Università della Svizzera italiana
Institute of Computing CI

Numerical Computing **2023**

Student: Jeferson Morales Mariciano
Discussed with: Leonardo Birindelli, Michele Dalle Rive, Martin Lettry, Sofia d'Atri

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Bonus assignment

Due date: Wednesday, 22 November 2023, 11:59 PM

Numerical Computing 2023 — Submission Instructions

(Please, notice that following instructions are mandatory: submissions that don't comply with, won't be considered)

- Assignments must be submitted to iCorsi (i.e. in electronic format).
- Provide both executable package and sources (e.g. C/C++ files, MATLAB). If you are using libraries, please add them in the file. Sources must be organized in directories called:

  Project_number_lastname_firstname

  and the file must be called:

  project_number_lastname_firstname.zip

  project_number_lastname_firstname.pdf
- The TAs will grade your project by reviewing your project write-up, and looking at the implementation you attempted, and benchmarking your code's performance.
- You are allowed to discuss all questions with anyone you like; however: (i) your submission must list anyone you discussed problems with and (ii) you must write up your submission independently.

1. Exercise: Inconsistent systems of equations [10 points]

Consider the following inconsistent systems of equations:

$$A_1 x = b_1, A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 2}, b_1 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} \tag{1}$$

$$A_2 x = b_2, A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 3}, b_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} \tag{2}$$

Find the least squares solution x^* and compute Euclidean norm of the residual, SE, RMSE.

1.

$$A^\top A x^* = A^\top b$$

$$A^\top A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^\top b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} x^* = \begin{bmatrix} 11 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3x_1 = 11 \\ 0x_2 = 0 \end{cases} \rightarrow \begin{cases} x_1 = \frac{11}{3} \\ x_2 \in \mathbb{R} \end{cases}$$

$$r = b - Ax^* = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{11}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \frac{11}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

$$euclidean\_norm = ||r||_2 = \sqrt{\sum_{i=1}^{n} r_i^2} = \frac{\sqrt{42}}{3} \approx 2.1602$$

$$SE = ||r||_2^2 = \sum_{i=1}^{n} r_i^2 = \frac{42}{9} \approx 4.6665$$

$$RMSE = \frac{euclidean\ norm}{\sqrt{\#equations}} = \frac{\frac{\sqrt{42}}{3}}{\sqrt{3}} = \frac{\sqrt{126}}{9} \approx 1.2472$$

2.

$$A^\top A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

$$A^\top b = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix} x^* = \begin{bmatrix} 9 \\ 10 \\ 9 \end{bmatrix}$$

$$\begin{cases} 3x_1 + 3x_2 + 2x_3 = 9 \\ 3x_1 + 6x_2 + 3x_3 = 10 \\ 2x_1 + 3x_2 + 3x_3 = 9 \end{cases}$$

$$E_{32}\Big(-\frac{1}{3}\Big) \cdot E_{21}\Big(-1\Big) \cdot E_{31}\Big(-\frac{2}{3}\Big) \cdot \begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

$$\begin{cases} 3x_1 + 3x_2 + 2x_3 = 9 \\ 3x_2 + x_3 = 1 \\ \frac{4}{3}x_3 = \frac{8}{3} \end{cases} \quad \rightarrow \begin{cases} x_1 = 2 \\ x_2 = -\frac{1}{3} \\ x_3 = 2 \end{cases}$$

$$r = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -\frac{1}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 10 \\ 12 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$euclidean\_norm = \frac{\sqrt{3}}{3} \approx 0.5774$$

$$SE = \frac{1}{3} \approx 0.3333$$

$$RMSE = \frac{\frac{\sqrt{3}}{3}}{\sqrt{4}} = \frac{\sqrt{3}}{6} \approx 0.2887$$

Exercise 2: Polynomials models for least squares [20 points]

In this exercise, we consider two small datasets about the crude oil (*crudeOil.txt*) and kerosene (*kerosene.txt*) production by year in Europe in the period $1980 - 2012$. By solving the following tasks, we will try to fit the data with different polynomial models and determine the best one.

Source code of exercises are in *./ex2* folder.

a

The *leastSquares.m* function is implemented and correctly returning the solution computed manually in Exercise 1 when checked with *ex2a.m* script.
Solution x^*, Euclidean norm of the residual, SE, RMSE are computed and printed in the console.

b

Both crude oil and kerosene data during the period $1980 - 2011$ got their least square solution and metrics in the script *linearModel.m* using a linear model.
Table 1 summarize x* and metrics for both datasets.
Plot 1 refers to crude oil dataset case and Plot 2 to kerosene dataset case.

