although we chose the first column to represent the departing variable, we could have chosen the third column. Try reworking the problem with this choice to see that you obtain the same solution.

EXAMPLE 5 *A Business Application: Media Selection*

The advertising alternatives for a company include television, radio, and newspaper advertisements. The costs and estimates for audience coverage are given in Table 9.2

TABLE 9.2

	Television	*Newspaper*	*Radio*
Cost per advertisement	\$ 2,000	\$ 600	\$ 300
Audience per advertisement	100,000	40,000	18,000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is \$18,200. How many advertisements should be run in each of the three types of media to maximize the total audience?

Solution To begin, we let x_1, x_2, and x_3 represent the number of advertisements in television, newspaper, and radio, respectively. The objective function (to be maximized) is therefore

$$z = 100{,}000x_1 + 40{,}000x_2 + 18{,}000x_3 \qquad \textbf{Objective function}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. The constraints for this problem are as follows.

$$\begin{aligned}
2000x_1 + 600x_2 + 300x_3 &\leq 18{,}200\\
x_2 &\leq 10\\
x_3 &\leq 0.5(x_1 + x_2 + x_3)\\
x_1 &\geq 0.1(x_1 + x_2 + x_3)
\end{aligned}$$

A more manageable form of this system of constraints is as follows.

$$\left.\begin{aligned}
20x_1 + 6x_2 + 3x_3 &\leq 182\\
x_2 &\leq 10\\
-x_1 - x_2 + x_3 &\leq 0\\
-9x_1 + x_2 + x_3 &\leq 0
\end{aligned}\right\} \quad \textbf{Constraints}$$

Thus, the initial simplex tableau is as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	*Basic Variables*	
20	6	3	1	0	0	0	182	s_1	← *Departing*
0	1	0	0	1	0	0	10	s_2	
−1	−1	1	0	0	1	0	0	s_3	
−9	1	1	0	0	0	1	0	s_4	
−100,000	−40,000	−18,000	0	0	0	0	0		
↑ *Entering*									

Now, to this initial tableau, we apply the simplex method as follows.

$$
\begin{array}{cccccccc|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & b & \text{Basic Variables} \\
\hline
1 & \frac{3}{10} & \frac{3}{20} & \frac{1}{20} & 0 & 0 & 0 & \frac{91}{10} & x_1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 10 & s_2 \leftarrow \text{Departing} \\
0 & -\frac{7}{10} & \frac{23}{20} & \frac{1}{20} & 0 & 1 & 0 & \frac{91}{10} & s_3 \\
0 & \frac{37}{10} & \frac{47}{20} & \frac{9}{20} & 0 & 0 & 1 & \frac{819}{10} & s_4 \\
\hline
0 & -10{,}000 & -3{,}000 & 5{,}000 & 0 & 0 & 0 & 910{,}000 & \\
 & \uparrow & & & & & & & \\
 & \text{Entering} & & & & & & &
\end{array}
$$

$$
\begin{array}{cccccccc|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & b & \text{Basic Variables} \\
\hline
1 & 0 & \frac{3}{20} & \frac{1}{20} & -\frac{3}{10} & 0 & 0 & \frac{61}{10} & x_1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 10 & x_2 \\
0 & 0 & \frac{23}{20} & \frac{1}{20} & \frac{7}{10} & 1 & 0 & \frac{161}{10} & s_3 \leftarrow \text{Departing} \\
0 & 0 & \frac{47}{20} & \frac{9}{20} & -\frac{37}{10} & 0 & 1 & \frac{449}{10} & s_4 \\
\hline
0 & 0 & -3{,}000 & 5{,}000 & 10{,}000 & 0 & 0 & 1{,}010{,}000 & \\
 & & \uparrow & & & & & & \\
 & & \text{Entering} & & & & & &
\end{array}
$$

$$
\begin{array}{cccccccc|l}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & b & \text{Basic Variables} \\
\hline
1 & 0 & 0 & \frac{1}{23} & -\frac{9}{23} & -\frac{3}{23} & 0 & 4 & x_1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 10 & x_2 \\
0 & 0 & 1 & \frac{1}{23} & \frac{14}{23} & \frac{20}{23} & 0 & 14 & x_3 \\
0 & 0 & 0 & \frac{8}{23} & -\frac{118}{23} & -\frac{47}{23} & 1 & 12 & s_4 \\
\hline
0 & 0 & 0 & \frac{118{,}000}{23} & \frac{272{,}000}{23} & \frac{60{,}000}{23} & 0 & 1{,}052{,}000 &
\end{array}
$$

From this tableau, we see that the maximum weekly audience for an advertising budget of \$18,200 is

$z = 1{,}052{,}000$ **Maximum weekly audience**

and this occurs when $x_1 = 4$, $x_2 = 10$, and $x_3 = 14$. We sum up the results here.

Media	*Number of Advertisements*	*Cost*	*Audience*
Television	4	\$ 8,000	400,000
Newspaper	10	\$ 6,000	400,000
Radio	14	\$ 4,200	252,000
Total	28	\$18,200	1,052,000

SECTION 9.3 ❑ EXERCISES

In Exercises 1–4, write the simplex tableau for the given linear programming problem. You do not need to solve the problem. (In each case the objective function is to be maximized.)

1. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\le 8 \\ x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2. Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 - x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

3. Objective function:
$z = 2x_1 + 3x_2 + 4x_3$
Constraints:
$$\begin{aligned} x_1 + 2x_2 \qquad &\le 12 \\ x_1 \qquad + x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

4. Objective function:
$z = 6x_1 - 9x_2$
Constraints:
$$\begin{aligned} 2x_1 - 3x_2 &\le 6 \\ x_1 + x_2 &\le 20 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In Exercises 5–8, explain why the linear programming problem is *not* in standard form as given.

5. (Minimize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

6. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 6 \\ 2x_1 - x_2 &\le -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 + 3x_3 &\le 5 \\ 2x_1 \qquad - 2x_3 &\ge 1 \\ x_2 + x_3 &\le 0 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

8. (Maximize)
Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\ge 4 \\ 2x_1 + x_2 &\ge 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

In Exercises 9–20, use the simplex method to solve the given linear programming problem. (In each case the objective function is to be maximized.)

9. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + 4x_2 &\le 8 \\ x_1 + x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

10. Objective function:
$z = x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\le 6 \\ 3x_1 + 2x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

11. Objective function:
$z = 5x_1 + 2x_2 + 8x_3$
Constraints:
$$\begin{aligned} 2x_1 - 4x_2 + x_3 &\le 42 \\ 2x_1 + 3x_2 - x_3 &\le 42 \\ 6x_1 - x_2 + 3x_3 &\le 42 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

12. Objective function:
$z = x_1 - x_2 + 2x_3$
Constraints:
$$\begin{aligned} 2x_1 + 2x_2 &\le 8 \\ x_3 &\le 5 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

13. Objective function:
$z = 4x_1 + 5x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\le 10 \\ 3x_1 + 7x_2 &\le 42 \\ x_1, x_2 &\ge 0 \end{aligned}$$

14. Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + 3x_2 &\le 15 \\ 2x_1 - x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

15. Objective function:
$z = 3x_1 + 4x_2 + x_3 + 7x_4$
Constraints:
$$\begin{aligned} 8x_1 + 3x_2 + 4x_3 + x_4 &\le 7 \\ 2x_1 + 6x_2 + x_3 + 5x_4 &\le 3 \\ x_1 + 4x_2 + 5x_3 + 2x_4 &\le 8 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

16. Objective function:
$z = x_1$
Constraints:
$$\begin{aligned} 3x_1 + 2x_2 &\le 60 \\ x_1 + 2x_2 &\le 28 \\ x_1 + 4x_2 &\le 48 \\ x_1, x_2 &\ge 0 \end{aligned}$$

17. Objective function:
$z = x_1 - x_2 + x_3$
Constraints:
$$\begin{aligned} 2x_1 + x_2 - 3x_3 &\le 40 \\ x_1 \qquad + x_3 &\le 25 \\ 2x_2 + 3x_3 &\le 32 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

18. Objective function:
$z = 2x_1 + x_2 + 3x_3$
Constraints:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 59 \\ 2x_1 \qquad + 3x_3 &\le 75 \\ x_2 + 6x_3 &\le 54 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

19. Objective function:
$z = x_1 + 2x_2 - x_4$
Constraints:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 \qquad &\le 24 \\ 3x_2 + 7x_3 + x_4 &\le 42 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

20. Objective function:
$z = x_1 + 2x_2 + x_3 - x_4$
Constraints:
$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &\le 60 \\ x_2 + 2x_3 + 5x_4 &\le 50 \\ 2x_1 + 3x_2 \qquad + 6x_4 &\le 72 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

21. A merchant plans to sell two models of home computers at costs of \$250 and \$400, respectively. The \$250 model yields a profit of \$45 and the \$400 model yields a profit of \$50. The merchant estimates that the total monthly demand will not exceed 250 units. Find the number of units of each model that should be stocked in order to maximize profit. Assume that the merchant does not want to invest more than \$70,000 in computer inventory. (See Exercise 21 in Section 9.2.)

22. A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking. Find the number of acres of each fruit that should be planted to maximize profit, assuming that the profit is \$140 per acre for crop A and \$235 per acre for B. (See Exercise 22 in Section 9.2.)

23. A grower has 50 acres of land for which she plans to raise three crops. It costs \$200 to produce an acre of carrots and the profit is \$60 per acre. It costs \$80 to produce an acre of celery and the profit is \$20 per acre. Finally, it costs \$140 to produce an acre of lettuce and the profit is \$30 per acre. Use the simplex method to find the number of acres of each crop she should plant in order to maximize her profit. Assume that her cost cannot exceed \$10,000.

24. A fruit juice company makes two special drinks by blending apple and pineapple juices. The first drink uses 30% apple juice and 70% pineapple, while the second drink uses 60% apple and 40% pineapple. There are 1000 liters of apple and 1500 liters of pineapple juice available. If the profit for the first drink is \$0.60 per liter and that for the second drink is \$0.50, use the simplex method to find the number of liters of each drink that should be produced in order to maximize the profit.

25. A manufacturer produces three models of bicycles. The time (in hours) required for assembling, painting, and packaging each model is as follows.

	Model A	*Model B*	*Model C*
Assembling	2	2.5	3
Painting	1.5	2	1
Packaging	1	0.75	1.25

The total time available for assembling, painting, and packaging is 4006 hours, 2495 hours and 1500 hours, respectively. The profit per unit for each model is \$45 (Model A), \$50 (Model B), and \$55 (Model C). How many of each type should be produced to obtain a maximum profit?

26. Suppose in Exercise 25 the total time available for assembling, painting, and packaging is 4000 hours, 2500 hours, and 1500 hours, respectively, and that the profit per unit is \$48 (Model A), \$50 (Model B), and \$52 (Model C). How many of each type should be produced to obtain a maximum profit?

27. A company has budgeted a maximum of \$600,000 for advertising a certain product nationally. Each minute of television time costs \$60,000 and each one-page newspaper ad costs \$15,000. Each television ad is expected to be viewed by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company's market research department advises the company to use at most 90% of the advertising budget on television ads. How should the advertising budget be allocated to maximize the total audience?

28. Rework Exercise 27 assuming that each one-page newspaper ad costs \$30,000.

29. An investor has up to \$250,000 to invest in three types of investments. Type A pays 8% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 14% annually and has a risk factor of 0.10. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-fourth of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

30. An investor has up to \$450,000 to invest in three types of investments. Type A pays 6% annually and has a risk factor of 0. Type B pays 10% annually and has a risk factor of 0.06. Type C pays 12% annually and has a risk factor of 0.08. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-half of the total portfolio is to be allocated to Type A investments and at least one-fourth of the portfolio is to be allocated to Type B investments. How much should be allocated to each type of investment to obtain a maximum return?

31. An accounting firm has 900 hours of staff time and 100 hours of reviewing time available each week. The firm charges \$2000 for an audit and \$300 for a tax return. Each audit requires 100 hours of staff time and 10 hours of review time, and each tax return requires 12.5 hours of staff time and 2.5 hours of review time. What number of audits and tax returns will bring in a maximum revenue?

32. The accounting firm in Exercise 31 raises its charge for an audit to \$2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)
Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 - 3x_2 &\le 1\\ -x_1 + 2x_2 &\le 4\\ x_1, x_2 &\ge 0 \end{aligned}$$

34. (Maximize)
Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} -x_1 + x_2 &\le 20\\ -2x_1 + x_2 &\le 50\\ x_1, x_2 &\ge 0 \end{aligned}$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)
Objective function:
$z = 2.5x_1 + x_2$
Constraints:
$$\begin{aligned} 3x_1 + 5x_2 &\le 15\\ 5x_1 + 2x_2 &\le 10\\ x_1, x_2 &\ge 0 \end{aligned}$$

36. (Maximize)
Objective function:
$z = x_1 + \frac{1}{2}x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\le 20\\ x_1 + 3x_2 &\le 35\\ x_1, x_2 &\ge 0 \end{aligned}$$

C **37.** Use a computer to maximize the objective function
$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
subject to the constraints
$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\le 65\\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\le 96\\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\le 80 \end{aligned}$$
where $x_1, x_2, x_3, x_4 \ge 0$.

C **38.** Use a computer to maximize the objective function
$z = 1.2x_1 + x_2 + x_3 + x_4$
subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\ge b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\ge b_2\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\ge b_m \end{aligned}$$

where $x_i \ge 0$ and $b_i \ge 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.

32. The accounting firm in Exercise 31 raises its charge for an audit to \$2500. What number of audits and tax returns will bring in a maximum revenue?

In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 33 and 34.

33. (Maximize)
Objective function:
$z = x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 - 3x_2 &\le 1 \\ -x_1 + 2x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

34. (Maximize)
Objective function:
$z = x_1 + 3x_2$
Constraints:
$$\begin{aligned} -x_1 + x_2 &\le 20 \\ -2x_1 + x_2 &\le 50 \\ x_1, x_2 &\ge 0 \end{aligned}$$

If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 35 and 36.

