Math 170-Worksheet 5
2023-2024 Spring

February 24, 2025

1. Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

   (a) $r(t) = \langle t, 2-t, 2t\rangle$ (b) $r(t) = \langle \sin \pi t, t, \cos \pi t\rangle$

2. At what points does the curve $r(t) = t\mathbf{i} + (2t - t^2)\mathbf{k}$, intersect the paraboloid $z^2 = x^2 + y^2$?

3. Show that the curve with parametric equations $x = t^2,\ y = 1 - 3t,\ z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

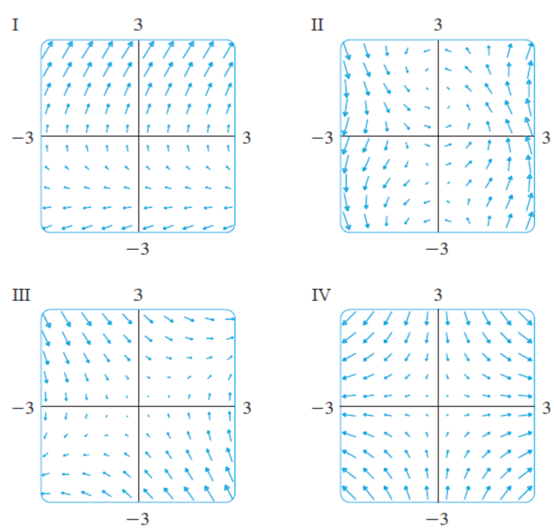
4. Find a vector function that represents the curve of intersection of the two surfaces.

   (a) The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

   (b) The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

5. At what point do the curves $u(t) = \langle t, 1-t, 3+t^2\rangle$ and $v(s) = \langle 3-s, s-2, s^2\rangle$ intersect? Find their angle of intersection.

6. Match the vector fields $\mathbf{F}$ with the plots labeled I–IV. Give reasons for your choices.



(a) $\mathbf{F}(x,y) = \langle x, -y \rangle$
(b) $\mathbf{F}(x,y) = \langle y, x-y \rangle$
(c) $\mathbf{F}(x,y) = \langle y, y+2 \rangle$
(d) $\mathbf{F}(x,y) = \langle \cos(x+y), x \rangle$

7. Evaluate the line integral, where C is the given curve.
   (a) $\int_C y^3 ds$, $C: x = t^3,\ y = t, 0 \le t \le 2$
   (b) $\int_C x \sin y ds$, C is the line segment from $(0,3)$ to $(4,6)$.
   (c) $\int_C (x+2y)dx + x^2 dy$, C consists of the line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$.
   (d) $\int_C x^2 + y^2 dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.
   (e) $\int_C z^2 dx + x^2 dy + y^2 dz$, C is the line segment from $(1,0,0)$ to $(4,1,2)$.

8. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $r(t)$.
   (a) $\mathbf{F}(x,y) = xy\mathbf{i} + 3y^2\mathbf{j}$, $r(t) = 11t^4\mathbf{i} + t^3\mathbf{j}$, $0 \le t \le 1$
   (b) $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos x\mathbf{j} + xz\mathbf{k}$, $r(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

9. Find the work done by the force field $\mathbf{F}(x,y) = x\mathbf{i} + (y+2)\mathbf{j}$ in moving an object along an arc of the cycloid $r(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$, $0 \le t \le 2\pi$.

10. Find the work done by the force field $\mathbf{F}(x,y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$ from the point $(1,1)$ to $(2,4)$.

11. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.
    (a) $\mathbf{F}(x,y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$
    (b) $\mathbf{F}(x,y) = (e^x \cos y)\mathbf{i} + (e^x \sin y)\mathbf{j}$
    (c) $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$
    (d) $\mathbf{F}(x,y) = (\ln y + 2xy^3)\mathbf{i} + (3x^2y^2 + x/y)\mathbf{j}$
    (e) $\mathbf{F}(x,y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$
    (f) $\mathbf{F}(x,y,z) = \mathbf{i} + \sin z\mathbf{j} + y\cos z\mathbf{k}$.
    (g) $\mathbf{F}(x,y,z) = e^x \sin yz\mathbf{i} + ze^x \cos yz\mathbf{j} + ye^x \cos yz\mathbf{k}$.

