Exercises Computational Mechanics (4MC10) – Lecture 1

Exercise 1

Are the following statements true or false?

a. $10^{100}h = O(h)$ as $h \to 0$

b. $|\ln(|h|)|\, h = O(h)$ as $h \to 0$

c. If $f(h) = O(h^p)$ and $g(h) = O(h^{p+1})$ then $f(h) + g(h) = O(h^p)$ as $h \to 0$

d. If $f(h) = O(h^p)$ and $g(h) = O(h^{p+1})$ then $f(h) + g(h) = O(h^{p+1})$ as $h \to 0$

e. If $\lim_{h\to 0} f(h) = 0$ then $f(h) = o(1)$ as $h \to 0$

f. If $f(h) = o(1)$ as $h \to 0$ then $\lim_{h\to 0} f(h) = 0$

g. $e^h - (1 + h + h^2/2) = O(h^3)$ as $h \to 0$

Exercise 2

Consider the functions $f(x) = e^x + e^{-x} + \sin(x)$ and $g(x) = \cos(x) + x^{1/2}\sin(x)$.

a. Determine the Taylor polynomial $p_4(x)$ of degree 4 of $f(x)$ at $x = 0$

b. Determine the order of approximation of $p_4(x)$ as an approximation to $f(x)$ near $x = 0$, i.e., what is the largest number p such that $f(x) - p_4(x) = O(x^p)$ as $x \to 0$

c. What is the Taylor polynomial $p_1(x)$ of degree 1 of $g(x)$ near $x = 0$?

d. What is the Taylor polynomial $p_2(x)$ of degree 2 of $g(x)$ near $x = 0$?

Exercise 3

The finite-difference schemes

$$D_h u(x) = \frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} \tag{1}$$

and

$$\tilde{D}_h u(x) = \frac{u(x+h) - u(x)}{h} \tag{2}$$

represent approximations to the first-order derivative $Du(x)$.

a. Show by means of Taylor-series expansions that $D_h u(x)$ according to (1) provides an approximation to $Du(x)$ with error $O(h^2)$ as $h \to 0$.

b. Show by means of Taylor-series expansions that $\tilde{D}_h u(x)$ according to (2) provides an approximation to $Du(x)$ with error $O(h)$ as $h \to 0$.

c. Which approximation to the first-order derivative Du is more accurate (for small step size h): $D_h u$ or $\tilde{D}_h u$?

d. Take the sample function $u(x) = e^x \cos(2x) + \sin(\sin(2x))$. Determine $Du(0)$ and complete the table below (Hint: use Matlab).

h	$\lvert D_h u(0) - Du(0)\rvert$	$\lvert \tilde{D}_h u(0) - Du(0)\rvert$
2^0	$2.426059258e+00$	$4.342132040e+00$
2^{-1}	$8.8786022402e-01$	$1.727135908e+00$
2^{-2}		
2^{-3}		
2^{-4}		
2^{-5}		
2^{-6}		

e. How do the results in the above table support the claim that the orders of approximation of $D_h u(x)$ according to (1) and of $\tilde{D}_h u(x)$ according to (2) as an approximation to $Du(x)$ are $O(h^2)$ and $O(h)$, respectively, as $h \to 0$?

f. Make a plot of $\log_2 |D_h u(0) - Du(0)|$ versus $\log_2 |h|$ for $h = 2^0, 2^{-1}, \ldots, 2^{-6}$. How does the plot convey that $|D_h u(x) - Du(x)| = O(h^2)$ as $h \to 0$?

Exercise 4

For any sufficiently smooth function, the finite-difference scheme $D_h^n u(x)$ in (3) yields an approximation to the nth-order derivative $D^n u(x)$ of $u(x)$:

$$D_h^n u(x) = \frac{u(x-3h) - 4u(x-2h) + 6u(x-h) - 4u(x) + u(x+h)}{h^4} \tag{3}$$

a. What is n ?

b. What is the order of approximation provided by (3) as an approximation to $D^n u(x)$ as $h \to 0$, i.e. what is the largest number p such that $|D_h^n u(x) - D^n u(x)| = O(h^p)$ as $h \to 0$?