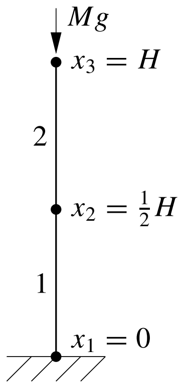
Exercises Computational Mechanics (4MC10) – Lecture 7

⋆ **Exercise 1**

Discretise the weak forms derived in Exercise 2 of Lecture 6 by introducing shape functions $\underset{\sim}{N}(x)$. Determine the system matrices $\underline{K}$, $\underset{\sim}{f}$ and $\underset{\sim}{q}$ of the resulting discrete set of equations $\underline{K}\,\underset{\sim}{u} = \underset{\sim}{f} + \underset{\sim}{q}$, where $\underset{\sim}{f}$ corresponds to the distributed 'forcing' term and $\underset{\sim}{q}$ results from the boundary term.

Exercise 2

Consider again the displacement field $u(x)$ in the control tower of Exercise 3 of the previous lecture. We discretise the domain $(0, H)$ by two linear finite elements, see Figure 1.



$H = 100$ m
$A = 5.0$ m^2
$\rho = 2.5 \cdot 10^3$ kg/m^3
$E = 2.0 \cdot 10^{10}$ Pa
$M = 1.00 \cdot 10^5$ kg
$g = 9.81$ m/s^2

Figure 1: Finite element mesh of the control tower consisting of two linear elements (left) and parameter values of the problem (right)

a. Sketch the shape functions associated with nodes 1, 2 and 3.

b. Verify that, using these shape functions, the weak form which was derived in Exercise 3 of the previous lecture can be rewritten in terms of the nodal displacements $\underset{\sim}{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\mathrm{T}}$ as

$$\underline{K}\,\underset{\sim}{u} = \underset{\sim}{f} + \underset{\sim}{q}$$

with $\underline{K}$ a 3×3 matrix and $\underset{\sim}{f}$, $\underset{\sim}{q}$ column matrices with 3 entries.

c. Determine expressions for $\underline{K}$, $\underset{\sim}{f}$, and $\underset{\sim}{q}$ in terms of the constants E, A, M, ρ, g and the element size $h = H/2$.

d. Implement the system matrices obtained above in a MATLAB file `controltower.m`; use the parameter values given in Figure 1.

e. Extend your MATLAB file by implementing the partitioning according to given and unknown degrees of freedom.

f. Solve the set of equations for the displacements $\underset{\sim}{u}$; compute the reaction force F_0 from the remaining equation.

g. Determine the highest compressive stress in the pillar as predicted by the finite element program.

h. Adapt your program such that it works for an arbitrary number of elements, m. Compare the results for several values of m to the analytical solution.

i. If the maximum allowable compressive stress in the pillar equals $\sigma_a = 2.5$ MPa, is this stress exceeded?

j. Do you see any advantage in using quadratic finite element shape functions in this problem?

⋆ Exercise 3

We consider in this exercise a linear system of equations which is given in matrix form as

$$\underline{K}\,\underset{\sim}{u} = \underset{\sim}{q}$$

with the matrices $\underline{K}$, $\underset{\sim}{u}$ and $\underset{\sim}{q}$ given by

$$\underline{K} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \qquad \underset{\sim}{q} = \begin{bmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ q_5 \end{bmatrix}$$

and u_i ($i = 1, 2, \ldots 5$), q_1, q_5 yet to be determined.
Write a MATLAB program which appropriately partitions the system and computes all components of $\underset{\sim}{u}$ and $\underset{\sim}{q}$ for each of the following cases:

a. $u_1 = 0$ and $q_5 = 1$

b. $u_1 = 0$ and $u_5 = 1$

c. $q_1 = 0$ and $u_5 = 1$

d. $q_1 = 0$ and $q_5 = 0$; what is wrong in this case?

⋆ Exercise 4

Consider again the heat exchanger pipe of Exercise 4 of Lecture 6. The weak form of the stationary heat conduction equation derived in this exercise can be written as

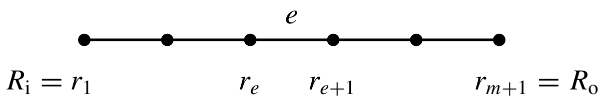
$$\int_{R_i}^{R_o} \frac{\mathrm{d}\phi}{\mathrm{d}r}\,\lambda\, r\,\frac{\mathrm{d}T}{\mathrm{d}r}\,\mathrm{d}r = \phi(R_i)Q_i - \phi(R_o)Q_o$$

where Q_i and Q_o are the heat fluxes at the inside and outside of the tube, which are yet unknown. The temperatures at the inside and outside, on the other hand, are assumed to be known and given by T_i and T_o.

a. Show that substituting finite element interpolations $\phi^h(r) = \underset{\sim}{N}^T(r)\,\underset{\sim}{\phi}$ and $T^h(r) = \underset{\sim}{N}^T(r)\,\underset{\sim}{T}$ for $\phi(r)$ and $T(r)$ results in a linear algebraic system of equations of the form $\underline{K}\,\underset{\sim}{T} = \underset{\sim}{q}$.

