ETH zürich

Discrete Mathematics **2024**

Student: Jeferson Morales Mariciano <jmorale@ethz.ch>

Assignment 5

Due date: Thursday, 24 October 2024, 23:59

Exercise 5.5, Properties of Relations ($\star$) (8 Points)

Prove or disprove the following claims:

a) A relation ρ on a set A is symmetric on A if and only if ρ^2 is symmetric on A.

b) If ρ is a relation on a set A that is symmetric and antisymmetric, then it musta hold $\rho = \mathsf{id}_A$.

c) Define the relations ρ_1 and rho_2 on $\mathbb{Z}$ as

$$a \,\rho_1\, b \iff b = a + 1, \qquad a \,\rho_2\, b \iff b \equiv_2 a.$$

Then for $\rho = \rho_1 \cup \rho_2$ it holds $\rho^2 = \mathbb{Z} \times \mathbb{Z}$.

a)

The claim is false, a counterexample follows:

$$\begin{aligned} \rho &= \{(a, c), (b, d), (c, b), (d, a)\} \\ \rho^2 &= \{(a, b), (b, a), (c, d), (d, c)\} \end{aligned}$$

The relation ρ^2 is symmetric, but ρ is not, which disprove the claim if and only if ($\iff$) from the right to left part ($\Longleftarrow$).

b)

The claim is false, a counterexample follows:

$$\rho = \emptyset \neq \mathsf{id}_A$$

The relation ρ is both symmetric and antisymmetric. However, ρ is not the identity relation as the claim states.

c)

The claim is true, and it can be proven as follows:
we will decompose the claim through composition of implications from which will lead to a clear satisfiability problem that show is satisfiable using a proof by case distinction.

Composition of Implications

Case Distinction

Check that the statement is always satisfiable for all cases of a, b, i.e. that there always exist some suitable c in the universe to satisfy the claim.

The cases are represented by the following matrix:

a	b	c
0	0	0
0	1	$s(a)$
1	0	$s(a)$
1	1	1

where $s(x)$ is the successor of x in the universe. Finally, it follows that the claim is always satisfiable, meaning is true for all cases of $a, b \in \mathcal{U} = \mathbb{Z} \times \mathbb{Z}$. Concluding, $\rho^2 = \mathbb{1} = \mathbb{Z} \times \mathbb{Z}$.