Exercises Computational Mechanics (4MC10) – Lecture 10

⋆ **Exercise 1**

The following fragment of MATLAB/TENSORLAB code constructs and plots a single, bilinear master element Q:

```
e  = cartesianbasis2d('e1', 'e2');
e1 = e(1);
e2 = e(2);

xie = [ -e1 - e2
         e1 - e2
         e1 + e2
        -e1 + e2 ];
Q   = [ 1  2  3  4 ];

femplot(xie, Q, 'Nodes', 'on')
```

a. Write down the shape functions $\underset{\sim}{N}^e(\vec{\xi})$ of the element.

b. Compute the value of the shape functions in the origin O.

c. Compute the value of the shape functions in a point P given by $\vec{\xi}_P = \frac{1}{2}\vec{e}_1 + \vec{e}_2$.

The shape function values $\underset{\sim}{N}^e(\vec{\xi})$ in a point $\vec{\xi}$, for instance $\vec{\xi} = \vec{\xi}_O = \vec{0}$, can be determined in MATLAB using the TENSORLAB statements

```
xi  = 0*e1 + 0*e2

xi1 = dot(xi, e1);
xi2 = dot(xi, e2);

Ne  = [ 1/4*(1-xi1)*(1-xi2)
        1/4*(1+xi1)*(1-xi2)
        1/4*(1+xi1)*(1+xi2)
        1/4*(1-xi1)*(1+xi2) ]
```

d. Add these statements to the above program and verify that they give the correct values in O and in P, as well as in the four nodes of the element.

e. Determine the shape function gradients $\vec{\nabla}_\xi \underset{\sim}{N}^e$ as a function of $\vec{\xi}$ by (manually) differentiating the shape functions as defined above.

f. Compute $\vec{\nabla}_\xi \underset{\sim}{N}^e$ in O and in P.

g. Add the computation of the shape function gradients as derived above to the MATLAB/TENSORLAB program and use it to reproduce the analytical results obtained above.

h. Use the program to compute the shape function gradients with respect to $\vec{\xi}$ in all four nodes $\vec{\xi}_i$ of the element.

⋆ **Exercise 2**

In this exercise we study the interpolation properties of a single quadratic master element of the Lagrange family.

a. Define and plot a bi-quadratic master element Q_{L} in the same fashion as in Exercise 1. Add also the computation of the shape functions.

b. Use your program to verify that the shape functions satisfy the requirement $N_i^e(\vec{\xi}_j) = \delta_{ij}$.

c. In particular, compute the values of the shape functions at the centre of the element, i.e. at $\vec{\xi} = \vec{\xi}_O = \vec{0}$.

In the following we use the bi-quadratic shape functions to interpolate nodal values $\underset{\sim}{u}^e$ given by

$$\underset{\sim}{u}^e = \begin{bmatrix} 11 & 14 & 12 & 17 & 15 & 13 & 18 & 16 & 19 \end{bmatrix}^{\mathrm{T}}$$

according to

$$u^h(\vec{\xi}) = \underset{\sim}{N}^{e\mathrm{T}}(\vec{\xi})\, \underset{\sim}{u}^e$$

d. Use your program to compute the value of $u^h(\vec{\xi})$ at the centre of the element, $u^h(\vec{0})$. Could you have predicted this value?

e. Compute $u^h(\vec{\xi}_P)$, where $\vec{\xi}_P = \frac{1}{2}\vec{e}_1 + \vec{e}_2$.

Exercise 3

Repeat Exercise 2, but now for a single eight-node serendipity element Q_{S}. Use only the first eight nodal values of $\underset{\sim}{u}^e$. Compare your results to those obtained for the Lagrange element; explain the differences and similarities.

Exercise 4

The simple finite element discretisation of Figure 1 consists of two elements which form a rectangular region in terms of the global coordinates x_1 and x_2. The element on the left, indicated by L, is a four-node linear isoparametric element; the right element, R, is an eight-node quadratic isoparametric element. The (global) positions of the nodes, as well as the node numbering, are indicated in the figure.
We consider an approximate solution $u^h(\vec{x})$ which is defined as

$$u^h(\vec{x}) = \underset{\sim}{N}^{\mathrm{T}}(\vec{x})\, \underset{\sim}{u}$$

where the column matrix $\underset{\sim}{N}(\vec{x})$ contains the shape functions associated with Nodes 1–10 and $\underset{\sim}{u}$ the nodal values.

a. Write down the shape functions, in local coordinates, of Element L, $\underset{\sim}{N}^L(\vec{\xi})$. Also give the corresponding nodal positions $\underset{\sim}{\vec{x}}^L$.

