Investigation about Orbit of Spacecraft around Mercury

Extended Essay: Physics

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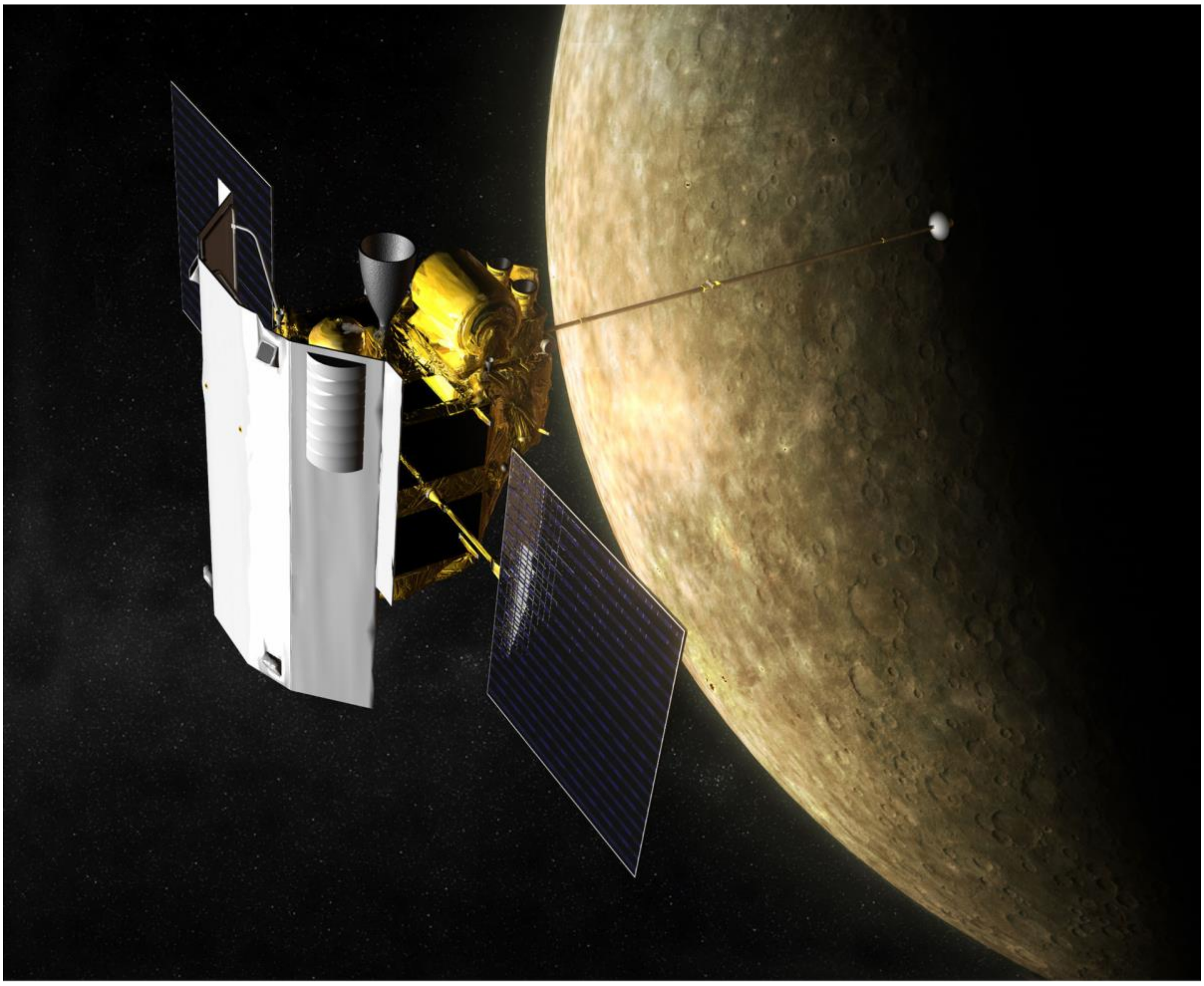

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1. Introduction

This essay will discuss the natural frozen orbit and artificial frozen orbit of the spacecraft around Mercury, and the research question is **How could a frozen orbit around Mercury be designed**?

Mercury is one of the major planet of solar system. The investigation is significant for humankind. Therefore, the design of the orbit of spacecraft around Mercury is important. In history there are three spacecraft which investigated Mercury before. The most well-known spacecraft is MESSENGER. It was a NASA robotic spacecraft that orbited the planet Mercury between 2011 and 2015. The spacecraft was launched aboard a Delta II rocket in August 2004 to study Mercury's chemical composition, geology, and magnetic field.[1] The mission of MESSENGER opened the gate to investigate Mercury further and it will be recorded in history.

In the year of 2017, I attended the Space Settlement Design Competition. One contest aimed to design a space station on Mercury to collect minerals. My task on the team was to develop the orbit of a space station. At that time, I am not able to design a perfect orbit of the space station because the model is very complex and need advanced knowledge of both mechanics and mathematics. However, I remembered this problem. I started to search background information and read books about space mechanics and mathematics. After a long time of the investigation, I think that I am able to tackle this problem, so I took this problem as the research question of my extended essay.

In 1959, Kozai[2] proposed the method of mean orbital elements. This method uses orbital elements as variables. It constructs the power series solutions of the equations of perturbation. This method helps the investigation of the long-term change of orbits. After Kozai proposed the method, Brouwer[3] proposed the method of mean orbital elements via Von-Zeipel transformation. This method constructed on Hamilton's canonical equation which is represented by Delaunay variables. Then, Hori[4] and Deprit[5] investigated the construction of canonical transformation via Lie series. The generation function of Lie transformation is an explicit function. It facilitates the use of it in the orbital mechanics.

In this essay, I use the method which is talked above to discuss the natural frozen orbit. And use the discussion about natural frozen orbit as a foundation to design the artificial orbit. In chapter 4, I propose a method of using continuous force to control the orbit. At the end of this chapter, I evaluate the effectiveness of the method via the calculation of examples.

2. The value of parameters used in the essay and the list of symbol

Parameters	*Value*
Standard gravity parameter of Sun $\mu_s\ (N \cdot m^2/kg)$	1.3271×10^{20}
Standard gravity parameter of Mercury $\mu\ (N \cdot m^2/kg)$	2.2032×10^{15}
The semi-major axis of Mercury $a_s\ (m)$	5.7909×10^{10}
Eccentricity of Mercury e_s	0.2056
The period of revolution (day)	87.9691
The period of rotation (day)	58.6460
Axial Tilt (°)	$2.04' \pm 0.08'$
Average radius $R\ (m)$	2.4398×10^6
Zonal harmonics term J_2	6.0×10^{-5}
Zonal harmonics term J_3	3.0×10^{-5}

(Table 1 The value of parameters used in the essay)[6 7 8]

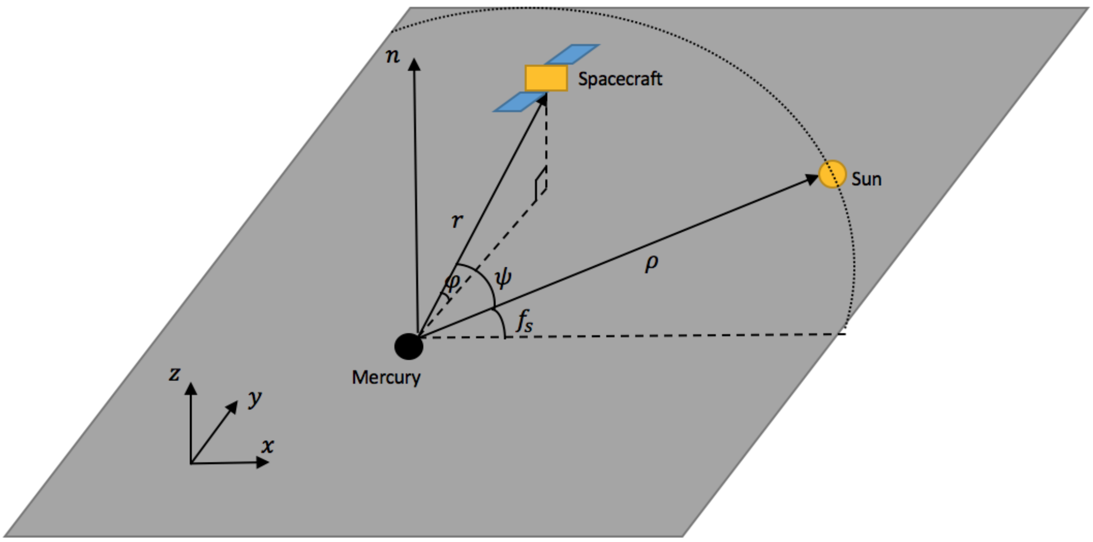
SYMBOL	MEANING
$\boldsymbol{a}$	The semi-major axis of orbit of spacecraft
$\boldsymbol{a_s}$	The semi-major axis of revolution of Mercury
$\boldsymbol{e}$	The eccentricity of the orbit of spacecraft
$\boldsymbol{e_s}$	The eccentricity of the revolution of Mercury
$\boldsymbol{i}$	It is the dip angle of the orbit of spacecraft
$\boldsymbol{\Omega}$	It is the right ascension of the ascending node of the spacecraft
$\boldsymbol{\omega}$	It is the argument of the perigee of the spacecraft
$\boldsymbol{f}$	It is the true anomaly of the spacecraft
$\boldsymbol{f_s}$	It is the true anomaly of the revolution of Mercury
$\boldsymbol{l, g, h, L, G, H}$	Delaunay variables
$\boldsymbol{r}$	The vector from Mercury to spacecraft
$\boldsymbol{\rho}$	The vector from Mercury to Sun
$\boldsymbol{\psi}$	It is the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$
$\boldsymbol{\varphi}$	Declination of the spacecraft to Mercury
$\boldsymbol{\mu_s}'$	The parameter of perturbation
$\boldsymbol{R}$	The average radius of Mercury
$\boldsymbol{R_{Mercury}}$	It is the perturbation function of Mercury
$\boldsymbol{R_{Sun}}$	It is the perturbation function of Sun
$\mathbf{H}$	The Hamilton's function
$\boldsymbol{P_k}$	Legendre polynomials
$\boldsymbol{u}$	It is the argument of orbit around Mercury

(Table 2 Symbols and explanations)

3. Natural Frozen Orbit

3.1 Hamilton's System

The model of the spacecraft around the orbit of Mercury could be considered as a two-body model which affected by the perturbing force of the Sun. Therefore, the perturbing forces which are considered by the essay will be the perturbing forces from the shape of Mercury and perturbing forces of Sun. This essay will assume that the plane of rotation of Mercury coincide with the plane of revolution of Mercury because the axial tilt of Mercury is only $2.04' \pm 0.08'$. And this essay will consider the zonal terms J_2 and J_3 for the non-spherical perturbation of Mercury.



(Fig 1 the schematic diagram of the system)

The model only considers the perturbative forces. Thus, there is no dissipative force in this model.

Hamilton's function of the spacecraft in this situation

$$\mathbf{H} = V_0 + R_{Mercury} + R_{Sun} \quad 9$$

And, it is easy to get the following functions by deduction

$$V_0 = \frac{1}{2}v^2 - \frac{\mu}{r}$$

μ is the standard gravitational parameter of Mercury.

