最佳化 HW2　　RE6121011 孫仁瓏

1、 • (Homework) Prove the convexity of the following functions:

- $-\log(x)$ on the positive real line;
- $\exp(x)$ on the real line;
- x^4 on the real line;
- $\|\boldsymbol{x}\|_2^2$ on the n-dimensional space $\mathbb{R}^n$ (for any $n \geq 2$).

Note that the convexity of the domain of each function should be verified.

- A set C is said to be a convex set if for any $\mathbf{x}_1, \mathbf{x}_2 \in C$, the line segment connecting $\mathbf{x}_1$ and $\mathbf{x}_2$ also belongs to C; precisely,

$$\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C, \quad \forall \theta \in [0,1],\ \boldsymbol{x}_1, \boldsymbol{x}_2 \in C.$$

- A function f is called a convex function if the following two conditions hold true:
  1. The function domain D is a convex set.
  2. For $\lambda \in [0,1],\ \boldsymbol{x}, \boldsymbol{y} \in D$,

$$f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}). \tag{5.1}$$

• $f(x) = -\log(x)$, $x \in R^+$

1) R^+ is a convex set :

For any $x_1, x_2 \in R^+$,

$\theta x_1 + (1-\theta)x_2 \in R^+$, $\forall \theta \in [0,1]$

2) eqn (5.1) :

For $\lambda \in [0,1]$, $x_1, x_2 \in R^+$

$$\lambda f(x_1) + (1-\lambda) f(x_2) = -\lambda \log(x_1) - (1-\lambda)\log(x_2)$$
$$= -[\lambda \log(x_1) + (1-\lambda)\log(x_2)]$$
$$= -\log(x_1^{\lambda} \cdot x_2^{1-\lambda})$$

$$f(\lambda x_1 + (1-\lambda)x_2) = -\log(\lambda x_1 + (1-\lambda)x_2)$$

$$\boxed{\begin{array}{l}\text{加權算术不等式：}\\ p_1 x_1 + \cdots + p_n x_n \geq x_1^{p_1} \cdots x_n^{p_n}\ ,\ \forall x_i \in R^+,\ \forall i=1,\dots,n,\ \sum_{i=1}^{n} p_i = 1\end{array}}$$

$\because \lambda x_1 + (1-\lambda)x_2 \geq x_1^{\lambda} x_2^{1-\lambda}$

$\therefore -\log(\lambda x_1 + (1-\lambda)x_2) \leq -\log(x_1^{\lambda} x_2^{1-\lambda})$

$\Rightarrow \lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$

By 1), 2) , $\forall x \in R^+$, $f(x) = -\log(x)$ is a convex function #

• $f(x) = e^x$, $x \in R$

1) R is a convex set :

For any $x_1, x_2 \in R$,

$\theta x_1 + (1-\theta)x_2 \in R$, $\forall \theta \in [0,1]$

2) equ (5.1):

For $\lambda \in [0,1]$, $x_1, x_2 \in R$

$\lambda f(x_1) + (1-\lambda) f(x_2) = \lambda e^{x_1} + (1-\lambda) e^{x_2}$

$f(\lambda x_1 + (1-\lambda) x_2) = e^{\lambda x_1 + (1-\lambda) x_2}$

加权算术不等式:

$p_1 x_1 + \cdots + p_n x_n \geq x_1^{p_1} \cdots x_n^{p_n}$, $\forall x_i \in R^+$, $\forall i = 1, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$

$\because e^k \in R^+$, $\forall k \in R$

$\therefore \lambda e^{x_1} + (1-\lambda) e^{x_2} \geq e^{\lambda x_1} \cdot e^{(1-\lambda) x_2} = e^{\lambda x_1 + (1-\lambda) x_2}$

$\Rightarrow \lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$

By 1), 2), $f(x) = e^x$ is a convex function. ※

- $f(x) = x^4$, $x \in R$

1) R is a convex set:

For any $x_1, x_2 \in R$,

$\theta x_1 + (1-\theta) x_2 \in R$, $\forall \theta \in [0,1]$

2) $f'' \geq 0$:

$f(x) = x^4$, $f'(x) = 4x$, $f''(x) = 4 \geq 0$

By 1), 2), $f(x) = x^4$ is a convex function ※

- $f(x) = \|x\|_2^2$, $x \in R^n$, $n \geq 2$

1) R^n is a convex set:

For any $\begin{cases} x_1 = (x_{11}, x_{12}, \cdots, x_{1n}) \in R^n, n \geq 2 \\ x_2 = (x_{21}, x_{22}, \cdots, x_{2n}) \in R^n, n \geq 2 \end{cases}$