b. Based on these expressions, show that the isoparametric map for Element L reads

$$\vec{x} = \vec{\xi} + \vec{e}_1 + \vec{e}_2$$

c. Show that on the right edge of Element L, i.e. for $x_1 = 2$, the approximate solution $u^h(\vec{x})$ can be written in terms of x_2 as

$$u^h(\vec{x}) = u^{hL}(x_2) = \tfrac{1}{2}(2 - x_2)\, u_3 + \tfrac{1}{2} x_2\, u_5$$

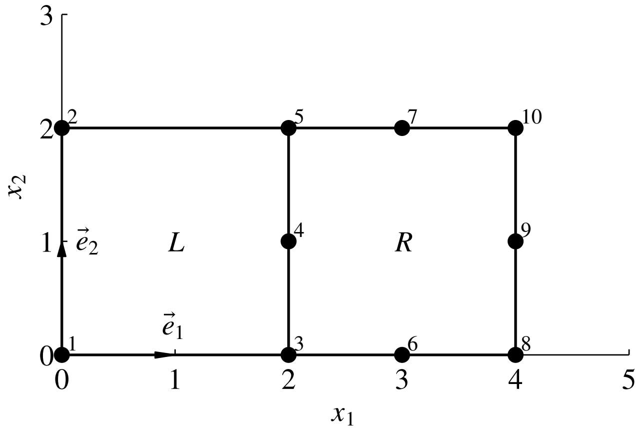

Figure 1: Finite element discretisation consisting of a bilineair element (L) and a quadratic element of the serendipity family (R); the numbers to the top-right of each node are the global node numbers

d. Derive the corresponding expression for the left edge of Element R, $u^{hR}(x_2)$.

e. Use the results obtained in Questions c. and d. to explain why this combination of the two types of elements is problematic.

f. Demonstrate that this problem may be removed by enforcing in the computation that

$$u_4 = \frac{1}{2}(u_3 + u_5)$$

⋆ Exercise 5

Consider the two-dimensional isoparametric master element of the Lagrange family as sketched in Figure 2. As indicated in the figure, this element has 16 nodes.

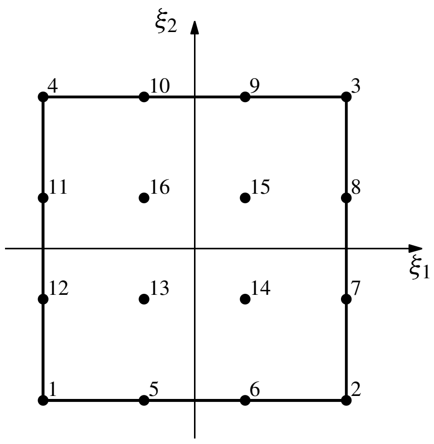

Figure 2: 16-node isoparametric element of the Lagrange family; the node numbers are indicated

a. Give the local coordinate vector $\vec{\xi}_4$ which indicates the position of node 4.

b. Give the coordinate vector $\vec{\xi}_{16}$ for node 16.

c. Write down the shape function $N_4(\vec{\xi})$ in terms of the local coordinates ξ_1 and ξ_2.

d. Write down shape function $N_{16}(\vec{\xi})$.

e. Plot $N_4(\vec{\xi})$ and $N_{16}(\vec{\xi})$ as functions of ξ_1 along the top edge of the element, i.e. for $\xi_2 = 1$. Verify their values in nodes 4, 3, 9 and 10.

f. Make a similar plot of $N_4(\vec{\xi})$ and $N_{16}(\vec{\xi})$ for $\xi_2 = \frac{1}{3}$.

Answers

Exercise 1

a. $\underset{\sim}{N}^e(\vec{\xi}) = \begin{bmatrix} \frac{1}{4}(1-\xi_1)(1-\xi_2) \\ \frac{1}{4}(1+\xi_1)(1-\xi_2) \\ \frac{1}{4}(1+\xi_1)(1+\xi_2) \\ \frac{1}{4}(1-\xi_1)(1+\xi_2) \end{bmatrix}$

b. $\underset{\sim}{N}^e(\vec{0}) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^{\mathrm{T}}$

c. $\underset{\sim}{N}^e(\vec{\xi}_P) = \begin{bmatrix} 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}^{\mathrm{T}}$