The non-spherical function of Mercury

$$R_{Mercury} = \frac{\mu}{r}J_2\left(\frac{R}{r}\right)^2 P_2(sin\varphi) + \frac{\mu}{r}J_3\left(\frac{R}{r}\right)^3 P_3(sin\varphi)$$

P_2 is the Legendre polynomial.

The third-body perturbation

$$R_{Sun} = -\frac{\mu_s}{|\boldsymbol{r}-\boldsymbol{\rho}|} + \frac{\mu_s}{\boldsymbol{\rho}^3}\boldsymbol{r}\cdot\boldsymbol{\rho} = -\frac{\mu_s}{|\boldsymbol{r}-\boldsymbol{\rho}|} + \frac{\mu_s}{\rho^3}rcos\psi$$

$\mu_s$is the standard gravitational parameter of Sun.

And take notice of $\frac{1}{|r-\rho|}$, $\frac{1}{|r-\rho|}$ could be expanded as a Legendre polynomial when r>> ρ.

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{\rho}|} = \frac{1}{\rho}\sum_{j=0}^{\infty}\left(\frac{r}{\rho}\right)^j P_j(cos\psi) \quad 10$$

Because $P_k = \frac{1}{2^k k!}\frac{d^k}{dx^k}[(x^2-1)^k]$

Thus,

$$R_{Sun} = -\frac{\mu_s}{|\boldsymbol{r}-\boldsymbol{\rho}|} + \frac{\mu_s}{\rho^3}rcos\psi = -\frac{\mu_s}{\rho}\sum_{j=2}^{\infty}\left(\frac{r}{\rho}\right)^j P_j(cos\psi)$$

The deduction of the above equation ignores the 0-degree term $-\frac{\mu_s}{\boldsymbol{\rho}}$, because it does not consist the variables of status. It cannot affect the solution of the Hamilton canonical equation.

And it could be simply deduced by calculation that the term $\frac{\mu_s}{a_s}(\frac{r}{a_s})^3$ is less than $\frac{1}{1000}$ of $\frac{\mu_s}{a_s}(\frac{r}{a_s})^2$. Thus, the third-body perturbation only need to consider the first and second term.

Therefore,

$$R_{Sun} = -\frac{\mu_s}{\rho}\left(\frac{r}{\rho}\right)^2 P_2(cos\psi)$$

It is easy to deduce the following relationships geometrically

$$\boldsymbol{r} = r\begin{bmatrix} cos\Omega\, cos(\omega + f) - sin\Omega\, sin(\omega + f)cosi \\ sin\Omega\, cos(\omega + f) + cos\Omega\, sin(\omega + f)\, cos\, i \\ sin(\omega + f)sini \end{bmatrix}$$

$$r = \frac{a(1 - e^2)}{1 + ecosf}$$

And,

$$\boldsymbol{\rho} = \rho\begin{bmatrix} cosf_s \\ sinf_s \\ 0 \end{bmatrix}$$

$$\rho = \frac{a_s(1 - {e_s}^2)}{1 + e_s cosf}$$

$$sin\varphi = \frac{z}{r} = sin\,(\omega + f)sini$$

$$cos\psi = \frac{1}{r\rho}\boldsymbol{r} \cdot \boldsymbol{\rho}$$

Then, substitute those equations above into Hamilton's function and use *u* to represent $\omega + f$.

$$\mathbf{H} = -\frac{\mu}{2a} - \frac{\mu}{4r}J_2\left(\frac{R}{r}\right)^2 (2 - 3sin^2 i + 3sin^2 i\ cos2u)$$

$$-\frac{\mu}{8r}J_3\left(\frac{R}{r}\right)^3 (12sini \cdot sinu - 15sin^3 i \cdot sinu + 5sin^3 i \cdot sin3u)$$

$$-\frac{\mu_s}{16\rho}J_3\left(\frac{r}{\rho}\right)^3 (4 - 6sin^2 i + 6sin^2 i \cdot cos2u + 6sin^2 i \cdot cos(2\Omega - 2f_s)$$

$$+ 3(1 + cosi)^2 \cos(2\Omega + 2\mathrm{u} - 2f_s) + 3(1 - cosi)^2 \cdot \cos\,(2\Omega - 2\mathrm{u} - 2f_s))$$

3.2 Equalization of Orbit

The Hamilton canonical equations

$$\begin{cases} \dfrac{dq_j}{dt} = \dfrac{\partial \mathbf{H}}{\partial p_j} \\ \dfrac{dp_j}{dt} = -\dfrac{\partial \mathbf{H}}{\partial q_j} \end{cases} \quad 11$$

(j is an integer)

Thus, in this model

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial \mathbf{H}}{\partial \dot{x}}, \dfrac{d\dot{x}}{dt} = -\dfrac{\partial \mathbf{H}}{\partial x} \\ \dfrac{dy}{dt} = \dfrac{\partial \mathbf{H}}{\partial \dot{y}}, \dfrac{d\dot{y}}{dt} = -\dfrac{\partial \mathbf{H}}{\partial y} \\ \dfrac{dx}{dt} = \dfrac{\partial \mathbf{H}}{\partial \dot{x}}, \dfrac{d\dot{z}}{dt} = -\dfrac{\partial \mathbf{H}}{\partial z} \end{cases}$$

For making the system more intuitive, introduce Delaunay variables into this model.

$$\begin{cases} l = M \\ g = \omega \\ h = \Omega \\ L = \sqrt{\mu a} \\ G = \sqrt{\mu a(1 - e^2)} \\ H = \sqrt{\mu a(1 - e^2)}cosi \end{cases} \quad 12$$

There is a canonical transformation between $[l, g, h]$, $[L, G, H]$ and $r, \dot{r}$.

Therefore, the Hamilton canonical equations could be written as the form below.

$$\begin{cases} \dot{l} = \frac{\partial \mathbf{H}}{\partial \dot{L}}, \dot{L} = -\frac{\partial \mathbf{H}}{\partial x} \\ \dot{g} = \frac{\partial \mathbf{H}}{\partial \dot{G}}, \dot{G} = -\frac{\partial \mathbf{H}}{\partial g} \\ \dot{h} = \frac{\partial \mathbf{H}}{\partial \dot{H}}, \dot{H} = -\frac{\partial \mathbf{H}}{\partial h} \end{cases}$$

Back to Hamilton's function, it is easy to see that there are two periodical terms, one of it is true anomaly f which is relate to the motion of the spacecraft and the other one is the true anomaly f_s which is relate to the motion of mercury around the Sun. These periodical terms are not good for analyzing the long-term motion of the spacecraft. For solving this problem, I will use the Lie transformation to equalize the two periodical terms respectively.

Firstly, use the functions below[13] to equalize f

$$\overline{\left(\frac{a}{r}\right)} = 1, \overline{\left(\frac{a}{r}\right)^p sınqf} = 0$$

$$\overline{\left(\frac{a}{r}\right)^p cosqf} = 0 \ (p \geq 2, q \geq p-1)$$

$$\overline{\left(\frac{a}{r}\right)^p cosqf} = (1-e^2)^{-(p-\frac{3}{2})} \sum_{n(2)=q}^{(p-2)-\delta} \binom{p-2}{n} \binom{n}{\frac{1}{2}(n-q)} \left(\frac{e}{2}\right)^n$$

After the first equalization, the original Hamilton's equation changes to the form below.

$$\bar{\mathbf{H}} = -\frac{1}{2}\frac{\mu^2}{\bar{L}^2} + \frac{1}{4}\frac{\mu^4 J_2 R^2}{\bar{L}^3\bar{G}^3}\left(1 - 3\frac{\bar{H}^2}{\bar{G}^2}\right) + \frac{3}{8}\frac{\mu^5 J_3 R^3}{\bar{L}^3\bar{G}^5}\sqrt{1 - \frac{\bar{G}^2}{\bar{L}^2}}sin\bar{g}\sqrt{1 - \frac{\bar{H}^2}{\bar{G}^2}}\left(1 - 5\frac{\bar{H}^2}{\bar{G}^2}\right)$$

$$-\frac{1}{16}\frac{\mu_s\bar{L}^4}{\mu^2\rho^3}(-2 + 6\frac{\bar{H}^2}{\bar{G}^2} + 15\left(1 - \frac{\bar{G}^2}{\bar{L}^2}\right)\left(1 - \frac{\bar{H}^2}{\bar{G}^2}\right)cos2\bar{g}$$

$$+ 3\left(1 - \frac{\bar{H}^2}{\bar{G}^2}\right)\left(5 - 3\frac{\bar{G}^2}{\bar{L}^2}\right)cos\left(-2f_s + 2\bar{h}\right)$$

$$+ \frac{15}{2}\left(1 - \frac{\bar{G}^2}{\bar{L}^2}\right)\left(1 + \frac{\bar{H}}{\bar{G}}\right)^2 cos\left(-2f_s + 2\bar{h} + 2\bar{g}\right)$$

$$+ \frac{15}{2}\left(1 - \frac{\bar{G}^2}{\bar{L}^2}\right)\left(1 - \frac{\bar{H}}{\bar{G}}\right)^2 cos\left(-2f_s + 2\bar{h} - 2\bar{g}\right))$$

In this equation $\bar{h}, \bar{g}$, $\bar{G}, \bar{L}, \bar{H}$ is the Delaunay variables after equalization. After the first equalization, there is no term which contain f.

Then proceed the second equalization, use the functions[14] below

$$\overline{\left(\frac{\rho}{a_s}\right)^2} = 1 + \frac{3}{2}{e_s}^2$$

$$\overline{\left(\frac{\rho}{a_s}\right)^p sinqf_s} = 0$$

$$\overline{(\frac{\rho}{a_s})^2 cos2f_s} = \frac{5}{2}{e_s}^2$$

After the second equalization Hamilton's function

$$\bar{\bar{\mathbf{H}}} = -\frac{1}{2}\frac{\mu^2}{\bar{\bar{L}}^2} + \frac{1}{4}\frac{\mu^4 J_2 R^2}{\bar{\bar{L}}^3\bar{\bar{G}}^3}\left(1 - 3\frac{\bar{\bar{H}}^2}{\bar{\bar{G}}^2}\right) + \frac{3}{8}\frac{\mu^5 J_3 R^3}{\bar{\bar{L}}^3\bar{\bar{G}}^5}\sqrt{1 - \frac{\bar{\bar{G}}^2}{\bar{\bar{L}}^2}}sin\bar{\bar{g}}\sqrt{1 - \frac{\bar{\bar{H}}^2}{\bar{\bar{G}}^2}}\left(1 - 5\frac{\bar{\bar{H}}^2}{\bar{\bar{G}}^2}\right)$$

$$-\frac{1}{16}\frac{\mu_s\bar{\bar{L}}^4}{\mu^2{a_s}^3(1 - {e_s}^2)^{\frac{3}{2}}}(\left(3\frac{\bar{\bar{H}}^2}{\bar{\bar{G}}^2} - 1\right)\left(5 - 3\frac{\bar{\bar{G}}^2}{\bar{\bar{L}}^2}\right)$$

$$+ 15\left(1 - \frac{\bar{\bar{H}}^2}{\bar{\bar{G}}^2}\right)\left(1 - \frac{\bar{\bar{G}}^2}{\bar{\bar{L}}^2}\right)cos2\bar{\bar{g}})$$

In this equation $\bar{\bar{g}}, \bar{\bar{L}}, \bar{\bar{G}}, \bar{\bar{H}}$ are the Delaunay variables after two equalizations. And it is interesting to note that the term of the right ascension of ascending node of the spacecraft $h(\Omega)$ is eliminated during the process of the elimination of term f_s. In fact, the term of the right ascension will be eliminated when the rotation of star coincides with the revolution plane of the star. In general view, the function will still contain the term $h(\Omega)$.

