ETH zürich

Discrete Mathematics **2024**

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Assignment 1 **Due date:** Thursday, 26 September 2024, 23:59

1. Exercise 1.5, Two New Logical Operators

We define two binary logical operators $\heartsuit$ and $\diamondsuit$ as follows:

A	B	$A \heartsuit B$
0	0	1
0	1	0
1	0	1
1	1	1

A	B	$A \diamondsuit B$
0	0	1
0	1	0
1	0	0
1	1	1

a) ($\star$)

Are $\heartsuit$ and $\diamondsuit$ commutative, i.e., does it hold

$$A \heartsuit B \equiv B \heartsuit A \quad \text{and} \quad A \diamondsuit B \equiv B \diamondsuit A?$$

Argue by comparing function tables.

Let's compare the function tables of the $\heartsuit$ operator to check commutativity,

$A \heartsuit B$	$B \heartsuit A$
1	1
0	1
1	0
1	1

The function tables are not equivalent, i.e. $A \heartsuit B \not\equiv B \heartsuit A$, because the $\heartsuit$ operator has an ordering dependency on the propositional symbols: only when the left propositional symbol is false, the result is false. That's why when swapping the order of the symbols for the operators, the function tables are not equivalent. E.g. for $A = 0$ and $B = 1$, $A \heartsuit B = 0$ and $B \heartsuit A = 1$, hence <u>not commutative</u>.

Afterwards, let's compare the function tables of the $\diamondsuit$ operator to check commutativity,

$A \diamondsuit B$	$B \diamondsuit A$
1	1
0	0
0	0
1	1

The function tables are equivalent, i.e. $A \diamondsuit B \equiv B \diamondsuit A$, because the $\diamondsuit$ operator has no ordering dependency on the propositional symbols: swapping the order of the symbols for the operators does not change the function tables. E.g. for $A = 0$ and $B = 1$, $A \diamondsuit B = 0$ and $B \diamondsuit A = 0$, hence <u>commutative</u>.

b) ($\star$)

Prove or disprove that

$$(\neg A \heartsuit B) \diamondsuit (B \diamondsuit C) \equiv \neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$$

by computing and comparing the function tables of the left-hand-side and the right-hand-side formulas.

From the function table of the 3 propositional symbols A, B, and C,

A	B	C
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

we compute the left-hand-side formula as follows,

$\neg A$	$\neg A \heartsuit B$	$B \heartsuit C$	$(\neg A \heartsuit B) \diamondsuit (B \heartsuit C)$
1	1	1	1
1	1	0	0
1	1	1	1
1	1	1	1
0	1	1	1
0	1	0	0
0	0	1	0
0	0	1	0

and the right-hand-side formula as follows,

$A \diamondsuit B$	$\neg(A \diamondsuit B)$	$A \diamondsuit C$	$\neg(A \diamondsuit C)$	$\neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$
1	0	1	0	1
1	0	0	1	0
0	1	1	0	1
0	1	0	1	1
0	1	0	1	1
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1

by comparing the left-hand-side and right-hand-side function tables, we can see that they are not equivalent, i.e.

$(\neg A \heartsuit B) \diamondsuit (B \diamondsuit C)$	$\neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$
1	1
0	0
1	1
1	1
1	1
0	1
0	0
0	1

$$(\neg A \heartsuit B) \diamondsuit (B \diamondsuit C) \not\equiv \neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$$

c) ($\star\star$)

Let F be a formula with the following function table:

A	B	C	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Find a formula G containing only the logical operators $\heartsuit$ and $\diamondsuit$, in which the propositional symbols A, B, and C all appear exactly once, and such that $G \equiv F$. No justification is required.

Let G be defined as $G \coloneqq (C \diamondsuit A) \heartsuit B$. Then, the function table of G is as follows,

$C \diamondsuit A$	$(C \diamondsuit A) \heartsuit B$		G	F
1	1		1	1
0	1		1	1
1	1		1	1
0	0	$\longrightarrow$	0	0
0	1		1	1
1	1		1	1
0	0		0	0
1	1		1	1

by comparing the function tables of F and G, we can see that $G \equiv F$. All constraints are satisfied: only logical operators $\heartsuit, \diamondsuit$ and propositional symbols A, B, C appear exactly once.