# Dynamic Model of a Scara Robot.

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## Abstract—

In this Project we are going to develop the dynamic model of a Scara robot which has 4 degrees of freedom. And then, we will apply all the values calculated to a basic model CAD of a Scara robot, these will be obtained by the manipulation of the CAD model. Then, with a Python program we will find numeric values to get real data about this model. International System units will being use in this project. As a final product, we obtain the tensorial form of the dynamic model and the numeric values according to our CAD model.

## I. Introduction

This document presents the necessary calculations to obtain the dynamic model of a Scara robot in its tensor form. This was done for the subject of "robotics" in the Mechatronic Engineering career at the "Instituto Tecnológico de Toluca" as part of the final evaluation. And we as members of the team made the effort to present this material in English, being that we are native Spanish speakers, in order to obtain greater dissemination within the internet communities that talk about topics like this.

There is a basic CAD model of a Scara robot, with which we are going to evaluate the values, obtained by software (such as moments of inertia, distances, etc.), for the result, to have a complete dynamic model of the entire robot with these specific values.

As research methodology used in this work, the quantitative methodology of the mathematical model was used.

As a limitation in the understanding of this project, we have the need that the reader must previously know several concepts for a good understanding; these concepts are mostly from physics and linear algebra.

In this first stage, the "Solid Works" software will be used, but its use can be extrapolated to any other CAD software that has the necessary tools.

## II. Dynamic model

### A. Scara diagrams.

First thing is first. The Scara robot diagrams that represent its position within a reference frame, in order to have the necessary references to make the necessary calculations. These diagrams are presented in Fig. 1 and Fig. 2.

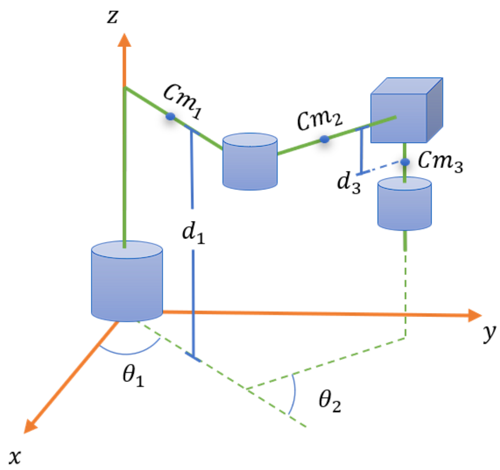


*Fig. 1 3D diagram of Scara robot*

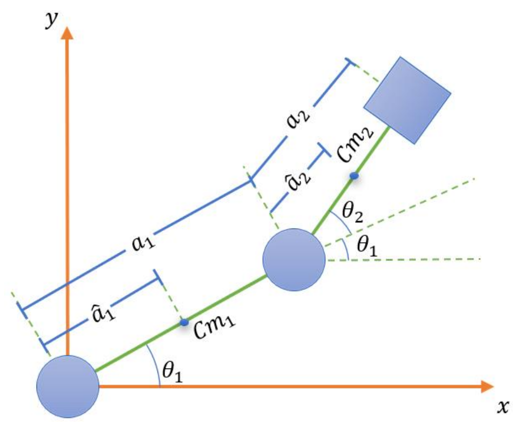


*Fig. 2 2D diagram of Scara robot*

## B. Position vectors of centers of mass

To calculate the position vectors of each center of mass of our robot, we will consider the structure what is seen in the Fig. 1 and Fig. 2. It is important to mention that since links 3 and 4 of our Scara are mechanically interlocked, it is being considered that both links share the same center of mass. Then, the position vectors result as shown in ( 39).

$$Cm_1 = \vec{O}_1 = \begin{bmatrix} \hat{a}_1 \cos(\theta_1) \\ \hat{a}_1 sen(\theta_1) \\ d_1 \end{bmatrix}$$

$$Cm_2 = \vec{O}_2 = \begin{bmatrix} a_1 cos(\theta_1) + \hat{a}_2 \cos(\theta_1 + \theta_2) \\ a_1 sen(\theta_1) + \hat{a}_2 \,\mathrm{sen}(\theta_1 + \theta_2) \\ d_1 \end{bmatrix} \tag{1}$$

$$Cm_3 = \vec{O}_3 = \begin{bmatrix} a_1 cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\ a_1 sen(\theta_1) + a_2 \mathrm{sen}(\theta_1 + \theta_2) \\ d_1 - d_3 \end{bmatrix}$$

## C. Kinetic Energy

To calculate the Kinetic Energy, we need to work something about its definition ( 39).