35. (Maximize)
Objective function:
$z = 2.5x_1 + x_2$
Constraints:
$$\begin{aligned} 3x_1 + 5x_2 &\le 15 \\ 5x_1 + 2x_2 &\le 10 \\ x_1, x_2 &\ge 0 \end{aligned}$$

36. (Maximize)
Objective function:
$z = x_1 + \frac{1}{2}x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\le 20 \\ x_1 + 3x_2 &\le 35 \\ x_1, x_2 &\ge 0 \end{aligned}$$

C **37.** Use a computer to maximize the objective function
$z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
subject to the constraints
$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\le 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\le 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\le 80 \end{aligned}$$
where $x_1, x_2, x_3, x_4 \ge 0$.

C **38.** Use a computer to maximize the objective function
$z = 1.2x_1 + x_2 + x_3 + x_4$
subject to the same set of constraints given in Exercise 37.

9.4 THE SIMPLEX METHOD: MINIMIZATION

In Section 9.3, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\ge b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\ge b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\ge b_m \end{aligned}$$

where $x_i \ge 0$ and $b_i \ge 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

In Example 5 in Section 9.2, we used geometric methods to solve the following minimization problem.

Minimization Problem: Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \qquad \text{Objective function}$$

subject to the following constraints

$$\left.\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The first step in converting this problem to a maximization problem is to form the augmented matrix for this system of inequalities. To this augmented matrix we add a last row that represents the coefficients of the objective function, as follows.

$$\begin{bmatrix} 60 & 60 & \vdots & 300 \\ 12 & 6 & \vdots & 36 \\ 10 & 30 & \vdots & 90 \\ \cdots & \cdots & \cdots & \cdots \\ 0.12 & 0.15 & \vdots & 0 \end{bmatrix}$$

Next, we form the **transpose** of this matrix by interchanging its rows and columns.

$$\begin{bmatrix} 60 & 12 & 10 & \vdots & 0.12 \\ 60 & 6 & 30 & \vdots & 0.15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 300 & 36 & 90 & \vdots & 0 \end{bmatrix}$$

Note that the rows of this matrix are the columns of the first matrix, and vice versa. Finally, we interpret the new matrix as a *maximization* problem as follows. (To do this, we introduce new variables, y_1, y_2, and y_3.) We call this corresponding maximization problem the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\leq 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\leq 0.15 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$.

As it turns out, the solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as follows.

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*	
60	12	10	1	0	0.12	s_1	← *Departing*
60	6	30	0	1	0.15	s_2	
−300	−36	−90	0	0	0		
↑ *Entering*							

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*	
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1	
0	−6	20	−1	1	$\frac{3}{100}$	s_2	← *Departing*
0	24	−40	5	0	$\frac{3}{5}$		
		↑ *Entering*					

y_1	y_2	y_3	s_1	s_2	b	*Basic Variables*
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
			↑ x_1	↑ x_2		

Thus, the solution of the dual maximization problem is $z = \frac{33}{50} = 0.66$. This is the same value we obtained in the minimization problem given in Example 5, in Section 9.2. The x-values corresponding to this optimal solution are obtained from the entries in the bottom row corresponding to slack variable columns. In other words, the optimal solution occurs when $x_1 = 3$ and $x_2 = 2$.

The fact that a dual maximization problem has the same solution as its original minimization problem is stated formally in a result called the **von Neumann Duality Principle,** after the American mathematician John von Neumann (1903–1957).

Theorem 9.2
The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

Solving a Minimization Problem

We summarize the steps used to solve a minimization problem as follows.

Solving a Minimization Problem

A minimization problem is in standard form if the objective function $w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem we use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ & & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_1 & c_2 & \ldots & c_n & 0 \end{array}\right]$$

2. Form the **transpose** of this matrix.

$$\left[\begin{array}{cccc:c} a_{11} & a_{21} & \ldots & a_{m1} & c_1 \\ a_{12} & a_{22} & \ldots & a_{m2} & c_2 \\ & & & & \\ a_{1n} & a_{2n} & \ldots & a_{mn} & c_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_1 & b_2 & \ldots & b_m & 0 \end{array}\right]$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function given by $z = b_1y_1 + b_2y_2 + \cdots + b_my_m$ subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n \end{aligned}$$

where $y_1 \geq 0, y_2 \geq 0, \ldots,$ and $y_m \geq 0$.

4. Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w. Moreover, the values of $x_1, x_2, \ldots,$ and x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

We illustrate the steps used to solve a minimization problem in Examples 1 and 2.

EXAMPLE 1 *Solving a Minimization Problem*

Find the minimum value of

$$w = 3x_1 + 2x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution The augmented matrix corresponding to this minimization problem is

$$\left[\begin{array}{cc:c} 2 & 1 & 6 \\ 1 & 1 & 4 \\ \hdashline 3 & 2 & 0 \end{array}\right].$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\left[\begin{array}{cc:c} 2 & 1 & 3 \\ 1 & 1 & 2 \\ \hdashline 6 & 4 & 0 \end{array}\right].$$

This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2y_1 + y_2 &\leq 3 \\ y_1 + y_2 &\leq 2 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \geq 0$ and $y_2 \geq 0$. We now apply the simplex method to the dual problem as follows.

y_1	y_2	s_1	s_2	b	Basic Variables	
2	1	1	0	3	s_1	← *Departing*
1	1	0	1	2	s_2	
−6	−4	0	0	0		
↑ *Entering*						

y_1	y_2	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	y_1
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	s_2 ← *Departing*
0	-1	3	0	9	
	↑ *Entering*				

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	-1	1	y_1
0	1	-1	2	1	y_2
0	0	2	2	10	
		↑ x_1	↑ x_2		

From this final simplex tableau, we see that the maximum value of z is 10. Therefore, the solution of the original minimization problem is

$$w = 10$$ **Minimum Value**

and this occurs when

$$x_1 = 2 \text{ and } x_2 = 2.$$

Both the minimization and the maximization linear programming problems in Example 1 could have been solved with a graphical method, as indicated in Figure 9.19. Note in Figure 9.19 (a) that the maximum value of $z = 6y_1 - 4y_2$ is the same as the minimum value of $w = 3x_1 + 2x_2$, as shown in Figure 9.19 (b). (See page 515.)

EXAMPLE 2 *Solving a Minimization Problem*

Find the minimum value of

$$w = 2x_1 + 10x_2 + 8x_3$$ **Objective function**

subject to the constraints

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \end{aligned}\right\}$$ **Constraints**

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Solution The augmented matrix corresponding to this minimization problem is

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \\ \hdashline 2 & 10 & 8 & 0 \end{array}\right].$$

Thus, the matrix corresponding to the dual maximization problem is given by the following transpose.

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 10 \\ 1 & 2 & 2 & 8 \\ \hdashline 6 & 8 & 4 & 0 \end{array}\right]$$

This implies that the dual maximization problem is as follows.

