ELSEVIER

European Journal of Operational Research 88 (1996) 3–12

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Invited Review

Stochastic vehicle routing

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Accepted January 1995

Abstract

The purpose of this review article is to provide a summary of the scientific literature on stochastic vehicle routing problems. The main problems are described within a broad classification scheme and the most important contributions are summarized in table form.

Keywords: Stochastic vehicle routing

1. Introduction

The classical *Vehicle Routing Problem* (VRP) is defined on a graph $G=(V,A)$ where $V=\{v_1, v_2,\ldots,v_n\}$ is a set of vertices and $A=\{(v_i,v_j): i \neq j,\ v_i,\ v_j \in V\}$ is the arc set. Vertex v_1 represents a *depot* at which are based m identical vehicles, while the remaining vertices correspond to *cities* or *customers*. A matrix $C=(c_{ij})$ is defined on A. The coefficients c_{ij} represent *distances*, *travel costs* or *travel times*. Here we use these terms interchangeably. The number of vehicles can be a given constant or a decision variable. Each vehicle has the same capacity Q. The VRP is the problem of constructing m vehicle routes of minimum total cost starting and ending at the depot, such that each remaining vertex is visited exactly once by one vehicle, and satisfying some side constraints. Here are the most common types of constraints.

(1) *Capacity constraints*: each city v_i has a demand d_i and the total demand of any route may not exceed the vehicle capacity. In these problems, vehicles make collections or deliveries at all customers, and we exclude from consideration the case where these two types of operations are combined. Delivery and collection problems are symmetrical with one another and equivalent from a modeling point of view. Here problems will be described in terms of collections.

(2) *Duration constraints*: the total length of each route may not exceed a preset constant L.

(3) *Time window constraints*: each vertex v_i must be visited within a time interval $[a_i, b_i]$. For recent survey articles on the VRP see Laporte (1992), Desrosiers et al. (1996) and Fisher (1996). A bibliography is provided in Laporte and Osman (1995)

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SSDI 0377-2217(95)00050-X

Stochastic Vehicle Routing Problems (SVRPs) arise whenever some elements of the problem are random. Common examples are stochastic demands and stochastic travel times. Sometimes, the set of customers to be visited is not known with certainty. In such a case, each customer has a probability p_i of being present. Stochastic VRPs differ from their deterministic counterpart in several fundamental respects. The concept of a solution is different, several fundamental properties of deterministic VRPs no longer hold in the stochastic case, and solution methodologies are considerably more intricate. Since they combine the characteristics of stochastic and integer programs, SVRPs are often regarded as computationally intractable. True enough, only relatively small instances can be solved to optimality and good heuristics are hard to design and assess. Yet, the study of SVRPs has gained in popularity in recent years and much progress has been made towards understanding the structure of these problems and the computational behaviour of related algorithms.

The purpose of this article is to provide a summary of the scientific literature on SVRPs. We begin in Section 2 by outlining the basic solution concepts and methodologies for these problems. Each of the following six sections is devoted to a particular type of SVRP. The conclusion follows in Section 9.

2. Solution concepts and algorithms

Stochastic VRPs can be cast within the framework of stochastic programming. Stochastic programs are modeled in two stages. In a first stage, a planned or "a priori" solution is determined. The realizations of the random variables are then disclosed and, in a second stage, a *recourse* or corrective action is then applied to the first stage solution. The recourse usually generates a cost or a saving that may have to be considered when designing the first stage solution.

To illustrate, consider the VRP with stochastic demands, i.e., each d_i is replaced by a random variable ξ_i. The first stage solution for this problem would consist of a set of m vehicle routes such that each customer is visited exactly once. After the first stage solution has been determined, the actual demands are revealed. It may then be impossible to implement the first stage solution as planned since the total demand of a route may exceed the capacity, i.e., route failures may occur. A possible second stage policy would be to follow each route as planned until the vehicle capacity becomes attained or exceeded, return to the depot to unload, and then resume collections at the customer on the planned route where route failure occurred. In this case, the recourse action consists of performing a return trip to the depot.

A stochastic program is usually modeled either as a *chance constrained program* (CCP) or as a *stochastic program with recourse* (SPR). In CCPs, one seeks a first stage solution for which the probability of failure is constrained to be below a certain threshold. A CCP solution does not take into account the cost of corrective actions in case of failure. In SPRs, the aim is to determine a first stage solution that minimizes the expected cost of the second stage solution: this cost is made up of the cost of the first stage solution, plus the expected net cost of recourse. SPRs are typically more difficult to solve than are CCPs, but their objective function is more meaningful.