**Definition:**

$$K_i(q,\dot{q}) = \frac{1}{2} m_i\, \vec{V}_i^T\, \vec{V}_i + \frac{1}{2}\, \vec{w}_i^T\, \mathrm{I}_i\, \vec{w}_i \tag{2}$$

But we will change something, we will use the definition of the inertia tensor relative to an inertia frame $\mathrm{I}_i = R_i^{i-1} I_i {R_i^{i-1}}^T$

Then, the definition of $K_i(q,\dot{q})$ is like appears in ( 39)

$$K_i(q,\dot{q}) = \frac{1}{2} m_i\, \vec{V}_i^T\, \vec{V}_i + \frac{1}{2}\, \vec{w}_i^T\, R_i^{i-1} \mathrm{I}_i {R_i^{i-1}}^T\, \vec{w}_i \tag{3}$$

We are going to use ( 39) to calculate its values for each link of the robot, but we are going to calculate each component of this equation separately, so that it is easier to transcribe all the equations during the process.

## D. Kinetic energy for each linkage.

**<u>Important:</u>** We can see that the sine and cosine functions are involved in the position vectors of the centers of mass; Since from now on the results will become extensive, we are going to reduce the expressions sine and cosine to an S and a C respectively, each one followed by a subscript that alludes to the angle of theta that they have as argument. If there are two numbers in the subscript, a sum of those angles is being indicated in the function argument.

A) Linkage 1

$$\begin{aligned}\frac{1}{2} m_1\, \vec{V}_1^T\, \vec{V}_1 = \frac{1}{2} m_1 \begin{bmatrix} -\hat{a}_1 S_1\, \dot{\theta}_1 \\ \hat{a}_1 C_1\, \dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} -\hat{a}_1 S_1\, \dot{\theta}_1 \\ \hat{a}_1 C_1\, \dot{\theta}_1 \\ 0 \end{bmatrix} &= \frac{1}{2} m_1 \left[\hat{a}_1^2\, \dot{\theta}_1^2 (S_1^2 + C_1^2)\right] \\ &= \frac{1}{2} m_1\, \hat{a}_1^2\, \dot{\theta}_1^2\end{aligned} \tag{4}$$

$$\begin{aligned}&\frac{1}{2}\, \vec{w}_1^T\, R_1^0\, I_1\, {R_1^0}^T\, \vec{w}_1 \\ &= \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix} \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\ &= \frac{1}{2} I_{1zz}\, \dot{\theta}_1^2\end{aligned} \tag{5}$$

Then, for linkage 1; we must add ( **4**) and ( 5), so we obtain ( **6**).

$$K_1(q,\dot{q}) = \frac{1}{2}\, \dot{\theta}_1^2 (m_1\, \hat{a}_1^2 + I_{1zz}) \tag{6}$$

B) Linkage 2

$$\frac{1}{2}m_2\,\vec{V}_2^T\,\vec{V}_2=\frac{1}{2}m_2\begin{bmatrix}-a_1S_1\,\dot{\theta}_1-\hat{a}_2S_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ a_1C_1\,\dot{\theta}_1-\hat{a}_2C_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ 0\end{bmatrix}^T\begin{bmatrix}-a_1S_1\,\dot{\theta}_1-\hat{a}_2S_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ a_1C_1\,\dot{\theta}_1-\hat{a}_2C_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ 0\end{bmatrix} \quad (7)$$

( 39) has a very large result, but it can be reduced to get what is showed in ( 39)

$$\frac{1}{2}m_2\,\vec{V}_2^T\,\vec{V}_2=\frac{1}{2}m_2\left[a_1^2\,\dot{\theta}_1^2+\hat{a}_2^2(\dot{\theta}_1+\dot{\theta}_2)^2+2a_1\hat{a}_2\dot{\theta}_2(\,\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\right] \quad (8)$$

For previous experience, we know how the structure of the result is, so we do not see mandatory to write all the body of the equation ( 39).

$$\frac{1}{2}\,\vec{w}_2^T\,R_2^1\,I_2\,R_2^{1^T}\,\vec{w}_2=\frac{1}{2}I_{2zz}(\,\dot{\theta}_1+\dot{\theta}_2)^2 \quad (9)$$