e. $\vec{\nabla}_\xi \underset{\sim}{N}^e = \begin{bmatrix} -\frac{1}{4}(1-\xi_2)\vec{e}_1 - \frac{1}{4}(1-\xi_1)\vec{e}_2 \\ \frac{1}{4}(1-\xi_2)\vec{e}_1 - \frac{1}{4}(1+\xi_1)\vec{e}_2 \\ \frac{1}{4}(1+\xi_2)\vec{e}_1 + \frac{1}{4}(1+\xi_1)\vec{e}_2 \\ -\frac{1}{4}(1+\xi_2)\vec{e}_1 + \frac{1}{4}(1-\xi_1)\vec{e}_2 \end{bmatrix}$

f. $\vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{0}) = \begin{bmatrix} -\frac{1}{4}\vec{e}_1 - \frac{1}{4}\vec{e}_2 \\ \frac{1}{4}\vec{e}_1 - \frac{1}{4}\vec{e}_2 \\ \frac{1}{4}\vec{e}_1 + \frac{1}{4}\vec{e}_2 \\ -\frac{1}{4}\vec{e}_1 + \frac{1}{4}\vec{e}_2 \end{bmatrix}$ $\quad \vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{\xi}_P) = \begin{bmatrix} -\frac{1}{8}\vec{e}_2 \\ -\frac{3}{8}\vec{e}_2 \\ \frac{1}{2}\vec{e}_1 + \frac{3}{8}\vec{e}_2 \\ -\frac{1}{2}\vec{e}_1 + \frac{1}{8}\vec{e}_2 \end{bmatrix}$

h. $\vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{\xi}_1) = \begin{bmatrix} -\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2 \\ \frac{1}{2}\vec{e}_1 \\ \vec{0} \\ \frac{1}{2}\vec{e}_2 \end{bmatrix}$ $\quad \vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{\xi}_2) = \begin{bmatrix} -\frac{1}{2}\vec{e}_1 \\ \frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2 \\ \frac{1}{2}\vec{e}_2 \\ \vec{0} \end{bmatrix}$

$\vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{\xi}_3) = \begin{bmatrix} \vec{0} \\ -\frac{1}{2}\vec{e}_2 \\ \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 \\ -\frac{1}{2}\vec{e}_1 \end{bmatrix}$ $\quad \vec{\nabla}_\xi \underset{\sim}{N}^e(\vec{\xi}_4) = \begin{bmatrix} -\frac{1}{2}\vec{e}_2 \\ \vec{0} \\ \frac{1}{2}\vec{e}_1 \\ -\frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 \end{bmatrix}$

Exercise 2

c. $\underset{\sim}{N}^e(\vec{0}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$

d. $u^h(\vec{0}) = u_9 = 19$

e. $u^h(\vec{\xi}_P) = 15.875.$

Exercise 3

c. $\underset{\sim}{N}^e(\vec{0}) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{\mathrm{T}}$

d. $u^h(\vec{0}) = 17.5$

e. $u^h(\vec{\xi}_P) = 15.875.$

Exercise 4

a. $\underset{\sim}{\mathrm{N}}^L(\vec{\xi}) = \begin{bmatrix} \frac{1}{4}(1-\xi_1)(1-\xi_2) \\ \frac{1}{4}(1+\xi_1)(1-\xi_2) \\ \frac{1}{4}(1+\xi_1)(1+\xi_2) \\ \frac{1}{4}(1-\xi_1)(1+\xi_2) \end{bmatrix}$ $\qquad$ $\underset{\sim}{\vec{\mathrm{x}}}^L = \begin{bmatrix} \vec{0} \\ 2\vec{e}_1 \\ 2\vec{e}_1 + 2\vec{e}_2 \\ 2\vec{e}_2 \end{bmatrix}$

d. $u^{hR}(x_2) = -\frac{1}{2}(x_2-1)(2-x_2)\,u_3 + x_2(2-x_2)\,u_4 + \frac{1}{2}x_2(x_2-1)\,u_5$

e. A discontinuity may occur in $u^h(\vec{x})$.

Exercise 5

a. $\vec{\xi}_4 = -\vec{e}_1 + \vec{e}_2$.

b. $\vec{\xi}_{16} = -\frac{1}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2$.

c. $N_4(\vec{\xi}) = -\frac{81}{256}\,(\xi_1+\frac{1}{3})(\xi_1-\frac{1}{3})(\xi_1-1)\,(\xi_2+1)(\xi_2+\frac{1}{3})(\xi_2-\frac{1}{3})$

d. $N_{16}(\vec{\xi}) = -\frac{729}{256}\,(\xi_1+1)(\xi_1-\frac{1}{3})(\xi_1-1)\,(\xi_2+1)(\xi_2+\frac{1}{3})(\xi_2-1)$