Dual Maximization Problem: Find the maximum value of

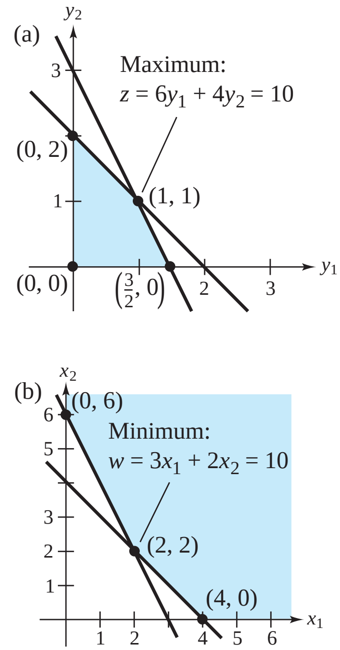
$$z = 6y_1 + 8y_2 + 4y_3 \qquad \textbf{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} y_1 \qquad\quad - \ y_3 &\le 2 \\ y_1 + \ y_2 + 2y_3 &\le 10 \\ y_1 + 2y_2 + 2y_3 &\le 8 \end{aligned}\right\} \qquad \textbf{Dual constraints}$$

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$. We now apply the simplex method to the dual problem as follows.

Figure 9.19



y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
1	0	−1	1	0	0	2	s_1	
1	1	2	0	1	0	10	s_2	
1	2	2	0	0	1	8	s_3	← *Departing*
−6	−8	−4	0	0	0	0		
	↑ *Entering*							

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
1	0	−1	1	0	0	2	s_1	← *Departing*
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2	
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2	
−2	0	4	0	0	4	32		
↑ *Entering*								

y_1	y_2	y_3	s_1	s_2	s_3	b	*Basic Variables*
1	0	-1	1	0	0	2	y_1
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	s_2
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	y_2
0	0	2	2	0	4	36	
			$\uparrow$	$\uparrow$	$\uparrow$		
			x_1	x_2	x_3		

From this final simplex tableau, we see that the maximum value of z is 36. Therefore, the solution of the original minimization problem is

$$w = 36 \qquad \textbf{Minimum Value}$$

and this occurs when

$$x_1 = 2, \; x_2 = 0, \text{ and } \; x_3 = 4.$$

Applications

EXAMPLE 3 ***A Business Application: Minimum Cost***

A small petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and it can produce 400 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 400 barrels of medium-grade oil, and 500 barrels of low-grade oil each day.

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Solution To begin, we let x_1 and x_2 represent the number of days the two refineries are operated. Then the total cost is given by

$$C = 20{,}000x_1 + 25{,}000x_2. \qquad \textbf{Objective function}$$

The constraints are given by

$$\left.\begin{array}{rl} \text{(High-grade)} & 400x_1 + 300x_2 \geq 25{,}000 \\ \text{(Medium-grade)} & 300x_1 + 400x_2 \geq 27{,}000 \\ \text{(Low-grade)} & 200x_1 + 500x_2 \geq 30{,}000 \end{array}\right\} \quad \textbf{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The augmented matrix corresponding to this minimization problem is

$$\begin{bmatrix} 400 & 300 & \vdots & 25{,}000 \\ 300 & 400 & \vdots & 27{,}000 \\ 200 & 500 & \vdots & 30{,}000 \\ \cdots & \cdots & \vdots & \cdots \\ 20{,}000 & 25{,}000 & \vdots & 0 \end{bmatrix}.$$

The matrix corresponding to the dual maximization problem is

$$\begin{bmatrix} 400 & 300 & 200 & \vdots & 20{,}000 \\ 300 & 400 & 500 & \vdots & 25{,}000 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 25{,}000 & 27{,}000 & 30{,}000 & \vdots & 0 \end{bmatrix}.$$

We now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
400	300	200	1	0	20,000	s_1
300	400	500	0	1	25,000	s_2 ← *Departing*
−25,000	−27,000	−30,000	0	0	0	
		↑ *Entering*				

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
280	140	0	1	$-\frac{2}{5}$	10,000	s_1 ← *Departing*
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3
−7,000	−3,000	0	0	60	1,500,000	
↑ *Entering*						

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
			↑ x_1	↑ x_2		

From the third simplex tableau, we see that the solution to the original minimization problem is

$$C = \$1{,}750{,}000 \qquad \text{Minimum cost}$$

and this occurs when $x_1 = 25$ and $x_2 = 50$. Thus, the two refineries should be operated for the following number of days.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for this number of days, the company will have produced the following amounts of oil.

$$\begin{aligned} \text{High-grade oil:} &\quad 25(400) + 50(300) = 25{,}000 \text{ barrels} \\ \text{Medium-grade oil:} &\quad 25(300) + 50(400) = 27{,}500 \text{ barrels} \\ \text{Low-grade oil:} &\quad 25(200) + 50(500) = 30{,}000 \text{ barrels} \end{aligned}$$

Thus, the original production level has been met (with a surplus of 500 barrels of medium-grade oil).

SECTION 9.4 ❑ EXERCISES

In Exercises 1–6, determine the dual of the given minimization problem.

1. Objective function:
$w = 3x_1 + 3x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. Objective function:
$w = 2x_1 + x_2$
Constraints:
$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Objective function:
$w = 4x_1 + x_2 + x_3$
Constraints:
$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 23 \\ x_1 + x_3 &\geq 10 \\ 8x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

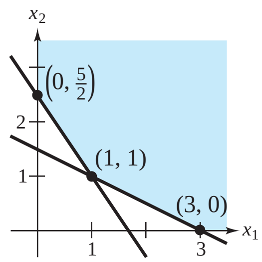
4. Objective function:
$w = 9x_1 + 6x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\geq 5 \\ 2x_1 + 2x_2 &\geq 8 \\ 2x_1 + x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. Objective function:
$w = 14x_1 + 20x_2 + 24x_3$
Constraints:
$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 7 \\ x_1 + 2x_2 + x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

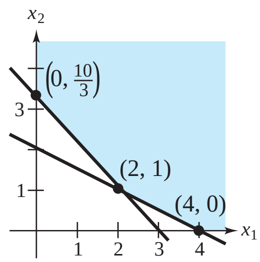
6. Objective function:
$w = 9x_1 + 4x_2 + 10x_3$
Constraints:
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\geq 6 \\ 6x_1 + x_2 + x_3 &\geq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 7–12, (a) solve the given minimization problem by the graphical method, (b) formulate the dual problem, and (c) solve the dual problem by the graphical method.