$\theta x_1 + (1-\theta) x_2$

$= (\theta x_{11} + (1-\theta) x_{21}, \theta x_{12} + (1-\theta) x_{22}, \cdots, \theta x_{1n} + (1-\theta) x_{2n}) \in R^n, n \geq 2, \forall \theta \in [0,1]$

2) equ (5.1):

For $\lambda \in [0,1]$, $x_1, x_2 \in R^n$

Let $h(x) = x^2$, $g(x) = \|x\|_2 \Rightarrow h \circ g = \|x\|_2^2$

$$\begin{aligned} \lambda g(x_1) + (1-\lambda) g(x_2) &= \lambda \|x_1\|_2 + (1-\lambda) \|x_2\|_2 \\ &= \|\lambda x_1\|_2 + \|(1-\lambda) x_2\|_2 \\ &\geq \|\lambda x_1 + (1-\lambda) x_2\|_2 \quad \text{by 三角不等式} \\ &= g(\lambda x_1 + (1-\lambda) x_2) \end{aligned}$$

$\Rightarrow g(x) = \|x\|_2$ is a convex function

$h'(x)=2x$, $h''(x)=2>0$

$\Rightarrow h(x)=x^2$ is a convex function

$\because h(x)$ & $g(x)$ is convex function

$\therefore h\circ g$ is a convex function

By 1), 2), $f(x)=\|x\|_2^2$, $x\in\mathbb{R}^n$, $n\geq 2$ is a convex function ※

2、

- (Homework) Prove that the set $C=\{\mathbf{X}\mid\mathbf{X}\succeq\mathbf{A}_i,\ i=1,\ldots,m\}$ (for symmetric $\mathbf{A}_i\in\mathbb{S}^n$) is also a convex set.

  [Hint: the intersection of some convex sets is still a convex set.]

Let $C=\bigcap_{i=1}^{m}C_i$, $C_i=\{X\mid X\succeq A_i\}$

Given $M\in S^n$, $M\succeq 0$ means M is a PSD matrix $\Rightarrow v^TMv\geq 0$, $\forall v\in\mathbb{R}^n$

$X\succeq A_i$ means that $X-A_i\succeq 0$ is PSD

Given $X,Y\in C_i$, $\theta\in[0,1]$

$$v^T(\theta X+(1-\theta)Y-A_i)v$$
$$=\theta v^T(X-A_i)v+(1-\theta)v^T(Y-A_i)v$$
$$\succeq\theta 0+(1-\theta)\cdot 0=0$$

meaning that $\theta X+(1-\theta)Y\succeq A_i$ as well.

So, C_i is convex, $\forall i=1,\cdots,m$

Also, we know if A & B are convex sets, then $A\cap B$ is also a convex set.

and we know C_i is a convex set $\forall i=1,\cdots,m$

So, $\bigcap_{i=1}^{m}C_i=C=\{X\mid X\succeq A_i, i=1,\cdots,m\}$ is a convex set. ※

3、

- (Homework) Prove the equality in (5.7).

  [Hint: you can assume that $\mathcal{S}_2=\mathbb{R}^m$.]

$$z^{q+1}:=\operatorname{prox}_{\frac{1}{c}f_2}(\boldsymbol{A}\boldsymbol{x}^q-\boldsymbol{d}^q).$$

ADMM handles convex optimization problems of the following form

$$\min_{x\in S_1,\,z\in S_2} f_1(x)+f_2(z),\quad \text{s.t. } z=Ax$$

Where $f_1:\mathbb{R}^n\to\mathbb{R}$ and $f_2:\mathbb{R}^m\to\mathbb{R}$ are convex functions,

A is an $m\times n$ matrix and $S_1\subseteq\mathbb{R}^n$, $S_2\subseteq\mathbb{R}^m$ are convex sets.

Tranform the formula into

$$\min_{\substack{x\in S_1\\ z\in S_2}} L(x,z,d) \triangleq \min_{\substack{x\in S_1\\ z\in S_2}} f_1(x)+f_2(z)+\frac{c}{2}\|Ax-z-d\|_2^2$$

$$\Rightarrow z^{q+1} = \arg\min_{z\in S_2} L(x^q, z, d^q)$$

$$= \arg\min_{z\in S_2} f_1(x)+f_2(z)+\frac{c}{2}\|Ax^q-z-d^q\|_2^2$$

$$= \arg\min_{z\in S_2} f_2(z)+\frac{c}{2}\|Ax^q-z-d^q\|_2^2 \quad (\because f_1(x) \text{ is constant for } z)$$

$$= \arg\min_{z\in S_2} f_2(z)+\frac{c}{2}\|v-z\|_2^2 \quad (\text{let } v = Ax^q-d^q)$$

$$= \arg\min_{z\in S_2} \frac{1}{c}f_2(z)+\frac{1}{2}\|v-z\|_2^2 \quad (\text{multiply by } \tfrac{1}{c})$$