Figure 1: Crude oil dataset - linear model

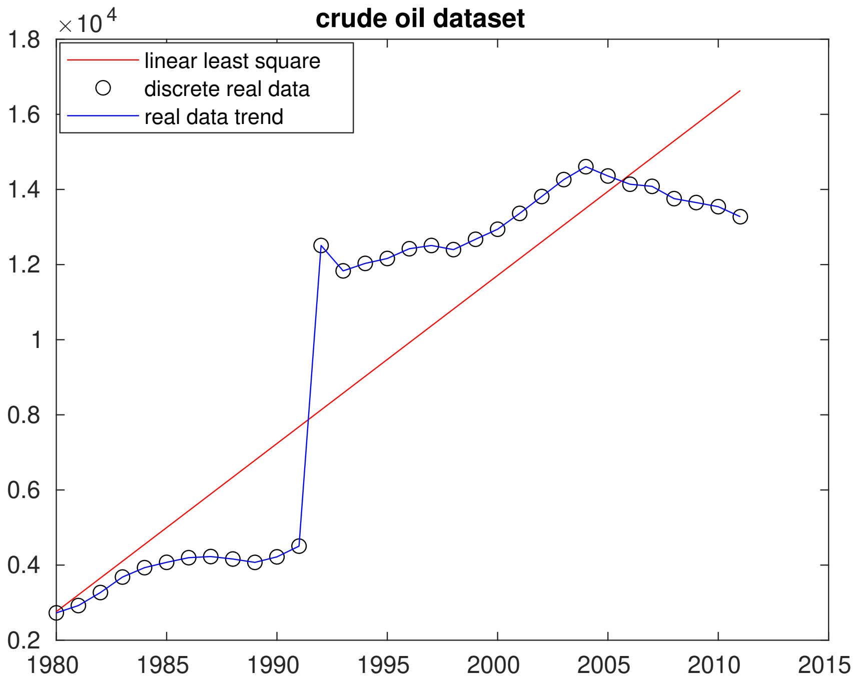
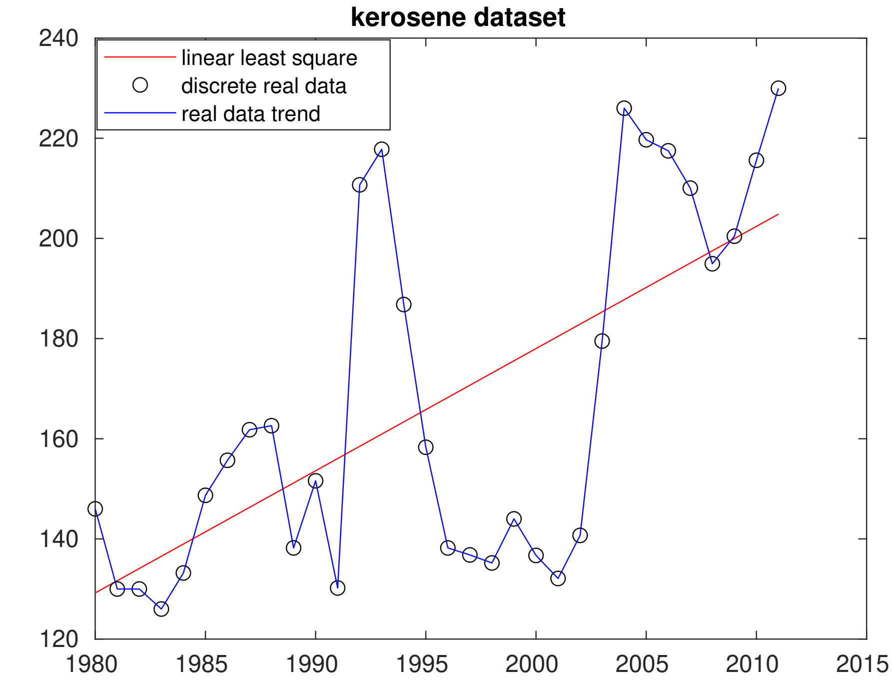


Figure 2: Kerosene dataset - linear model



c

The same procedure of the previous task is repeated using a quadratic model in *quadraticModel.m* script.
Table 2 summarize x* and metrics for both datasets.
Plot 3 refers to crude oil dataset case and Plot 4 to kerosene dataset case.

Figure 3: Crude oil dataset - quadratic model

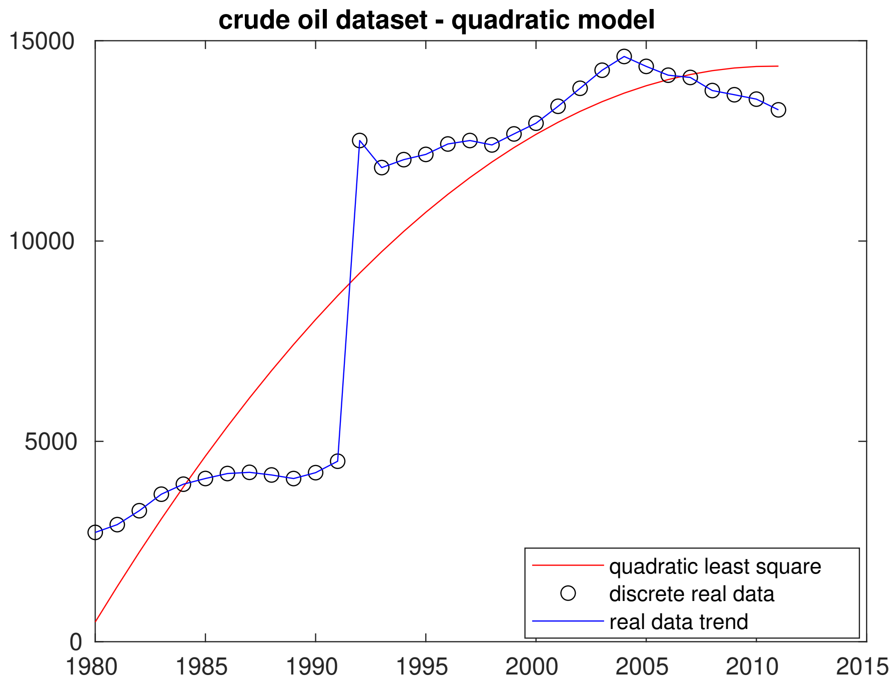
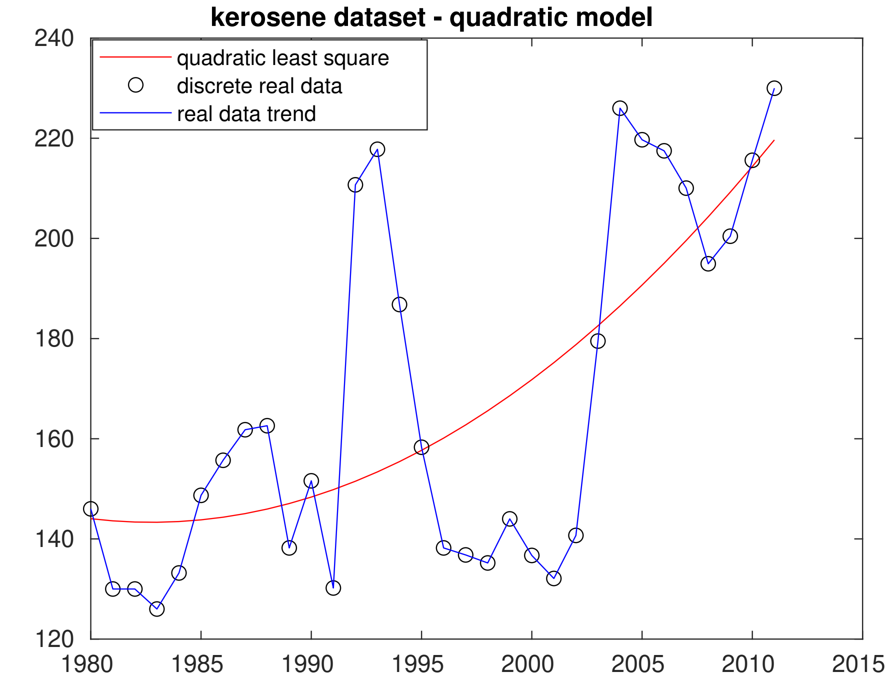


Figure 4: Kerosene dataset - quadratic model



d

The same procedure of the previous task is repeated using a cubic model in *cubicModel.m* script. Table 4 summarize x* and metrics for both datasets.
Plot 5 refers to crude oil dataset case and Plot 6 to kerosene dataset case.

Figure 5: Crude oil dataset - cubic model

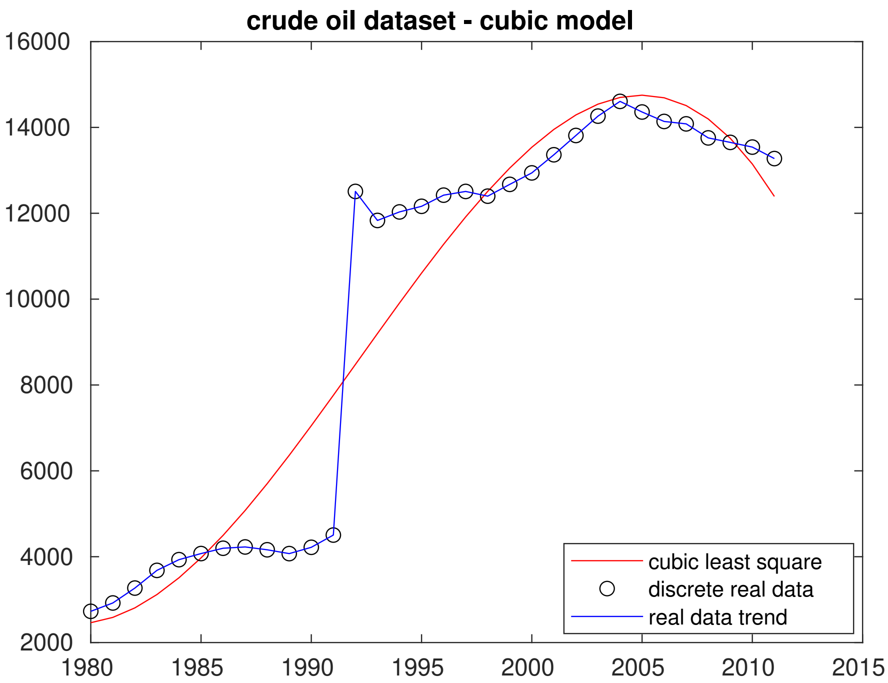
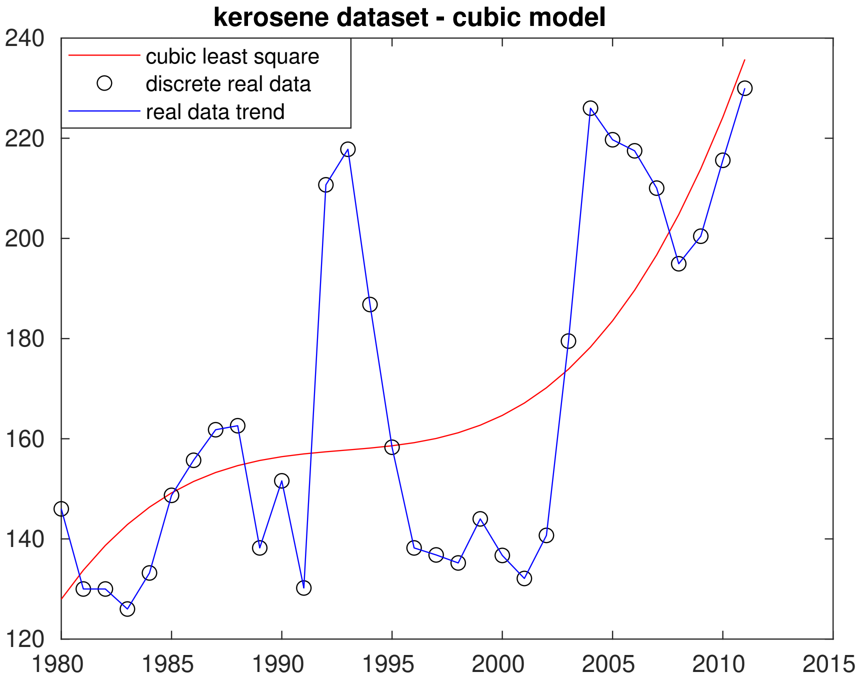


