ODE Reference Information

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Terminology

Types

There are a few types of ordinary differential equations (ODE), including:

- Separable;
- Linear/non-linear;
- First-order, second-order.

Solutions

Regarding the solutions of ODEs, there are, among others:

- Explicit/implicit solutions;
- Fixed point(s);
- Locally/globally stable fixed points; unstable fixed points

Analytical skills needed

To solve all ODEs in this assignment you only need the following 4 skills.

1. Solve an ODE by 'separation of variables', which you already learned in the first part of the course in the *CA Analytics* slide deck. The basic steps of this technique are:

   (a) *Separate the variables.* All $x(t)$ (or just x for short) on one side, all t on the other side. E.g.:

   $$\frac{dx}{dt} = x,$$
   $$\frac{1}{x} dx = 1 \cdot dt.$$

   (b) *Integrate both sides.* This simply means: put an integration symbol on both side. E.g.:

   $$\int \frac{1}{x} dx = \int 1 \cdot dt.$$

Then use high-school calculus and/or a list of known integral solutions[1] to solve the integrals. The solution of the example (collecting all integration constants into one constant C):

$$\ln x = t + C$$

(c) *Solve for $x(t)$.* Find a (nice) formula for $x(t)$. That is, use basic algebra to get $x(t)$ on one side and everything else on the other side. In the example, exponentiate both sides and done! I also rearrange a bit for aesthetics:

$$x = e^{t+C},$$
$$x = e^t e^C,$$
$$x = C_0 \cdot e^t$$

In the last step I collected the constant term e^C and made (redefined) a new constant C_0, just for simplicity. You can also leave the original form if you wish. As long as the result of this step is an equation of the form $x = ...$ and no x appears on the right side.

(d) *Specialize to a specific initial condition.* We cannot plot the above solution yet because we do not know the initial conditions of the ODE. That is, we know that $x(t)$ grows exponentially as function of t, but we do not know yet where this growth starts – i.e., we do not know the value for $x(0)$ (so $t = 0$). But suppose we are given the initial condition $x(0) = 2$. Then we can find the value for the unknown C_0, and then we are ready to plot the solution.

$$x(0) = C_0 \cdot e^0,$$
$$2 = C_0 \cdot 1,$$
$$C_0 = 2.$$

And thus the final solution (for the given initial condition $x(0) = 2$) that can be plotted using any software:

$$x(t) = 2 \cdot e^t.$$

One straightforward way to plot is to make an array of values for t in, for this example, the range $[0, 3]$, say with step size 0.1. Then compute the array of corresponding x values using the solution above. Then use `matplotlib` as usual to plot the x array versus the t array, like `plt.plot(t_array, x_array, '-o')`. For other ODEs, play around with the range for t and the step size to see what looks nice and shows the important features.

[1] To solve an integral you may use find and use the appropriate equation in e.g. http://www.wikiwand.com/en/List_of_integrals_of_rational_functions.

2. Find fixed point(s) of an ODE. A fixed point is a value for x such that x does not change anymore (remains 'fixed'). This means that the first derivative of x equals zero. Easy! The first derivative is precisely the given ODE to begin with! So all we need to do is set the ODE equal to 0 and then solve it, meaning, find values for x which satisfy this equation. Using the above simple example:

$$\frac{dx}{dt} = x = 0.$$

   This is particularly simple: only $x = 0$ satisfies this equation. Thus we found that there is only one fixed point for this ODE, namely $x = 0$.

3. Determine stability of the fixed points. A fixed point $x(t) = F$ is called 'locally stable' iff $x(t) = F \pm \epsilon$ (where $\epsilon \neq 0$) moves towards F whenever it is (very) close to F already (so 'small enough' but still non-zero ϵ). It is called 'globally stable' if $x(t)$ will always converge to F no matter the initial condition, so for any value of ϵ. Finally, a fixed point is called 'unstable' if $x(t)$ moves away from the fixed point, no matter how close $x(t)$ is to F – with the single exception of exactly $x(t) = F$ of course.