12. Find a function f such that $\mathbf{F} = \nabla f$ and use that to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.
    (a) $\mathbf{F}(x,y) = (xy^2)\mathbf{i} + (x^2y)\mathbf{j}$, where C: $r(t) = \langle t + \sin(\frac{\pi t}{2}), t + \cos(\frac{\pi t}{2}) \rangle$ for $t \in [0,1]$
    (b) $\mathbf{F}(x,y,z) = (yz)\mathbf{i} + (xz)\mathbf{j} + (xy + 2z)\mathbf{k}$, where C is the line segment from $(1,0,-2)$ to $(4,6,3)$.
    (c) $\mathbf{F}(x,y,z) = (yze^{xz})\mathbf{i} + (e^{xz})\mathbf{j} + (xye^{xz})\mathbf{k}$, where C: $r(t) = (t^2+1)\mathbf{i} + (t^2-1)\mathbf{j} + (t^2-2t)\mathbf{k}$ for $t \in [0,2]$
    (d) $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$, C is the arc of the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.
    (e) $\mathbf{F}(x,y,z) = \sin y\mathbf{i} + (x\cos y + \cos z)\mathbf{j} - y\sin z\mathbf{k}$, $C: \mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 0 \leqslant t \leqslant \dfrac{\pi}{2}$.

13. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.
    (a) $\int_C xy^2 dx + x^2 y dy$, C is the triangle with vertices $(0,0)$, $(2,2)$ and $(2,4)$

(b) $\int_C y^3dx - x^3dy$, C is the circle $x^2 + y^2 = 4$

(c) $\int_C (x-y)dx + (x+y)dy$ where C is the circle with center the origin and radius 2.

(d) $\int_C (y + e^{\sqrt{x}})dx + (2x + cosy^2)dy$ where C is the boundary of the region enclosed by the parabolas $y = x^2, x = y^2$.

(e) $\int_C y^4dx + 2xy^3dy$ where C is the ellipse $x^2 + 2y^2 = 2$.

14. Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

(a) $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x\rangle$, where C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$.

(b) $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y\rangle$, where C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

(c) $\mathbf{F}(x, y) = \langle e^{-x} + y^2, e^{-y} + x^2\rangle$, C consists of the arc of the curve $y = \cos x$, from $(-\frac{\pi}{2}, 0)$ to $(\frac{\pi}{2}, 0)$ and the line segment from $(\frac{\pi}{2}, 0)$ to $(-\frac{\pi}{2}, 0)$.

15. Find curl and divergence of the given vector field.

(a) $\mathbf{F}(x, y, z) = xy^2z^3\mathbf{i} + x^3yz^2\mathbf{j} + x^2y^3z\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + \sin(xz)\mathbf{j} + \sin(xy)\mathbf{k}$

16. Determine whether $\mathbf{F}$ is conservative or not, if so, find f such that $\nabla f = \mathbf{F}$.

(a) $\mathbf{F}(x, y, z) = xyz^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = \mathbf{i} + \sin(z)\mathbf{j} + y\cos(z)\mathbf{k}$

17. Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$

(a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. (Compute the integral over upper and lower halves of circle $x^2 + y^2 = 1$ from (1,0) to (-1,0))

18. Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression meaningful. If not, explain why. If so, state whether it is a scalar field or vector field.

a) curl(grad f) b) div (div f)

c) grad(curl f) d) grad (div **F**)

e) (grad f)$\times$ (div **F**) f) div (curl (grad f))

19. Is there any vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} =< x \sin y, \cos y, z - xy >$? Explain.

20. Is there any vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} =< xyz, -y^2z, yz^2 >$? Explain.

21. Prove the following identities by assuming that appropriate partial derivatives exist and are continuous.

a) curl ($\mathbf{F}+\mathbf{G}$) = curl ($\mathbf{F}$)+ curl ($\mathbf{G}$)

b) curl ($f\mathbf{F}$) $= f$curl ($\mathbf{F}$) $+ \nabla(f) \times \mathbf{F}$.

c) div ($\nabla f \times \nabla g$) $= 0$.

d) curl (curl $\mathbf{F}$)= grad (div $\mathbf{F}$)-$\nabla^2\mathbf{F}$.