Exercises Computational Mechanics (4MC10) – Lecture 13

Exercise 1

We focus on the accuracy and stability properties of numerical time-integration methods for linear first-order ODEs, based on the simple mass-damper-spring system in Figure 1. The spring exerts a force on the mass that is proportional to the displacement, acting in the opposite direction. The damper exerts a force proportional to velocity, also acting in the opposite direction. The equation of motion for this system therefore reads:

$$m\ddot{x} = -kx - c\dot{x} \tag{1}$$

with m the mass, k the spring constant and c the damping coefficient. For convenience, we set $m = 1$. Equation (1) is supplemented with initial conditions

$$x(0) = \alpha_0, \qquad \dot{x}(0) = \alpha_1 \tag{2}$$

for given initial displacement α_0 and initial velocity α_1.

a. Define the velocity by $v = \dot{x}$. Write the equation of motion (1) and the initial condition (2) in first-order form.

b. Denoting by $\boldsymbol{A}$ the system matrix of the first-order system, derive the eigenvalues of $\boldsymbol{A}$.

c. Determine the values of k and c for which the system matrix $\boldsymbol{A}$ has: a) 2 real eigenvalues; b) 2 complex conjugate eigenvalues; c) 1 real eigenvalue with multiplicity 2.

d. For which values of the damping coefficient c is the ODE (1) stable?

e. What is the period of the undamped mass-spring system corresponding to (1) with $c = 0$?

Exercise 2

We consider a finite-difference approximation of the solution to the first-order system associated with (1), based on point-wise approximations $(x, v)(t_i) \approx (x, v)_i^h$ on a grid with points $t_i = ih$ $(i = 0, 1, 2, \ldots)$ with step size h. To condense the notation, we collect $(x, v)_i^h$ in a vector $\boldsymbol{q}_i^h = (x, v)_i^h$. The *forward Euler scheme* for the first-order system reads:

$$\frac{\boldsymbol{q}_{i+1}^h - \boldsymbol{q}_i^h}{h} = \boldsymbol{A}\boldsymbol{q}_i^h \qquad \text{for } i = 0, 1, 2, \ldots \tag{3}$$

a. Extract from (3) an explicit expression for $\boldsymbol{q}_{i+1}^h$ in terms of $\boldsymbol{q}_i^h$.

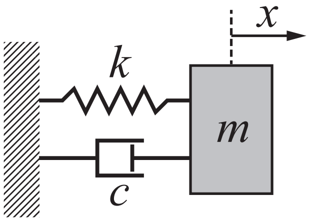

Figure 1: Illustration of the mass-damper-spring system

b. Is the forward Euler scheme (3) an explicit or an implicit scheme?

c. What is the amplification factor associated with (3)?

d. Download the `MATLAB` script `stability_region.m` and run it to plot the stability region of the forward Euler scheme.

e. Consider the eigenvalues $\lambda_{1,2}$ of the mass-damper-spring system for $k = 10$ and $c = 2$. Plot $h\lambda_{1,2}$ for $h = 2^{-1}, 2^{-2}, 2^{-3}$ in the stability-region plot of the forward Euler scheme.

f. For which of the above value(s) of the step size h is the forward Euler scheme stable? Explain!

g. Now consider the undamped case $k = 10$, $c = 0$. For which values of the step size h is the forward Euler scheme stable?

Exercise 3

We consider a finite-difference approximation of the solution to the first-order system associated with (1), based on the *trapezoidal method*:

$$\frac{\boldsymbol{q}_{i+1}^h - \boldsymbol{q}_i^h}{h} = \boldsymbol{A}\left(\frac{\boldsymbol{q}_{i+1}^h + \boldsymbol{q}_i^h}{2}\right) \qquad \text{for } i = 0, 1, 2, \ldots \tag{4}$$

a. Extract from (4) an explicit expression for $\boldsymbol{q}_{i+1}^h$ in terms of $\boldsymbol{q}_i^h$.

b. Is the trapezoidal method (4) an explicit or an implicit scheme?

c. What is the amplification factor associated with (3)?

d. Adapt the `MATLAB` script `stability_region.m` and run it to plot the stability region of the trapezoidal scheme.

e. Consider the eigenvalues $\lambda_{1,2}$ of the mass-damper-spring system for $k = 10$ and $c = 2$. Plot $h\lambda_{1,2}$ for $h = 2^{-1}, 2^{-2}, 2^{-3}$ in the stability-region plot of the trapezoidal scheme.

f. For which of the above value(s) of the step size h is the trapezoidal scheme stable? Explain!

g. Now consider the undamped case $k = 10$, $c = 0$. For which values of the step size h is the forward Euler scheme stable?

Exercise 4

Next, we verify the above theoretical findings numerically in `MATLAB`

a. Create a `MATLAB` script which applies the forward Euler scheme to the first-order form of the mass-damper-spring system.

b. Take $k = 10$, $c = 2$, $\alpha_0 = 0$, $\alpha_1 = 1$. Consider the time interval $(0, 10)$. Run the `MATLAB` script with stepsizes $h = 2^{-1}, 2^{-2}, 2^{-3}$. Plot the displacement versus time t. What do you observe?

c. Adapt your `MATLAB` script such that it applies the trapezoidal method instead of the forward Euler scheme.

d. Take $k = 10$, $c = 2$, $\alpha_0 = 0$, $\alpha_1 = 1$. Consider the time interval $(0, 10)$. Run the `MATLAB` script with stepsizes $h = 2^{-1}, 2^{-2}, 2^{-3}$. Plot the displacement versus time t. What do you observe?