Figure 6: Kerosene dataset - cubic model



e

Compare the linear, quadratic and cubic models on the basis of the quality metrics computed above, by creating a table containing the results for the two models. Which one of the three models would you pick for the crude oil data? And for the kerosene?

Table 1: Crude oil and kerosene dataset metrics using linear model

/	x^*	euclidean norm	SE	RMSE
oil	$(1.0e+05) \cdot \begin{bmatrix} -8.8339 & 0.0045 \end{bmatrix}$	1.1471e+04	1.3159e+08	2.0278e+03
kerosene	$(1.0e+03) \cdot \begin{bmatrix} -4.7013 & 0.0024 \end{bmatrix}$	151.0222	2.2808e+04	26.6972

Table 2: Crude oil and kerosene dataset metrics using quadratic model

/	x^*	euclidean norm	SE	RMSE
oil	$(1.0e+07) \cdot \begin{bmatrix} -5.9181 & 0.0059 & -0.0000 \end{bmatrix}$	9.5826e+03	9.1827e+07	1.6940e+03
kerosene	$(1.0e+05) \cdot \begin{bmatrix} 3.7596 & -0.0038 & 0.0000 \end{bmatrix}$	145.3013	2.1112e+04	25.6859

Table 3: Crude oil and kerosene dataset metrics using cubic model

/	x^*	euclidean norm	SE	RMSE
oil	$(1.0e+10) \cdot \begin{bmatrix} 1.1560 & -0.0017 & 0.0000 & -0.0000 \end{bmatrix}$	7.9179e+03	6.2692e+07	1.3997e+03
kerosene	$(1.0e+07) \cdot \begin{bmatrix} -9.4357 & 0.0142 & -0.0000 & 0.0000 \end{bmatrix}$	138.4756	1.9175e+04	24.4793

The best model for both crude oil and kerosene datasets is the cubic model, because it has the lowest lowest metrics overall: euclidean norm, SE and RMSE.

Such result can be due the rigidness of the linear model, which is not able to fit the data well, and the flexibility and adaptability of the cubic model which is the best to describe the dynamic evolution of the datasets.

Provide an estimate of the crude oil and kerosene production in 2012 **by using the three models and compare the values obtained with the real values reported in the data source. Comment on your results.**

Table 4: Estimate of 2012 for crude oil and kerosene dataset

dataset	real sample	linear	quadratic	cubic
oil	13,111.91	17,081.92	14,344.11	11,472.59
kerosene	267.89	207.28	225.16	248.57

Analyzing the data, it is seen clearly that for kerosene dataset the cubic model is the best one to fit the data than the results returned from the rest, which are really far from the real sample. Such behavior can be explained by the fact that the kerosene dataset is the most dynamic and scattered one.
For crude oil, the quadratic model is the one that best fit the real sample of 2012, but indeed the cubic model is also very accurate and in general, the models are not so far from the real sample.

Exercise 3: Analysis of periodic data [20 points]

In this exercise, we consider the following dataset about the mean temperature in Switzerland between January 1864 and March 2021 included in the (*temperature.txt*) file. Aim is to capture the periodic behaviour of the data by using periodic models.

