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Optimization Methods **2024**

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Discussed with:

Assignment 4

Due date: Monday, 3 June 2024, 12:00 AM

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1. Exercise (20/100)

Consider the quadratic function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(\mathbf{x}) = 7x^2 + 4xy + y^2 \tag{1}$$

where $\mathbf{x} = (x, y)^T$.

1. Write this function in canonical form, i.e. $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$, where A is a symmetric matrix.
2. Descrive briefly how the Conjugate Gradient (CG) Method works and discuss whether it is suitable to minimize f from equation 1. Explain your reasoning in detail (max. 30 lines).

1. Answer

The function written in canonical form correspond to:

$$\begin{aligned}
f(\mathbf{x}) &= 7x^2 + 4xy + y^2 \\
&= \begin{bmatrix} 7x + 2y & 2x + y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}
\end{aligned}$$

With $\mathbf{b} = \mathbf{0}$, $c = 0$, and A being clearly a symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 14 & 4 \\ 4 & 2 \end{bmatrix}$$

Let's verify if A is positive define as required by the quadratic form:

$$
\begin{aligned}
\det(\lambda I - \mathbf{A}) &= \begin{vmatrix} \lambda - 14 & -4 \\ -4 & \lambda - 2 \end{vmatrix} \\
&= \lambda^2 - 16 + 12 \\
&\Rightarrow \lambda_{1,2} = 8 \pm 2\sqrt{13} > 0
\end{aligned}
$$

Finally, since all eigenvalues are positive, $\mathbf{A}$ is SPD.

2. Answer

The CG method is an algorithm to find the numerical solution of linear equation systems, with specific application to the minimization of quadratic functions whose Hessian matrix is symmetric and positive definite.

The CG method is appropriate and effective for minimizing the quadratic function.

2. Exercise (20/100)

Consider the following constrained minimization problem for $\mathbf{x} = (x, y, z)^T$

$$
\begin{aligned}
&\min_{\mathbf{x}} f(\mathbf{x}) := -3x^2 + y^2 + 2z^2 + 2(x + y + z) \\
&\text{subject to} \quad c(\mathbf{x}) = x^2 + y^2 + z^2 - 1 = 0
\end{aligned} \tag{2}
$$

Write down the Lagrangian function and derive th KKT conditions for 2

3. Exercise (60/100)

1. Read the chapter on Simplex method, in particular the section 13.3 The Simplex Method, in Numerical Optimization, Nocedal and Wright. Explain how the method works, with a particular attention to the search direction.

2. Consider the following contrained minimization problem, $\mathbf{x} = (x_1, x_2)^T$;

$$
\min_{\mathbf{x}} f(\mathbf{x}) := 4x_1 + 3x_2 \tag{3}
$$

subect to:

$$
\begin{aligned}
6 - 2x_1 - 3x_2 &\geq 0 \\
3 + 3x_1 - 2x_2 &\geq 0 \\
5 - 2x_2 &\geq 0 \\
4 - 2x_1 - x_2 &\geq 0 \\
x_2 &\geq 0 \\
x_1 &\geq 0
\end{aligned} \tag{4}
$$

a) Sketch the feasible region for this problem.

b) Which are the basic feasible points of the problem 3? Compute them by hand using the geometrical interpretation and find the optimal point $\mathbf{x}^*$ that minimizes f subject to the constraints.

c) Prove that the first order necessary conditions holds for the optimal point.