In the two process of equalization, there are generating functions.

$$V = \frac{1}{n\sqrt{1-e^2}} \int \left(\frac{r}{a}\right)^2 (\boldsymbol{H} - \bar{\boldsymbol{H}}) df$$

$$W = \frac{1}{n_s\sqrt{1-{e_s}^2}} \int \left(\frac{\rho}{a_s}\right)^2 (\bar{\boldsymbol{H}} - \bar{\bar{\boldsymbol{H}}}) df_s$$

In these equations, n is the speed of the spacecraft, $n = \sqrt{\frac{\mu}{a^3}}$ and n_s is the angular speed of the revolution, $n_s = \sqrt{\frac{\mu_s}{{a_s}^3}}$.

In this two equalization, the transformation of the Delaunay variables

$$\begin{cases} \bar{l} = \bar{\bar{l}} + \dfrac{\partial W}{\partial \bar{\bar{L}}}, & \bar{L} = \bar{\bar{L}} \\ \bar{g} = \bar{\bar{g}} + \dfrac{\partial W}{\partial \bar{\bar{G}}}, & \bar{G} = \bar{\bar{G}} - \dfrac{\partial W}{\partial \bar{\bar{g}}} \\ \bar{h} = \bar{\bar{h}} + \dfrac{\partial W}{\partial \bar{\bar{H}}}, & \bar{H} = \bar{\bar{H}} \end{cases}$$

$$\begin{cases} l = \bar{l} + \dfrac{\partial V}{\partial \bar{L}}, L = \bar{L} - \dfrac{\partial V}{\partial \bar{l}} \\ g = \bar{g} + \dfrac{\partial V}{\partial \bar{G}}, G = \bar{G} - \dfrac{\partial V}{\partial \bar{g}} \\ h = \bar{h} + \dfrac{\partial V}{\partial \bar{H}}, H = \bar{H} \end{cases}$$

For simplification, the Delaunay variables after equalization will be written as

$[l, g, h, L, G, H]$.

Back to Hamilton's function, the function only has g, L, G, H four variables. Thus, the canonical equation changes to

$$\begin{cases} \dot{l} = \dfrac{\partial \mathbf{H}}{\partial L}, & \dot{L} = 0 \\ \dot{g} = \dfrac{\partial \mathbf{H}}{\partial G}, & \dot{G} = -\dfrac{\partial \mathbf{H}}{\partial g} \\ \dot{h} = \dfrac{\partial \mathbf{H}}{\partial H}, & \dot{H} = 0 \end{cases}$$

It is easy to see that *L* and *H* are constants. It means that the second row of these equations could be decoupled with equations of other two rows.

Substitute Hamilton's function into the equations above.

I obtained that

$$\dot{g} = \frac{3}{4}J_2 \frac{\mu^4 R^2}{L^3 G^6}(5H^2 - G^2) + \frac{3}{8}J_3 \frac{\mu^5 R^3 sing}{G^9 L^4 \sqrt{L^2 - G^2}\sqrt{G^2 - H^2}}(41H^2L^2G^2 - 40H^4L^2 - 35H^2G^4 + 35H^4G^2 - 5G^4L^2 + 4G^6) + \frac{3}{8}\mu_s \frac{L^2}{\mu^2 {a_s}^3 (1 - {e_s}^2)^{\frac{3}{2}} G^3}(5H^2L^2 - G^4 - 5H^2L^2 cos2g + 5G^4 cos2g)$$

$$\dot{G} = -\frac{3}{8}J_3 \frac{\mu^5 R^3 \sqrt{L^2 - G^2}\sqrt{G^2 - H^2 cosg}}{L^4 G^8}(G^2 - 5H^2) - \frac{15}{8}\mu_S \frac{L^2 sin2g}{\mu^2 {a_s}^3 (1 - {e_s}^2)^{\frac{3}{2}} G^2}(G^2 - H^2)(L^2 - G^2)$$

These equations are enough to work out the change of g and G with time. But compare with the change of g and G it is more important to prove the existence of specific solutions and the stability of these solutions. Thus, it necessary to find the libration point of Hamilton's canonical equation.

3.3 The Frozen Orbit

After the equalization, the analyzation of long-term motion of spacecraft is possible. The following content is about the frozen orbit of the spacecraft

For simplification, in this part and following part of this essay $\bar{\bar{a}}, \bar{\bar{e}},\ \bar{\bar{\imath}}, \bar{\bar{\omega}}$ will be written as a, e, i, ω.

Firstly, start at the simplest situation $\dot{G} = 0$

Because of

$$\dot{G} = -\frac{3}{8} J_3 \frac{\mu^5 R^3 \sqrt{L^2 - G^2}\sqrt{G^2 - H^2 cosg}}{L^4 G^8}(G^2 - 5H^2)$$

$$-\frac{15}{8}\mu_S \frac{L^2 sin2g}{\mu^2 {a_s}^3 (1 - {e_s}^2)^{\frac{3}{2}} G^2}(G^2 - H^2)(L^2 - G^2)$$

There are two equations which could be easily obtained.

$$cosg = 0$$

$$sing = -\frac{1}{10}\frac{J_3}{\mu_s}\frac{\mu^7 R^3 {a_s}^3 (1 - {e_s}^2)^{\frac{3}{2}}(G^2 - 5H^2)}{L^6 G^6 \sqrt{G^2 - H^2}\sqrt{L^2 - G^2}}$$

Thus,

$$g = \pm\frac{\pi}{2}, \qquad g = arcsin\left(\frac{c_2}{c_1}\right), \qquad g = \pi - arcsin\,(\frac{c_2}{c_1})$$

In these solutions above

$$c_1 = 10\mu_s L^6 G^6 \sqrt{G^2 - H^2}\sqrt{L^2 - G^2}$$

$$c_2 = -\mu^7 J_3 R^3 {a_s}^3 (1-{e_s}^2)^{\frac{3}{2}} (G^2 - 5H^2)$$

Then, substitute all the solutions in $\dot{g} = 0$, then it could get the solution of the *G* of the frozen orbit. Because, the function of $\dot{g}$ contain the term $sin\, g$and $cos\, 2g$, thus $g = arcsin\left(\frac{c_2}{c_1}\right)$ and $g = \pi - arcsin\,(\frac{c_2}{c_1})$ have the same solution of *G*.

In the second condition $L^2 - G^2 = 0$ (e=0), it could get $\dot{G} = 0$ too. For investigating the existence of frozen orbit at this situation and eliminating the singularity of g, use the canonical transformation.

$$\begin{cases} \eta = \sqrt{L^2 - G^2} sing \\ \xi = \sqrt{L^2 - G^2} cosg \end{cases} \quad 15$$

When *e=0 (*$\eta = 0, \xi = 0$*)*

$$\left.\frac{\partial \mathbf{H}}{\partial \eta}\right|_{\eta=0,\xi=0} = \frac{3}{8} \frac{\mu^5 R^3 \sqrt{L^2 - H^2}(L^2 - 5H^2)}{L^{12}}$$

$$\left.\frac{\partial \mathbf{H}}{\partial \xi}\right|_{\eta=0,\xi=0} = 0$$

Bring in a new parameter $\sigma = \frac{H}{L} = \sqrt{1-e^2} cosi$, the equations above mean that there is frozen orbit if and only if $\sigma = 1\ (i = 0°) or\ \sigma = \sqrt{\frac{1}{5}} (i = 64.3°)$ when $J_3 \neq 0$. In other word the non-spherical perturbation limits the existence of frozen orbit when eccentricity is 0.

For the Hamilton system that the essay is discussing, the track of the spacecraft is a closed curve in the 2-D phase plane[16]. It means there is no asymptotic stability libration point. If the libration point (g_0, G_0) is the local maximum or local minimum of Hamilton's system, this point is Lyapunov stable. Otherwise, the libration point is not stable.

Thus, use the Hessian matrix to verify it is extremum value or not.

For libration point (g_0, G_0)

$$\mathrm{H}(g_0, G_0) = \begin{bmatrix} \frac{\partial^2 \mathbf{H}}{\partial g^2} & \frac{\partial^2 \mathbf{H}}{\partial g \partial G} \\ \frac{\partial^2 \mathbf{H}}{\partial G \partial g} & \frac{\partial^2 \mathbf{H}}{\partial G^2} \end{bmatrix}_{(g_0, G_0)} \qquad 17$$

(H means the Hessian matrix, it is not a Delaunay variable)

when H is positive-definite or negative-definite matrix, H have the local extreme value and the orbit is stable. If H is an indefinite matrix, the orbit is not stable.

3.3 Brief Summary of the chapter

In this chapter, I investigated the orbit of the spacecraft around Mercury in natural conditions. According to the space condition of the spacecraft, I design a Hamilton's system depends on the perturbation of both Mercury and Sun. Then I bring in the Delaunay variables for making the system more intuitive and simplifying the calculation. I decided to equalize Hamilton's function for eliminating the periodical terms in the function because periodical terms are adverse to investigate the long-term change of orbital elements. I obtained Hamilton's canonical equations, and got a system of single degree of freedom.

For the system of single degree of freedom, I listed the conditions of the existence of libration points. I discussed the one simple example of frozen orbit. And I brought in Hessian matrix for judging the stability of frozen orbits. Due to the limitation of words and content, it is not possible to discuss this issue deeper. This chapter lay a solid foundation for the discussion of next chapter----artificial frozen orbit.

4. Artificial Frozen Orbit

The height of perigee of the orbit around mercury will change significantly because of the influence of the third body perturbation. It possible to collide with mercury. In the real-life situation, the orbit of the spacecraft will be edit periodically because the condition to form a frozen orbit around mercury is rigor. So the following content will discuss the artificial frozen orbit around the mercury to suit the real-life demand.

4.1 Objective function

The conditions to form frozen orbit are $A[J_2, J_{3,}R, \mu_S, a_s, e_s]$. If those parameters could be modified artificially, we could get any orbit that we want for fulfilling the demand of the space mission. We could use the thrust of the spacecraft to change the original perturbation parameter $A[J_2, J_{3,}R, \mu_S, a_s, e_s]$ to $A'[J_2', J_3', R', \mu_S', a_s', e_s']$.

Now we could write the function of motion of the spacecraft.

$$\ddot{r} = -\frac{\mu}{r^3}\boldsymbol{r} + f(\boldsymbol{r}, t, A) + \mathbf{u}(\mathrm{t})$$

$$= -\frac{\mu}{r^3}\boldsymbol{r} + f(\boldsymbol{r}, t, A) + f(\boldsymbol{r}, t, A') - f(\boldsymbol{r}, t, A)$$

$$= -\frac{\mu}{r^3}\boldsymbol{r} + f(\boldsymbol{r}, t, A')$$

In the equation, $f(\boldsymbol{r}, t, A) = -\begin{bmatrix} \frac{\partial \mathbf{H}}{\partial x} \\ \frac{\partial \mathbf{H}}{\partial y} \\ \frac{\partial \mathbf{H}}{\partial z} \end{bmatrix}$ is the perturbation acceleration of the original parameter. And in this function $\mathbf{H}$ is Hamilton's function before equalization. And

$\boldsymbol{u}(t) = \begin{bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{bmatrix} = f(\boldsymbol{r}, t, A') - f(\boldsymbol{r}, t, A)$ is the acceleration which could be control of the spacecraft. It is the difference between the original perturbation parameter and the required perturbation parameter. Thus, the required perturbation parameter could be perceived as controlled parameter of a continuous force. If $A' = [J_2', J_3', R', \mu_S', a_s', e_s']$ fulfills the condition of the formation of frozen orbit, the artificial frozen orbit could be formed.

