
Assignment 5

Due date: Thursday, 24 October 2024, 23:59

Exercise 5.5, Properties of Relations (★)

(8 Points)

Prove or disprove the following claims:

- a) A relation ρ on a set A is symmetric on A if and only if ρ^2 is symmetric on A .
- b) If ρ is a relation on a set A that is symmetric and antisymmetric, then it must hold $\rho = \text{id}_A$.
- c) Define the relations ρ_1 and ρ_2 on \mathbb{Z} as

$$a \rho_1 b \iff b = a + 1, \quad a \rho_2 b \iff b \equiv_2 a.$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$.

a)

The claim is false, a counterexample follows:

$$\begin{aligned} \rho &= \{(a, c), (b, d), (c, b), (d, a)\} \\ \rho^2 &= \{(a, b), (b, a), (c, d), (d, c)\} \end{aligned}$$

The relation ρ^2 is symmetric, but ρ is not, which disproves the claim if and only if (\iff) from the right to left part (\Leftarrow).

b)

The claim is false, a counterexample follows:

$$\rho = \emptyset \neq \text{id}_A$$

The relation ρ is both symmetric and antisymmetric. However, ρ is not the identity relation as the claim states.

c)

The claim is true, and it can be proven as follows:

we will decompose the claim through composition of implications from which will lead to a clear satisfiability problem that shows is satisfiable using a proof by case distinction.