Then, for linkage 2; we must add ( **8**) and ( 39), so we obtain ( 39).

$$K_2(q,\dot{q})=\frac{1}{2}m_2a_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\hat{a}_2^2(\,\dot{\theta}_1+\dot{\theta}_2)^2+m_2a_1\hat{a}_2C_2(\,\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1+\frac{1}{2}I_{2zz}(\,\dot{\theta}_1+\dot{\theta}_2)^2 \quad (10)$$

C) Linkage 3

$$\frac{1}{2}m_3\,\vec{V}_3^T\,\vec{V}_3=\frac{1}{2}m_3\begin{bmatrix}-a_1S_1\dot{\theta}_1-a_2S_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ a_1C_1\dot{\theta}_1+a_2C_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ -\dot{d}_3\end{bmatrix}^T\begin{bmatrix}-a_1S_1\dot{\theta}_1-a_2S_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ a_1C_1\dot{\theta}_1+a_2C_{12}(\,\dot{\theta}_1+\dot{\theta}_2)\\ -\dot{d}_3\end{bmatrix} \quad (11)$$

Note: $d_3$ was derivate because it is a variable; we must remember that there is a prismatic joint in our robot, and that defines de value of $d_3$.

We can see the result of solve ( 39) in ( 39).

$$\frac{1}{2}m_3\left[\dot{\theta}_1^2a_1^2+a_2^2(\,\dot{\theta}_1+\dot{\theta}_2)^2+2\dot{\theta}_1a_1a_2(\,\dot{\theta}_1+\dot{\theta}_2)C_2+\dot{d}_3^2\right] \quad (12)$$

$$\frac{1}{2}\,\vec{w}_3^T\,R_3^2\,I_3\,R_3^{2^T}\,\vec{w}_3=\frac{1}{2}I_{3zz}(\,\dot{\theta}_1+\dot{\theta}_2-\dot{\theta}_4)^2 \quad (13)$$

We must remember that links 4 and 3 are mechanically coupled; and since link 4 presents a rotational movement, this rotation influences the angular velocity of link 3.

$$\begin{aligned}K_3(q,\dot{q})=&\frac{1}{2}m_3\theta_1^2a_1^2+\frac{1}{2}m_3a_2^2(\,\dot{\theta}_1+\dot{\theta}_2)^2+2a_1a_2C_2\dot{\theta}_1(\,\dot{\theta}_1+\dot{\theta}_2)+\dot{d}_3^2\\&+\frac{1}{2}I_{3zz}(\,\dot{\theta}_1+\dot{\theta}_2-\dot{\theta}_4)^2\end{aligned} \quad (14)$$

If links 3 and 4 are mechanically coupled; these last equations are valid for both links.

## E. Potential Energy for each linkage

The definition of potential energy for the linkage i is presented in ( 39).

$$u_i(q) = m_i g(d) \tag{15}$$

Where:

$g$ -> Force of gravity

$d$ -> Distance between the center of mass of the link and the XY plane, measured through the Z axis.

A) Linkage 1

$$u_1(q) = m_1 \boldsymbol{g}(\boldsymbol{d_1}) \tag{16}$$

B) Linkage 2

$$u_2(q) = \boldsymbol{m_2} g(d_1) \tag{17}$$

C) Linkage 3

$$u_3(q) = \boldsymbol{m_3} g(d_1 - \boldsymbol{d_3}) \tag{18}$$

## F. Total kinetic energy

Now, to obtain the total energies, we must add everything obtained in each link in a single equation.

$$\begin{aligned} K(q,\dot{q}) = {} & \boldsymbol{\frac{1}{2}} m_1 \boldsymbol{\hat{a}_1^2} \dot{\theta}_1^2 + \boldsymbol{\frac{1}{2} I_{1zz}} \dot{\theta}_1^2 + \boldsymbol{\frac{1}{2} m_2 \hat{a}_2^2} (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 \boldsymbol{a_1} \hat{a}_2 C_2 \boldsymbol{\dot{\theta}_2} (\dot{\theta}_1 + \dot{\theta}_2) \\ & + \frac{1}{2} I_{2zz} (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_3 \boldsymbol{a_1^2} \dot{\theta}_1^2 + \boldsymbol{\frac{1}{2} m_3} a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ & + m_3 \boldsymbol{a_1} a_2 \boldsymbol{C_2 \dot{\theta}_1} (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} m_3 \boldsymbol{\dot{d}_3^2} + \frac{1}{2} I_{3zz} (\dot{\theta}_1 + \dot{\theta}_2 - \dot{\theta}_4)^2 \end{aligned} \tag{19}$$

