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SECTION F

PROBLEM SET # 13

PROBLEM 1

$$f_{x,y} = \begin{cases} 2e^{-(u+v)} & 0 < u < v < \infty \\ 0 & \text{otherwise} \end{cases}$$

a) ~~E[Y] =~~ MMSE, c^*, Y
$\to$ MSE

$$E[Y] = \int_0^\infty \int_0^v v f_{x,y}(u,v)\,du\,dv$$

$$= \int_0^\infty \int_0^v v(2e^{-u})(e^{-v})\,du\,dv$$

$$= \int_0^\infty 2v(e^{-v} - e^{-2v})\,dv = \boxed{3/2}$$

MSE:

$Var(Y) = E[(Y-3/2)^2] = E[Y^2] - E[Y]^2$

~~Var(Y) = 5/4~~

~~= 1~~

$$E[Y]^2 = \int_0^\infty \int_0^v v^2 \cdot 2e^{-u} e^{-v}\,du\,dv$$

$$= \int_0^\infty \left(2v^2(e^{-v} - e^{-2v})\right) - 7/2$$

$$= \frac{7}{2} - (3/2)^2 = \boxed{5/4}$$

b) MMSE $\Rightarrow g^*(x)$
MSE

$g^*(x) = E[Y|X]$

$g^*(u) = E[Y|X=u]$

$$= \int_{-\infty}^{\infty} v \cdot f(v|u)\,dv = \int_u^\infty v \cdot e^{u} e^{-v}$$

$$\boxed{g^*(x) = g^*(u) = u+1}$$

$$\boxed{MSE = 1}$$

c) $L^*(x)$

$$L^*(x) = f(x) = \hat{E}[Y|X]$$

$$= \frac{cov(X,Y)}{Var(X)}(X - E[X])$$

$+ E[Y]$

$$E[X,Y] = \iint uv f_{xy}(u,v)\,du\,dv$$

$$= 2\int_0^{\infty}\int_0^{u} uve^{-v}e^{-u}\,du\,dv = 1$$

$E[XY] - E[X]E[Y]$

$= 1/4 \Rightarrow Cov(X, Y)$

SO:

$L^*(X)$

$= \frac{(1/4)}{(1/4)}(x - 1/2) + 3/2$

$L^*(x)$ $\boxed{= x + 1}$

$\boxed{MSE = 1}$

QUESTION 3

$L(x_1, x_2) = a + bx_1 + cx_2$

a) $z = E[(y - bx_1 - cx_2 - a)^2]$

$\cdot\ y - bx_1 - cx_2$

$a = E[z] = E[y - bx_1 - cx_2]$

$= E[y] - bE[x_1] - cE[x_2]$

$= E[z]$

$E[(z - E[y])^2] = E[(z - E(z))^2]$

$= Var(z) = Var(y - bx_1 - cx_2)$

$= cov(y - bx_1 - cx_2,\ y - bx_1 - cx_2)$

$= Var(y) - b\,cov(x, y)$

$- c\,cov(y, x_2) - b\,cov(x, y)$

$+ b^2 var(x_1) + b\,cov(x_1, x_2)$

$- c\,cov(x_2, y)$

$= Var(y) - 2cov(y, x_2) - 2b$

$cov(x, y) + b^2 var(x_1) + c^2 var(x_2)$

Set:

$a = E[y]$

$b = \dfrac{cov(x, y)}{Var(x_1)}$

$c = \dfrac{cov(y, x_2)}{var(x_2)}$

b)

we have:

$Var(y) - 2\dfrac{cov(y, x_2)}{Var(x_2)}\,cov(y, x_2)$

$- 2\dfrac{cov(y, x_1)}{Var(x_1)}\,cov(y, x_1)$

$+ \dfrac{cov^2(x, y)}{Var(x_1)}$

$= Var(y) + \dfrac{cov^2(x, y)}{Var(x_1)}$

$+ \dfrac{cov^2(x, x_2)}{Var(x_2)} - \dfrac{2cov^2(y, x_2)}{Var(x_2)}$

$- \dfrac{2cov^2(y, x_1)}{Var(x_1)}$

so we set:

$Var(y) - \dfrac{cov^2(y, x_2)}{Var(x_2)}$

$- \dfrac{cov^2(y, x_1)}{Var(x_1)}$

Question 3

a) Expectation, $E[Y]$

$= 3/2 \times 100$

$= 300/2$

$= \boxed{150}$

b) Using CLT

Normalized

$P\left\{ \frac{S_n - 150}{\sqrt{100/12}} \le \frac{150}{\sqrt{100/12}} \right\}$

$= \Phi(1.73)$

$\boxed{= 0.0418}$

Question 4

a) $[\hat{p} - 0.05, \hat{p} + 0.05]$

$\Rightarrow$ so $P\{|\hat{p} - p| \ge 0.05\}$

$\cancel{= P\left\{ \frac{|x - np|}{\sqrt{np(1-p)}} \ge \frac{0.05\sqrt{n}}{\sqrt{p(1-p)}} \right\}}$

$\cancel{\le Var}$

$P\left\{ \left| \frac{S_n}{n} - \mu \right| \ge a \right\} \le \frac{Var(S_n/n)}{a^2}$

$= \frac{c}{n a^2}$ $\quad 1 - 0.99 = \frac{1}{a^2}$

$a^2 = 100$

so $a = 10$

$\frac{a}{2\sqrt{n}} = 0.05$

$= \left(\frac{10}{2(0.05)} \right)^2$

$\boxed{n = 10,000}$

b)

$$P\{|\hat{p}-p| \geq 0.05\}$$

$$= P\left\{\frac{|X-np|}{\sqrt{np(1-p)}} \geq \frac{0.05\sqrt{n}}{\sqrt{p(1-p)}}\right\}$$

$$\leq P\left\{\frac{|X-np|}{\sqrt{np(1-p)}} \geq 2(0.05)\sqrt{n}\right\}$$

so:

$$P\left(\frac{|X-np|}{\sqrt{np(1-p)}} > c\right) = 2Q(c)$$

$$P\left\{\frac{|X-np|}{\sqrt{np(1-p)}} > 0.1\sqrt{n}\right)$$

$$= 2Q(0.1\sqrt{n})$$

$$\Downarrow$$

$$= 0.01$$

$$Q(0.1\sqrt{n}) = 0.005$$

$$Q(2.58) = 0.005$$

$$n = \left(\frac{2.58}{0.1}\right)^2 = 665$$

QUESTION 5

$\mu_x = 0$, $\mu_y = -1$, $\sigma^2_x = 1$

$\rho_{x,y} = -1/2$

a) $\operatorname{cov}(X+Y, X-Y) = 0$

$= \operatorname{cov}(X,X) - \operatorname{cov}(X,Y) - \operatorname{cov}(Y,Y)$

$+ \operatorname{cov}(X,Y)$

$\operatorname{Var}(X) - \operatorname{Var}(Y) = 0$

$\operatorname{Var}(X) = \operatorname{Var}(Y)$

$\Rightarrow \sigma_y^2 = 1$

b) $P(2X+Y>0)$

~~Var(Z)~~

$\operatorname{Var}(2X+Y) = \operatorname{cov}(2X+Y, 2X+Y)$

$= 4\operatorname{cov}(X,Y) + 4\operatorname{Var}(X)$

$+ \operatorname{Var}(Y)$

$\operatorname{cov}(X,Y) = \rho_{xy} \cdot \sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}$

$= (-1/2)\sqrt{1 \cdot 1} = -1/2$

$4(1) - 2 + 1 = 3$

$P(2X+Y>0)$

$= P\left(\frac{2X+Y+1}{\sqrt{3}} > \frac{1}{\sqrt{3}}\right)$

$= Q(0.58)$

$\boxed{= 0.281}$

c) $P(2X+Y>0 \mid X-Y=0)$

$E[X] = 1 + E[Y]$

$Var(X) - 2Cov(X,Y) + Var(Y)$

$= 3$

$Cov(Z,W) = 2Cov\,Var(X) - Cov(X,Y)$

$- Var(Y) = 3/2$

$E[Z|W] + \frac{Cov(W,Z)}{Var(W)}(w - \mu_w)$

$= -3/2$

$Var(Z)(1 - \rho_{z,w}^2) = 3(3/4)$

$= 9/4$

$\frac{P(2X+Y>0)}{P(Y=0)}$

$= Q\left(\frac{3/2}{3/2}\right) = Q(1)$

$\boxed{= 0.1587}$