Exercises Computational Mechanics (4MC10) – Lecture 8

⋆ **Exercise 1**

Derive the element shape functions $\underset{\sim}{N}^e(\xi)$ for an isoparametric cubic element.

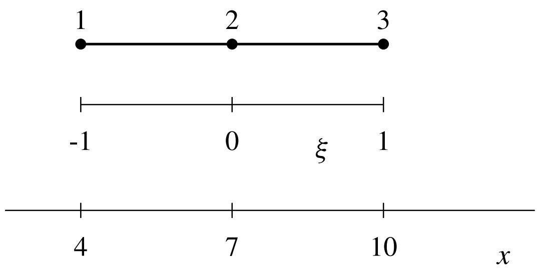
⋆ **Exercise 2**

We consider the single, quadratic isoparametric element depicted in Figure 1. The shape functions associated with this element read

$$\underset{\sim}{N}^e(\xi) = \begin{bmatrix} -\frac{1}{2}(1-\xi)\xi \\ (1-\xi)(1+\xi) \\ \frac{1}{2}\xi(1+\xi) \end{bmatrix}$$

The element is part of a bigger, physical mesh. In terms of the global coordinate system associated with this mesh, the element nodes are situated at positions (see also the figure)

$$\underset{\sim}{x}^e = \begin{bmatrix} 4 & 7 & 10 \end{bmatrix}^{\mathrm{T}}$$



Figuur 1: Second-order isoparametric element; ξ denotes the local coordinate, whereas x is the global coordinate.

a. Determine the shape function derivatives with respect to the local coordinate, $\mathrm{d}\underset{\sim}{N}^e/\mathrm{d}\xi$.

b. Write down the isoparametric mapping $x(\xi)$.

c. Compute the Jacobian $J(\xi)$ as a function of ξ. Could you have predicted the result?

d. Compute the shape function derivatives with respect to the global coordinate, $\mathrm{d}\underset{\sim}{N}^e/\mathrm{d}x$, as a function of ξ.

e. Compute the integral

$$I = \int_4^{10} x \,\mathrm{d}x$$

f. Rewrite the integral I in terms of ξ using the isoparametric transformation. Verify that evaluating this integral gives the same result as above.

★ **Exercise 3**

A function $g(\xi)$ is given on the domain $(-1, 1)$ by

$$g(\xi) = \xi^2(1 - \xi^2)$$

We denote the exact integral on the entire domain by

$$I = \int_{-1}^{1} g(\xi)\,\mathrm{d}\xi$$

and approximations of it obtained by Gauss quadrature by I_s, where s denotes the number of Gauss points that is used.

a. Compute the integral I analytically.

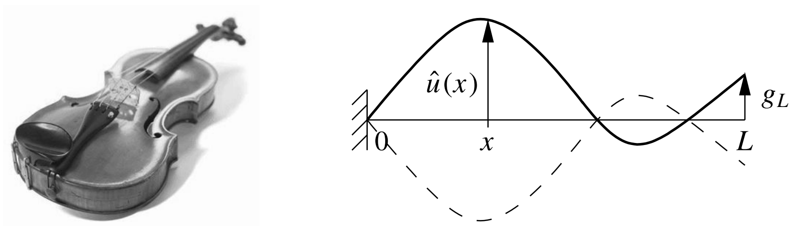
b. Determine the approximations I_1, I_2 and I_3 given by the one, two and three-point Gauss schemes. For which s does the result become exact? Does this agree with Table 2.1 in the lecture notes?

★ **Exercise 4**

The vibration of strings, for instance in a musical instrument such as the violin in Figure 2, is governed by the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

In this equation, x is a spatial coordinate along the string and t denotes the time; $u(x, t)$ denotes the transversal displacement and c is the wave velocity in the string (see also Figure 2).



$L = 400$ mm
$g_L = 1$ mm
$\lambda = 50$ mm

Figuur 2: String instrument (left), schematic representation of a single vibrating string (centre) and parameter values for the numerical implementation (right)

If we assume the vibration to be harmonic in time, the displacement $u(x, t)$ can be written as

$$u(x, t) = \hat{u}(x)\cos(\omega t)$$

with ω a given (angular) frequency and $\hat{u}(x)$ a function which gives the vibration amplitude at each position x. After substitution of this relation, the wave equation can be rewritten as:

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} + \frac{1}{\lambda^2}\,\hat{u}(x) = 0$$

with the wavelength λ defined as $\lambda = c/\omega$. This equation is known as the one-dimensional Helmholtz equation.

We consider the part of the string between its fixation on the left (at $x = 0$) and the excitation by the pole on the right ($x = L$). This excitation is assumed to take the form of a harmonic displacement with amplitude $\hat{u}(L) = g_L$ where g_L is a known constant.

a. Derive the weak form of the Helmholtz equation on the interval $(0, L)$.

b. Show that discretising the weak form according to Galerkin's method results in a linear system of the form

$$\underset{\sim}{\mathrm{K}}\,\underset{\sim}{\hat{\mathrm{u}}} = \underset{\sim}{\mathrm{q}}$$

Determine expressions for $\underset{\sim}{\mathrm{K}}$ and $\underset{\sim}{\mathrm{q}}$.

c. Write down the expression for the element matrix of an element e and rewrite it in terms of a local coordinate ξ using the isoparametric transformation.

d. Replace the integral in the expression obtained above by Gauss quadrature using s quadrature points.

e. Implement the numerically integrated element matrix for a discretisation by m uniform linear elements and a two-point quadrature rule in a MATLAB program `violinstringlinear`. Use the parameter values as given in Figure 2. Check your code for a single element.

f. Add the assembly, partitioning and solving to your program and generate numerical solutions for different numbers of elements. Compare these numerical solutions with the exact solution

$$\hat{u}(x) = g_L \frac{\sin(x/\lambda)}{\sin(L/\lambda)}$$

g. Examine the influence of using a more accurate or a less accurate integration rule.