## G. Total potential energy

$$u(q,\dot{q}) = \boldsymbol{m_1} g(d_1) + \boldsymbol{m_2} g(d_1) + \boldsymbol{m_3} g(d_1 - \boldsymbol{d_3}) \tag{20}$$

## H. Lagrange

In ( 39) we presented the result with all the reductions that can be applied without making more difficult the next steps.

$$L(q,\dot{q}) = \frac{\mathbf{1}}{\mathbf{2}} m_1 \widehat{\boldsymbol{a}}_{\mathbf{1}}^{\mathbf{2}} \dot{\theta}_1^2 + \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{I}_{\mathbf{1zz}} \dot{\theta}_1^2 + \frac{\mathbf{1}}{\mathbf{2}} \dot{\theta}_1^2 \boldsymbol{a}_{\mathbf{1}}^{\mathbf{2}} (m_2 + \boldsymbol{m}_{\mathbf{3}}) + \frac{1}{2} \widehat{\boldsymbol{a}}_{\mathbf{2}}^{\mathbf{2}} m_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$
$$+ m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_{\mathbf{2}} \boldsymbol{C}_{\mathbf{2}} \dot{\boldsymbol{\theta}}_{\mathbf{1}} (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} I_{2zz} (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_3 \boldsymbol{a}_{\mathbf{2}}^{\mathbf{2}} (\dot{\theta}_1 + \dot{\theta}_2)^2$$
$$+ m_3 \dot{\boldsymbol{\theta}}_{\mathbf{1}} a_1 \boldsymbol{a}_{\mathbf{2}} \boldsymbol{C}_{\mathbf{2}} (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} \boldsymbol{m}_{\mathbf{3}} \dot{d}_3^2 + \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{I}_{\mathbf{3zz}} (\dot{\theta}_1 + \dot{\theta}_2 - \dot{\theta}_4)^2$$
$$- \boldsymbol{d}_{\mathbf{1}} g (m_1 + m_2) - m_3 \boldsymbol{g} (\boldsymbol{d}_{\mathbf{1}} - \boldsymbol{d}_{\mathbf{3}}) \tag{21}$$

## I. Euler-Lagrange equation (motion equation)

The Euler-Lagrange equation is showed in ( 39).

$$\mathbf{T}_i = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial t}} \left[ \frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \dot{\theta}_i}} \right] - \frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \theta_i}} \tag{22}$$

So, we must calculate equation ( 39) at each link to obtain the equation of the minimum torque needed to move the link

A) Linkage 1

$$\frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \theta_1}} = 0 \tag{23}$$

$$\frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \dot{\theta}_1}} = \boldsymbol{m}_{\mathbf{1}} \ddot{\theta}_1 \widehat{\boldsymbol{a}}_{\mathbf{1}}^{\mathbf{2}} + I_{1zz} \ddot{\boldsymbol{\theta}}_{\mathbf{1}} + \boldsymbol{a}_{\mathbf{2}} \ddot{\theta}_1 (m_2 + m_3) + m_2 \widehat{\boldsymbol{a}}_{\mathbf{2}}^{\mathbf{2}} (\ddot{\theta}_1 + \ddot{\theta}_{\mathbf{2}})$$
$$+ m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_2 \boldsymbol{C}_{\mathbf{2}} (2\ddot{\theta}_1 + \ddot{\theta}_2) - m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_2 \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2)$$
$$+ \boldsymbol{I}_{\mathbf{2zz}} (\ddot{\theta}_1 + \ddot{\theta}_2) + m_3 \boldsymbol{a}_{\mathbf{1}} a_2 \boldsymbol{C}_{\mathbf{2}} (2\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$- m_3 \boldsymbol{a}_{\mathbf{1}} a_2 \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) + \boldsymbol{m}_{\mathbf{3}} a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + \boldsymbol{I}_{\mathbf{3zz}} (\ddot{\theta}_1 + \ddot{\boldsymbol{\theta}}_{\mathbf{2}} - \ddot{\boldsymbol{\theta}}_{\mathbf{4}})$$
$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial t}} \left[ \frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \dot{\theta}_1}} \right] = \boldsymbol{m}_{\mathbf{1}} \ddot{\theta}_1 \widehat{\boldsymbol{a}}_{\mathbf{1}}^{\mathbf{2}} + I_{1zz} \ddot{\boldsymbol{\theta}}_{\mathbf{1}} + \boldsymbol{a}_{\mathbf{2}} \ddot{\theta}_1 (m_2 + m_3) + m_2 \widehat{\boldsymbol{a}}_{\mathbf{2}}^{\mathbf{2}} (\ddot{\theta}_1 + \ddot{\theta}_{\mathbf{2}})$$
$$+ m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_2 [\boldsymbol{C}_{\mathbf{2}} (2\ddot{\theta}_1 + \ddot{\theta}_2) - \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2)] + \boldsymbol{I}_{\mathbf{2zz}} (\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$+ m_3 \boldsymbol{a}_{\mathbf{1}} \boldsymbol{a}_{\mathbf{2}} [\boldsymbol{C}_{\mathbf{2}} (2\ddot{\theta}_1 + \ddot{\theta}_2) - \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2)] + \boldsymbol{m}_{\mathbf{3}} a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$+ \boldsymbol{I}_{\mathbf{3zz}} (\ddot{\theta}_1 + \ddot{\boldsymbol{\theta}}_{\mathbf{2}} - \ddot{\boldsymbol{\theta}}_{\mathbf{4}}) \tag{24}$$

