Discrete Mathematics **2024**

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Assignment 11 **Due date:** Thursday, 5 December 2024, 23:59

Exercise 11.4, Combining Proof Systems ($\star$) (8 Points)

Let

$$\Sigma = (\mathcal{S}, \mathcal{P}, \tau, \phi)$$

be a complete and sound proof system.

a) Define $\mathcal{P}'$ and ϕ' so that

$$\Sigma' = (\mathcal{S} \times \mathcal{S} \times \mathcal{S}, \mathcal{P}', \tau', \phi')$$

is a complete and sound proof system (and prove it!), where

$$\tau'((s_1, s_2, s_3)) = 1 \iff \text{ at least 2 among } \tau(s_1), \tau(s_2), \tau(s_3) \text{ are equal to 1}$$

b) Let

$$\overline{\Sigma} = (\mathcal{S}^2, \overline{\mathcal{P}}, \overline{\tau}, \overline{\phi})$$

be a complete and sound proof system with

$$\begin{aligned} \overline{\tau}((s_1, s_2)) = 1 \iff & \text{exactly 1 of the statements is true in } \Sigma, \\ & \text{that is, } \tau(s_1) = 1 \text{ or } \tau(s_2) = 1, \text{but not both.} \end{aligned} \tag{1}$$

Define $\mathcal{P}^*$ and ϕ^* so that $\Sigma^* = (\mathcal{S}, \mathcal{P}^*, \tau^*, \phi^*)$ is a complete and sound proof system (and prove it!), where

$$\tau^*(s) = 1 \iff \tau(s) = 0$$

a)

Remark $\mathcal{S}' = \mathcal{S} \times \mathcal{S} \times \mathcal{S} = \{(s_1, s_2, s_3) \mid \forall i \in \{1,2,3\}\ s_i \in \mathcal{S}\}$, where $|\mathcal{S}'| = |\mathcal{S}|^3$.
Let

$$\begin{aligned} &\mathcal{P}' = \mathcal{P} \times \mathcal{P} \times \mathcal{P} = \{(p_1, p_2, p_3) \mid \forall i \in \{1,2,3\}\ p_i \in \mathcal{P}\} \\ &\phi' : \mathcal{S}' \times \mathcal{P}' \to \{0,1\}, \quad s' = (s_1, s_2, s_3) \in \mathcal{S}',\ p' = (p_1, p_2, p_3) \in \mathcal{P}' \\ &(s', p') \mapsto \begin{cases} 1 & \text{if } \forall i \in \{1,2,3\}\ (\phi(s_i, p_i) = 1 \ \land \text{ at least 2 of } \tau(s_i) = 1) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Completeness:
Assume $\tau'(s') = 1$. Then, at least 2 of $\tau(s_1), \tau(s_2), \tau(s_3)$ must equal 1 by definition of τ' in **a)**.
By completeness of proof system Σ, every true statement has a proof, i.e. $\forall i \in \{1,2,3\},\ \exists p_i \in \mathcal{P},\ \phi(s_i, p_i) = 1$.
Hence, by grouping any 3 arbitrary statements in 3-tuples there must exist at least a corresponding arbitrary grouping of proofs in 3-tuples proving (at least 2) such statements, i.e. $\forall s' \in \mathcal{S}',\ s' = (s_1, s_2, s_3),\ \exists p' \in \mathcal{P}',\ p' = (p_1, p_2, p_3) \ \land\ \phi'(s', p') = 1$.
Thus, Σ' is complete.

Soundness: Assume $\phi'(s', p') = 1$. Then, $\forall i \in \{1, 2, 3\}$ $\phi(s_i, p_i) = 1$ and at least 2 of $\tau(s_1), \tau(s_2), \tau(s_3)$ equal 1 by definition of ϕ'.
By soundness of proof system Σ, no false statement has a valid proof proving it true, i.e. $\forall i \in \{1, 2, 3\},\ \forall p_i \in \mathcal{P},\ \forall s_i \in \mathcal{S},\ \phi(s_i, p_i) = 1 \implies \tau(s_i) = 1$.
Hence, no proof for false statement exists, specifically satisfying τ' definition of having at least 2 of $\tau(s_i) = 1$.
Thus, Σ' is sound.

b)

Completeness: **Soundness**: