Exercises Computational Mechanics (4MC10) – Lecture 9

⋆ **Exercise 1**

Derive the weak forms of the following partial differential equations, which are defined on a volume V with surface S:

a. $\nabla^2 u = 0$

b. $\vec{\nabla}\cdot\Big((\vec{x}\vec{x} + \boldsymbol{I})\cdot\vec{\nabla}u\Big) + \vec{x}\cdot\vec{x} - 1 = 0$

c. $\nabla^2 u - \lambda u(\vec{x}) = 0$

d. $\nabla^2 u - \vec{v}\cdot\vec{\nabla}u = 0$ with $\vec{v}$ a constant vector

e. $\nabla^4 u = 0$

⋆ **Exercise 2**

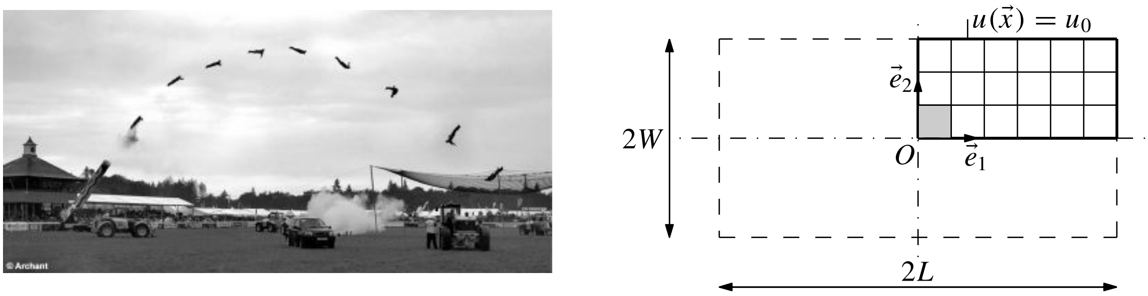
Discretising the weak forms which have been derived in Exercise 1 results in a linear system

$$\underset{\sim}{\mathrm{K}}\,\underset{\sim}{\mathrm{u}} = \underset{\sim}{\mathrm{f}} + \underset{\sim}{\mathrm{q}}$$

for each of the problems. Derive expressions for the matrices $\underset{\sim}{\mathrm{K}}$, $\underset{\sim}{\mathrm{f}}$ and $\underset{\sim}{\mathrm{q}}$ in these systems in terms of a set of shape functions $\underset{\sim}{\mathrm{N}}(\vec{x})$ which can be assumed to satisfy all the necessary continuity requirements.

Exercise 3

Human cannonballs at the end of their flight often land in a net – see the picture on the left in Figure 1. To be sure that the cannonball and net do not touch the ground upon landing, we analyse the displacement of the net due to the cannonball's impact.



Figuur 1: Stroboscopic image of human cannonball in flight (left) and sketch of the net seen from above (right)

The length and width of the net are denoted $2L$ and $2W$ respectively. If we assume that the cannonball lands perfectly vertically at the centre of the net, the problem becomes double-symmetric and we can limit our analysis to one quarter of the net, as sketched on the right in Figure 1. Positions within this rectangular domain $V : (0, L) \times (0, W)$ are indicated by the position vector $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$, with $\vec{e}_1$ and $\vec{e}_2$ Cartesian basis vectors as indicated in the sketch. The net's height with respect to the ground

is denoted $u(\vec{x})$; its displacement in horizontal direction is neglected. The net is fixed on all edges at a height $u(\vec{x}) = u_0$. It is pre-tensioned by an isotropic force per unit length σ which is assumed to be constant, even upon deformation of the net.
Before the cannonball's impact, the shape of the net is determined by its own weight. In this case the height $u(\vec{x})$ satisfies the partial differential equation

$$\vec{\nabla}\cdot\left(\sigma\vec{\nabla}u\right) - \rho g = 0$$

where ρ is the mass per unit area of the net and g the gravitational constant.

a. Which boundary condition must be imposed at $x_1 = L$ and at $x_2 = W$?

b. Show that on the remaining two edges of V, i.e. for $x_1 = 0$ and for $x_2 = 0$, the following natural boundary condition must hold:

$$q(\vec{x}) = \vec{n}(\vec{x})\cdot\left(\sigma\vec{\nabla}u\right) = 0$$

with $n(\vec{x})$ the unit normal to the boundary. Hint: which implications does the symmetry assumption have for these edges?

c. Derive the weak formulation of the above equilibrium equation.

d. Employ Galerkin's method to derive the discretised equilibrium equations

$$\underset{\sim}{\underline{K}}\,\underset{\sim}{u} = \underset{\sim}{f} + \underset{\sim}{q}$$

where $\underset{\sim}{N}(\vec{x})$ contains a set of shape functions; give expressions for $\underset{\sim}{\underline{K}}$, $\underset{\sim}{f}$ and $\underset{\sim}{q}$.

We finally consider the net's deformation upon impact of the human cannonball. We assume that upon impact he or she exerts a force of $3Mg$, where M is the mass of the person, and that this additional load is distributed uniformly on an area of $\frac{2}{9}LW$ – cf. the shaded region in the sketch of Figure 1.

e. Determine the force per unit of area which must be added to ρg for the shaded part of V in the definition of $\underset{\sim}{f}$ in order to account for the cannonball's impact.

⋆ Exercise 4

A position vector $\vec{x}$ and a displacement vector $\vec{u}$ are defined in a Cartesian basis $\{\vec{e}_1, \vec{e}_2\}$ as

$$\vec{x} = 4\,\vec{e}_1 + 2\,\vec{e}_2 \qquad \vec{u} = -\vec{e}_1 + 2\,\vec{e}_2$$

a. Carefully read the TENSORLAB tutorial.

b. Define the basis vectors $\vec{e}_1$ and $\vec{e}_2$ using

```
e = cartesianbasis2d('e1', 'e2');
e1 = e(1);
e2 = e(2);
```

c. Compute the products

```
dot(e1, e1)
dot(e1, e2)
```

and verify the results

d. Define the vectors $\vec{x}$ and $\vec{u}$ by typing

```
x = 4*e1 + 2*e2
u =  -e1 + 2*e2
```

e. Verify that evaluating the inner products $\vec{x} \cdot \vec{e}_i$ $(i = 1, 2)$ gives the components of $\vec{x}$ with respect to the basis $\{\vec{e}_1, \vec{e}_2\}$, i.e. the coordinates x_1 and x_2.

f. Compute the inner product of $\vec{x}$ and $\vec{u}$ using TENSORLAB and verify the result.

g. Compute the length (norm) $\|\vec{u}\|$ of $\vec{u}$ using TENSORLAB.

h. Plot the vector $\vec{u}$ in point $\vec{x}$ using

```
quiver(x, u, 0)
```

and verify that the colour of the vector corresponds with its length.

i. Compute the dyadic product `x * u` and verify the result.

Exercise 5

A vector column $\underset{\sim}{\vec{a}}$ is defined in a two-dimensional Cartesian basis $\{\vec{e}_1, \vec{e}_2\}$ as

$$\underset{\sim}{\vec{a}} = \begin{bmatrix} 3\vec{e}_1 - \vec{e}_2 \\ \vec{e}_1 + 3\vec{e}_2 \\ 5\vec{e}_1 \end{bmatrix}$$

Furthermore, the second-order tensor $\boldsymbol{C}$ is given by

$$\boldsymbol{C} = 2\vec{e}_1\vec{e}_1 - \vec{e}_1\vec{e}_2 - \vec{e}_2\vec{e}_1 + \vec{e}_2\vec{e}_2$$

a. Compute by hand the product $\boldsymbol{C} \cdot \underset{\sim}{\vec{a}}$.

b. In MATLAB/TENSORLAB define the vector column $\underset{\sim}{\vec{a}}$ and the tensor $\boldsymbol{C}$. Compute the above product and compare the result with your own calculation.

c. Which kind of quantity do you expect as the result of the product $\underset{\sim}{\vec{a}}^{\mathrm{T}} \cdot \boldsymbol{C} \cdot \underset{\sim}{\vec{a}}$? A single value, column matrix or matrix consisting of scalar values, vectors or tensors?

d. Compute the above product with TENSORLAB and check your answer.

e. Compute with TENSORLAB the product $\underset{\sim}{\mathbf{D}} \cdot \underset{\sim}{\vec{a}}$ where

$$\underset{\sim}{\mathbf{D}} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \frac{1}{5}\boldsymbol{C} \end{bmatrix}$$

and $\boldsymbol{O}$ is the second-order zero tensor.

Answers

Exercise 1

a. $$\int\limits_V \vec{\nabla}\phi\cdot\vec{\nabla}u \,\mathrm{d}V = \int\limits_S \phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

b. $$\int\limits_V \vec{\nabla}\phi\cdot(\vec{x}\vec{x}+\boldsymbol{I})\cdot\vec{\nabla}u \,\mathrm{d}V = \int\limits_V \phi(\vec{x})\,(\vec{x}\cdot\vec{x}-1)\,\mathrm{d}V + \int\limits_S \phi(\vec{x})\,\vec{n}(\vec{x})\cdot(\vec{x}\vec{x}+\boldsymbol{I})\cdot\vec{\nabla}u \,\mathrm{d}S$$

c. $$\int\limits_V \Big(\vec{\nabla}\phi\cdot\vec{\nabla}u + \phi(\vec{x})\lambda u(\vec{x})\Big)\,\mathrm{d}V = \int\limits_S \phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

d. $$\int\limits_V \Big(\vec{\nabla}\phi\cdot\vec{\nabla}u - \vec{\nabla}\phi\cdot\vec{v}\,u(\vec{x})\Big)\,\mathrm{d}V = \int\limits_S \Big(\phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u - \phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{v}\,u(\vec{x})\Big)\,\mathrm{d}S$$

or

$$\int\limits_V \Big(\vec{\nabla}\phi\cdot\vec{\nabla}u + \phi(\vec{x})\,\vec{v}\cdot\vec{\nabla}u\Big)\,\mathrm{d}V = \int\limits_S \phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

e. $$\int\limits_V \nabla^2\phi\,\nabla^2 u \,\mathrm{d}V = \int\limits_S \Big(\vec{\nabla}\phi\cdot\vec{n}(\vec{x})\,\nabla^2 u - \phi(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}\big(\nabla^2 u\big)\Big)\,\mathrm{d}S$$