7. Objective function:
$w = 2x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 3x_1 + 2x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$



8. Objective function:
$w = 14x_1 + 20x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 7x_1 + 6x_2 &\geq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

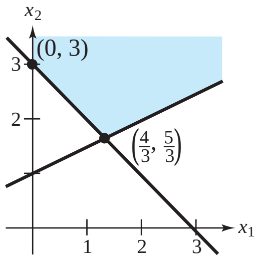


9. Objective function:

$w = x_1 + 4x_2$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ -x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

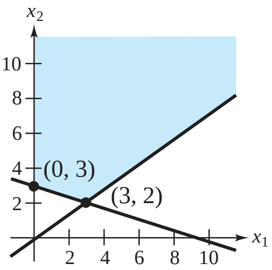


10. Objective function:

$w = 2x_1 + 6x_2$

Constraints:

$$\begin{aligned} -2x_1 + 3x_2 &\geq 0 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

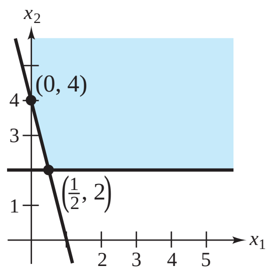


11. Objective function:

$w = 6x_1 + 3x_2$

Constraints:

$$\begin{aligned} 4x_1 + x_2 &\geq 4 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

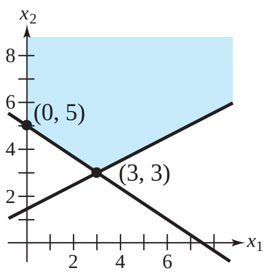


12. Objective function:

$w = x_1 + 6x_2$

Constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\geq 15 \\ -x_1 + 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$



In Exercises 13–20, solve the given minimization problem by solving the dual maximization problem with the simplex method.

13. Objective function:

$w = x_2$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 10 \\ -6x_1 + 5x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

14. Objective function:

$w = 3x_1 + 8x_2$

Constraints:

$$\begin{aligned} 2x_1 + 7x_2 &\geq 9 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

15. Objective function:

$w = 2x_1 + x_2$

Constraints:

$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

16. Objective function:

$w = 2x_1 + 2x_2$

Constraints:

$$\begin{aligned} 3x_1 + x_2 &\geq 6 \\ -4x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

17. Objective function:

$w = 8x_1 + 4x_2 + 6x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 6 \\ 4x_1 + x_2 + 3x_3 &\geq 7 \\ 2x_1 + x_2 + 4x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

18. Objective function:

$w = 8x_1 + 16x_2 + 18x_3$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\geq 4 \\ -4x_1 + 3x_2 - x_3 &\geq 1 \\ x_1 - x_2 + 3x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

19. Objective function:

$w = 6x_1 + 2x_2 + 3x_3$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 28 \\ 6x_1 + x_3 &\geq 24 \\ 3x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. Objective function:

$w = 42x_1 + 5x_2 + 17x_3$

Constraints:

$$\begin{aligned} 3x_1 - x_2 + 7x_3 &\geq 5 \\ -3x_1 - x_2 + 3x_3 &\geq 8 \\ 6x_1 + x_2 + x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 21–24, two dietary drinks are used to supply protein and carbohydrates. The first drink provides 1 unit of protein and 3 units of carbohydrates in each liter. The second drink supplies 2 units of protein and 2 units of carbohydrates in each liter. An athlete requires 3 units of protein and 5 units of carbohydrates. Find the amount of each drink the athlete should consume to minimize the cost and still meet the minimum dietary requirements.

21. The first drink costs \$2 per liter and the second costs \$3 per liter.

22. The first drink costs \$4 per liter and the second costs \$2 per liter.

23. The first drink costs \$1 per liter and the second costs \$3 per liter.

24. The first drink costs \$1 per liter and the second costs \$2 per liter.

In Exercises 25–28, an athlete uses two dietary drinks that provide the nutritional elements listed in the following table.

Drink	*Protein*	*Carbohydrates*	*Vitamin D*
I	4	2	1
II	1	5	1

Find the combination of drinks of minimum cost that will meet the minimum requirements of 4 units of protein, 10 units of carbohydrates, and 3 units of vitamin D.

25. Drink I costs \$5 per liter and drink II costs \$8 per liter.

26. Drink I costs \$7 per liter and drink II costs \$4 per liter.

27. Drink I costs \$1 per liter and drink II costs \$5 per liter.

28. Drink I costs \$8 per liter and drink II costs \$1 per liter.

29. A company has three production plants, each of which produces three different models of a particular product. The daily capacities (in thousands of units) of the three plants are as follows.

	Model 1	*Model 2*	*Model 3*
Plant 1	8	4	8
Plant 2	6	6	3
Plant 3	12	4	8

The total demand for Model 1 is 300,000 units, for Model 2 is 172,000 units, and for Model 3 is 249,500 units. Moreover, the daily operating cost for Plant 1 is \$55,000, for Plant 2 is \$60,000, and for Plant 3 is \$60,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

30. The company in Exercise 29 has lowered the daily operating cost for Plant 3 to \$50,000. How many days should each plant be operated in order to fill the total demand, and keep the operating cost at a minimum?

31. A small petroleum company owns two refineries. Refinery 1 costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 200 barrels of medium-grade oil, and 150 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$30,000 per day to operate, and it can produce 300 barrels of high-grade oil, 250 barrels of medium-grade oil, and 400 barrels of low-grade oil each day. The company has orders totaling 35,000 barrels of high-grade oil, 30,000 barrels of medium-grade oil, and 40,000 barrels of low-grade oil. How many days should the company run each refinery to minimize its costs and still meet its orders?

32. A steel company has two mills. Mill 1 costs \$70,000 per day to operate, and it can produce 400 tons of high-grade steel, 500 tons of medium-grade steel, and 450 tons of low-grade steel each day. Mill 2 costs \$60,000 per day to operate, and it can produce 350 tons of high-grade steel, 600 tons of medium-grade steel, and 400 tons of low-grade steel each day. The company has orders totaling 100,000 tons of high-grade steel, 150,000 tons of medium-grade steel, and 124,500 tons of low-grade steel. How many days should the company run each mill to minimize its costs and still fill the orders?

C **33.** Use a computer to minimize the objective function

$$w = x_1 + 0.5x_2 + 2.5x_3 + 3x_4$$

subject to the constraints

$$\begin{aligned} 1.5x_1 + x_2 \phantom{{}+6x_3} + 2x_4 &\geq 35 \\ 2x_2 + 6x_3 + 4x_4 &\geq 120 \\ x_1 + x_2 + x_3 + x_4 &\geq 50 \\ 0.5x_1 \phantom{{}+x_2} + 2.5x_3 + 1.5x_4 &\geq 75 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

C **34.** Use a computer to minimize the objective function

$$w = 1.5x_1 + x_2 + 0.5x_3 + 2x_4$$

subject to the same set of constraints given in Exercise 33.

9.5 THE SIMPLEX METHOD: MIXED CONSTRAINTS

In Sections 9.3 and 9.4, we looked at linear programming problems that occurred in *standard form.* The constraints for the maximization problems all involved $\leq$ inequalities, and the constraints for the minimization problems all involved $\geq$ inequalities.

Linear programming problems for which the constraints involve *both* types of inequalities are called **mixed-constraint** problems. For instance, consider the following linear programming problem.

