Exercises Computational Mechanics (4MC10) – Lecture 2

Exercise 1

We consider a string of unit length which is subjected to a load $f(x)$ per unit length, and which is suspended at both ends. The vertical displacement $u(x)$ of the string is described by the differential equation

$$Lu := -T\,D^2u = f \qquad \text{on the interval } (0,1) \tag{1}$$

with T the tension in the string, subject to the boundary conditions

$$Bu := \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

For convenience, we take $T = 1$. A finite-difference approximation of the second-order derivative $D^2u(x)$ in (1) is provided by:

$$D_h^2 u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \tag{3}$$

a. Use Taylor expansions to prove that $D_h^2u(x)$ is an approximation of $D^2u(x)$ and establish the order of accuracy of $D_h^2u(x)$ as an approximation to $D^2u(x)$ (as $h \to 0$).

b. To determine the finite-difference approximation (3), it is convenient to write the relation between $u(x + jh)$ $(j = -1, 0, 1)$ and the derivatives $D^0u(x), \ldots, D^2u(x)$ in matrix form. Complete the following matrix form of the Taylor expansions (fill in the dots):

$$\begin{pmatrix} u(x-h) \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 1 & \cdot & 1 \\ \cdot & 0 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} u(x) \\ \cdot \\ D^2u(x)\,h^2/2 \end{pmatrix} + O\big(h^{(\cdot)}\big) \tag{4}$$

c. Determine the inverse of the matrix in problem 1b (use Maple or `MATLAB` or do it by hand) to find the expressions for the approximations to the derivatives.

d. Based on (4), one would expect a different order of accuracy of $D_h^2u(x)$ than derived in question 1a. What is the difference, and why does it occur?

e. What is the finite-difference approximation of the first-order derivative according to (4)?

Exercise 2

To construct a finite-difference approximation to (1)–(2), we divide the interval $(0,1)$ corresponding to the extent of the string into N subintervals of length $h = 1/N$. We approximate the value of $u(x_i)$ at $x_i = ih$ $(i = 0, 1, \ldots, N)$ by a number u_i^h. Upon inserting the approximation D_h^2 for D^2 in equation (1), we obtain for $i = 1, 2, \ldots, N-2, N-1$:

$$-\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} = f(x_i). \tag{5}$$

a. Write the equations at x_1 and x_2 in matrix form (fill in the dots).

$$-\frac{1}{h^2}\begin{pmatrix} \cdot & -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} u_0^h \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \cdot \end{pmatrix}$$

b. We use the boundary conditions to eliminate u_0^h. For u_0^h we simply obtain:

$$u_0^h = 0. \tag{6}$$

Use (6) to eliminate u_0^h from the equations at x_1 (fill in the dots).

$$-\frac{1}{h^2}\begin{pmatrix} -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} u_1^h \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \cdot \end{pmatrix}$$

c. The boundary condition at $x = 1$ can be used in a similar manner to eliminate u_N^h from the equation at x_{N-1}. Determine the corresponding equations at x_{N-2} and x_{N-1} (fill in the dots).

$$-\frac{1}{h^2}\begin{pmatrix} 0 & 1 & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ \cdot \\ u_{N-2}^h \\ u_{N-1}^h \end{pmatrix} = \begin{pmatrix} f(x_{N-2}) \\ \cdot \end{pmatrix}$$

d. Write down the system of equations (5) for $N = 7$ with u_0^h and u_N^h eliminated. That is, write the system for $u_1^h, u_2^h, \dots, u_6^h$, making use of (5) and the equations that you derived under 2b and 2c (fill in the dots).

$$\underbrace{-\frac{1}{h^2}\begin{pmatrix} -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}}_{=A^h}\begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \\ u_4^h \\ u_5^h \\ u_6^h \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \\ f(x_6) \end{pmatrix}$$

e. Write a `MATLAB` script that generates the above matrix A^h, but now for variable N and step size $h = 1/N$.