   Local stability of a fixed point can be determined in the following two different ways. You may use either one.

   (a) *Use the second derivative (exact option).* Observe that local stability means the following two things. If $x(t) < F$ then the first derivative dx/dt should be positive (to increase $x(t)$ toward F). Conversely, if $x(t) > F$ then the first derivative dx/dt should be negative (to decrease $x(t)$ toward F). This means that the second derivative (derivative of first derivative, which we write as d^2x/dt^2) must be negative, i.e., $d^2x/dt^2 < 0$. That's it! So, using the same reasoning you can verify easily that a positive second derivative implies that the fixed point is unstable.[2]

   Let's do it for the running example $dx/dt = x$. The second derivative is $d^2x/dt^2 = 1$, which is positive, so this fixed point is unstable. This implies that *only* if we set exactly $x(0) = 0$ then we reach the fixed point; any other initial condition, no matter how close to zero, will move away from 0.

   (b) *Graph nearby initial conditions (approximate option).* In the running example, simply plot two solutions for two 'nearby' initial conditions, say $x(0) = 0.1$ and $x(0) = -0.1$. You will see that both curves move away from 0, no matter how close to 0 you set the initial conditions. This means that the fixed point is unstable. Just to be sure you did your math right you can also plot a third curve with initial condition $x(0) = 0$, just to verify that this curve will stay perfectly horizontal. If the solution were stable then both curves would converge to the fixed point 0.

[2]You may wonder: what if the second derivative equals 0? Well then you usually have a semi-stable fixed point, which is stable on one side but unstable on the other side. But not necessarily! It can also be e.g. that any initial condition simply remains fixed, in which case there are infinitely many fixed points but none of them are considered stable.

4. *Numerical approximation of a solution.* Plotting a solution means putting t on the horizontal axis and $x(t)$ on the vertical axis. You can find this curve analytically as described above, but more often than not computational scientists use so-called numerical approximations.[3] By this we mean that we use an algorithm that results in an approximate curve for $x(t)$, which we can plot and should look very similar to the analytical solution (if any). 'Numerical' just means that we use the (finite) floating-point operations of the computer to get our result, instead of (exact) mathematical derivation like above.

   There is a very simple algorithm to plot an approximate solution, with any desired accuracy (at the cost of more computation). It is called the Euler algorithm. Most instructive is to watch the 10-minute video:
   https://www.khanacademy.org/math/differential-equations/first-order-differential-equa eulers-method-tutorial/v/eulers-method. There is also reading material mentioned below on the subject.

Background material

These videos and reading material give you a basic introduction of ODEs and how they are solved. You should be able to just start with the exercises below, but it is of course instructive to review this material at any time, or to use it as reference.

Short video lectures

- First 3 videos of https://youtu.be/-_POEWfygmU?list=PL96AE8D9C68FEB902
- https://www.khanacademy.org/math/differential-equations/first-order-differential-equa logistic-differential-equation/v/logistic-differential-equation-intuition
- https://www.khanacademy.org/math/differential-equations/first-order-differential-equa eulers-method-tutorial/v/eulers-method

Reading material

- http://mathinsight.org/ordinary_differential_equation_introduction
- http://www.stewartcalculus.com/data/CALCULUS%20Concepts%20and%20Contexts/upfiles/3c3-LinearDiffEqns_Stu.pdf. It is only needed to understand the definition of what a 'linear' ODE[4] is (first two equations); the way to solve such ODEs (using 'integrating factors') is optional to read (or to use).
- http://tutorial.math.lamar.edu/Classes/DE/Modeling.aspx. This is a general introduction to how modeling is done and how ODEs are created to model things; this page does not explain (well) how to actually solve ODEs. Perhaps needless to say, but you do not need to study hard on the example problems that are listed; just make sure you understand the main point.
- http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx
- http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx[5]

[3] This is because most ODEs encountered in realistic applications cannot be (easily) analytically solved.
[4] Remind yourself of what linear and non-linear functions are, then you'll see a parallel.
[5] Here they use the term 'equilibrium point' where I use 'fixed point', but it is the same thing.