Actually, for a group of average orbital element, there are infinite number of groups of perturbation parameter could form the frozen orbit. Thus, my work could not only find those perturbation parameters, I should optimize the parameters. And find the best group of perturbation parameters $\mathrm{A}'^{*}[J_2'^{*}, J_3'^{*}, R'^{*}, \mu_S'^{*}, a_s'^{*}, e_s'^{*}]$.

For a normal space mission, the purpose of the optimization is to save the fuel. This function could be used as the objective function of fuel for finding the best perturbation parameters.

$$\mathrm{A}'^{*} = \mathrm{argmin}\{\int_0^T u(t)^2 dt\}$$

In this function $u(t)^2 = u_x(t)^2 + u_y(t)^2 + u_z(t)^2$. T is the time to maintain the manual control, and T is a period of the revolution of Mercury.

For a specific artificial frozen orbit, the objective function $\int_0^T u(t)^2 dt$ could illustrate the consume of fuel effectively. Obviously, the minimum usages of fuel are different when the average orbital elements are different. It is not rigor to use the usage of fuel of a single orbit to illustrate the method of control. Thus, we need to establish a serial of standards to determine the efficiency of the method of control in different orbits.

Now back to the issue, it is easy to find that when the required perturbation parameter $J_2', J_3', \mu_S' = 0$ the system is a two-body system which has no perturbation. In this situation, the orbital elements except true anomaly are constant. This is the specific situation of frozen orbit, and it is the optimal situation for spacecraft. Now, assume $A_0{}'[0,0,R,0,a_s,e_s]$, and the consume function of fuel of it is $\int_0^T u_0(t)^2 dt$. I use this situation as a standard to define a new objective function.

$$\mathrm{A'^*} = \mathrm{argmin}\{\frac{\int_0^T u^*(t)^2 dt}{\int_0^T u_0(t)^2 dt}\}$$

This equation means the ratio between optimal usage of fuel and the usage of fuel in two-body system. For a specific group of orbital element, $\int_0^T u_0(t)^2 dt$ is a constant value. So the optimal solution of this function is the same as the first object function. The output of this objective function has a range $(0,1]$, because $\mathrm{A'^*}$ is the optimal solution of $\mathrm{A'}$, and $A_0{}'$ is one of the subset of $\mathrm{A'}$. Obviously, there is an inverse relationship between the value of the objective function and the efficiency of the control.

4.2 The Choices of parameters

Back to Hamilton's function. For making it more intuitive, the function is written as the form of orbital elements.

$$\overline{\overline{\mathbf{H}}} = -\frac{\mu}{2a} + \frac{1}{4}\frac{\mu J_2 R^2}{a^3(1-e^2)^{\frac{3}{2}}}(1-3cos^2 i) + \frac{3}{8}\frac{\mu J_3 R^3}{a^4(1-e^2)^{\frac{5}{2}}} esinwsini(1-5cos^2 i)$$
$$-\frac{1}{16}\frac{\mu_s a^2}{{a_s}^3(1-{e_s}^2)^{\frac{3}{2}}}[(6cos^2 i-2)\left(1+\frac{3}{2}e^2\right)+\frac{5}{2}(6-6cos^2 i)e^2 cos2w]$$

As we can see from the function, the degree of the radius of Mercury of in term of J_2 is 2, and in term of J_3 is 3. If I change the value of R, the variations of the value of J_2 and J_3 are different. And the range of value of these 3 terms R, J_2, J_3 are $[0, +\infty), (-\infty, +\infty), (-\infty, +\infty)$. For the third-body perturbation we also can get the functions of (μ_s, a_s, e_s). The functions are $\mu_s, \frac{1}{a_s{}^3}, \frac{1}{(1-e_s{}^2)^{\frac{3}{2}}}$, and it have the same influence to the perturbation term. And the range of value of (μ_s, a_s, e_s) are $[0, +\infty), [0, +\infty), [1, +\infty)$.

Now back to Hamilton's function before equalization

$$\mathbf{H} = -\frac{\mu}{2a} - \frac{\mu}{4r} J_2 \left(\frac{R}{r}\right)^2 (2 - 3sin^2 i + 3sin^2 i\ cos2u)$$
$$-\frac{\mu}{8r} J_3 \left(\frac{R}{r}\right)^3 (12 sin i sin u - 15 sin^3 i sin u + 5 sin^3 i sin 3u) - \frac{\mu_s}{16\rho} J_3 \left(\frac{r}{\rho}\right)^3 (4$$
$$- 6sin^2 i + 6sin^2 i cos2u + 6sin^2 i cos(2\Omega - 2f_s) q$$
$$+ 3(1 + cos i)^2\ cos(2\Omega + 2u - 2f_s) + 3(1 - cos i)^2 cos\ (2\Omega - 2u - 2f_s))$$

The degree of non-spherical perturbation parameters R, J_2, J_3 are same as Hamilton's function after equalization. Therefore, we can choose 2 of these 3 parameters to find the best solution. For making it more intuitive, I chose parameters $J_2{}'$ and $J_3{}'$. And the range of value of these two parameters are $(-\infty, +\infty)$, it is more flexible than the other one. For the part of third-body perturbation, in this function, the degree of μ_S and $\frac{1}{a_s{}^3}$ are the same as Hamilton's function after equalization. But the degree of the eccentricity ratio is different, and the function above include the term f_s which changes periodically. Thus, the influences of perturbation parameters are different. It is not possible to use one of it to take the place of the other one. For optimizing the calculation, I still chose $\mu_S{}'$ as perturbation parameter. Even though I could only obtain the second best solution via it.

4.3 The Condition of existence of artificial frozen orbit

As the discussion above, for a specific initial value of orbit

$$\begin{cases} r_0 = r(l_0, g_0, h_0, L_0, G_0, H_0) \\ \dot{r_0} = \dot{r}(l_0, g_0, h_0, L_0, G_0, H_0) \end{cases}$$

Now, substitute $\dot{\mathrm{G}}(g) = 0$ into $\dot{g} = 0$, then we could get the condition of frozen orbit.

$$\begin{cases} a_1\mu_s + a_2J_2 \pm a_3J_3 = 0, \ \ \omega = \pm\dfrac{\pi}{2} \\ b_1{\mu_s}^2 + b_2J_2\mu_s + b_3{J_3}^2 = 0, \ \ sin\omega = \dfrac{c_2}{c_1} \end{cases}$$

In this groups of equations,

$$\begin{cases} a_1 = \dfrac{3}{4}\dfrac{{L_0}^4}{\mu^2{a_s}^3(1-{e_s}^2)^{\frac{3}{2}}}(5\dfrac{{H_0}^2}{{G_0}^3} - 3\dfrac{G_0}{{L_0}^2}) \\ a_2 = \dfrac{3}{4}\mu^4R^2\dfrac{5{H_0}^2 - {G_0}^2}{{G_0}^6{L_0}^3} \\ a_3 = \dfrac{3}{8}\mu^5R^3(\dfrac{41{H_0}^2{L_0}^2{G_0}^2 - 40{H_0}^4{L_0}^2 - 35{H_0}^2{G_0}^4 + 35{H_0}^4{G_0}^2 - 5{G_0}^4{L_0}^2 + 4{G_0}^6}{G^{10}{L_0}^5\sqrt{1-\dfrac{{G_0}^2}{{L_0}^2}}\sqrt{1-\dfrac{{H_0}^2}{{G_0}^2}}}) \\ b_1 = \dfrac{3}{2}\dfrac{{L_0}^4G_0}{\mu^2({a_s}^3(1-{e_s}^2)^{\frac{3}{2}})^2} \\ b_2 = \dfrac{3}{4}\dfrac{\mu^4R^2}{{a_s}^3(1-{e_s}^2)^{\frac{3}{2}}}\dfrac{5{H_0}^2 - {G_0}^2}{{G_0}^{15}{L_0}^{10}} \\ b_3 \end{cases}$$

It is easy to see that the relationships between J_2, J_3, μ_S are clear. When $\omega = \pm\frac{\pi}{2}$, the relationship between J_2, J_3, μ_S is linear. When $sin\omega = \frac{c_2}{c_1}$, the relationship is quadratic polynomial. Because of this group of equations, we could use other two parameters to express the other parameter. And I found that order of magnitudes of J_2, J_3 is nearly same. But the order of magnitude of μ_S is bigger. For optimizing parameters, I use J_2, J_3 to express μ_S.

Then, the function of motion change to

$$\ddot{r} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{f}(\boldsymbol{r}, t, A')$$

$$A' = [J_2{}', J_3{}', R, \mu_S{}'(J_2{}', J_3{}'), a_S, e_s]$$

Thus, the initial three parameters become two parameters after optimization.

4.4 The constraints

The real non-spherical parameters of Mercury are bigger than 0. And the third body perturbation parameter is bigger than 0.

The following table could be easily deduced.

Type	$\omega = \frac{\pi}{2}$	$\omega = -\frac{\pi}{2}$	$sin\omega = \frac{c_2}{c_1}$
Constraint	$-\frac{a_2 J_2{}' + a_3 J_3{}'}{a_1} \geq 0$	$-\frac{a_2 J_2{}' + a_3 J_3{}'}{a_1} \geq 0$	$\Delta = b_2{}^2 J'_2 - 4 b_1 b_3 J'_2{}^2$ $max\ (\frac{-b_2 J'_2 \pm \sqrt{\Delta}}{2b_1}) \geq 0$ $-1 < \frac{c_2}{c_1} < 1$

(Table 3 The constraints)

4.5 Method which is used

There are many methods to establish an artificial orbit. The simplest way is use the force which is equal to perturbation force to offset the perturbation force. Under this condition, the orbit of spacecraft is a Keplerian orbit. As the information that we know, this method will consume more fuel than the method that required. And there is another method which is depend on the rule of optimal control of orbit. This method will set a required orbit and the range of error, then calculate the optimal control rule which consumes least fuel. It is possible to find an approach of control which consumes less fuel. But there is a problem, it cannot make sure that the final orbit is the required orbit, and it is not possible to estimate the mean value of orbital elements. Compare with this method, the method of using continuous small force to change the parameters of perturbation which used by this essay has a clear physical meaning and the orbital element is a certain value. In the following content, I will use complex method[18] to optimize the parameters and find the ideal artificial orbit.

4.6 Analyzation of Examples

Then, I will use a group of examples to show the method is effective.

As we know from the chapter 3, for the natural frozen orbit, there are three types of ω: $\omega = \frac{\pi}{2}$, $\omega = -\frac{\pi}{2}$, and $sin\omega = \frac{c_2}{c_1}$. Of course, the characteristics of natural frozen orbits suit the artificial frozen orbit.