$$= \arg\min_{z\in S_2} \frac{1}{c}f_2(z)+\frac{1}{2}\|z-v\|_2^2 \quad (\|v-z\|_2^2 = \|z-v\|_2^2)$$

$$= \mathrm{prox}_{\frac{1}{c}f_2}(v) \quad (\because \mathrm{prox}_f(v) = \arg\min_x[f(x)+\tfrac{1}{2}\|x-v\|_2^2])$$

$$= \mathrm{prox}_{\frac{1}{c}f_2}(Ax^q-d^q) \#$$

4、

- (Homework) Please implement the above CVX command. Please demo your code based on the FIR filter parameterized by $\{h_i\}_{i=-n}^{n}$.

  [Hint: you can freely specify a suitable size (n, P), frequency samples $\{\omega_1,\ldots,\omega_P\} \in [0,\pi]$, as well as the desired frequency response $H_{\text{des}}(\omega_p)$ with "symmetricity" (i.e., $h_i = h_{-i}$).]

```
>> n = 5;
P = 10;
omega = linspace(0, pi, P);
Hdes = cos(rand(P, 1));

cvx_begin
    variables h(n + 1) t;

    minimize(t);

    subject to
    for p = 1 : P
        a=0;
        for i=2:n+1
            a=a+h(i) * cos(omega(p) * (i-1));
        end
        abs(Hdes(p) - h(1) - 2 * a) <= t;
    end
cvx_end

Calling SDPT3 4.0: 30 variables, 17 equality constraints
   For improved efficiency, SDPT3 is solving the dual problem.
------------------------------------------------------------

 num. of constraints = 17
 dim. of socp   var  = 20,     num. of socp blk  = 10
 dim. of linear var  = 10
 number of nearly dependent constraints = 1
 To remove these constraints, re-run sqlp.m with OPTIONS.rmdepconstr = 1.
*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 version  predcorr  gam  expon  scale_data
    NT      1      0.000   1       0
it pstep dstep pinfeas dinfeas  gap      prim-obj      dual-obj    cputime
```

```
-------------------------------------------------------------------
 0|0.000|0.000|1.5e+01|1.0e+01|1.1e+03| 0.000000e+00  0.000000e+00| 0:0:00| chol  1  1
 1|1.000|1.000|7.1e-04|1.0e-01|1.5e+01|-4.730565e-03 -1.455363e+01| 0:0:00| chol  1  1
 2|1.000|0.953|1.9e-07|1.4e-02|6.9e-01|-1.018598e-02 -6.752683e-01| 0:0:00| chol  1  1
 3|0.946|0.986|5.8e-08|1.2e-03|2.0e-01|-9.310840e-02 -2.910260e-01| 0:0:00| chol  1  1
 4|0.894|0.877|1.7e-08|2.3e-04|3.6e-02|-1.225229e-01 -1.581126e-01| 0:0:00| chol  1  1
 5|0.909|1.000|8.8e-09|1.0e-05|1.0e-02|-1.375094e-01 -1.479250e-01| 0:0:00| chol  1  1
 6|0.957|1.000|1.2e-08|1.0e-06|1.6e-03|-1.414233e-01 -1.430077e-01| 0:0:00| chol  1  1
 7|0.891|0.991|7.5e-09|1.1e-07|1.7e-04|-1.421724e-01 -1.423386e-01| 0:0:00| chol  1  1
 8|1.000|0.912|1.5e-07|2.0e-08|3.0e-05|-1.422611e-01 -1.422908e-01| 0:0:00| chol  1  1
 9|0.995|1.000|5.7e-09|2.3e-09|6.0e-07|-1.422758e-01 -1.422764e-01| 0:0:00| chol  1  1
10|0.994|0.997|4.2e-10|4.3e-10|7.7e-09|-1.422761e-01 -1.422761e-01| 0:0:00|
  stop: max(relative gap, infeasibilities) < 1.49e-08
-------------------------------------------------------------------
 number of iterations   = 10
 primal objective value = -1.42276097e-01
 dual   objective value = -1.42276103e-01
 gap := trace(XZ)       = 7.71e-09
 relative gap           = 6.00e-09
 actual relative gap    = 5.15e-09
 rel. primal infeas (scaled problem)   = 4.24e-10
 rel. dual     "        "       "      = 4.35e-10
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 9.1e-01, 8.4e-01, 5.8e-01
 norm(A), norm(b), norm(C) = 1.3e+01, 2.0e+00, 3.7e+00
 Total CPU time (secs)  = 0.07
 CPU time per iteration = 0.01
 termination code       =  0
 DIMACS: 4.2e-10  0.0e+00  8.0e-10  0.0e+00  5.1e-09  6.0e-09
-------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +0.142276
```