Exercises Computational Mechanics (4MC10) – Lecture 3

Exercise 1

We consider a string of unit length which is subjected to a load $f(x)$ per unit length, and which is free at both ends. The tension in the string is normalized to 1. The vertical displacement $u(x)$ of the string is then described by the differential equation

$$Lu := -D^2u = f \qquad \text{on the interval } (0,1) \tag{1}$$

The boundary conditions are of so-called *Neumann* type:

$$Bu = g \tag{2}$$

with

$$Bu := \begin{pmatrix} Du(0) \\ Du(1) \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

Given a grid with point x_i ($i = 0, 1, 2, \ldots, N$) and step size $h = 1/N$, in conjunction with (1) and (2), we consider the finite-difference scheme and discrete boundary conditions:

$$L^h u^h = f^h \tag{4a}$$

$$B^h u^h = g^h \tag{4b}$$

with

$$(L^h u^h)_i = -\frac{u^h_{i+1} - 2u^h_i + u^h_{i-1}}{h^2} \qquad \text{for } i = 1, 2, \ldots, N \tag{5a}$$

$$B^h u^h = \begin{cases} -\dfrac{3u^h_i - 4u^h_{i+1} + u^h_{i+2}}{2h} & \text{for } i = 0 \\ \dfrac{3u^h_i - 4u^h_{i-1} + u^h_{i-2}}{2h} & \text{for } i = N \end{cases} \tag{5b}$$

and

$$f^h_i = f(x_i) \qquad \text{for } i = 1, 2, \ldots, N \tag{6a}$$

$$g^h = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6b}$$

a. Show that Equations (4)–(6) yield a consistent approximation of the boundary value problem (1)–(3).

b. What is the order of approximation of the discrete equations (4)–(6) as and approximation to (1)–(3)?

c. Write a `MATLAB` script that generates the matrix A^h corresponding to L^h and B^h,

$$A^h = \begin{pmatrix} -\frac{3}{2h} & \frac{2}{h} & -\frac{1}{2h} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{2h} & -\frac{2}{h} & \frac{3}{2h} \end{pmatrix} \tag{7}$$

for variable N and step size $h = 1/N$.

h	$\left\|I^h(Lu)) - L^h(I^h u)\right\|_{\mathrm{RMS}}$	$\dfrac{\left\|I^h(Lu)) - L^h(I^h u)\right\|_{\mathrm{RMS}}}{\left\|I^{2h}(Lu)) - L^{2h}(I^{2h} u)\right\|_{\mathrm{RMS}}}$
2^{-3}	6.507537666286199e-01	–
2^{-4}	1.724009332947653e-01	2.649249872005016e-01
2^{-5}		
2^{-6}		
2^{-7}		
2^{-8}		
2^{-9}		

Table 1: RMS value of $I^h(Lu) - L^h(I^h u)$ for $h = 2^{-3}, 2^{-4}, \ldots, 2^{-9}$ and the ratio between consecutive RMS values.

h	$\left\|B^h(I^h u)\right\|_{\mathrm{RMS}}$	$\dfrac{\left\|B^h(I^h u)\right\|_{\mathrm{RMS}}}{\left\|B^{2h}(I^{2h} u)\right\|_{\mathrm{RMS}}}$
2^{-3}	3.431457505076201e-01	–
2^{-4}	4.635460456560381e-02	1.350872173035243e-01
2^{-5}		
2^{-6}		
2^{-7}		
2^{-8}		
2^{-9}		

Table 2: RMS value of $B^h(I^h u)$ for $h = 2^{-3}, 2^{-4}, \ldots, 2^{-9}$ and the ratio between consecutive RMS values.

d. Use the method of manufactured solutions to verify the correctness of the `MATLAB` implementation. Use sample function
$$u(x) = \left(\frac{1 - \cos(2\pi x)}{2}\right)$$
and compute the RMS values
$$\|I^h(Lu) - L^h(I^h u)\|_{\mathrm{RMS}}, \qquad \|B^h(I^h u)\|_{\mathrm{RMS}} \tag{8}$$
(note that $Bu = 0$). Complete Tables 1 and 2.

Exercise 2

To illustrate that (4)–(6) is unstable, we attempt to solve a problem corresponding to $f = 1$.

a. Implement the right-hand-side vector corresponding to $f = 1$ in the `MATLAB` script. To obtain the corresponding displacement, we solve the system $A \cdot U = b$. In `MATLAB` this can be done by using the backslash operator, `>> U=A\b`. Comment on the `MATLAB` output.

b. The matrix A^h corresponding to (4)–(6) is singular. There is a nonzero vector v such that $Av = 0$. This vector can be obtained in `MATLAB` by means of `>> v = null(A)`. Determine the vector v for $h = 2^{-4}$. What does the output represent?

The instability of (4)–(6) is related to instability of (1)–(2): Due to the Neumann condition on both the left and right side of the string, if v is a constant function, then $Lv = 0$ and $Bv = 0$. Hence, if u is a solution to (4)–(6), then so is $u + cv$ for any constant c.

c. Replace the Neumann condition at $x = 0$ by the Dirichlet condition:

$$u(0) = 0 \tag{9}$$

Modify the first row of the matrix A^h in the implementation in `MATLAB` in accordance with (9), and determine the solution by means of `>> U=A\b`.