$$\mathbf{T}_1 = \boldsymbol{m}_{\mathbf{1}} \ddot{\theta}_1 \widehat{\boldsymbol{a}}_{\mathbf{1}}^{\mathbf{2}} + I_{1zz} \ddot{\boldsymbol{\theta}}_{\mathbf{1}} + \boldsymbol{a}_{\mathbf{2}} \ddot{\theta}_1 (m_2 + m_3) + m_2 \widehat{\boldsymbol{a}}_{\mathbf{2}}^{\mathbf{2}} (\ddot{\theta}_1 + \ddot{\theta}_{\mathbf{2}})$$
$$+ [m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_2 + + m_3 \boldsymbol{a}_{\mathbf{1}} \boldsymbol{a}_{\mathbf{2}}] \boldsymbol{C}_{\mathbf{2}} (2\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$- [m_2 \boldsymbol{a}_{\mathbf{1}} \hat{a}_2 + + m_3 \boldsymbol{a}_{\mathbf{1}} \boldsymbol{a}_{\mathbf{2}}] \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_2 [2\dot{\theta}_1 + \dot{\theta}_2] + \boldsymbol{I}_{\mathbf{2zz}} (\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$+ \boldsymbol{m}_{\mathbf{3}} a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + \boldsymbol{I}_{\mathbf{3zz}} (\ddot{\theta}_1 + \ddot{\boldsymbol{\theta}}_{\mathbf{2}} - \ddot{\theta}_{\mathbf{4}}) \tag{25}$$

B) Linkage 2

$$\frac{\boldsymbol{\partial L(q,\dot{q})}}{\boldsymbol{\partial \theta_2}} = -\boldsymbol{m}_{\mathbf{2}} a_1 \widehat{\boldsymbol{a}}_{\mathbf{2}} \boldsymbol{S}_{\mathbf{2}} \dot{\boldsymbol{\theta}}_{\mathbf{1}} (\dot{\theta}_1 + \dot{\theta}_2) - m_3 \boldsymbol{a}_{\mathbf{1}} a_2 \boldsymbol{S}_{\mathbf{2}} \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_{\mathbf{2}}) \tag{26}$$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{\theta}}_2} = m_2\hat{\boldsymbol{a}}_2^2(\dot{\theta}_1+\dot{\theta}_2) + m_2\boldsymbol{a}_1\hat{a}_2\boldsymbol{C}_2\dot{\theta}_1 + \boldsymbol{I}_{2zz}(\dot{\theta}_1+\dot{\theta}_2) + m_3\boldsymbol{a}_2^2(\dot{\theta}_1+\dot{\theta}_2) + m_3\boldsymbol{a}_1 a_2\boldsymbol{C}_2\dot{\theta}_1 + \boldsymbol{I}_{3zz}(\dot{\theta}_1+\dot{\theta}_2-\dot{\theta}_4) \quad (27)$$