	Example 1	Example 2	Example 3
a	*5,000~2,0000km*	*5,000~2,0000km*	*5,000~2,0000km*
e	*0.1~0.9*	*0.1~0.9*	*0.1~0.9*
i	*65°*	*80°*	*80°*
ω	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$arcsin\omega = \frac{c_2}{c_1}$

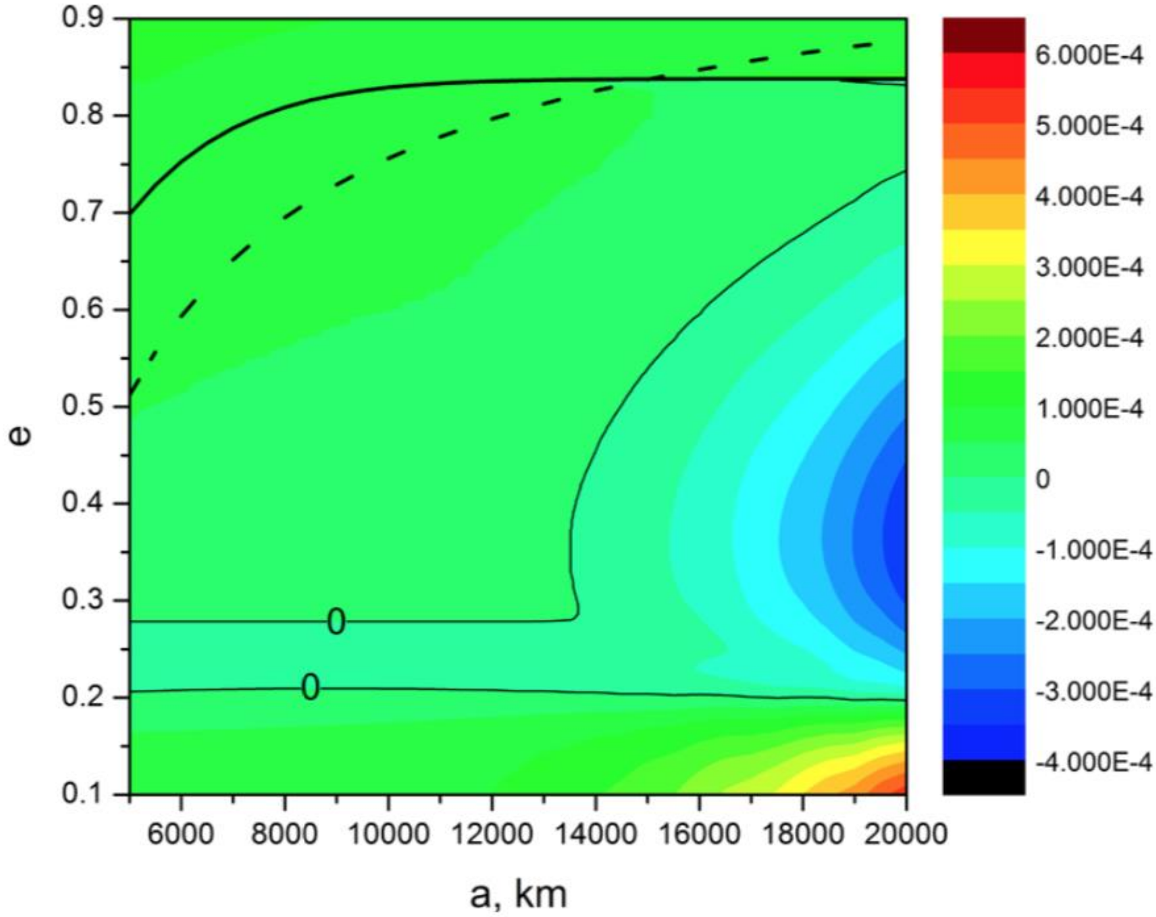
(Table 4 The orbital elements of examples)

In this table, a is the semi-major axis. And I chose relative high values of the dip angle of the orbit, because the near-polar orbits are most commonly used in the space mission.

I used MATLAB to draw the contour line of $J_2{}'$, $\mu_s{}'$ and objective function for analyzing the effectiveness of the method which is used in the essay.

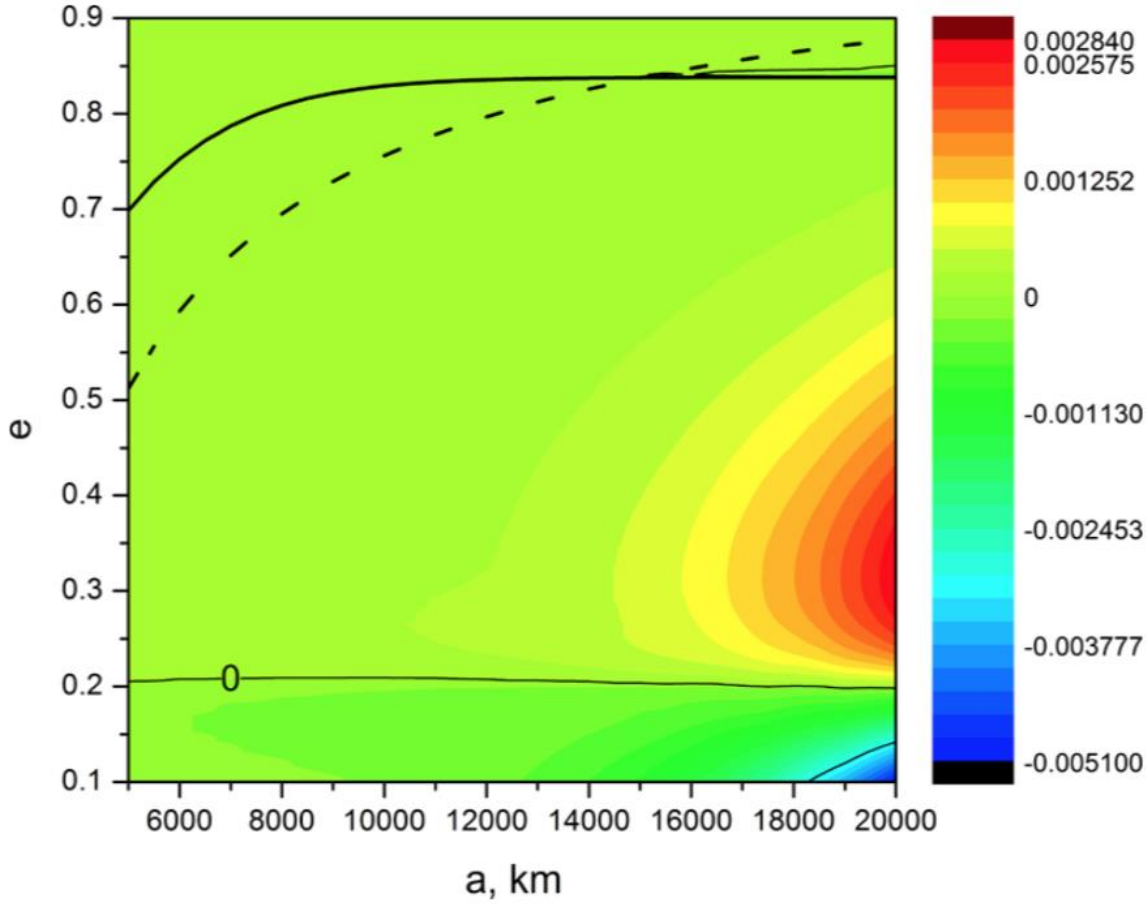
In the following graphs, the line is natural frozen orbit and the dash line is the line of collision.

Example 1



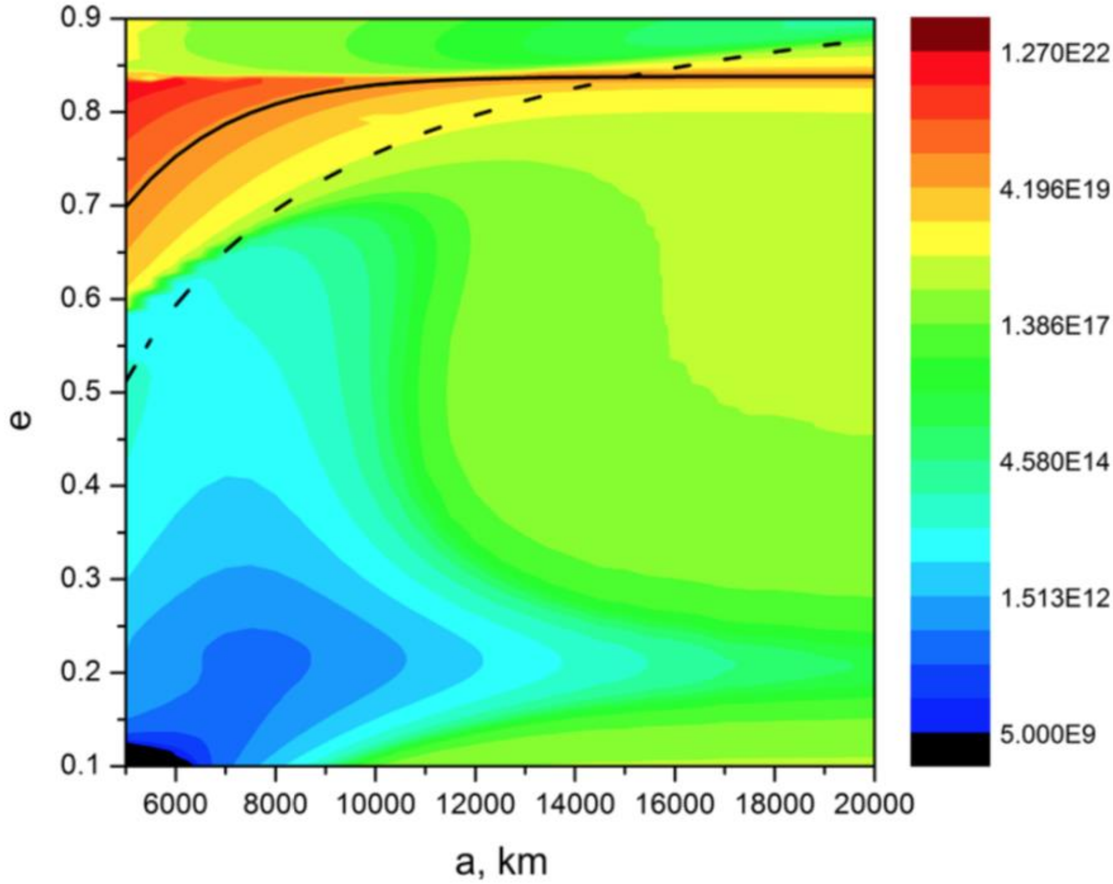
(Fig 2 The change of $J_2{}'$ with semi-major axis and eccentric ratio)

There is a contour line which has the value of 0, it means in some parts of this graph, $J_2{}'$ is less than 0. In most case, the $J_2{}'$ is bigger than 0. But $J_2{}'$ is effective because it is a control parameter.