For a given problem, corrective actions (or *recourse* policies) may take one of several forms. Consider again the VRP with stochastic demands. Instead of waiting for route failures to occur to perform return trips to the depot, one could plan "preventive breaks" at strategic points along the planned route, preferably when the vehicle is close to the depot and is near full capacity. Another form of recourse could be to reoptimize the remaining portion of the route upon each failure. Such sophisticated recourse policies are of course more involved than simple return trips, and their expected cost may be difficult to take into account in the first stage solution. The best choice of a recourse policy is also linked to the moment at which information is made available. For example, information about a customer's demand may become known upon arriving at the customer's location or before leaving the previous customer on the planned route. In the latter case, the range of recourse actions is wider – one may, for example, postpone the visit of a customer whose demand is too large. For a further discussion of recourse policies for this problem, see Dror, Laporte and Trudeau (1989).

SVRPs are usually modeled as mixed or pure integer stochastic programs, or as Markov decision processes. All known exact algorithms belong to the first category. First note that under some mild assumptions, several classes of chance constrained SVRPs can be transformed into equivalent deterministic VRPs (Stewart and Golden, 1983; Laporte, Louveaux and Mercure, 1989; Bastian and Rinnooy Kan, 1992). Exact algorithms for a number of SVRPs with recourse have been proposed by Laporte, Louveaux and Mercure (1989, 1992, 1994) and by Gendreau, Laporte and Séguin (1995). The last three articles describe implementations of the Integer L-Shaped method (Laporte and Louveaux, 1993), an exact algorithm applicable to a wide range of stochastic integer programs with recourse. This branch-and-cut method computes a first stage solution using a lower bound on the expected cost of recourse. At any feasible and non-dominated solution, the expected cost of recourse is computed exactly and an optimality cut is generated. The effect of such a cut is to replace in the objective function the lower bound on the solution cost by its true value, or force the branching scheme to move to another solution. As these cuts are sometimes weak, they should ideally be used in conjunction with lower bounding functionals. This method has proved successful on a variety of stochastic programs. Another line of research in the area of exact methods has been to exploit particular data structures in some SVRPs. For example, in the single-vehicle VRP with stochastic demands, the demand distribution may be such that at most one failure is likely to occur. In such a case one can easily compute an exact solution that has a very high probability of being optimal by solving a sequence of deterministic problems (Dror, Laporte and Louveaux, 1993). Finally, note that as a rule, dynamic programming does not extend naturally to SVRPs. As shown by Jaillet (1985), Jaillet and Odoni (1988) and Dror, Laporte and Trudeau (1989) the principle of optimality can no longer be verified in SVRPs. However, Carraway, Morin and Moskowitz (1989) have proposed a generalized dynamic algorithm concept that could, in theory at least, be applied to certain classes of SVRPs.

Most algorithms proposed for SVRPs are heuristics, typically adaptations of methods originally designed for the deterministic case. Adapting heuristics to the stochastic case is by no means straightforward as intricate probability computations are usually involved. For example, when merging two routes (as in the Clarke and Wright algorithm (1964), for example), it is usually necessary to compute the expected cost of recourse from any customer on that route to the depot. This type of computation can sometimes be rather complex. It is illustrated for a relatively simple problem in Laporte, Louveaux and Mercure (1989); for a more complicated case, see Bertsimas (1992) or Séguin (1994). Since exact solutions for most classes of SVRPs cannot be computed and lower bounds are usually quite poor, it is difficult to assess the quality of these heuristics. Most authors resort to comparing heuristics with one another. We now proceed to the study of particular types of SVRPs.

