ETH zürich

Discrete Mathematics **2024**

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Assignment 11 **Due date:** Thursday, 5 December 2024, 23:59

Exercise 11.4, Combining Proof Systems ($\star$) (8 Points)

Let

$$\Sigma = (\mathcal{S}, \mathcal{P}, \tau, \phi)$$

be a complete and sound proof system.

a) Define $\mathcal{P}'$ and ϕ' so that

$$\Sigma' = (\mathcal{S} \times \mathcal{S} \times \mathcal{S}, \mathcal{P}', \tau', \phi')$$

is a complete and sound proof system (and prove it!), where

$$\tau'((s_1, s_2, s_3)) = 1 \iff \text{ at least 2 among } \tau(s_1), \tau(s_2), \tau(s_3) \text{ are equal to 1}$$

b) Let

$$\overline{\Sigma} = (\mathcal{S}^2, \overline{\mathcal{P}}, \overline{\tau}, \overline{\phi})$$

be a complete and sound proof system with

$$\overline{\tau}((s_1, s_2)) = 1 \iff \begin{array}{l} \text{exactly 1 of the statements is true in } \Sigma, \\ \text{that is, } \tau(s_1) = 1 \text{ or } \tau(s_2) = 1, \text{but not both.} \end{array} \tag{1}$$

Define $\mathcal{P}^*$ and ϕ^* so that $\Sigma^* = (\mathcal{S}, \mathcal{P}^*, \tau^*, \phi^*)$ is a complete and sound proof system (and prove it!), where

$$\tau^*(s) = 1 \iff \tau(s) = 0$$

a)

Remark $\mathcal{S}' = \mathcal{S} \times \mathcal{S} \times \mathcal{S} = \{(s_1, s_2, s_3) \mid \forall i \in \{1,2,3\}\ s_i \in \mathcal{S}\}$, where $|\mathcal{S}'| = |\mathcal{S}|^3$.
Let

$$\mathcal{P}' = \mathcal{P} \times \mathcal{P} \times \mathcal{P} = \{(p_1, p_2, p_3) \mid \forall i \in \{1,2,3\}\ p_i \in \mathcal{P}\}$$
$$\phi' : \mathcal{S}' \times \mathcal{P}' \to \{0,1\}, \quad s' = (s_1, s_2, s_3) \in \mathcal{S}',\ p' = (p_1, p_2, p_3) \in \mathcal{P}'$$
$$(s', p') \mapsto \begin{cases} 1 & \text{if } \forall i \forall j \in \{1,2,3\}\ \phi(s_i, p_i) = 1\ \land\ \phi(s_j, p_j) = 1\ \land\ i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Completeness:
Assume $\tau'(s') = 1$. Then, at least 2 of $\tau(s_1), \tau(s_2), \tau(s_3)$ must equal 1 by definition of τ' in **a)**.
By completeness of proof system Σ, every true statement has a proof, i.e. $\forall i \in \{1,2,3\},\ \forall s_i \in \mathcal{S},\ \tau(s_i) = 1 \implies \exists p_i \in \mathcal{P},\ \phi(s_i, p_i) = 1$.
Since 2 among s_1, s_2, s_3 satisfy $\tau(s_i) = 1\ \land\ \tau(s_j) = 1$ by definition of ϕ' (assuming the exercise required distinct elements, thus $i \neq j$, which is a stronger case than repeated ones), it is possible to contruct the proof to verify correctness of at least 2 of such statements, i.e. $\forall s' \in \mathcal{S}',\ s' = (s_1, s_2, s_3),\ \tau'(s') = 1 \implies \exists p' \in \mathcal{P}',\ p' = (p_1, p_2, p_3)\ \land\ \phi'(s', p') = 1$, where 2 statements are

verified by the before mentioned completeness of Σ. Moreover, there are at least as many proofs as true statements since for each true statement there exist at least a proof. The definition of the proof system Σ' ensures it has at least 2 truth statements. When 2 true statements with their proofs and 1 false is proposed, then the third proof is just one of the already present, as the distinct elements required are only 2, the verification function will just yield false.
Thus, Σ' is complete.