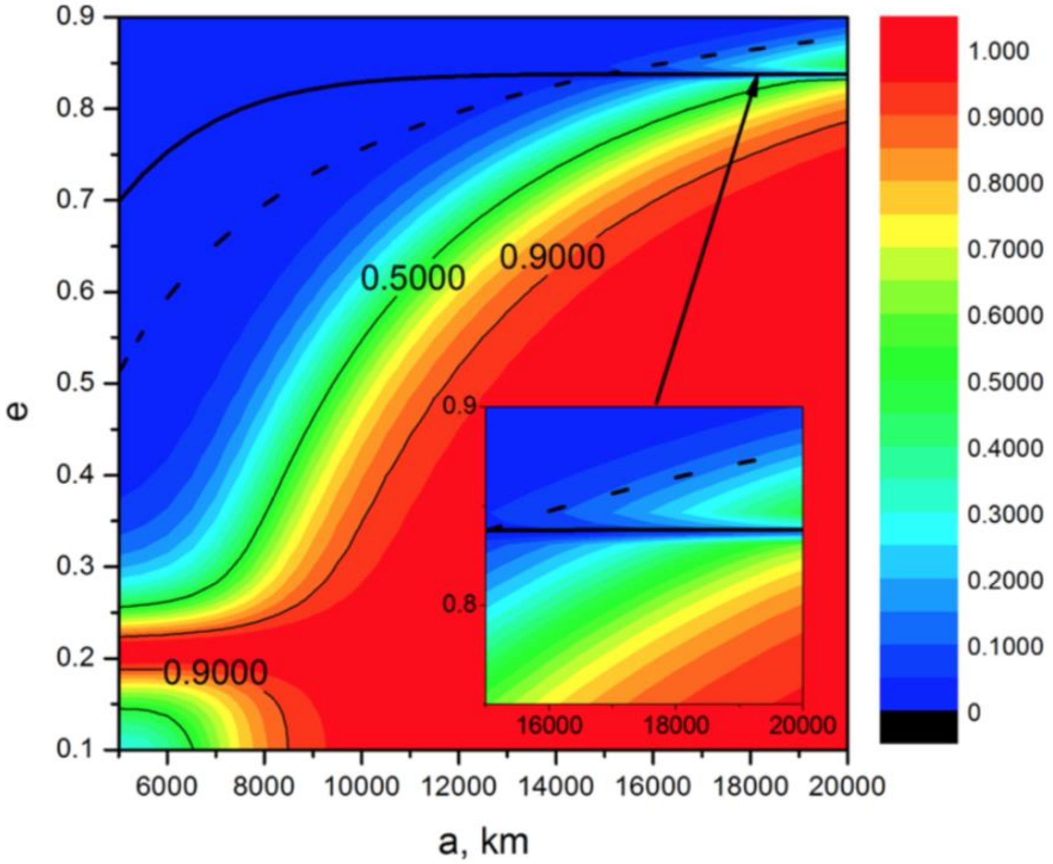
(Fig 3 The change of $J_3{}'$ with semi-major axis and eccentric ratio (logarithmic coordinates))

There are some parts of $J_3{}'$ smaller than 0. It is same as $J_2{}'$, these two parameters change more and more intensely when a increase and these two parameters across 0 when e= 0.2.



(Fig 4 The change of μ_s' with semi-major axis and eccentric ratio (logarithmic coordinates))

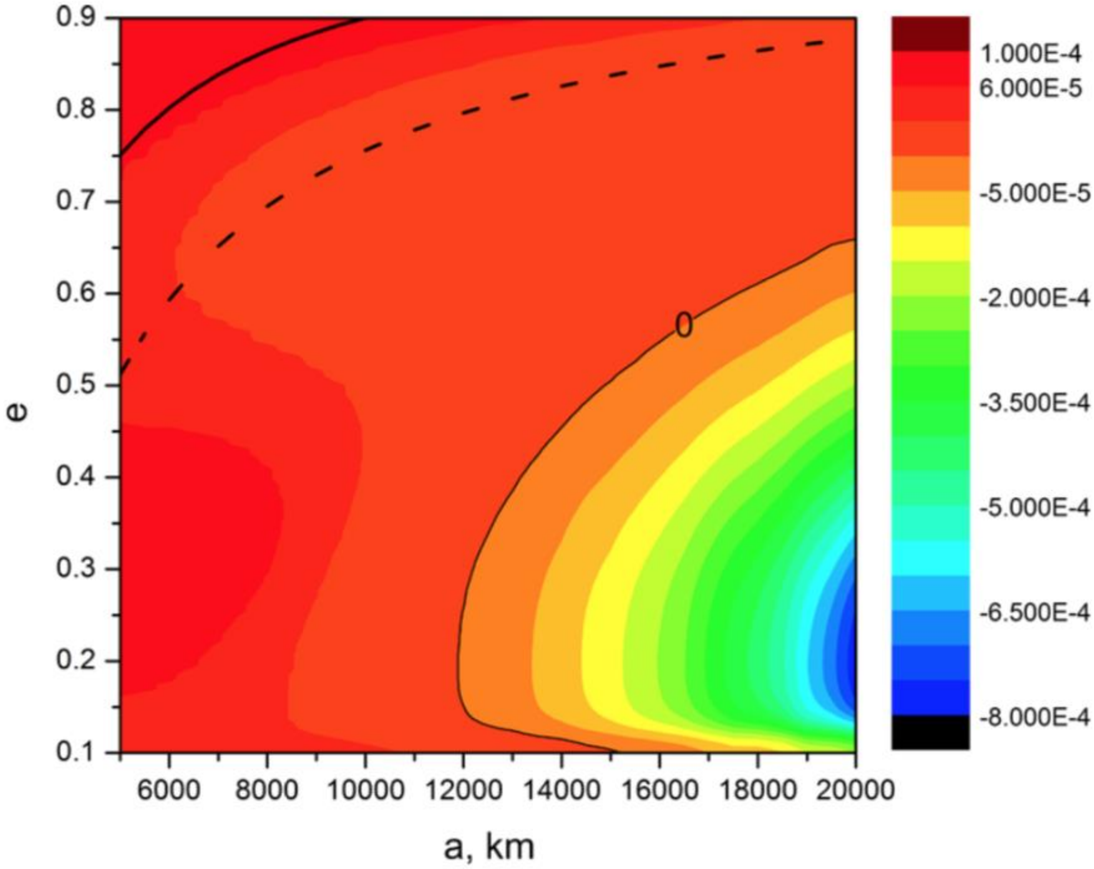
It is easy to see that most part of μ_s' is smaller than μ_s. And near the original frozen orbit the value of μ_s' near to the original value of μ_s



(Fig 5 The change of objective function with semi-major axis and eccentric ratio (logarithmic coordinates))

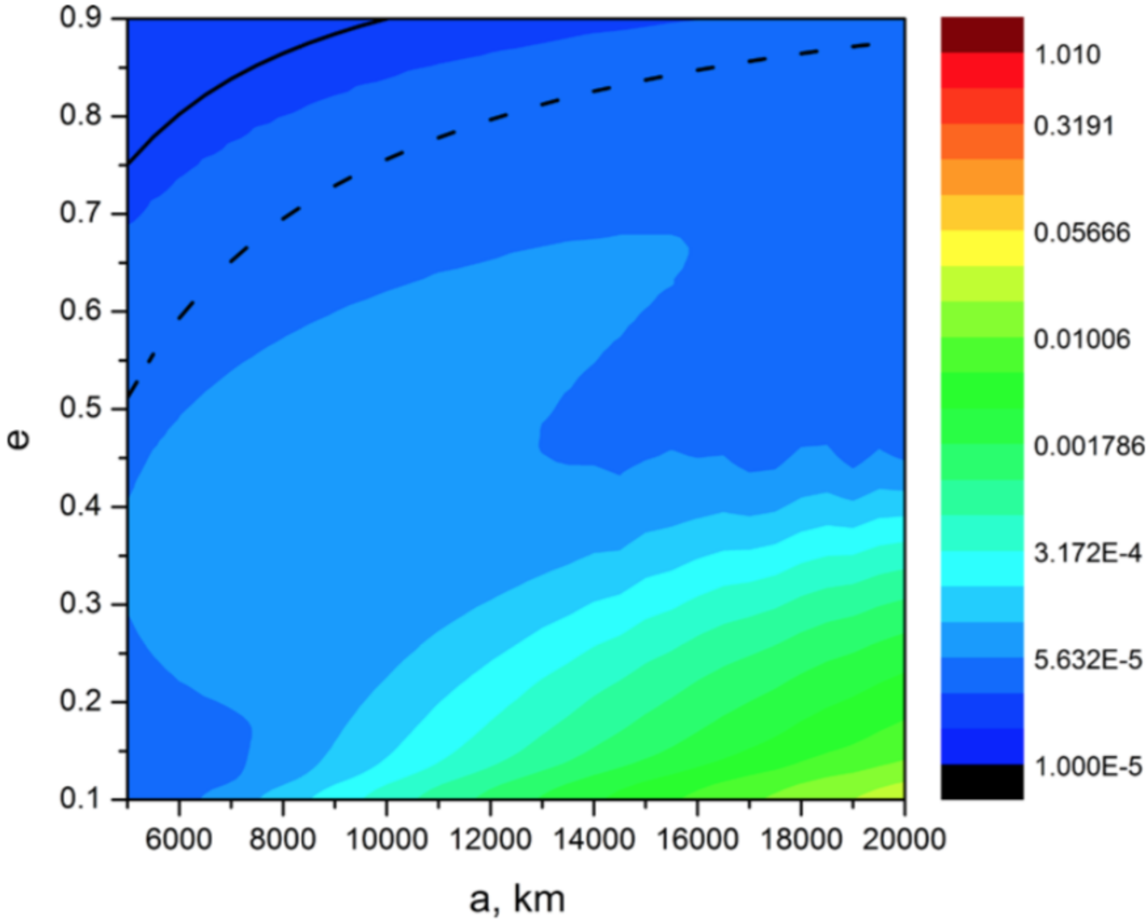
When *a* is small the effectiveness of artificial control is high. And when the objective function near the original orbit the value of the function will tend to 0.

Example 2



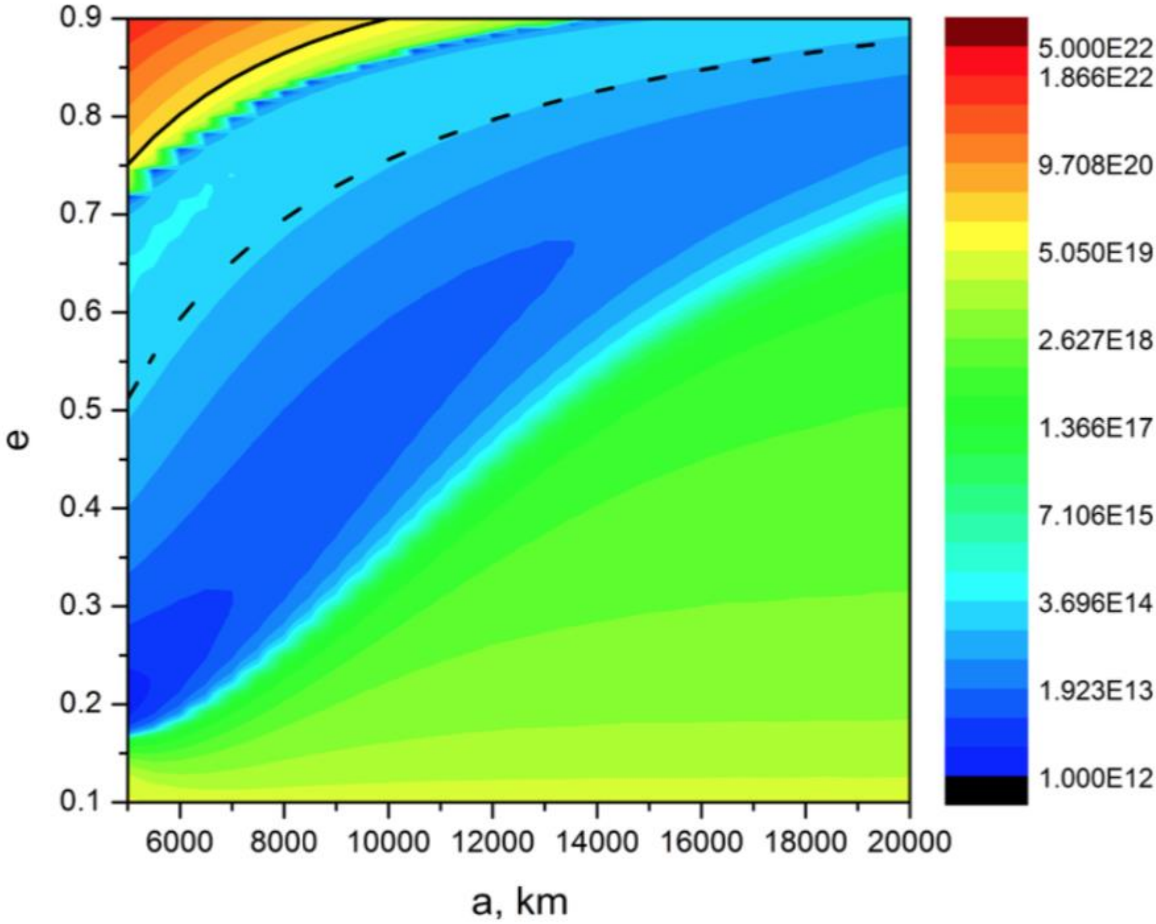
(Fig 6 The change of $J_2{}'$ with semi-major axis and eccentric ratio)

For most part of the graph, the value of $J_2{}'$ is less than the original value $J_2 = 6 \times 10^{-5}$. When *a* is big and *e* is small, $J_2{}'$ is negative, and the absolute value is big. When *e* is big the value of $J_2{}'$ is near to the value of J_2.