Exercise 2

a. $$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{d}V \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

b. $$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot(\vec{x}\vec{x}+\boldsymbol{I})\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{d}V \quad \underset{\sim}{\mathrm{f}} = \int\limits_{V^h} \underset{\sim}{\mathrm{N}}(\vec{x})\,(\vec{x}\cdot\vec{x}-1)\,\mathrm{d}V \quad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot(\vec{x}\vec{x}+\boldsymbol{I})\cdot\vec{\nabla}u \,\mathrm{d}S$$

c. $$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \Big(\vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}} + \underset{\sim}{\mathrm{N}}(\vec{x})\,\lambda\,\underset{\sim}{\mathrm{N}}^{\mathrm{T}}(\vec{x})\Big)\,\mathrm{d}V \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

d. $$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \Big(\vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}} - \vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{v}\,\underset{\sim}{\mathrm{N}}^{\mathrm{T}}(\vec{x})\Big)\,\mathrm{d}V \quad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \quad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \Big(\underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u - \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{v}\,\vec{u}(\vec{x})\Big)\,\mathrm{d}S$$

or

$$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \Big(\vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}} + \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{v}\cdot\vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\Big)\,\mathrm{d}V \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}u \,\mathrm{d}S$$

e. $$\underset{\approx}{\mathrm{K}} = \int\limits_{V^h} \nabla^2\underset{\sim}{\mathrm{N}}\,\nabla^2\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\,\mathrm{d}V \qquad \underset{\sim}{\mathrm{f}} = \underset{\sim}{0} \qquad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \Big(\vec{\nabla}\underset{\sim}{\mathrm{N}}\cdot\vec{n}(\vec{x})\,\nabla^2 u - \underset{\sim}{\mathrm{N}}(\vec{x})\,\vec{n}(\vec{x})\cdot\vec{\nabla}\big(\nabla^2 u\big)\Big)\,\mathrm{d}S$$

Exercise 3

a. $u(\vec{x}) = u_0$

c. $$\int\limits_V \sigma\,\vec{\nabla}\phi\cdot\vec{\nabla}u \,\mathrm{d}V = -\int\limits_V \phi(\vec{x})\,\rho g \,\mathrm{d}V + \int\limits_S \phi(\vec{x})\,q(\vec{x})\,\mathrm{d}S \quad \forall\phi(\vec{x})$$

d. $\underset{\sim}{\mathrm{K}} = \int\limits_{V^h} \sigma\, \vec{\nabla}\underset{\sim}{\mathrm{N}} \cdot \vec{\nabla}\underset{\sim}{\mathrm{N}}^{\mathrm{T}}\, \mathrm{d}V \qquad \underset{\sim}{\mathrm{f}} = -\int\limits_{V^h} \underset{\sim}{\mathrm{N}}(\vec{x})\, \rho g\, \mathrm{d}V \qquad \underset{\sim}{\mathrm{q}} = \int\limits_{S^h} \underset{\sim}{\mathrm{N}}(\vec{x})\, q(\vec{x})\, \mathrm{d}S$

e. $\dfrac{27}{2} \dfrac{Mg}{LW}$

Exercise 4

c. $\vec{e}_1 \cdot \vec{e}_1 = 1 \qquad \vec{e}_1 \cdot \vec{e}_2 = 0$

f. $\vec{x} \cdot \vec{u} = 0$

g. $\|\vec{u}\| \approx 2.24$

i. $\vec{u}\vec{v} = -4\,\vec{e}_1\vec{e}_1 + 8\,\vec{e}_1\vec{e}_2 - 2\,\vec{e}_2\vec{e}_1 + 4\,\vec{e}_2\vec{e}_2$

Exercise 5

a. $\boldsymbol{C} \cdot \underset{\sim}{\vec{\mathrm{a}}} = \begin{bmatrix} 7\vec{e}_1 - 4\vec{e}_2 \\ -\vec{e}_1 + 2\vec{e}_2 \\ 10\vec{e}_1 - 5\vec{e}_2 \end{bmatrix}$

c. A single scalar value

d. $\underset{\sim}{\vec{\mathrm{a}}}^{\mathrm{T}} \cdot \boldsymbol{C} \cdot \underset{\sim}{\vec{\mathrm{a}}} = 80$

e. $\underset{\sim}{\mathbf{D}} \cdot \underset{\sim}{\vec{\mathrm{a}}} = \begin{bmatrix} 7\vec{e}_1 - 4\vec{e}_2 \\ \vec{e}_1 - 2\vec{e}_2 \\ 2\vec{e}_1 - \vec{e}_2 \end{bmatrix}$