3. The Traveling Salesman Problem with Stochastic Customers (TSPSC)

In the *Traveling Salesman Problem with Stochastic Customers* (TSPSC), each vertex v_i is present with probability p_i. In the first stage a Hamiltonian tour through all vertices is constructed and the set of present vertices is then revealed. In the second stage solution, the tour is followed by simply skipping absent customers. This problem was introduced by Jaillet (1985) who described a number of mathematical models, bounds and theoretical properties. In particular, this study shows that an a priori solution obtained by solving a deterministic *Traveling Salesman Problem* (TSP) can be arbitrarily bad for the TSPSC. Another interesting property is that an optimal solution to a TSPSC defined in a plane may cross itself, contrary to what happens for the TSP (Flood, 1956). A number of heuristics using a nearest neighbour criterion or a savings criterion (Clarke and Wright, 1964) were implemented and tested by Jézéquel (1985) and by Rossi and Gavioli (1987). Later, Bertsimas (1988) and Bertsimas and Howell (1993) have further investigated some of the properties of TSPSCs and have proposed new heuristics. These include space filling curves (Bartholdi and Platzman, 1982), a probabilistic 2-opt edge interchange mechanism, and vertex moves within a tour. More recently, Laporte, Louveaux and Mercure (1994) have applied an Integer L-Shaped method to the TSPSC and have solved to optimality instances involving up to 50 vertices. The main contributions relative to the TSPSC are summarized in Table 1.

4. The Traveling Salesman Problem with Stochastic Travel Times (TSPST)

In the *Traveling Salesman Problem with Stochastic Travel Times* (TSPST) the c_{ij} coefficients represent travel times and are random variables. All authors who have treated this problem attempt to determine an a priori solution such that the probability of completing the tour within a given deadline is maximized. To our knowledge, no mathematical model has ever been presented for this problem. Kao (1978) proposes two heuristics for the TSPST. The first is based on dynamic programming while the second makes use of implicit enumeration. Travel time distributions must be such that the probability of a sum of random variables can be readily computed. Sniedovich (1981) has shown that the dynamic programming algorithm could yield sub-optimal solutions as the monotonicity property required by this method is not verified in the TSPST. Later, Carraway, Morin and Moskowitz (1989) have proposed a generalized dynamic programming algorithm to overcome this difficulty and have applied it to the TSPST.

5. The *m*-Traveling Salesman Problem with Stochastic Travel Times (*m*-TSPST)

The *m-Traveling Salesman Problem with Stochastic Travel Times* (*m*-TSPST) is simply the

Table 1
Contributions to the Traveling Salesman Problem with Stochastic Customers

Authors	Year	Distribution [a]	Characteristics and contributions [c]
Jaillet	1985	$p_i = p$ for all i [b]	Ph.D. thesis. Models, properties and bounds. Heuristics based on standard TSP methods. Enumerative exact algorithm. No computations results.
Jézéquel	1985	$p_i = p$ for all i	M.Sc. dissertation. Tests on various heuristics, including some described by Jaillet (1985).
Jaillet	1987	p_i	Properties and bounds. Description of several heuristics and computational results.
Rossi, Gavioli	1987	$p_i = p$ for all i	Tests on nearest neighbour and savings heuristics.
Jaillet, Odoni	1988	p_i	Mostly extracted from Jaillet (1985, 1987) and Jézéquel (1985).
Berman, Simchi-Levi	1988	p_i	Development of a lower bound.
Bertsimas	1988	p_i	Ph.D. thesis. Properties and bounds. Design and tests of several heuristics based on standard TSP methods.
Jaillet	1988	$p_i = p$ for all i [b]	Properties and bounds. Extracted from Jaillet (1985).
Bertsimas, Jaillet, Odoni	1990	$p_i = p$ for all i [b]	Properties, bounds and heuristics. Mostly extracted from Bertsimas (1988).
Bertsimas, Howell	1993	p_i	Properties, bounds and heuristics. Extracted from Bertsimas (1988).
Laporte, Louveaux, Mercure	1994	p_i	Model, properties and bounds. Exact Integer L-Shaped algorithm tested for $n \leq 50$.

[a] Almost exclusively Bernouilli distributions with parameters p_i. Other distributions are sometimes mentioned.
[b] Can be generalized to some extent.
[c] Various heuristics are always compared to one another.

m vehicle version of the TSPST, with all routes starting and ending at a common depot. Typically, the number of vehicles is a decision variable with an associated fixed cost. This problem is addressed in two papers. In each case, a deadline is imposed on each vehicle route and a penalty is incurred for late route completions. The study by Lambert, Laporte and Louveaux (1993) arises from a banking context. The problem is to design money collection routes through bank branches in the presence of stochastic travel times. Late arrival at the depot means that all money contained in the vehicle loses one day's interest. The authors propose a mathematical programming formulation and an adaptation of the Clarke and Wright (1964) savings algorithm. Laporte, Louveaux and Mercure (1992) consider stochastic service times at the vertices as well. Here the penalty for late arrival is proportional to the length of the delay. Formulations are proposed for the chance constrained and recourse versions of the problem. The latter problem is solved to optimality for $n = 10$, 15 and 20, using an Integer L-Shaped method.