Mixed-Constraint Problem: Find the maximum value of

$$z = x_1 + x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &\leq 50 \\ 2x_1 + x_2 &\geq 36 \\ x_1 + x_3 &\geq 10 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Since this is a maximization problem, we would expect each of the inequalities in the set of constraints to involve $\leq$. Moreover, since the first inequality does involve $\leq$, we can add a slack variable to form the following equation.

$$2x_1 + x_2 + x_3 + s_1 = 50$$

For the other two inequalities, we must introduce a new type of variable, called a **surplus variable,** as follows.

$$\begin{aligned} 2x_1 + x_2 - s_2 &= 36 \\ x_1 + x_3 - s_3 &= 10 \end{aligned}$$

Notice that surplus variables are *subtracted from* (not added to) their inequalities. We call s_2 and s_3 surplus variables because they represent the amount that the left side of the inequality exceeds the right side. Surplus variables must be nonnegative.

Now, to solve the linear programming problem, we form an initial simplex tableau as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
2	1	1	1	0	0	50	s_1	
2	1	0	0	−1	0	36	s_2	
1	0	1	0	0	−1	10	s_3	← *Departing*
−1	−1	−2	0	0	0	0		
		↑ *Entering*						

You will soon discover that solving mixed-constraint problems can be difficult. One reason for this is that we do not have a convenient feasible solution to begin the simplex method. Note that the solution represented by the initial tableau above.

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 50, -36, -10)$$

is not a feasible solution because the values of the two surplus variables are negative. In fact, the values $x_1 = x_2 = x_3 = 0$ do not even satisfy the constraint equations. In order to eliminate the surplus variables from the current solution, we basically use "trial and error." That is, in an effort to find a feasible solution, we arbitrarily choose new entering variables. For instance, in this tableau, it seems reasonable to select x_3 as the entering variable. After pivoting, the new simplex tableau becomes

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	1	0	1	0	1	40	s_1	
2	1	0	0	−1	0	36	s_2	← Departing
1	0	1	0	0	−1	10	x_3	
1	−1	0	0	0	−2	20		
	↑ Entering							

The current solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 10, 40, -36, 0)$ is still not feasible, so we choose x_2 as the entering variable and pivot to obtain the following simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
−1	0	0	1	1	1	4	s_1	← Departing
2	1	0	0	−1	0	36	x_2	
1	0	1	0	0	−1	10	x_3	
3	0	0	0	−1	−2	56		
					↑ Entering			

At this point, we finally obtained a feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 36, 10, 4, 0, 0).$$

From here on, we apply the simplex method as usual. Note that the entering variable here is s_3 because its column has the most negative entry in the bottom row. After pivoting one more time, we obtain the following final simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
−1	0	0	1	1	1	4	s_3
2	1	0	0	−1	0	36	x_2
0	0	1	1	1	0	14	x_3
1	0	0	2	1	0	64	

Note that this tableau is final because it represents a feasible solution *and* there are no negative entries in the bottom row. Thus, we conclude that the maximum value of the objective function is

$$z = 64 \qquad \textbf{Maximum value}$$

and this occurs when

$$x_1 = 0, \; x_2 = 36, \text{ and } x_3 = 14.$$

EXAMPLE 1 *A Maximization Problem with Mixed Constraints*

Find the maximum value of

$$z = 3x_1 + 2x_2 + 4x_3 \qquad \textbf{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5x_3 &\le 18 \\ 4x_1 + 2x_2 + 3x_3 &\le 16 \\ 2x_1 + x_2 + x_3 &\ge 4 \end{aligned}\right\} \qquad \textbf{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Solution To begin, we add a slack variable to each of the first two inequalities and subtract a surplus variable from the third inequality to produce the following initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
3	2	5	1	0	0	18	s_1	
4	2	3	0	1	0	16	s_2	
2	1	1	0	0	−1	4	s_3	← *Departing*
−3	−2	−4	0	0	0	0		
	↑ *Entering*							

As it stands, this tableau does not represent a feasible solution (because the value of s_3 is negative). Thus, we want s_3 to be the departing variable. We have no real guidelines as to which variable should enter the solution, but by trial and error, we discover that using x_2 as the entering variable produces the following tableau (which does represent a feasible solution).

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
−1	0	3	1	0	2	10	s_1	← *Departing*
0	0	1	0	1	2	8	s_2	
2	1	1	0	0	−1	4	x_2	
1	0	−2	0	0	−2	8		

Now, because this simplex tableau does represent a feasible solution, we proceed as usual, choosing the most negative entry in the bottom row to be the entering variable. (In this case, we have a tie, so we arbitrarily choose x_3 to be the entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-1	0	3	1	0	2	10	s_1	← *Departing*
0	0	1	0	1	2	8	s_2	
2	1	1	0	0	-1	4	x_2	
1	0	-2	0	0	-2	8		
		↑ *Entering*						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$	x_3	
$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{14}{3}$	s_2	← *Departing*
$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	x_2	
$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$		
					↑ *Entering*			

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	x_3
$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$	s_3
$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$	x_2
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	17	

Thus, the maximum value of the objective function is

$$z = 17$$

and this occurs when

$$x_1 = 0,\ x_2 = \tfrac{13}{2},\text{ and } x_3 = 1.$$

Mixed Constraints and Minimization

In Section 9.4, we discussed the solution of minimization problems in *standard form*. Minimization problems that are not in standard form are more difficult to solve. One technique that can be used is to change a mixed-constraint minimization problem to a mixed-constraint maximization problem by multiplying each coefficient in the objective function by -1. We demonstrate this technique in the following example.

EXAMPLE 2 *A Minimization Problem with Mixed Constraints*

Find the minimum value of

$$w = 4x_1 + 2x_2 + x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\le 14 \\ 3x_1 + x_2 + 5x_3 &\ge 4 \\ x_1 + 4x_2 + 3x_3 &\ge 6 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Solution First, we rewrite the objective function by multiplying each of its coefficients by -1, as follows.

$$z = -4x_1 - 2x_2 - x_3 \qquad \text{Revised objective function}$$

Maximizing this revised objective function is equivalent to minimizing the original objective function. Next, we add a slack variable to the first inequality and subtract surplus variables from the second and third inequalities to produce the following initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	3	4	1	0	0	14	s_1	
3	1	5	0	−1	0	4	s_2	← *Departing*
1	4	3	0	0	−1	6	s_3	
4	2	1	0	0	0	0		
	↑ *Entering*							

Note that the bottom row has the negatives of the coefficients of the revised objective function. Another way of looking at this is that for minimization problems (in nonstandard form), the bottom row of the initial simplex consists of the coefficients of the original objective function.

