最佳化 HW1　數據 114　徐仁瓏 RE6121011

- **Property 5** The set of $n \times n$ symmetric matrices $\mathbb{S}^n$, positive semidefinite (PSD) cone $\mathbb{S}^n_+$ and positive definite (PD) cone $\mathbb{S}^n_{++}$ are all convex sets. $\square$

*Proof:*

($\mathbb{S}^n$)

(Homework) Prove that $\mathbb{S}^n$ is convex.

Convex : $\sum_{i=1}^{n} \theta_i x_i \in C \quad , \forall x_i \in C \ , \ \forall \theta_i \geq 0 \ , \ \sum_{i=1}^{n} \theta_i = 1$

令 $A_\ell = \begin{bmatrix} a_{11}^{\ell} & a_{12}^{\ell} & \cdots & a_{1n}^{\ell} \\ a_{21}^{\ell} & a_{22}^{\ell} & \cdots & a_{2n}^{\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{\ell} & a_{n2}^{\ell} & \cdots & a_{nn}^{\ell} \end{bmatrix} \in S^n \ , \ \forall \ell = 1, \cdots, k$

令 $\theta_\ell \geq 0$ 且 $\sum_{\ell=1}^{k} \theta_\ell = 1$

則 $\sum_{\ell=1}^{k} \theta_\ell \cdot A_\ell = \theta_1 \cdot A_1 + \theta_2 A_2 + \cdots + \theta_k A_k$

$$= \begin{bmatrix} \sum_{\ell=1}^{k} \theta_\ell a_{11}^{\ell} & \sum_{\ell=1}^{k} \theta_\ell a_{12}^{\ell} & \cdots & \sum_{\ell=1}^{k} \theta_\ell a_{1n}^{\ell} \\ \sum_{\ell=1}^{k} \theta_\ell a_{21}^{\ell} & \sum_{\ell=1}^{k} \theta_\ell a_{22}^{\ell} & \cdots & \sum_{\ell=1}^{k} \theta_\ell a_{2n}^{\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\ell=1}^{k} \theta_\ell a_{n1}^{\ell} & \sum_{\ell=1}^{k} \theta_\ell a_{n2}^{\ell} & \cdots & \sum_{\ell=1}^{k} \theta_\ell a_{nn}^{\ell} \end{bmatrix}$$

$\because A_\ell \in S^n , \forall \ell = 1, \cdots, k \Rightarrow a_{ij}^{\ell} = a_{ji}^{\ell} \ , \ \forall \ell = 1, \cdots, k \ , \ \forall i, j = 1, \cdots, n$

$\therefore \sum_{\ell=1}^{k} \theta_\ell a_{ij}^{\ell} = \sum_{\ell=1}^{k} \theta_\ell a_{ji}^{\ell} \ , \ \forall i, j = 1, \cdots, n \quad \Rightarrow \sum_{\ell=1}^{k} \theta_\ell A_\ell \in S^n$

Therefore, $S^n$ is convex. #

- **Property 10** The set of $n \times n$ symmetric matrices $\mathbb{S}^n$, the set of PSD matrices $\mathbb{S}^n_+$ and the set of PD matrices $\mathbb{S}^n_{++} \cup \{\mathbf{0}\}$ are all cones. □

(Homework) Prove this property.

cone: $\{\theta v \mid v \in C, \theta \geq 0\} \subseteq C$

① $S^n$:

令 $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in S^n$ , $\theta \geq 0$

則 $\theta A = \begin{bmatrix} \theta a_{11} & \theta a_{12} & \cdots & \theta a_{1n} \\ \theta a_{21} & \theta a_{22} & \cdots & \theta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \theta a_{n1} & \theta a_{n2} & \cdots & \theta a_{nn} \end{bmatrix}$

又 $\because A \in S^n \Rightarrow a_{ij} = a_{ji}$ , $\forall i,j = 1,\cdots,n$

$\Rightarrow \theta a_{ij} = \theta a_{ji}$ , $\forall i,j = 1,\cdots,n$

$\Rightarrow \theta A \in S^n$

Therefore, $S^n$ is a cone.

② $S^n_+$:

def: A symmetric matrix $M$ with real entries is PSD if the real number $z^T M z$ is nonnegative for every nonzero real column vector $z$.

令 $M \in S^n_+ \Leftrightarrow \vec{z}^T M \vec{z} \geq 0$ , for any $\vec{z} \neq \vec{0}$

$\theta \geq 0$

則 $\vec{z}^T(\theta M)\vec{z} = \theta \vec{z}^T M \vec{z}$ ($\because \theta$ is a scalar)

又 $\vec{z}^T M \vec{z} \geq 0$ 且 $\theta \geq 0$

$\therefore \vec{z}^T(\theta M)\vec{z} = \theta \vec{z}^T M \vec{z} \geq 0$ , for any $\vec{z} \neq \vec{0}$

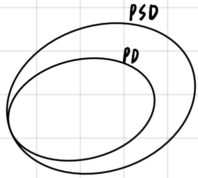
$\therefore \theta M \in S^n_+$

Therefore, $S^n_+$ is a cone.

③ $S_{++}^n \cup \{0\}$ :

def : A symmetric matrix $M$ with real entries is PD if the real number $z^T M z$ is positive for every nonzero real column vector $z$.

In fact, $S_{++}^n \subseteq S_+^n$




Thus, if $S_+^n$ is a cone, then $S_{++}^n$ is a cone.

令 $M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} \in \{0\}$ , 且 $\alpha \geq 0$

則 $\alpha M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n} \in \{0\}$

Therefore, $\{0\}$ is a cone.

$\because$ $S_{++}^n$ is a cone, and $\{0\}$ is a cone.

$\therefore$ $S_{++}^n \cup \{0\}$ is also a cone.

- Think about an interesting question:

  (Homework) Since convex hull, affine hull and conic hull of a finite set $\mathcal{S} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ can be written as ($\theta_i$'s are real)

$$\begin{aligned} \textbf{conv}\,\mathcal{S} &= \{\theta_1 \boldsymbol{x}_1 + \cdots + \theta_n \boldsymbol{x}_n \mid \theta_1 + \cdots + \theta_n = 1,\ \theta_i \geq 0\}, \\ \textbf{aff}\,\mathcal{S} &= \{\theta_1 \boldsymbol{x}_1 + \cdots + \theta_n \boldsymbol{x}_n \mid \theta_1 + \cdots + \theta_n = 1\}, \\ \textbf{conic}\,\mathcal{S} &= \{\theta_1 \boldsymbol{x}_1 + \cdots + \theta_n \boldsymbol{x}_n \mid \theta_i \geq 0\}, \end{aligned}$$

  one may conclude that $\textbf{conv}\,\mathcal{S} = \textbf{aff}\,\mathcal{S} \cap \textbf{conic}\,\mathcal{S}$. Is this conclusion correct? If so, please prove it; otherwise, give a counterexample.

Counterexample: $x_1 = (1,0,0)$ $x_2 = (0,1,0)$ $x_3 = (-1,0,0)$

conv $\{x_1, x_2, x_3\}$ is the triangle with vertices $x_1, x_2, x_3$

aff $\{x_1, x_2, x_3\}$ is the 2D plan passing through $x_1, x_2, x_3$

conic $\{x_1, x_2, x_3\}$ is the X-Y plan for $y \geq 0$