Soundness: Assume $\phi'(s', p') = 1$. Then, $\forall i \forall j \in \{1,2,3\}\ \phi(s_i, p_i) = 1 \ \wedge\ \phi(s_j, p_j) = 1 \ \wedge\ i \neq j$ by defintion of ϕ'.
By soundness of proof system Σ, no false statement has a valid proof proving it true, or equivalently, if a statement does not have a proof, it must be false, i.e. $\forall i \in \{1,2,3\},\ \forall p_i \in \mathcal{P},\ \forall s_i \in \mathcal{S},\ \phi(s_i, p_i) = 1 \implies \tau(s_i) = 1$.
Hence, no proof for false statement exists, specifically satisfying τ' definition of having at least 2 of $\tau(s_i) = 1$.
Thus, Σ' is sound.

b)

Let

$$\mathcal{P}^* = \mathcal{P} \times \mathcal{P} = \{(p_1, p_2) \mid \forall i \in \{1,2\}\ p_i \in \mathcal{P}\}$$
$$\phi' : \mathcal{S} \times \mathcal{P}^* \to \{0,1\}, \quad s \in \mathcal{S}',\ p^* = \overline{p} = (p_1, p_2) \in \mathcal{P}^*$$
$$(s, p^*) \mapsto \begin{cases} 1 & \text{if } \big(\exists t \in \mathcal{S},\ \exists \overline{p} \in \overline{\mathcal{P}},\ \overline{\phi}((s,t), \overline{p}) = 1\big) \ \wedge\ (\exists i \in \{1,2\},\ \phi(t, \overline{p_i}) = 1) \\ 0 & \text{otherwise} \end{cases}$$

Clarifying that the definition of the complete proof system $\overline{\Sigma}$ ensures that there is at least 1 true statement in order to use its $\overline{\tau}$, $t \in \mathcal{S}$ wants to be such a **true** statement we use to distinguish our statement s from the proof system Σ^* in order to be able to verify, together with $\overline{\phi}$, if our statement w.r.t. its truth function is either true or false. Such true statement is found in polynomial time by a linear scan among the statements and proofs and checking with $\phi(t, p)$. Otherwise, a tautology can be used as such in the set of rules admit so. Recall there are at least as many proofs as true statements from completeness of Σ.
Completeness: Assume $\tau^*(s) = 1$. Then, $\tau(s) = 0$ by definition of τ^*.
By soundness of Σ, no false statements have a proof. Using the clarified true statement t by checking $\phi(t, p) = 1$, we use it within $\overline{\phi}((s,t), \overline{p}) = 1$, By soundness of $\overline{\Sigma}$, $\forall p \forall s,\ \overline{\phi}((s,t), \overline{p}) = 1 \implies \overline{\tau}((s,t)) = 1$. By definition of $\overline{\tau}$, only one of s, t is true, and since we known from assumption premises that s is false because $\tau(s) = 0$, then t must be true.
Hence, we can construct $\overline{p} = p^*$ for $\phi^*(s, p^*) = 1$ because p^* for sure is not a proof for s. Thus, Σ^* is complete.

Soundness: Assuming $\phi^*(s, p^*) = 1$. Then, $\overline{\phi}((s,t), \overline{p}) = 1 \ \wedge\ \phi(t, \overline{p_i}) = 1$ for some i.
By soundness of Σ, $\phi(t, \overline{p_i}) = 1 \implies \tau(t) = 1$.
By soundness of $\overline{\Sigma}$, $\overline{\phi}((s,t), \overline{p}) = 1$ and $\tau(t) = 1$ implies $\tau(s) = 0$.
Hence, by definition of τ^*, $\tau(s) = 0 \iff \tau^*(s) = 1$. Thus, Σ^* is sound.