As with maximization problems with mixed constraints, this initial simplex tableau does not represent a feasible solution. By trial and error, we discover that we can choose x_2 as the entering variable and s_2 as the departing variable. After pivoting, we obtain the following tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
−7	0	−11	1	3	0	2	s_1
3	1	5	0	−1	0	4	x_2
−11	0	−17	0	4	−1	−10	s_3
−2	0	−9	0	2	0	−8	

From this tableau, we can see that the choice of x_2 as the entering variable was a good one. All we need to do to transform the tableau into one that represents a feasible solution is to multiply the third row by -1, as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-7	0	-11	1	3	0	2	s_1	
3	1	5	0	-1	0	4	x_2	
11	0	17	0	-4	1	10	s_3	← Departing
-2	0	-9	0	2	0	-8		
		↑ Entering						

Now that we have obtained a simplex tableau that represents a feasible solution, we continue with our standard pivoting operations as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$\frac{2}{17}$	0	0	1	$\frac{7}{17}$	$\frac{11}{17}$	$\frac{144}{17}$	s_1	
$-\frac{4}{17}$	1	0	0	$\frac{3}{17}$	$-\frac{5}{17}$	$\frac{18}{17}$	x_2	← Departing
$\frac{11}{17}$	0	1	0	$-\frac{4}{17}$	$\frac{1}{17}$	$\frac{10}{17}$	x_3	
$\frac{65}{17}$	0	0	0	$-\frac{2}{17}$	$\frac{9}{17}$	$-\frac{46}{17}$		
				↑ Entering				

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$\frac{2}{3}$	$-\frac{7}{3}$	0	1	0	$\frac{4}{3}$	6	s_1
$-\frac{4}{3}$	$\frac{17}{3}$	0	0	1	$-\frac{5}{3}$	6	s_2
$\frac{1}{3}$	$\frac{4}{3}$	1	0	0	$-\frac{1}{3}$	2	x_3
$\frac{11}{3}$	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	-2	

Finally, we conclude that the maximization value of the revised objective function is $z = -2$, and hence the minimum value of the original objective function is

$$w = 2$$

(the negative of the entry in the lower-right corner), and this occurs when

$$x_1 = 0, \quad x_2 = 0, \quad \text{and} \quad x_3 = 2.$$

Applications

EXAMPLE 3 ***A Business Application: Minimum Shipment Cost***

An automobile company has two factories. One factory has 400 cars (of a certain model) in stock and the other factory has 300 cars (of the model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The cost of shipping cars from the two factories to the customers is shown in Table 9.3.

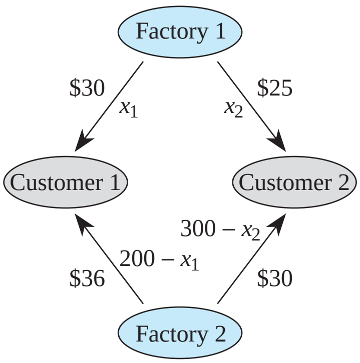
TABLE 9.3

	Customer 1	*Customer 2*
Factory 1	\$ 30	\$ 25
Factory 2	\$ 36	\$ 30

How should the company ship the cars in order to minimize the shipping cost?

Solution To begin, we let x_1 and x_2 represent the number of cars shipped from Factory 1 to the first and second customers, respectively. (See Figure 9.20.) The total cost of shipping is then given by

Figure 9.20



$$C = 30x_1 + 25x_2 + 36(200 - x_1) + 30(300 - x_2) = 16{,}200 - 6x_1 - 5x_2.$$

The constraints for this minimization problem are as follows.

$$\begin{aligned} x_1 + x_2 &\le 400 \\ (200 - x_1) + (300 - x_2) &\le 300 \quad \longrightarrow \quad x_1 + x_2 \ge 200 \\ x_1 &\le 200 \\ x_2 &\le 300 \end{aligned}$$

The corresponding maximization problem is to maximize $z = 6x_1 + 5x_2 - 16{,}200$. Thus, the initial simplex tableau is as follows.

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables	
1	1	1	0	0	0	400	s_1	
1	1	0	−1	0	0	200	s_2	← *Departing*
1	0	0	0	1	0	200	s_3	
0	1	0	0	0	1	300	s_4	
−6	−5	0	0	0	0	−16,200		
↑ *Entering*								

Note that the current z-value is $-16{,}200$ because the initial solution is

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 400, -200, 200, 300).$$

Now, to this initial tableau, we apply the simplex method as follows.

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables	
0	0	1	1	0	0	200	s_1	
1	1	0	−1	0	0	200	x_1	
0	−1	0	1	1	0	0	s_3	← Departing
0	1	0	0	0	1	300	s_4	
0	1	0	−6	0	0	−15,000		
			↑ Entering					

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables	
0	1	1	0	−1	0	200	s_1	← Departing
1	0	0	0	1	0	200	x_1	
0	−1	0	1	1	0	0	s_2	
0	1	0	0	0	1	300	s_4	
0	−5	0	0	6	0	−15,000		
	↑ Entering							

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables
0	1	1	0	−1	0	200	x_2
1	0	0	0	1	0	200	x_1
0	0	1	1	0	0	200	s_2
0	0	−1	0	1	1	100	s_4
0	0	5	0	1	0	−14,000	

From this tableau, we see that the minimum shipping cost is \$14,000. Since $x_1 = 200$ and $x_2 = 200$, we conclude that the number of cars that should be shipped from each factory is as shown in Table 9.4.

TABLE 9.4

	Customer 1	*Customer 2*
Factory 1	200 cars	200 cars
Factory 2	0	100 cars

SECTION 9.5 ❑ EXERCISES

In Exercises 1–6, add the appropriate slack and surplus variables to the system and form the initial simplex tableau.

1. (Maximize)
Objective function:
$w = 10x_1 + 4x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. (Maximize)
Objective function:
$w = 3x_1 + x_2 + x_3$
Constraints:
$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 10 \\ x_2 + 5x_3 &\geq 6 \\ 4x_1 - x_2 + x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

3. (Minimize)
Objective function:
$w = x_1 + x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 3x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. (Minimize)
Objective function:
$w = 2x_1 + 3x_2$
Constraints:
$$\begin{aligned} 3x_1 + x_2 &\geq 4 \\ 4x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. (Maximize)
Objective function:
$w = x_1 + x_3$
Constraints:
$$\begin{aligned} 4x_1 + x_2 &\geq 10 \\ x_1 + x_2 + 3x_3 &\leq 30 \\ 2x_1 + x_2 + 4x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

6. (Maximize)
Objective function:
$w = 4x_1 + x_2 + x_3$
Constraints:
$$\begin{aligned} 2x_1 + x_2 + 4x_3 &\leq 60 \\ x_2 + x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 7–12, use the given entering and departing variables to solve the given mixed constraint problem.

7. (Maximize)
Objective function:
$w = -x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\geq 3 \\ x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$
Entering x_2, departing s_1.

8. (Maximize)
Objective function:
$w = 2x_1 + x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\geq 4 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$
Entering x_1, departing s_1.

9. (Minimize)
Objective function:
$w = x_1 + 2x_2$
Constraints:
$$\begin{aligned} 2x_1 + 3x_2 &\leq 25 \\ x_1 + 2x_2 &\geq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$
Entering x_2, departing s_2.

