Supplementary Note: Algebraic Invariance of the $K = 4$ Safe Window

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DERIVATION OF THE INVARIANT SET

We examine the stability of the fiber-coupled cycle $\mathcal{C} = \{1, 4, 2\}$ under the skew-product map T. We define the "Safe Window" $\mathcal{W}_K$ as the set of fiber values n that result in a zero coupling term c when the base $w = 1$.

Lemma 1: The Carry Gate Condition

For a trajectory at $w_t = 1$ to map to $w_{t+1} = 4$ (remaining on the cycle), the update rule $w_{t+1} = 3(1) + 1 + c$ requires $c = 0$. Given the coupling definition $c = \lfloor Kn/p \rfloor$, this imposes the strict inequality:

$$c = 0 \iff Kn < p \iff 0 \leq n \leq \frac{p-1}{K} \tag{1}$$

We define this range $\mathcal{W}_K = [0, (p-1)/K]$ as the **Safe Window**. If $n_t \notin \mathcal{W}_K$ when $w_t = 1$, the trajectory is immediately expelled from $\mathcal{C}$.

Lemma 2: The Conditional Return Map

Assuming a trajectory resides on the cycle $\{1, 4, 2\}$ for a full period ($t \to t + 3$), the fiber n transforms as:

1. $w : 1 \to 4 \implies n \mapsto Kn \pmod{p}$ (Valid if $c = 0$)
2. $w : 4 \to 2 \implies n \mapsto n \cdot 2^{-1} \pmod{p}$
3. $w : 2 \to 1 \implies n \mapsto n \cdot 2^{-1} \pmod{p}$

The cumulative return map $R_K(n)$ is the composition:

$$R_K(n) \equiv n \cdot K \cdot \frac{1}{4} \equiv \frac{K}{4} n \pmod{p} \tag{2}$$

Theorem: Existence of Invariant Set for $K = 4$

For the specific case $K = 4$, we demonstrate that $\mathcal{W}_4$ is an invariant set.

1. **Identity Return:** With $K = 4$, the return map becomes $R_4(n) \equiv n \pmod{p}$. Thus, if $n_0 \in \mathcal{W}_4$, then $n_3 = n_0$. The fiber value is periodic with period 3.
2. **Gate Satisfaction:** For any $n \in \mathcal{W}_4$, we have $Kn < p$. Consequently, at the odd step ($w = 1$), the fiber maps to $4n$ without modular wraparound. The carry is $c = \lfloor 4n/p \rfloor = 0$.

Therefore, any trajectory initialized with $w_0 = 1$ and $n_0 \in \mathcal{W}_4$ satisfies the gate condition at $t = 0$ and, due to the identity return map, satisfies it for all future returns $t = 3k$. This proves the existence of a non-trivial invariant set.

Sieve Constraint for $K \neq 4$

For $K \neq 4$, the return map $R_K(n) \equiv \frac{K}{4} n$ is a non-identity permutation of $\mathbb{Z}_p^\times$. The Safe Window $\mathcal{W}_K$ is not invariant under R_K in general. Long-term survival requires the trajectory to satisfy the recurrence $R_K^k(n_0) \in \mathcal{W}_K$ for all $k \geq 1$. This constitutes a **Sieve Constraint**. In our computations (using $p \approx 10^9$ and $N = 60$ steps), no trajectories satisfied this constraint for tested values $K \in \{2, 3, 5, 6, 8, 16\}$, resulting in observed extinction.