1. trivial; since $\hat{H} = \sum_{\lambda,\nu} \varepsilon_{\lambda\nu} a_\lambda^\dagger a_\nu$

2. $\langle l,k|\hat{V}|i,j\rangle = \int d^3r_1 d^3r_2 d^3r_3 d^3r_4 \langle l,k|r_1 r_2\rangle\langle r_1 r_2|\hat{V}|r_3 r_4\rangle\langle r_3 r_4|i,j\rangle$

$= \int d^3r_1 d^3r_2\, \phi_l^*(r_1)\phi_k^*(r_2) V(|r_1-r_2|)\phi_i(r_2)\phi_j(r_1)$

which is the direct minus exchange.

and for $\hat{V} = \frac{1}{2}\int d^3r\, d^3r'\, \hat{\psi}^\dagger(r)\hat{\psi}^\dagger(r') V(r,r')\hat{\psi}(r')\hat{\psi}(r)$

$\hat{\psi}(r) = \sum_n \hat{a}_n \phi_n(r)$

$\hat{V} = \frac{1}{2}\sum V_{i,j,k,l}\, a_l^\dagger a_k^\dagger a_j a_i$

$\langle a_l^\dagger a_k^\dagger a_j a_i\rangle = \langle a_i^\dagger a_i a_j^\dagger a_j\rangle(\delta_{k,j}\delta_{l,i} - \delta_{j,l}\delta_{k,i})$

$\therefore \langle a_i^\dagger a_i a_j^\dagger a_j\rangle = \langle a_i^\dagger a_i\rangle\langle a_j^\dagger a_j\rangle$

$\therefore \langle a_l^\dagger a_k^\dagger a_j a_i\rangle = \langle a_l^\dagger a_i\rangle\langle a_k^\dagger a_j\rangle - \langle a_k^\dagger a_i\rangle\langle a_l^\dagger a_j\rangle$

3. $\hat{\psi}_\sigma(r) = \sum_n \hat{a}_{n,\sigma} \phi_n(r)$

$\because \begin{cases} [\hat{a}_{m,\sigma}, \hat{a}^{\dagger}_{n,\sigma'}]_\pm = \delta_{m,n}\delta_{\sigma,\sigma'} \\ [\hat{a}_{m,\sigma}, \hat{a}_{n,\sigma'}]_\pm = [a^{\dagger}_{m,\sigma}, a^{\dagger}_{n,\sigma'}]_\pm = 0 \end{cases}$

$$[\hat{\psi}^{\dagger}_\sigma(r), \hat{\psi}_{\sigma'}(r')]_\pm = \sum_{m,n} \phi^{*}_m(r)\phi_n(r')[a^{\dagger}_{m,\sigma}, \hat{a}_{n,\sigma'}]_\pm$$

$$= \delta_{\sigma,\sigma'}\delta^{(3)}(r-r')$$

$$\therefore [\hat{\psi}_\sigma(r), \hat{\psi}_{\sigma'}(r')]_\pm = \sum_{m,n} \phi^{*}_m(r)\phi_n(r')[\hat{a}_{m,\sigma}, \hat{a}_{n,\sigma'}]_\pm$$

$$= 0$$

similar to $[\hat{\psi}^{\dagger}_\sigma(r), \hat{\psi}^{\dagger}_{\sigma}(r')]_\pm$