10. (Minimize)
Objective function:
$w = 3x_1 + 2x_2$
Constraints:
$$\begin{aligned} x_1 + x_2 &\geq 20 \\ 3x_1 + 4x_2 &\leq 70 \\ x_1, x_2 &\geq 0 \end{aligned}$$
Entering x_1, departing s_1.

11. (Maximize)
Objective function:
$w = x_1 + x_2$
Constraints:
$$\begin{aligned} -4x_1 + 3x_2 + x_3 &\leq 40 \\ -2x_1 + x_2 + x_3 &\geq 10 \\ x_2 + x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$
Entering x_2, departing s_2.

12. (Maximize)
Objective function:
$w = x_1 + 2x_2 + 2x_3$
Constraints:
$$\begin{aligned} x_1 + x_2 &\geq 50 \\ 2x_1 + x_2 + x_3 &\leq 70 \\ x_2 + 3x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$
Entering x_2, departing s_1.

In Exercises 13–20, use the simplex method to solve the given problem.

13. (Maximize)
Objective function:
$w = 2x_1 + 5x_2$
Constraints:
$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

14. (Maximize)
Objective function:
$w = -x_1 + 3x_2$
Constraints:
$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 5x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

15. (Maximize)
Objective function:
$w = 2x_1 + x_2 + 3x_3$
Constraints:
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &\leq 85 \\ x_2 - 5x_3 &\geq 20 \\ 3x_1 + 2x_2 + 11x_3 &\geq 49 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

16. (Maximize)
Objective function:
$w = 3x_1 + 5x_2 + 2x_3$
Constraints:
$$\begin{aligned} 9x_1 + 4x_2 + x_3 &\leq 70 \\ 5x_1 + 2x_2 + x_3 &\leq 40 \\ 4x_1 + x_2 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

17. (Minimize)

Objective function:

$w = x_1 + x_2$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 25 \\ 2x_1 + 5x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

18. (Minimize)

Objective function:

$w = 2x_1 + 3x_2$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 &\leq 22 \\ x_1 + x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

19. (Minimize)

Objective function:

$w = -2x_1 + 4x_2 - x_3$

Constraints:

$$\begin{aligned} 3x_1 - 6x_2 + 4x_3 &\leq 30 \\ 2x_1 - 8x_2 + 10x_3 &\geq 18 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. (Minimize)

Objective function:

$w = x_1 + x_2 + x_3$

Constraints:

$$\begin{aligned} x_1 + 2x_2 + x_3 &\geq 30 \\ 6x_2 + x_3 &\leq 54 \\ x_1 + x_2 + 3x_3 &\geq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In Exercises 21–24, maximize the given objective function subject to the following constraints.

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ -x_1 + x_2 &\leq 3 \\ x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

21. $w = 2x_1 + x_2$

22. $w = x_1 + 2x_2$

23. $w = x_2$

24. $w = -x_1 - x_2$

In Exercises 25–28, maximize the given objective function subject to the following constraints.

$$\begin{aligned} 3x_1 + 2x_2 &\geq 6 \\ x_1 - x_2 &\leq 2 \\ -x_1 + 2x_2 &\leq 6 \\ x_1 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

25. $w = x_1 + x_2$

26. $w = x_1 - 2x_2$

27. $w = -4x_1 + x_2$

28. $w = 4x_1 - x_2$

In Exercises 29–32, a tire company has two suppliers, S_1 and S_2. S_1 has 900 tires on hand and S_2 has 800 tires on hand. Customer C_1 needs 500 tires and customer C_2 needs 600 tires. Minimize the cost of filling the orders subject to the given table (showing the shipping cost per tire).

29.

	C_1	C_2
S_1	0.60	1.20
S_2	1.00	1.80

30.

	C_1	C_2
S_1	0.80	1.00
S_2	1.00	1.20

31.

	C_1	C_2
S_1	1.20	1.00
S_2	1.00	1.20

32.

	C_1	C_2
S_1	0.80	1.00
S_2	1.00	0.80

33. An automobile company has two factories. One factory has 400 cars (of a certain model) in stock and the other factory has 300 cars (of the model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The cost of shipping cars from the two factories to the two customers is as follows.

	Customer 1	*Customer 2*
Factory 1	\$36	\$30
Factory 2	\$30	\$25

How should the company ship the cars in order to minimize the shipping cost?

34. Suppose in Exercise 33 that the shipping costs for each of the two factories are as follows.

	Customer 1	*Customer 2*
Factory 1	\$25	\$30
Factory 2	\$35	\$30

How should the company ship the cars in order to minimize the shipping cost?

35. A company has budgeted a maximum of \$600,000 for advertising a certain product nationally. Each minute of television time costs \$60,000 and each one-page newspaper ad costs \$15,000. Each television ad is expected to be viewed by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company's market research department advises the company to use at least 6 television ads and at least 4 newspaper ads. How should the advertising budget be allocated to maximize the total audience?

36. Rework Exercise 35 assuming that each one-page newspaper ad costs \$30,000.

In Exercises 37 and 38, use the following information. A computer company has two assembly plants, Plant A and Plant B, and two distribution outlets, Outlet I and Outlet II. Plant A can assemble 5000 computers in a year and Plant B can assemble 4000 computers in a year. Outlet I must have 3000 computers per year and Outlet II must have 5000 computers per year. The transportation costs from each plant to each outlet are indicated in the given table. Find the shipping schedule that will produce the minimum cost. What is the minimum cost?

37.

	Outlet I	*Outlet II*
Plant A	\$4	\$5
Plant B	\$5	\$6

38.

	Outlet I	*Outlet II*
Plant A	\$4	\$5
Plant B	\$6	\$4

CHAPTER 9 ❑ REVIEW EXERCISES

In Exercises 1–6, sketch a graph of the solution of the system of inequalities.

1. $\begin{aligned} x + 2y &\le 160 \\ 3x + y &\le 180 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

2. $\begin{aligned} 2x + 3y &\le 24 \\ 2x + y &\le 16 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

3. $\begin{aligned} 3x + 2y &\ge 24 \\ x + 2y &\ge 12 \\ 2 \le x &\le 15 \\ y &\le 15 \end{aligned}$

4. $\begin{aligned} 2x + y &\ge 16 \\ x + 3y &\ge 18 \\ 0 \le x &\le 25 \\ 0 \le y &\le 15 \end{aligned}$

5. $\begin{aligned} 2x - 3y &\ge 0 \\ 2x - y &\le 8 \\ y &\ge 0 \end{aligned}$

6. $\begin{aligned} x - y &\le 10 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

In Exercises 7 and 8, determine a system of inequalities that models the given description, and sketch a graph of the solution of the system.

7. A Pennsylvania fruit grower has 1500 bushels of apples that are to be divided between markets in Harrisburg and Philadelphia. These two markets need at least 400 bushels and 600 bushels, respectively.

8. A warehouse operator has 24,000 square meters of floor space in which to store two products. Each unit of product I requires 20 square meters of floor space and costs \$12 per day to store. Each unit of product II requires 30 square meters of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400.

In Exercises 9–14, find the minimum and/or maximum values of the given objective function by the graphical method.

9. Maximize: $z = 3x + 4y$