The domain (R_i, R_o) is now divided into m linear elements with a constant length $h = (R_o - R_i)/m$; the number m can for the moment be arbitrary. For $m = 5$ this mesh has been sketched in Figure 2.

b. Determine the contribution $\underline{K}^e$ of an arbitrary element e to the system matrix $\underline{K}$ in terms of λ and the radii r_e and r_{e+1}.



$\lambda = 80$ W/Km
$R_\mathrm{i} = 2$ mm
$R_\mathrm{o} = 20$ mm
$T_\mathrm{i} = 360$ K
$T_\mathrm{o} = 300$ K

Figure 2: Finite element discretisation of the heat exchanger tube for $m = 5$ (left) and parameter values for the numerical implementation (right)

c. Implement the element matrix $\underset{\sim}{K}^e$ and the assembly to the global matrix $\underset{\sim}{K}$ for arbitrary m in a MATLAB file `exchanger.m`; use the value for λ given in Figure 2.

d. Implement the partitioning of the system matrix for the two essential boundary conditions of the problem.

e. Implement the solving of the partitioned system, using the parameter values of Figure 2. Run the analysis for $m = 5$ elements and plot the temperature distribution.

f. Repeat the analysis for finer meshes. Does the temperature profile converge upon mesh refinement?

g. Compute the heat fluxes Q_i and Q_o. How many elements do you need to determine these quantities to within 1% accuracy?

Answers

Exercise 1

a. $\underline{\mathrm{K}} = \displaystyle\int_0^1 \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x} \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x}\, \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \underset{\sim}{\mathrm{N}}(x) \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_0^1$

b. $\underline{\mathrm{K}} = \displaystyle\int_0^1 \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x}\,(x+1)\, \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x}\, \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \int_0^1 \underset{\sim}{\mathrm{N}}(x)\,(x-1)\,\mathrm{d}x \qquad \underset{\sim}{\mathrm{q}} = \underset{\sim}{\mathrm{N}}(x)\,(x+1) \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_0^1$

c. $\underline{\mathrm{K}} = \displaystyle\int_0^1 \left(\frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x} \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x} + \underset{\sim}{\mathrm{N}}(x)\,\lambda\, \underset{\sim}{\mathrm{N}}^{\mathrm{T}}(x) \right) \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \underset{\sim}{\mathrm{N}}(x) \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_0^1$

d. $\underline{\mathrm{K}} = \displaystyle\int_0^1 \left(\frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x} \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x} - \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x}\,\lambda\, \underset{\sim}{\mathrm{N}}^{\mathrm{T}}(x) \right) \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \underset{\sim}{\mathrm{N}}(x) \left.\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \lambda u(x) \right)\right|_0^1$

or

$\underline{\mathrm{K}} = \displaystyle\int_0^1 \left(\frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x} \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x} + \underset{\sim}{\mathrm{N}}(x)\,\lambda\, \frac{\mathrm{d}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x} \right) \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \underset{\sim}{\mathrm{N}}(x) \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_0^1$

e. $\underline{\mathrm{K}} = \displaystyle\int_0^1 \frac{\mathrm{d}^2\underset{\sim}{\mathrm{N}}}{\mathrm{d}x^2} \frac{\mathrm{d}^2\underset{\sim}{\mathrm{N}}^{\mathrm{T}}}{\mathrm{d}x^2}\, \mathrm{d}x \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \left.\left(\frac{\mathrm{d}\underset{\sim}{\mathrm{N}}}{\mathrm{d}x} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \underset{\sim}{\mathrm{N}}(x) \frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \right)\right|_0^1$

Exercise 2

c. $\underline{\mathrm{K}} = \dfrac{EA}{h} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \underset{\sim}{\mathrm{f}} = -\tfrac{1}{2}\rho g A h \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \underset{\sim}{\mathrm{q}} = \begin{bmatrix} -F_0 \\ 0 \\ -Mg \end{bmatrix}$

f. $\underset{\sim}{\mathrm{u}} = \begin{bmatrix} 0 & -5.1 & -7.1 \end{bmatrix} \cdot 10^{-3}\ \mathrm{m} \qquad F_0 = -1.3 \cdot 10^7\ \mathrm{N}$

g. $\max(-\sigma) = 2.0 \cdot 10^6\ \mathrm{Pa}$

i. Yes

j. Yes, they would give the exact result in this case even for a single element

Exercise 3

a. $\underset{\sim}{\mathrm{u}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}^{\mathrm{T}} \qquad \underset{\sim}{\mathrm{q}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$

b. $\underset{\sim}{\mathrm{u}} = \begin{bmatrix} 0 & 0.25 & 0.50 & 0.75 & 1 \end{bmatrix}^{\mathrm{T}} \qquad \underset{\sim}{\mathrm{q}} = \begin{bmatrix} -0.25 & 0 & 0 & 0 & 0.25 \end{bmatrix}^{\mathrm{T}}$

c. $\underset{\sim}{\mathrm{u}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}} \qquad \underset{\sim}{\mathrm{q}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$

d. A singular matrix is obtained; no unique solution for $\underset{\sim}{\mathrm{u}}$ exists

Exercise 4

b. $\underline{\mathrm{K}}^e = \tfrac{1}{2}\lambda \dfrac{r_{e+1} + r_e}{r_{e+1} - r_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

f. Yes

g. $Q_\mathrm{i} = Q_\mathrm{o} = 2.0846\ \mathrm{kW/m} \qquad m = 12$