$$\begin{aligned} \mathrm{T}_2 = {} & [\boldsymbol{I}_{3zz} + \boldsymbol{m}_2\hat{\boldsymbol{a}}_2^2 + \boldsymbol{m}_2\boldsymbol{a}_1\hat{\boldsymbol{a}}_2\boldsymbol{C}_2 + \boldsymbol{m}_3\boldsymbol{a}_2^2 + \boldsymbol{m}_3\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{C}_2]\ddot{\boldsymbol{\theta}}_1 \\ & + [m_2\hat{a}_2^2 + m_3a_2^2 + I_{3zz}]\ddot{\theta}_2 + [\boldsymbol{I}_{3zz}]\ddot{\boldsymbol{\theta}}_4 \\ & + [\boldsymbol{m}_2\boldsymbol{a}_2\hat{\boldsymbol{a}}_2\boldsymbol{S}_2\dot{\boldsymbol{\theta}}_1 + \boldsymbol{m}_3\boldsymbol{a}_1\boldsymbol{a}_2\dot{\boldsymbol{\theta}}_1\boldsymbol{S}_2]\dot{\boldsymbol{\theta}}_2 \\ & + [m_2a_1\hat{a}_2\dot{\theta}_1S_2 - m_2a_1\hat{a}_2\dot{\theta}_2S_2 + m_3a_1a_2\dot{\theta}_1S_2 - m_3a_1a_2\dot{\theta}_2S_2]\dot{\theta}_1 \end{aligned} \quad (28)$$

### C) Linkage 3

We must consider two variables for this link, $d_3$ and $\theta_4$, so we have two sections here.

$\boldsymbol{\theta_4}$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{\theta}_4} = \mathbf{0} \quad (29)$$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{\theta}}_4} = \boldsymbol{I}_{3zz}[\dot{\theta}_1+\dot{\theta}_2-\dot{\theta}_4] \quad (30)$$

$$\frac{\partial}{\partial t}\left[\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{\theta}}_4}\right] = \boldsymbol{I}_{3zz}[\ddot{\theta}_1+\ddot{\boldsymbol{\theta}}_2-\ddot{\boldsymbol{\theta}}_4] \quad (31)$$

$$T_{3_{\theta 4}} = \boldsymbol{I}_{3zz}[\ddot{\theta}_1+\ddot{\boldsymbol{\theta}}_2-\ddot{\boldsymbol{\theta}}_4] = \mathbf{T}_4 \quad (32)$$

$\boldsymbol{d_3}$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{d}_3} = \boldsymbol{m}_3 g \quad (33)$$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{\theta}}_4} = \boldsymbol{m}_3\dot{d}_3 \quad (34)$$

$$\frac{\partial}{\partial t}\left[\frac{\partial \boldsymbol{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{\theta}}_4}\right] = \boldsymbol{m}_3\ddot{d}_3 \quad (35)$$

$$T_{3_{\boldsymbol{d3}}} = \boldsymbol{m}_3\ddot{d}_3 - \boldsymbol{m}_3 g = \mathbf{T}_3 \quad (36)$$

## J. Tensor form of the equation of motion

For this robot, the tensor form of the equation of motion is provided in ( 39). But as we have seen, we have torque equations with very long coefficients, therefore we are going to give the value of each coefficient of the matrix independently at the bottom of the matrix.

$$\begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\\ \mathbf{T}_3\\ \mathbf{T}_4\end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_3\\ \ddot{\boldsymbol{d}}_3\\ \ddot{\theta}_4\end{bmatrix} + \begin{bmatrix} \boldsymbol{c}_{11} & \boldsymbol{c}_{12} & \boldsymbol{c}_{13} & \boldsymbol{c}_{14} \\ \boldsymbol{c}_{21} & \boldsymbol{c}_{22} & \boldsymbol{c}_{23} & \boldsymbol{c}_{24} \\ \boldsymbol{c}_{31} & \boldsymbol{c}_{32} & \boldsymbol{c}_{33} & \boldsymbol{c}_{34} \\ \boldsymbol{c}_{41} & \boldsymbol{c}_{42} & \boldsymbol{c}_{43} & \boldsymbol{c}_{44}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\boldsymbol{d}}_3\\ \dot{\theta}_4\end{bmatrix} + \begin{bmatrix}g_1\\ g_2\\ g_3\\ g_4\end{bmatrix} \quad (40)$$