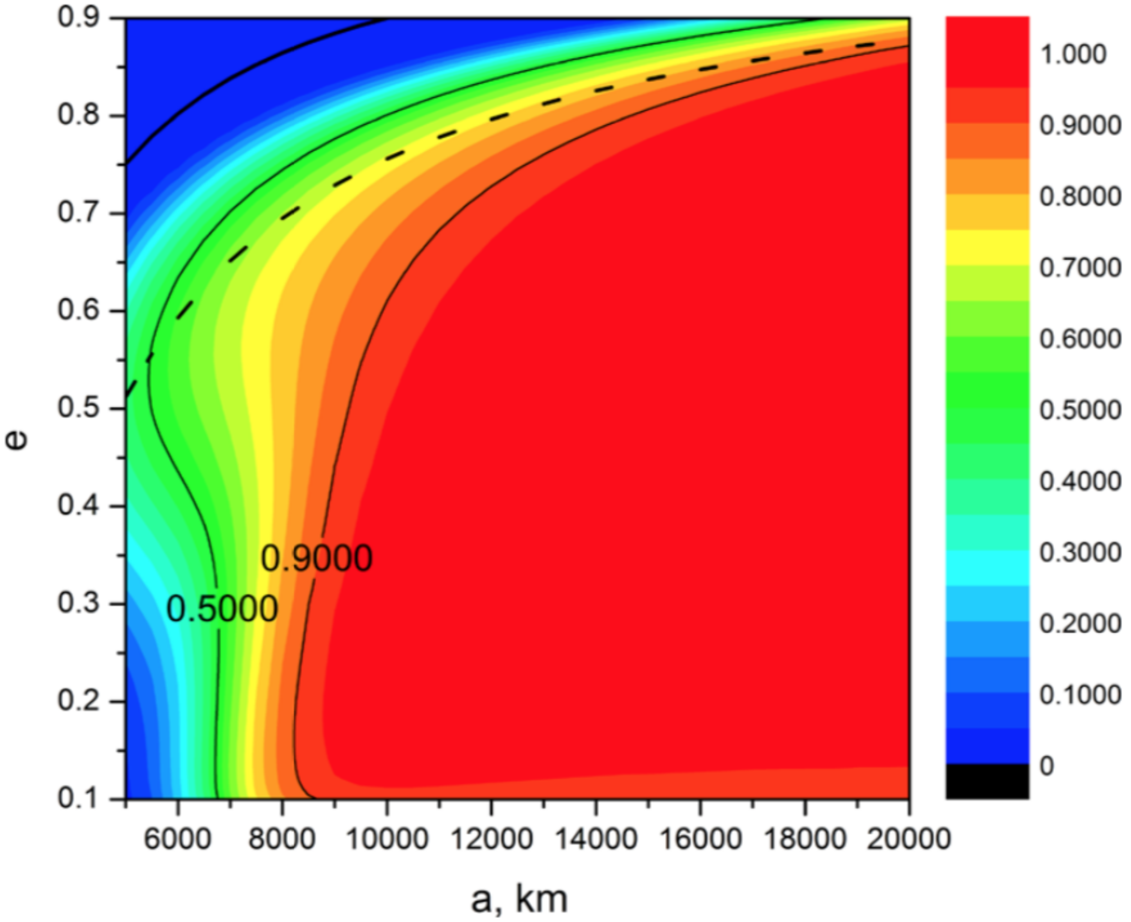
(Fig 7 The change of $J_3{}'$ with semi-major axis and eccentric ratio (logarithmic coordinates))

In this graph, $J_3{}'$ is bigger than 0. And the $J_3{}'$ will change significantly with *e* when a is big.



(Fig 8 The change of $\mu_s{}'$ with semi-major axis and eccentric ratio (logarithmic coordinates))

For the part below the original frozen orbit, the value of $\mu_s{}'$ is very small. For the area above frozen orbit, the value of $\mu_s{}'$ is bigger than μ_s.



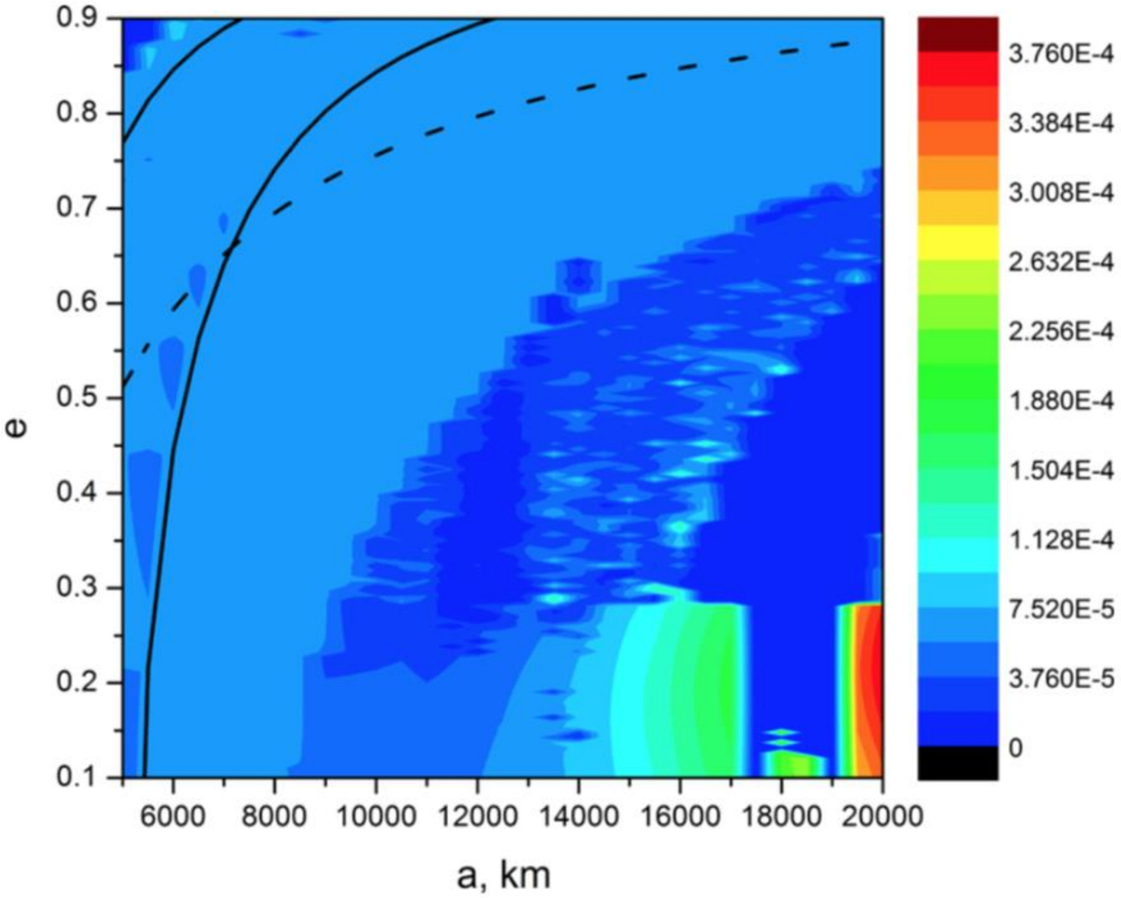
(Fig 9 The change of objective function with semi-major axis and eccentric ratio (logarithmic coordinates))

When a is small, especially when a<8000km, the efficiency of the method of control is high. But the value of objective function increase significantly with the increase of a.

Example, 3.

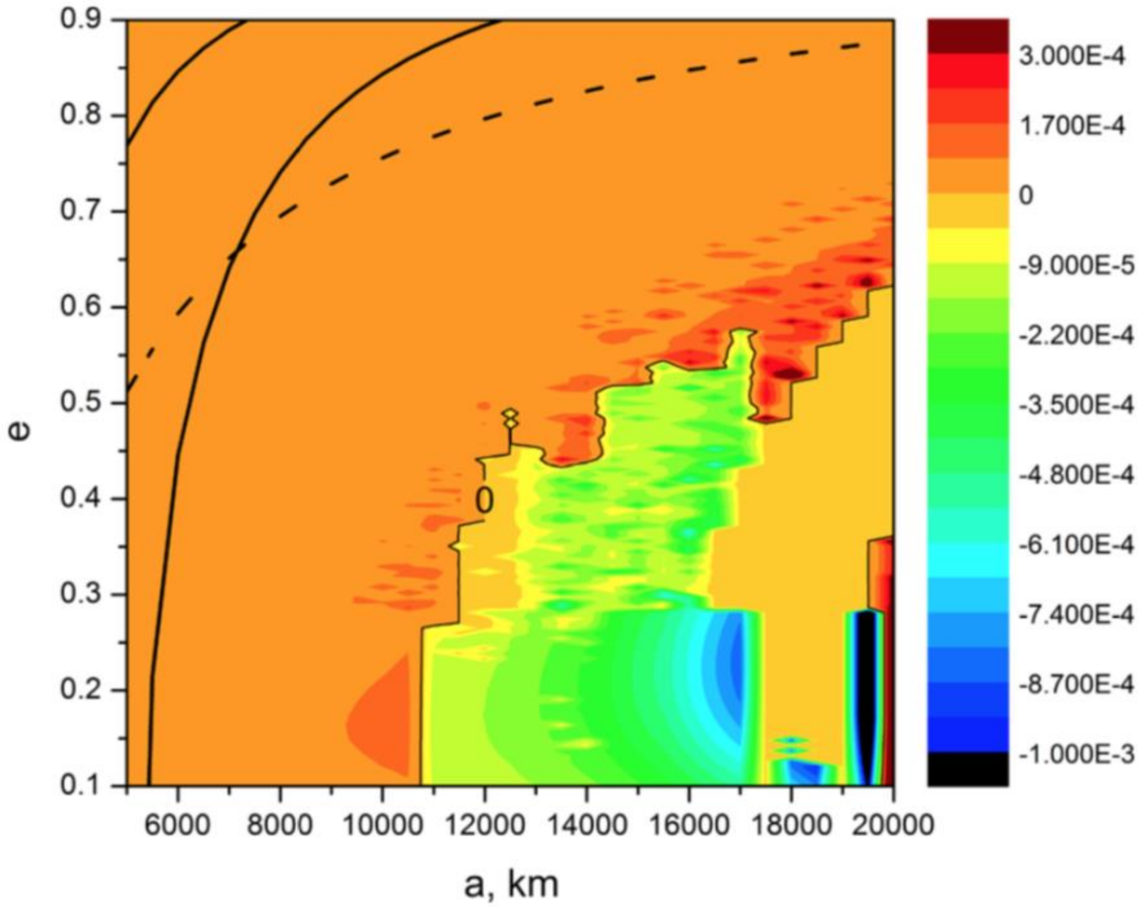
The graph of the example seems broken. Because there many constraint of this example.