6. The Vehicle Routing Problem with Stochastic Demands (VRPSD)

The *Vehicle Routing Problem with Stochastic Demands* (VRPSD) is without any doubt the most studied of all SVRPs. Here, customer demands are random variables usually (but not always) assumed to be independent. We now briefly describe the main contributions to this problem.

To our knowledge, Tillman (1969) was the first to propose an algorithm for the VRPSD, in the case where there are several depots. Penalties are incurred whenever vehicles are almost empty or filled over capacity. The algorithm proposed by Tillman is based on Clarke and Wright (1964). A second major article is due to Stewart and Golden (1983). It contains extensions and generalizations of previous results by Gheysens, Golden, Stewart and Yee (see Table 2). A chance constrained model and two recourse models are presented. The first of these recourse models uses a penalty proportional to the probability of exceeding the vehicle capacity. In the second recourse model, the penalty is proportional to the expected demand in excess of the vehicle capacity. Several demand distributions are considered and two heuristics are tested: one based on Clarke and Wright, and another one based on Lagrangean relaxation. A major contribution to the study of VRPSDs is Bertsimas' thesis (1988). The author derives several bounds, asymptotic results and other theoretical properties for the case where each demand is equal to 1 with probability p_i and to 0 with probability $1 - p_i$. A number of greedy heuristics are proposed and analyzed in an asymptotic sense. In the study by Laporte, Louveaux and Mercure (1989), the depot location is also a decision variable. This article considers more general demand distributions. Exact branch and cut algorithms are presented for the chance constrained version of the problem, and for a bounded penalty model in which the expected cost of recourse cannot exceed a certain percentage of the first stage solution value. Computational results are presented for problems involving up to 30 vertices. Séguin's thesis (1994) and the paper by Gendreau, Laporte and Séguin (1995) describe an Integer L-Shaped algorithm for the recourse model where the penalty functions is the cost of back and forth trips to the depot due to route failures. Depending on the value of some parameters, the problem can be solved to optimality for up to 70 vertices. Finally, we refer to the survey article of Dror, Laporte and Trudeau (1989) which describes a variety of operating and service policies, properties and models for the VRPSD. These authors prove that as in the case of the TSPSC, optimal routes may cross themselves. The main contributions relative to the VRPSD are summarized in Table 2.

7. The Vehicle Routing Problem with Stochastic Customers (VRPSC)

The *Vehicle Routing Problem with Stochastic Customers* (VRPSC) is a direct extension of the TSPSC: Customers are present with some probability but have deterministic demands. The vehicle capacity must be respected and return trips to the depot may be necessary whenever it becomes attained. As in the TSPSC, absent customers are skipped in the second stage solution. All articles except one only treat the case of unit demands. The best source of information on this problem is Bertsimas' thesis (1988) which describes several properties, bounds and heuristics. Waters (1989)