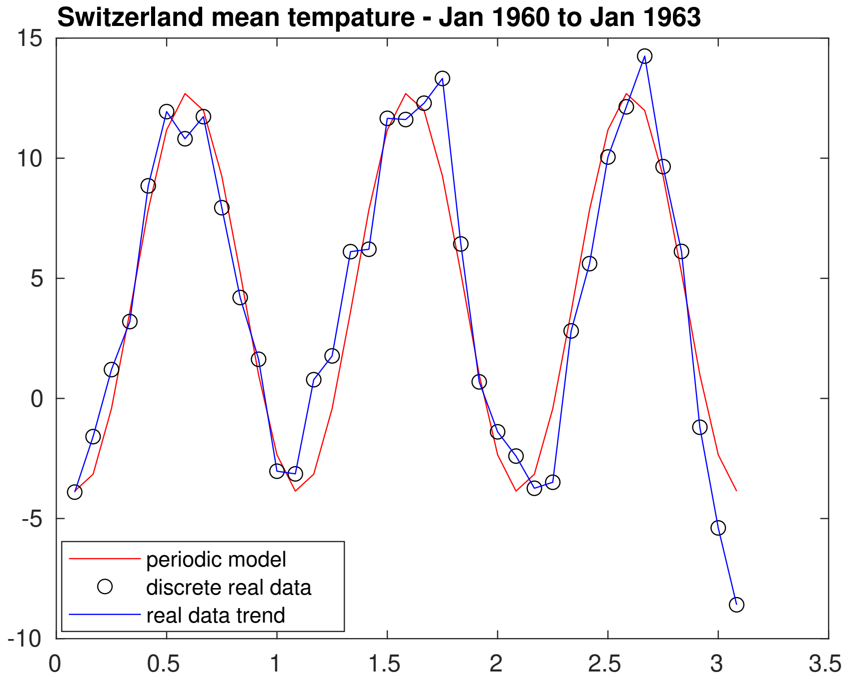
Source code of exercises are in *./ex3* folder.

a

Script file *periodicA.m* contains the solution.

