1. free energy of the seure paramagnet:

$$F = -k_B T \ln Z_1^N = N k_B T \ln 2 \cosh\left(\frac{\mu B}{k_B T}\right)$$

$$\therefore M = N\mu \tanh\left(\frac{\mu B}{k_B T}\right)$$

$$\chi = \frac{M}{B} = \frac{N\mu^2}{k_B T}$$

Polarizable singlet: $Z = -\alpha B^2$

$$\therefore \mu = -\frac{\partial Z}{\partial B} = \alpha B$$

$$\therefore M = N\mu = \alpha N B$$

2. $S_e \cdot S_d = \frac{\hbar^2}{4}$

Hamiltonian: $H_K^{eff} = \sum_\sigma \left(\sum_k \varepsilon_k a_{k\sigma}^+ a_{k\sigma} + \frac{J_0}{4V} \sum_{kk'} a_{k'\sigma}^+ a_{k\sigma}\right)$

$$|\Phi\rangle = \sum_{|k|>k_F} a_k b_k^+ |FS\rangle$$

$$\because b_k^+ = a_{k\uparrow}^+ \psi_{d\uparrow}^+$$

$\therefore$ the binding energy: $E_b = D e^{-2/J_0 N(0)}$

$\therefore$ Singlet is likely to occur.

3. $z = N(0)J_z$, $x = N(0)J_\perp$, $t = -\ln D/D_0$

$\therefore \frac{dz}{dt} = -x^2$, $\frac{dx}{dt} = -zx$

$\therefore x\,dx = z\,dz \rightarrow x^2 = z^2 + \lambda$

$\therefore \frac{dz}{dt} = z^2 \rightarrow \int_{z_0}^{z(t)} \frac{dz}{z^2} = -\int_0^t dt'$

$\therefore z(t) = \frac{z_0}{1+tz_0}$

$\frac{dz}{dt} = z^2 + k^2 \rightarrow \int_{z_0}^{z(t)} \frac{dz}{z^2+k^2} = -\int_0^t dt' = -t$

$\therefore$ the solution: $z(t) = k\tan(\arctan\frac{z_0}{k} - kt)$

$t \rightarrow \frac{\pi}{2k}(1 + \frac{2}{\pi}\arctan\frac{z_0}{k})$, $z(t) \rightarrow \infty$

$J_\perp = 0$, $J_z > 0$, ferromagnetic

$J_z < 0$, anti-ferromagnetic.

$J_z \rightarrow \infty$, perturbation theory breaks down.

4. Lippmann-Schwinger:

$$|\psi\rangle = |k\sigma\rangle + \frac{1}{E - H_0 + i\varepsilon} V|\psi\rangle$$

$$H_0|\psi\rangle = E|\psi\rangle, \quad T|k\sigma\rangle = V|\psi\rangle$$

$$\therefore T = V + V \frac{1}{E - H_0 + i\varepsilon} T$$

$$T = V + V \frac{1}{E - H_0 + i\varepsilon} V + V \frac{1}{E - H_0 + i\varepsilon} V \frac{1}{E - H_0 + i\varepsilon} V + \cdots$$

Scattering: $\frac{d\sigma}{d\Omega} \propto |\langle k'|T|k\rangle|^2$

where $|k\rangle = a_k^\dagger |0\rangle$

$\therefore$ the interaction potential V:

$$V = H_z + H_+ + H_- = -\frac{1}{2\hbar V} \sum_{kk'} \left[\sum_\sigma J_z S_d^z a_{k\sigma}^\dagger a_{k'\sigma} + J_+ S_d^+ a_{k\downarrow} a_{k'\uparrow} + J_- S_d^- a_{k\uparrow} a_{k'\downarrow} \right]$$

$\therefore$ expanding gives:

$$\left(\frac{-1}{2\hbar V}\right)^3 J_z J_+ J_- \sum_{k_1 \cdots k_6} S_d^z a_{k_1\sigma}^\dagger a_{k_2\sigma} \frac{1}{E - H_0 + i\varepsilon} S_d^+ a_{k_3\downarrow}^\dagger a_{k_4\uparrow} \frac{1}{E - H_0 + i\varepsilon} S_d a_{k_5\uparrow}^\dagger a_{k_6\downarrow}$$

$\therefore$ take $\delta J_z = \frac{J_z J_+ J_-}{16 V^2} \sum_{qk} \frac{1}{E + \varepsilon_k - \varepsilon_q} \frac{1}{E - \varepsilon_q + \varepsilon_k}$

$$\delta J_z = -\frac{J_z J_+ J_- \rho_0^2}{2} \frac{\delta D}{D} \int_0^D d\varepsilon \frac{1}{\varepsilon - E}$$

$\rho_0 = N(0)/2,$

$$\int_0^D d\varepsilon \frac{1}{\varepsilon - E} = \ln \frac{D-E}{-E}$$

$-E \to 0$

the integral is finite and will set to:

$$\int_0^D d\varepsilon \frac{1}{\varepsilon - E} \to 2\frac{D}{D} = 1$$

$$\therefore \frac{dJ_z}{d\ln D} = 2\rho_0 \left(J_+ J_- - \frac{1}{2}\rho_0 J_z J_+ J_- \right)$$