Table 2
Contributions to the Vehicle Routing Problem with Stochastic Demands

Authors	Year	Demand distribution [a]	Model [b]	Characteristics and contributions
Tillman	1969	P ($\lambda = 2$)	—	Savings heuristic, multi-depot.
Golden, Stewart	1978	P (λ_i)	CCP	Savings heuristic (penalty equal to sum of return trips).
Golden, Yee	1979	B, NB, G	CCP	Heuristic, correlated demands, theoretical results.
Yee, Golden	1980	Any	—	Dynamic programming to plan preventive return trips.
Stewart, Golden	1980	—	CCP	Extension of Golden and Stewart (1978).
Stewart	1981	—	SPR	Ph.D. thesis. Heuristics and Lagrangean method.
Stewart, Golden, Gheysens	1982	N	SPR	Continuation of Stewart and Golden (1983), heuristic.
Stewart, Golden	1983	Several	CCP, SPR	Model, savings heuristic, Lagrangean method, survey.
Bodin, Golden, Assad, Ball	1983	Several	CCP	Transformation of SVRP into deterministic VRP.
Dror, Trudeau	1986	Any, N	CCP, SPR	Properties, savings heuristic.
Jaillet	1987	Ber ($p_i = p$) could be generalized	—	One vehicle, 0–1 demands, properties.
Bertsimas	1988	Ber	—	Ph.D. thesis. One vehicle, 0–1 demands, properties, heuristics with worst-case performance.
Jaillet, Odoni	1988	Ber ($p_i = p$) could be generalized	—	One vehicle, 0–1 demands, properties.
Laporte, Louveaux, Mercure	1989	Several	CCP, BPM	Variable depot location, exact constraint relaxation algorithms, $n \leq 30$; bounded penalty model.
Dror, Laporte, Trudeau	1989	Any	CPP, SPR, M	Properties, models, survey.
Waters	1989	Any	SPR	Model.
Laporte, Louveaux	1990	Any	SPR	Model, bounds, preventive return trips.
Bertsimas, Jaillet, Odoni	1990	Ber ($p_i = p$)	—	One vehicle, 0–1 demands, properties, heuristic. Extracted from Bertsimas (1988).
Dror	1992	Any	SPR, M	Model.
Bouzaïene-Ayari, Dror, Laporte	1993	Any	—	Savings heuristic, split deliveries.
Bastian, Rinnooy Kan	1992	Any	CCP, SPR	Model: Time-dependent traveling salesman problem.
Bertsimas	1992	Discrete	—	Properties, bounds, heuristics. Extracts and generalizations of Bertsimas (1988).
Dror, Laporte, Louveaux	1993	Any	CCP, SPR	Restricted failures. Models, heuristic and exact algorithms.
Séguin	1994	Discrete Continuous	SPR	Ph.D. thesis. Exact Integer L-Shaped algorithm for discrete case, $n \leq 70$.
Gendreau, Laporte, Séguin	1995	Discrete	SPR	Extracted from Séguin (1994).

[a] P: Poisson; B: binomial; NB: negative binomial; G: gamma; N: normal; Ber: Bernoulli.
[b] CCP: chance constrained programming; SPR: stochastic programming with recourse; BPM: bounded penalty model; M: Markovian model.

considers non-binary demands and proposes an empirical comparison of three operating policies: (1) follow the planned route without skipping absent customers; (2) skip absent customers; (3) reoptimize the remaining route whenever the absence of a customer is revealed.

Two interesting properties stand out and apply to both the VRPSD and the VRPSC. (1) Even if travel costs are symmetrical (i.e., $c_{ij} = c_{ji}$ for all i, j), the overall solution cost is dependent on the direction of travel (Dror and Trudeau, 1986; Jaillet and Odoni, 1988). (2) Larger vehicle capacities may yield larger solution costs (Jaillet and Odoni, 1988). The main contributions relative to the VRPSC are summarized in Table 3.

8. The Vehicle Routing Problem with Stochastic Customers and Demands (VRPSCD)

The *Vehicle Routing Problem with Stochastic Customers and Demands* (VRPSCD) combines the VRPSC and the VRPSD. The problem is mentioned in early studies by Jézéquel (1985), Jaillet (1987), Jaillet and Odoni (1988) and Trudeau and Dror (1992). The definition proposed by Bertsimas (1992) seems the most interesting. In a first stage, one determines a set of routes starting and ending at the depot and visiting each customer exactly once. The set of customers with zero demand (absent customers) is then gradually revealed, but the positive demand of every remaining customer becomes known only when the vehicle arrives at the customer's location. In the second stage, the first stage routes are followed as planned, with the following two exceptions: (1) any absent customer is skipped; (2) whenever the vehicle capacity becomes exceeded, it returns to the depot to unload, and resumes collections starting at the last visited customer; if for any customer the vehicle capacity becomes exactly attained, the vehicle then returns to the depot and resumes collections at the next present customer along its route. This problem occurs naturally in less-than-truckload operations where carriers often make collections at a set of regular customers on a periodic basis, e.g., daily (see Delorme, Roy and Rousseau, 1988), but not all customers require the vehicle's visit every day and this can be known just before starting collections; these customers are simply dropped from the planned route. The quantity to be collected at a customer is random.