And in this example there are two frozen orbit.



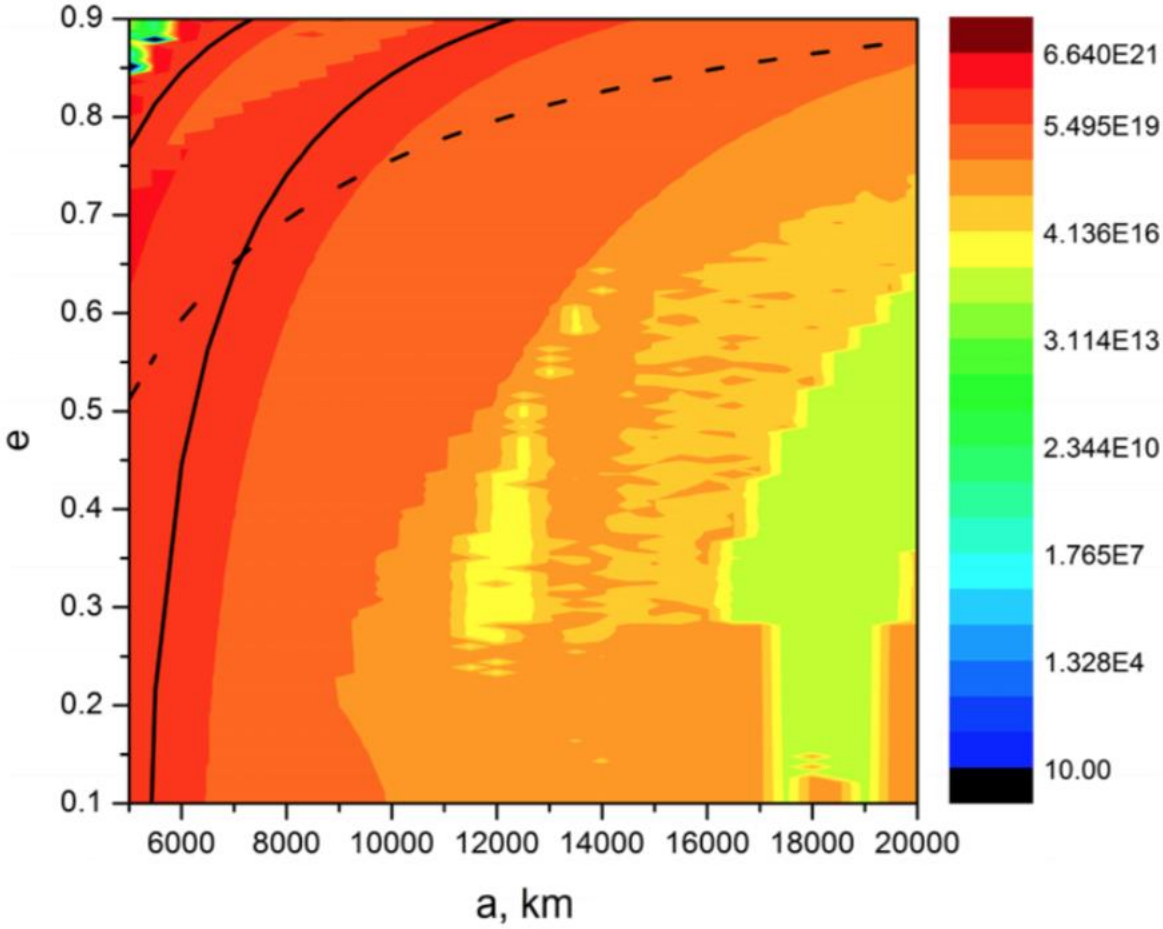
(Fig 10 The change of $J_2{}'$ with semi-major axis and eccentric ratio)

In this graph, the value of $J_2{}'$ is bigger than 0. And for most of the part of graph, the value of $J_2{}'$ is small. $J_2{}'$ is big if and only if a is big.



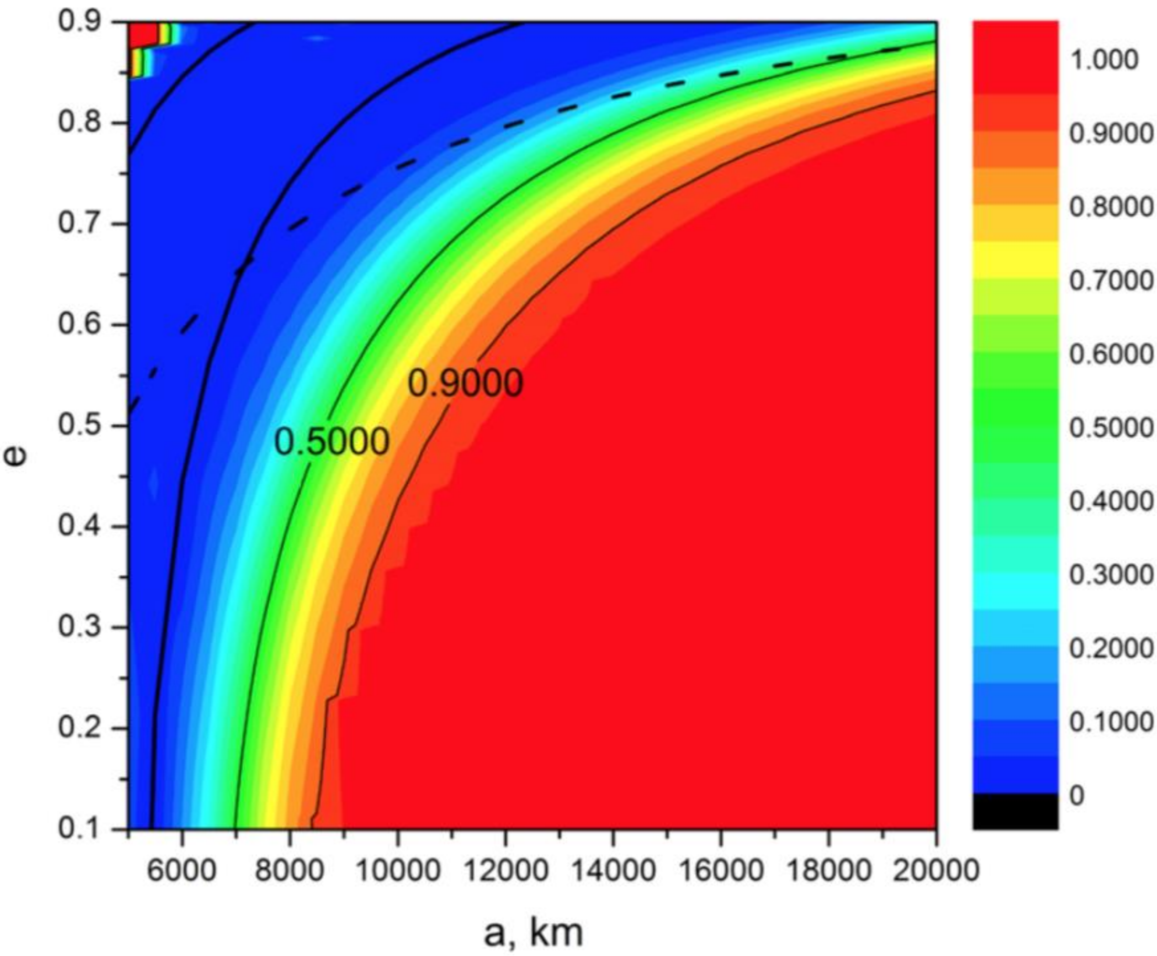
(Fig 11 The change of $J_3{}'$ with semi-major axis and eccentric ratio)

Obviously, the value of the $J_3{}'$ bigger than 0 when a is big and e is small. The other part of the graph the value of $J_3{}'$ is big.



(Fig 12 The change of $\mu_s{}'$ with semi-major axis and eccentric ratio (logarithmic coordinates))

Most part of $\mu_s{}'$ is smaller than the original value. When the value of a is small and e is big, $\mu_s{}'$ touch the boundary of $\mu_s{}' = 0$.



(Fig 13 The change of objective function with semi-major axis and eccentric ratio)

As we can see from the graph above, the value of objective function tends to 0 when it near the original function. And the efficient of control is high when *e* is big and *a* is small (except the top left corner)

For this example, c_2 and c_1 in $\omega = arcsin\frac{c_2}{c_1}$ are determined by required perturbation parameter $\mu_s{}'$ and $J_3{}'$. Thus, it is needed to add new constraint, if it is need to limit the range of value of ω.

As we can see from these three examples above, there are big area of low value of objective function in the figures about objective function. In other word, it means this method of using continuous small force to change the perturbation parameters is effective and flexible.

4.7 Brief Summary of the chapter

In this chapter, I design a new method to control the orbit----the using of small and continuous force. Base on the model of double equalization in chapter 2
, I chose J_2, J_3, μ_S as optimal parameters and I deduced the constraint equations for all the situation of frozen orbit. Then, I bring in complex method to find the most optimal parameters of perturbation.

I chose three examples to evaluate the effectiveness of the method which was designed by me. And use MATLAB to analyze the change of parameters of perturbation and objective function. The analyzation shows that the method which was designed is effective.

5. Conclusion and Evaluation

In the part of natural frozen orbit of the essay, I consider the perturbation force of Sun and the non-spherical force of Mercury. I got Hamilton's function in the form of Delaunay variables. And equalize this function twice for investigating the long-term change of the orbital elements, then I got a single degree of freedom system of (g, G). For this system, I figure out the condition of liberation point. And the Hessian matrix is used to judge the stability of the orbit.

In the second part of the essay, I design a method which could control the orbit by using continuous thrust. In this method the spacecraft could use the thrust to modify the magnitude of perturbation for meeting the condition of frozen orbit and maintain the constant of orbital elements. I discussed the optimization of parameters, and decided the optimal parameters. And deduced the constraint of examples. Then I use MATLAB to sketch the graph of the optimal parameter and the objective function. The analyzations of examples prove that the method which was designed by myself is effective.

For improving this essay, firstly, the model is not as precise as possible. For making the model more precise, the third-body perturbation need to consider more terms. Secondly, due to the limitation of words, I could only calculate 3 examples. But it is not enough. Thirdly, it is need to discuss the feasibility of the artificial orbital, including the simulation of the consumption of fuel and acceleration. And I need to compare the result of this method with other methods, for illustrating the effectiveness of the method.

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