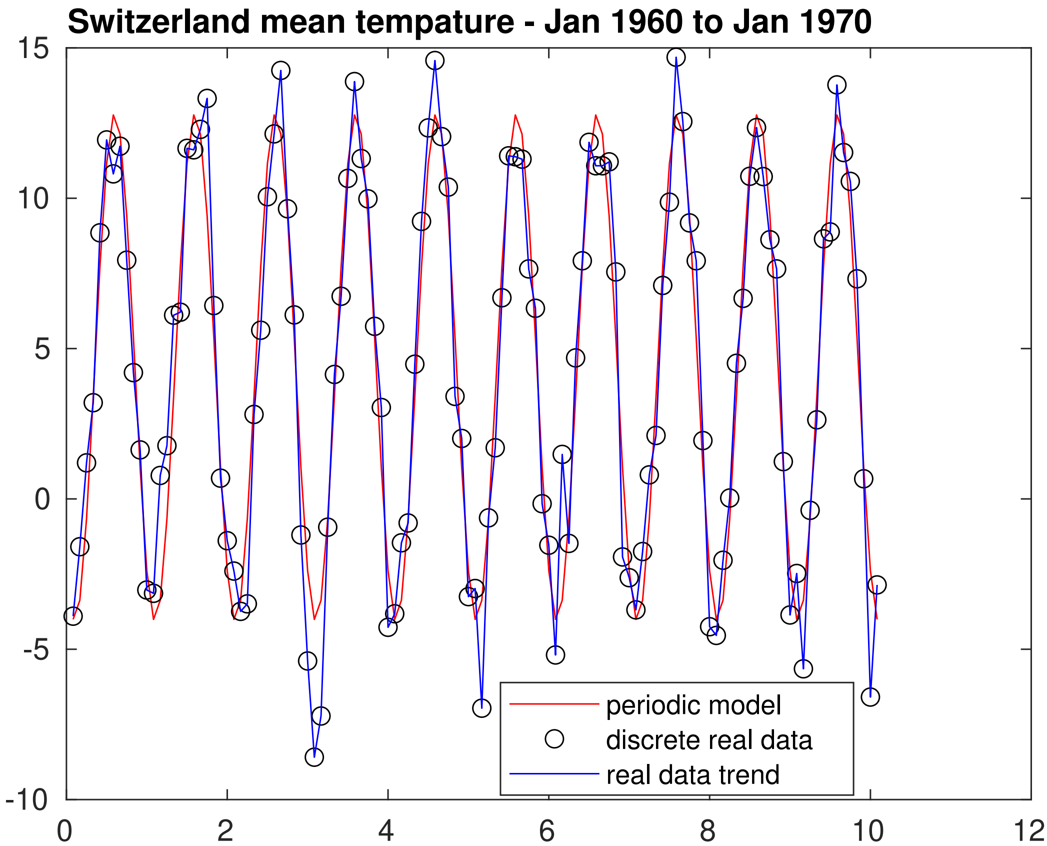
I

Figure 7: Switzerland Jan 1960 to Jan 1963 mean temperature



II

Figure 8: Switzerland Jan 1960 to Jan 1970 mean temperature

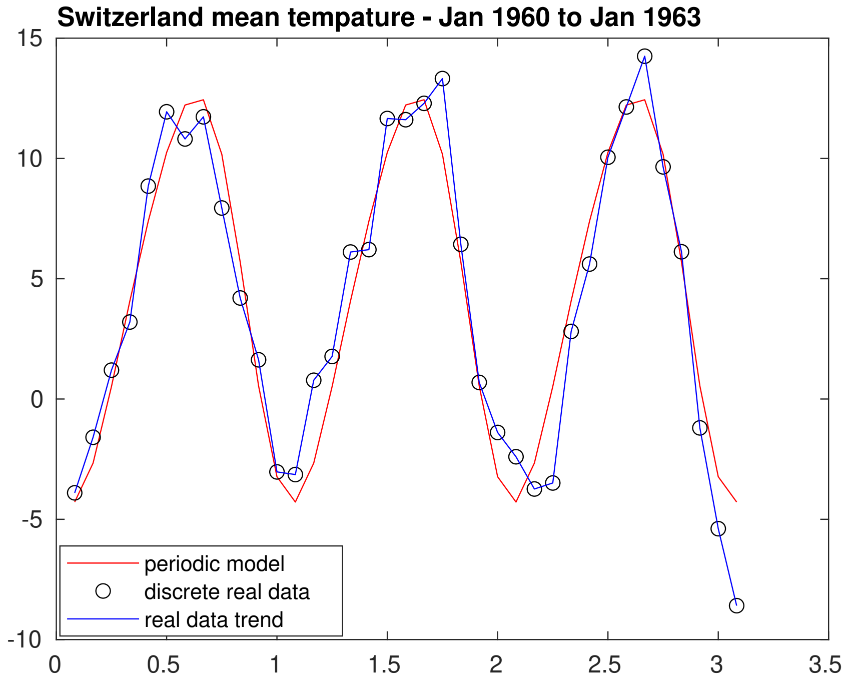


b

Script file *periodicB.m* contains the solution.

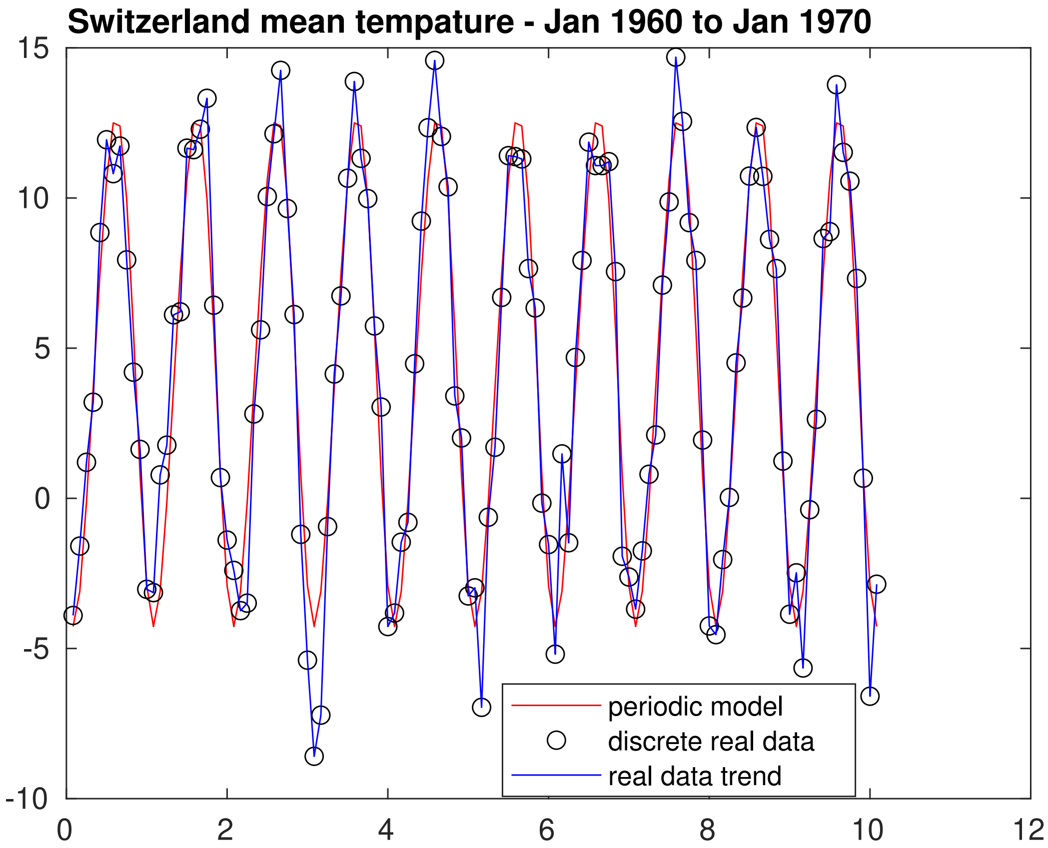
I

Figure 9: Switzerland Jan 1960 to Jan 1963 mean temperature



ii

Figure 10: Switzerland Jan 1960 to Jan 1970 mean temperature



c

Table 5: Metrics of the first periodic model

datasets	x^*	euclidean norm	SE	RMSE
Jan 1960 - Jan 1963	$\begin{bmatrix}4.4166 & -6.7605 & -4.8397\end{bmatrix}$	11.2223	125.9405	1.8449
Jan 1960 - Jan 1963	$\begin{bmatrix}4.3816 & -6.7751 & -5.0419\end{bmatrix}$	18.0385	325.3890	1.6399