The finite-difference operator L^h corresponding to the matrix A^h is given by

$$(L^h u^h)_i = \sum_{i=1}^{N-1} A_{ij}^h u_j^h \qquad \text{for } i = 1, 2, \dots, N-1 \tag{7}$$

Note that L^h acts only on the interior grid points, i.e., on $(u_1^h, u_2^h, \dots, u_{N-1}^h)$. To test the consistency and accuracy of the finite-difference scheme L^h and the correctness of the implementation of the corresponding matrix A^h, we apply the *method of manufactured solutions*. We use the sample function:

$$u(x) = e^{2x}\sin(2\pi x) \tag{8}$$

Note that (8) satisfies the boundary conditions (2).

h	$\left\|I^h(Lu)) - L^h(I^h u)\right\|_{\text{RMS}}$	$\dfrac{\left\|I^h(Lu)) - L^h(I^h u)\right\|_{\text{RMS}}}{\left\|I^{2h}(Lu)) - L^{2h}(I^{2h} u)\right\|_{\text{RMS}}}$
2^{-3}	5.961315018777608e+00	–
2^{-4}	1.655710671509594e+00	2.777425226303684e-01
2^{-5}		
2^{-6}		
2^{-7}		
2^{-8}		
2^{-9}		

Table 1: RMS value according to (12) for $h = 2^{-3}, 2^{-4}, \ldots, 2^{-9}$ and the ratio between consecutive RMS values.

f. Determine the second-order derivative $D^2u(x)$ of $u(x)$ in (8).

The injection $I^h v$ of any function $v(x)$ into the grid points is defined by $(I^h v)_i = v(x_i)$. If the finite-difference scheme L^h is consistent with the differential operator L in (1) it must hold that

$$I^h(Lu) - (L^h(I^h u)) = o(1) \quad \text{as } h \to 0; \tag{9}$$

for all admissible functions v; see the reader. In particular, (9) must hold for the sample function (8).

g. Show that (9) applied to the sample function $u(x)$, can be expressed in terms of the matrix A^h and the injection of the sample function, $I^h u$, and the injection of the negative of its second-order derivative, $I^h(Lu) = I^h(-D^2u)$, as:

$$(I^h(Lu))_i - \sum_{j=1}^{N-1} A^h_{ij}(I^h u)_j = o(1) \quad \text{as } h \to 0; \tag{10}$$

for $i = 1, 2, \ldots, N-1$.

h. Show that for any function v subject to the boundary conditions $v(0) = 0$ and $v(1) = 0$ it holds that:

$$(I^h(Lv))_i - (L^h(I^h v))_i = O(h^2) \quad \text{as } h \to 0 \tag{11}$$

i. To verify (11), compute the RMS value of $I^h(Lu)) - L^h(I^h u)$ for the sample function u, according to

$$\left\|I^h(Lu)) - L^h(I^h u)\right\|_{\text{RMS}} = \left(\frac{1}{N-1}\sum_{i=1}^{N-1}\Big((I^h(Lu))_i - (L^h(I^h u))_i\Big)^2\right)^{1/2} \tag{12}$$

for $h = 2^{-3}, \ldots, 2^{-9}$ and determine the ratio between consecutive RMS values. Complete Table 1.

j. How do the results in Table 1 confirm that the order of the finite-difference scheme L^h as an approximation to L is 2?

Exercise 3

Now that the finite-difference scheme and its implementation have been verified, we can proceed to use it to actually solve a string problem. Let us consider a vertically upward load per unit length given by

$$f(x) = 10^2 \, e^{-1000(x-(1/2))^2} \tag{13}$$

a. Implement the right-hand-side vector corresponding to (13) with components $f_i^h = f(x_i)$ in the `MATLAB` script. To obtain the corresponding displacement, we solve the system $A \cdot U = b$. In `MATLAB` this can be done by using the backslash operator, `>> U=A\b`. For $h = 2^{-9}$, plot u_i^h versus x_i.

b. The load $f(x)$ in (13) corresponds to a load that is concentrated in the middle of the string, pushing upward. This can be verified by plotting $f(x)$. Verify that the computed displacement of the string in the previous question makes sense.

c. Determine the displacement of the midpoint of the string for $h = 2^{-3}, \ldots, 2^{-9}$.

d. The exact value of the midpoint displacement up to 16 Digits is $u(1/2) = 1.351247804099482$. Determine the error in the approximate midpoint displacement for $h = 2^{-3}, \ldots, 2^{-9}$.

e. Verify that the order of accuracy of the approximation to the midpoint displacement is $p = 2$.

f. We shall now use Richardson extrapolation to determine an improved approximation of the midpoint displacement based on the computed displacements. Denoting the midpoint displacement obtained at mesh width h by $\hat{u}^h$, determine coefficients α and β such that $\alpha \hat{u}^h + \beta \hat{u}^{2h} = u(1/2) + O(h^3)$ (as $h \to 0$)

g. Use Richardson extrapolation to determine an improved approximation of $u(1/2)$, based on $\hat{u}^h$ for $h = 2^{-8}, 2^{-9}$. What is the error in the displacement obtained by Richardson extrapolation and how does it compare to the errors of $\hat{u}^h$ for $h = 2^{-8}, 2^{-9}$?