Exercise 5

We are now going to replace the linear elements used in the previous exercise by quadratic elements.

a. Copy the file `violinstringlinear.m` to `violinstringquadratic.m` and replace the linear shape functions by their quadratic counterparts; use two integration points per element.

b. Generate numerical solutions for different numbers of elements and compare them with the exact solution. Which type of elements do you consider to be more efficient for this problem, linear or quadratic?

c. Examine the influence of using a more accurate or a less accurate integration rule. What happens if you use only one integration point?

d. If the boundary conditions of the problem are changed such that a free-standing string is obtained, *i.e.* by applying two natural boundary conditions ($q_0 = q_L = 0$), can you still generate a numerical solution? Compare your answer with that to Exercise 1.h of Lecture 5. What is different here?

Answers

Exercise 1

$$\underset{\sim}{N}^e(\xi) = \begin{bmatrix} -\frac{9}{16}(\xi+\frac{1}{3})(\xi-\frac{1}{3})(\xi-1) \\ \frac{27}{16}(\xi+1)(\xi-\frac{1}{3})(\xi-1) \\ -\frac{27}{16}(\xi+1)(\xi+\frac{1}{3})(\xi-1) \\ \frac{9}{16}(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3}) \end{bmatrix}$$

Exercise 2

a. $\dfrac{\mathrm{d}\underset{\sim}{N}^e}{\mathrm{d}\xi} = \begin{bmatrix} \xi-\frac{1}{2} \\ -2\xi \\ \xi+\frac{1}{2} \end{bmatrix}$

b. $x = 7 + 3\xi$

c. $J(\xi) = 3$

d. $\dfrac{\mathrm{d}\underset{\sim}{N}^e}{\mathrm{d}x} = \dfrac{1}{3}\begin{bmatrix} \xi-\frac{1}{2} \\ -2\xi \\ \xi+\frac{1}{2} \end{bmatrix}$

e. $I = 42$

f. $I = \displaystyle\int_{-1}^{1} (7+3\xi)3\,\mathrm{d}\xi$

Exercise 3

a. $I = \frac{4}{15}$

b. $I_1 = 0 \qquad I_2 = \frac{4}{9} \qquad I_3 = \frac{4}{15}$

Exercise 4

a. $\displaystyle\int_0^L \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}\hat{u}}{\mathrm{d}x} - \frac{1}{\lambda^2}\phi(x)\,\hat{u}(x)\right)\mathrm{d}x = \phi(x)\frac{\mathrm{d}\hat{u}}{\mathrm{d}x}\bigg|_0^L$

b. $\underset{\sim}{K} = \displaystyle\int_0^L \left(\frac{\mathrm{d}\underset{\sim}{N}}{\mathrm{d}x}\frac{\mathrm{d}\underset{\sim}{N}^{\mathrm{T}}}{\mathrm{d}x} - \frac{1}{\lambda^2}\underset{\sim}{N}(x)\,\underset{\sim}{N}^{\mathrm{T}}(x)\right)\mathrm{d}x$

$\underset{\sim}{q} = \begin{bmatrix} q_0 & 0 & \dots & 0 & q_L \end{bmatrix}^{\mathrm{T}}$ with $q_0 = -\dfrac{\mathrm{d}\hat{u}}{\mathrm{d}x}\bigg|_{x=0} \qquad q_L = \dfrac{\mathrm{d}\hat{u}}{\mathrm{d}x}\bigg|_{x=L}$

c. $\underset{\sim}{K}^e = \displaystyle\int_{x_1^e}^{x_2^e} \left(\frac{\mathrm{d}\underset{\sim}{N}^e}{\mathrm{d}x}\frac{\mathrm{d}\underset{\sim}{N}^{e\mathrm{T}}}{\mathrm{d}x} - \frac{1}{\lambda^2}\underset{\sim}{N}^e(x)\,\underset{\sim}{N}^{e\mathrm{T}}(x)\right)\mathrm{d}x = \int_{-1}^{1} \left(\frac{\mathrm{d}\underset{\sim}{N}^e}{\mathrm{d}x}\frac{\mathrm{d}\underset{\sim}{N}^{e\mathrm{T}}}{\mathrm{d}x} - \frac{1}{\lambda^2}\underset{\sim}{N}^e(\xi)\,\underset{\sim}{N}^{e\mathrm{T}}(\xi)\right)J(\xi)\,\mathrm{d}\xi$

d. $\underset{\sim}{K}^e = \displaystyle\sum_{k=1}^{s} w_k \left(\frac{\mathrm{d}\underset{\sim}{N}^e}{\mathrm{d}x}\bigg|_{\xi=\xi_k}\frac{\mathrm{d}\underset{\sim}{N}^{e\mathrm{T}}}{\mathrm{d}x}\bigg|_{\xi=\xi_k} - \frac{1}{\lambda^2}\underset{\sim}{N}^e(\xi_k)\,\underset{\sim}{N}^{e\mathrm{T}}(\xi_k)\right)J(\xi_k)$

Exercise 5

b. Quadratic

c. Inaccurate solution; system matrix $\underline{K}$ becomes singular

d. Yes; this is a result of the fact that $\hat{u}(x)$ itself appears in the differential equation, next to its second-order derivative