Note: We must take care about all the numbers that participates in each element of the torque equations. At this point is too easy get a mistake.

$$
\begin{aligned}
m_{11} &= \boldsymbol{m_1}\hat{a}_1^2 + (\boldsymbol{m_2} + \boldsymbol{m_3})\boldsymbol{a_2} + 2m_3\boldsymbol{a_1}a_2\boldsymbol{C_2} + m_2\boldsymbol{\hat{a}_2^2} + m_3\boldsymbol{a_2^2} + 2m_2\boldsymbol{a_1}\hat{a}_2\boldsymbol{C_2} \\
&\qquad + 2a_1C_2(\boldsymbol{m_2}\boldsymbol{\hat{a}_2} + \boldsymbol{m_3}\boldsymbol{a_2}) + I_{1zz} + I_{2zz} + I_{3zz} \\
\boldsymbol{m_{12}} &= m_2\boldsymbol{\hat{a}_2^2} + m_2a_1\boldsymbol{\hat{a}_2}C_2 + \boldsymbol{m_3}a_2^2 + \boldsymbol{m_3}a_1\boldsymbol{a_2}C_2 + \boldsymbol{C_2}a_1(\boldsymbol{m_2}\boldsymbol{\hat{a}_2} + \boldsymbol{m_3}\boldsymbol{a_2}) + \boldsymbol{I_{2zz}} \\
&\qquad + I_{3zz} \\
\boldsymbol{m_{13}} &= 0 \\
\boldsymbol{m_{14}} &= I_{3zz} \\
\boldsymbol{m_{21}} &= m_2\boldsymbol{\hat{a}_2^2} + m_2\boldsymbol{C_2}a_1\boldsymbol{\hat{a}_2} + m_3\boldsymbol{a_2^2} + m_3\boldsymbol{C_2}a_1\boldsymbol{a_2} + I_{3zz} \\
\boldsymbol{m_{22}} &= m_2\boldsymbol{\hat{a}_2^2} + m_3\boldsymbol{a_2^2} + I_{3zz} \\
\boldsymbol{m_{23}} &= 0 \\
\boldsymbol{m_{24}} &= I_{3zz} \\
\boldsymbol{m_{31}} &= 0 \\
\boldsymbol{m_{32}} &= \boldsymbol{0} \\
\boldsymbol{m_{33}} &= m_3 \\
\boldsymbol{m_{34}} &= \boldsymbol{0} \\
\boldsymbol{m_{41}} &= \boldsymbol{I_{3zz}} \\
\boldsymbol{m_{42}} &= I_{3zz} \\
\boldsymbol{m_{43}} &= \boldsymbol{0} \\
\boldsymbol{m_{44}} &= \boldsymbol{I_{3zz}}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
c_{11} &= -\boldsymbol{2m_2}S_2\boldsymbol{a_1}\hat{a}_2 - \boldsymbol{2m_3}S_2\boldsymbol{a_1}a_2 \\
c_{12} &= -m_2\boldsymbol{S_2}a_1\boldsymbol{\hat{a}_2} - m_3\boldsymbol{S_2}a_1\boldsymbol{a_2} \\
c_{13} &= 0 \\
c_{14} &= 0 \\
c_{21} &= -m_2\boldsymbol{\dot{\theta}_2}a_1\boldsymbol{\hat{a}_2} - \boldsymbol{S}_2a_1[m_3\boldsymbol{\dot{\theta}_2}a_2 + \boldsymbol{m_2}\dot{\theta}_1\boldsymbol{\hat{a}_2} + m_3\boldsymbol{\dot{\theta}_1}a_2] \\
c_{22} &= a_1\boldsymbol{S_2}\left[m_2\boldsymbol{\hat{a}_2}\dot{\theta}_1 + \boldsymbol{m_3}a_2\boldsymbol{\dot{\theta}_1}\right] \\
c_{23} &= 0 \\
c_{24} &= 0 \\
c_{31} &= 0 \\
c_{32} &= 0 \\
c_{33} &= 0 \\
c_{34} &= 0 \\
c_{41} &= 0 \\
c_{42} &= 0 \\
c_{43} &= 0 \\
c_{44} &= \boldsymbol{0}
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
g_1 &= 0 \\
g_2 &= 0 \\
g_3 &= -m_3\boldsymbol{g} \\
g_4 &= \boldsymbol{0}
\end{aligned}
\tag{39}
$$