The VRPSCD is an exceedingly difficult problem. Even computing the value of the objective function is hard. Bertsimas (1992) provides a recursive expression for this, as well as bounds, asymptotic results and an analysis of several reoptimization policies. The function evaluation described in Séguin (1994) and in Gendreau, Laporte and Séguin (1995, 1996) is slightly less restrictive in that it allows for full or split deliveries. Note that only one of the versions of the VRPSCD studied by Jaillet and Odoni (1988) and Bertsimas (1992) is in fact a true VRPSCD: this is when absent customers are not visited (otherwise, their problem reduces to a VRPSD). Another study is that of Benton and Rossetti (1992) who consider the case where route reoptimizations are allowed as demands become known.

Séguin (1994) and Gendreau, Laporte and Séguin (1995) proposed the first exact algorithm

Table 3
Contributions to the Vehicle Routing Problem with Stochastic Customers

Authors	Year	Characteristics and contributions
Jézéquel	1985	M.Sc. dissertation. Notion of risk. TSP based heuristic described.
Jaillet	1987	Properties, partially generalized distributions of customers' presence.
Jaillet, Odoni	1988	Extracted from Jaillet (1987).
Bertsimas	1988	Ph.D. thesis. Properties, bounds, asymptotic results and heuristic with worst-case performance.
Waters	1989	Non-binary demands. Comparison of three operating policies.
Bertsimas, Jaillet, Odoni	1990	Survey of Jézéquel (1985), Jaillet (1987), Bertsimas (1988) and Jaillet and Odoni (1988).
Bertsimas	1992	Extracted from Bertsimas (1988).

Table 4
Contributions to the Vehicle Routing Problem with Stochastic Customers and Demands

Authors	Year	Characteristics and contributions
Jézéquel	1985	M.Sc. dissertation. Description of a TSP based heuristic. No test results.
Jaillet	1987	Creation of customers with non-unitary demands by agglomeration from customers with Bernouilli demands.
Jaillet, Odoni	1988	Extracted from Jaillet (1987)
Trudeau, Dror	1992	The VRPSCD is indirectly addressed within the context of inventory-routing. The expected cost of recourse is computed under the assumptions of split deliveries and continuous demand distributions.
Benton, Rossetti	1992	Comparison of several operating policies.
Bertsimas	1992	Asymptotic results, bounds, properties. Recursion formula for the computation of expected cost.
Séguin	1994	Ph.D. thesis. Models, bounds and properties. First exact algorithm using an Integer L-Shaped method ($n \leq 46$). Tabu search heuristic (optimal in 89.45% of cases; average deviation from optimality: 0.38%).
Gendreau, Laporte, Séguin	1995	Integer L-Shaped exact algorithm. Extracted from Séguin (1994).
Gendreau, Laporte, Séguin	1996	Tabu search heuristic. Extracted from Séguin (1994).

for the VRPSDC. Again, it uses an Integer L-Shaped method. Solutions are reported for instances involving up to 46 vertices. One interest of these studies is to show that stochastic customers are a far more complicating factor than stochastic demands. Another interest is that they provide for the first time optimal solutions against which heuristics can be measured. One such heuristic was recently proposed by Gendreau, Laporte and Séguin (1996). It constitutes the first application of tabu search (see, e.g., Glover, 1993) in a stochastic VRP setting. A novelty in this algorithm is the use of a proxy function to avoid computing at each iteration the exact expected cost of a candidate solution. On a set of 825 instances ranging from 6 to 46 vertices, it produces an optimal solution in 89.45% of all cases. The average deviation from optimality is only 0.38% and in 97.8% of all cases it is smaller than 5%. We have summarized in Table 4 the main scientific contributions relative to the VRPSCD.

9. Conclusion

The study of SVRPs is a relatively new and fast growing research area which should gain in importance with the spread of real-time vehicle guidance systems and the need to provide updated instructions to drivers. As compared to the development of research in the deterministic case (see, e.g., Laporte, 1992), research on SVRPs is rather underdeveloped. Two promising research avenues are the development of exact algorithms based on the Integer L-Shaped method, and the construction of tabu search heuristics.

Acknowledgments

The first two authors were partially funded by the Canadian Natural Sciences and Engineering Research Council (grants OGP0038816 and OGP0039682), and by the Quebec Government (FCAR grant 93ER0289). René Séguin was supported by FCAR, by Ultramar Canada, and by the Brussels-Capital Region. This support is gratefully acknowledged.

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