ETH zürich

Discrete Mathematics **2024**

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Assignment 7 **Due date:** Thursday, 7 November 2024, 23:59

Exercise 7.3, Properties of Greastest Common Divisors and Least Common Multiple $(\star)$ (8 Points)

Prove or disprove the following properties. Only use the definitions of ideals, gcd and lcm, and don't use the results from Section 4.3.3 in the lecture.

a) For all positive integers a, b

$$(a) \cup (b) = (\gcd(a, b)).$$

b) For all positive integers a, b

$$(a) \cap (b) = (\operatorname{lcm}(a, b)).$$

a)

The claim is false, a counterexample follows:

$$\begin{aligned}
&a = 6,\ b = 10,\\
&\gcd(6, 10) = \gcd(2 \cdot 3, 2 \cdot 5) = 2\\
&(2) = \{u \cdot 2 \mid u \in \mathbb{Z}\} = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 14, \pm 16, \pm 18, \pm 20, \ldots\}\\
&(6) \cup (10) = \{v \cdot 6 \mid v \in \mathbb{Z}\} \cup \{w \cdot 10 \mid w \in \mathbb{Z}\} = \{0, \pm 6, \pm 10, \pm 12, \pm 18, \pm 20, \ldots\}\\
&(6) \cup (10) \subseteq (2) \quad \wedge \quad (2) \nsubseteq (6) \cup (10) \implies (6) \cup (10) \neq (2).
\end{aligned}$$

The ideals in (2) have multiples of 2 which are all even numbers as $4, 8, 14$ which are not multiples of 6 or 10, hence there cannot be in the union of (6) and (10).