Table 6: Metrics of the second periodic model

datasets	x^*	euclidean norm	SE	RMSE
Jan 1960 - Jan 1963	$\begin{bmatrix}4.4284 & -6.7401 & -4.8279 & -0.9191\end{bmatrix}$	10.5138	110.5400	1.7285
Jan 1960 - Jan 1963	$\begin{bmatrix}4.3838 & -6.7713 & -5.0397 & -0.5321\end{bmatrix}$	17.5594	308.3312	1.5963

Given the following data, it is seen that the second periodic model is the best one to fit the data, since the metrics are lower than the first one.
The difference is not by magnitudes of order, but it is still significant.
The x coordinates for plotting the data are phased in 12 (twelves) to follow the periodicity of the

data modelling years.
Hence, adding more terms to the model was beneficial to fit the data better, the second model is preferred over the first one. Model results are satisfactory, but not perfect, since the data is not perfectly periodic.
A way to improve the model is to add more terms to the model, but the data among the phases are scattered and it could be that in the future the more years we analyze, the more the phase goes out our model due to **climate change**, which will also need to be taken into account.

Exercise 4: Data linearization and Levenberg-Marquardt method for the exponential model [20 points]

Source code of exercises are in *./ex4* folder.

The file *nuclear.txt* contains the data on the nuclear electric power consumption by year in China in the period $1999 - 2006$. We consider the power law model:

$$y_i = \alpha_1 x_i^{\alpha_2}$$

a

Find the least squares best fit by using data linearization and compute the RMSE both of the log-linearized model and of the original exponential model. Include in your report all the computations and the necessary steps, as explained in the slides of the tutorial.

Using data linearization of the power law model we obtain the following least square best fit:

$$\alpha^* = \begin{bmatrix} 10.7026 & 0.7549 \end{bmatrix}$$

Table 7: RMSE of models

models	RMSE
log linearized model	0.20423
power law model	4.8141

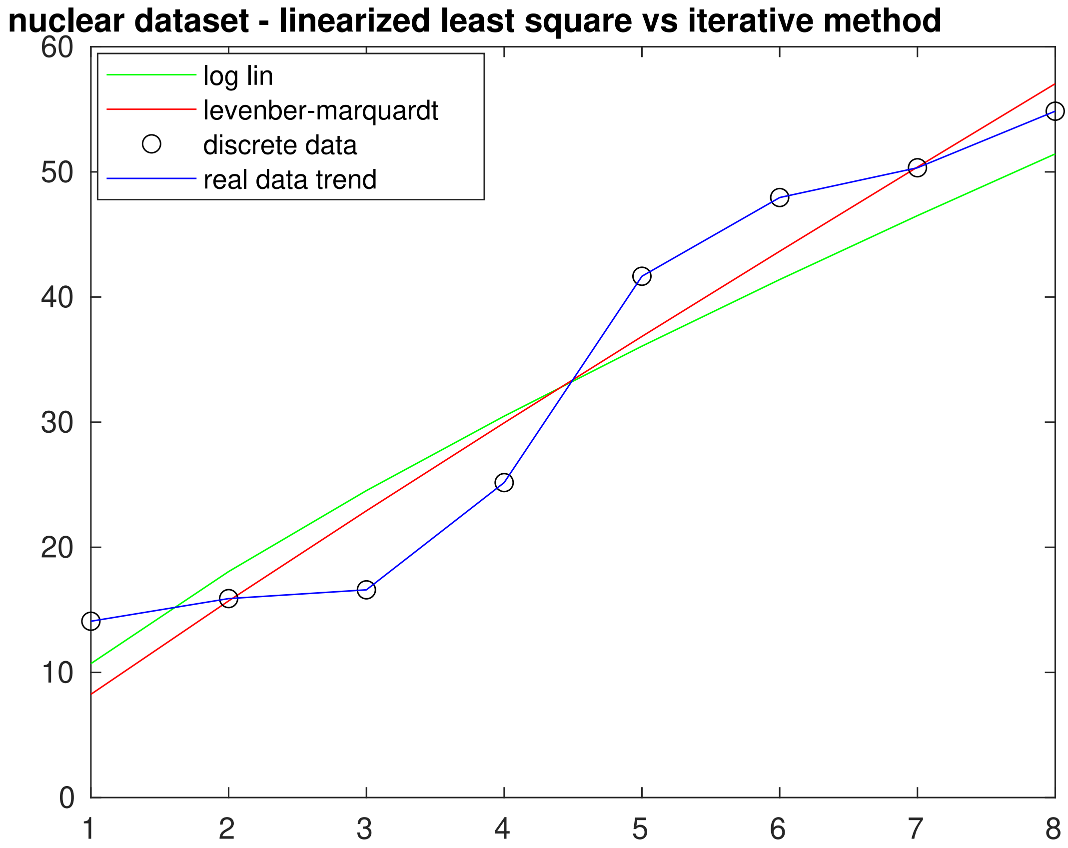
b

Write a function levenbergMarquardt() in which you implement the Levenberg-Marquardt algorithm for solving nonlinear least squares problems. Following again the slides of the tutorial, show how you can formulate the problem in order to solve it with Levenberg-Marquardt method and compute analytically all the necessary quantities. Finally, write a script ex4b.m in which you use the function levenbergMarquardt() to fit the data points and compute the RMSE.

$$\alpha^* = \begin{bmatrix} 8.2502 & 0.9299 \end{bmatrix}$$

c

Figure 11: nuclear dataset - power law model



Given the following metrics:

The Levenberg - Marquardt is preferred since it shows slightly lower metrics than means that the model is better to fit the data.
The plot above shows a better slope orientation for the levenberg Marquardt model than the log linearized power law model.

Exercise 5: Tikhonov regularization [15 points]

a

Given the Tikhonov regularization denominated as $f(x)$:

$$f(x) = \min_x ||Ax - b||_2^2 + \alpha||x||_2^2$$

Find its gradient:

$$\nabla f(x) = 2A^\top Ax - 2A^\top b + 2\alpha x$$

x is defined as:

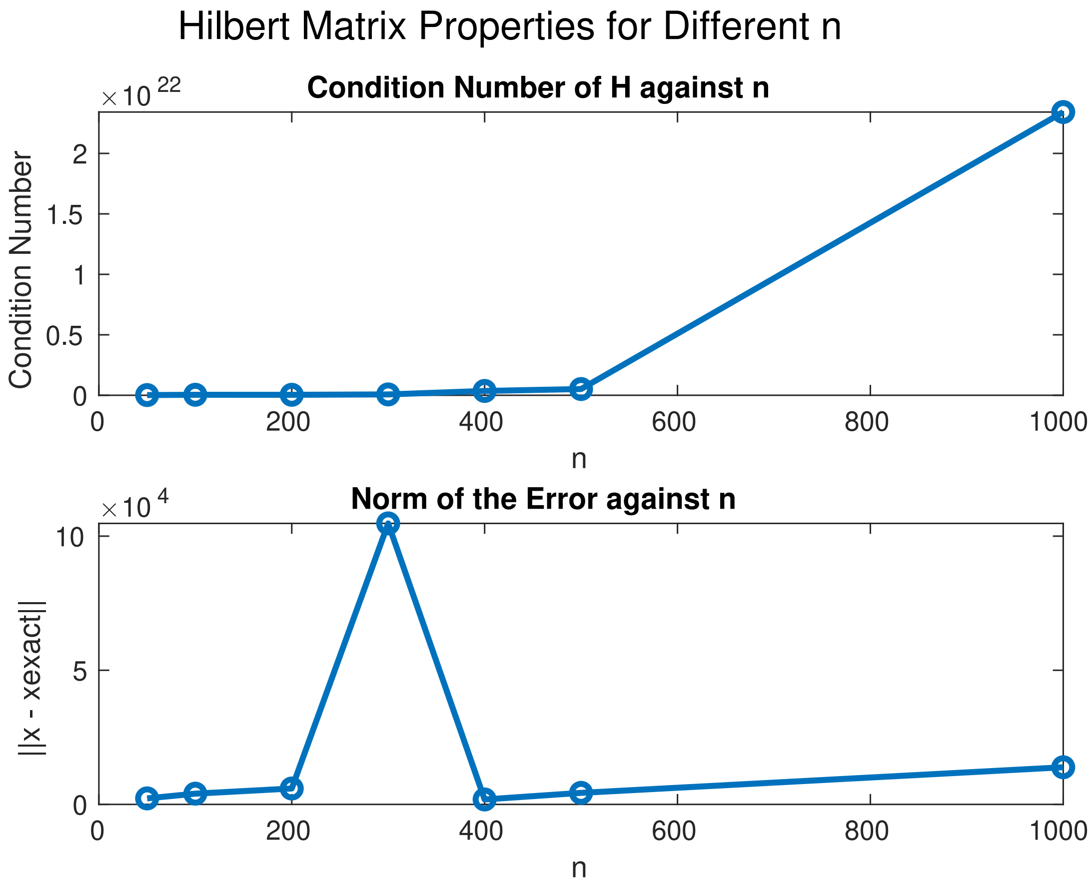
$$x = (A^\top A + \alpha I)^{-1} A^\top b$$

Where I is the identity matrix.
This solution equals the minimum of the Tikhonov regularization, when the derivative is set to 0.
The additional regularization term $\alpha||x||_2^2$ encourages small x_i values to avoid overfitting.

b

Produce also two figures in which you plot: the condition number of H (use cond() in Matlab) against n; the norm of the error $||x_{exact} - x||_2$ against n

Figure 12: Plot using Tikhonov regularization



c

To visualize the results, produce two figures in which you plot: the norm of the error $||x_{exact} - x_{reg}||_2$ against the values of α; $||H_x - b||_2 against ||x||_2$ for the different values of α. Comment your results

Figure 13: Plot using Tikhonov reagularization with different α values