1. $L_j = \int_{-\infty}^{\infty} dx\, x^j \frac{e^x}{(e^x+1)^2}$

$\frac{e^x}{(e^x+1)^2}$ is an even function

$\therefore$ ~~for even j~~, $L_{2n+1}=0$

for $j=2n$, use $\Gamma(z)=\int_0^\infty \frac{dx}{x} x^z e^{-x}$

$\therefore L_{2n} = 2\int_0^\infty x^{2n} \frac{e^x}{(e^{-x}+1)^2} = 2\int_0^\infty x^{2n} \frac{e^{-x}}{(e^{-x}+1)^2}$

and $\frac{1}{(1+e^{-x})^2} = \sum_{m=1}^{\infty} m(-1)^{m-1} e^{-(m-1)x}$

$\therefore L_{2n} = 2\Gamma(2n+1) \sum_{m=1}^{\infty} \frac{(-1)^m}{m^{2n}} = 2(2n)!\left(1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\cdots\right)$

$\therefore L_{2n} = 2(2n)!\left[\left(1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right) - \frac{2}{2^{2n}}\left(1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots\right)\right]$

recall the Rieman zeta function:

$L_{2n} = 2(2n)!\left(1-\frac{2}{2^{2n}}\right)\zeta(2n) = 2(2^{2n-1}-1)\pi^{2n}B_n$

B_e is the Bernoulli numbers

2. $n = \int_0^\infty d\varepsilon\, N(\varepsilon) f(\varepsilon)$, $n = \frac{N}{V}$

$\therefore n \approx \int_0^\mu d\varepsilon\, N(\varepsilon) + \frac{\pi^2}{6}(k_BT)^2 N'(\varepsilon)\big|_{\varepsilon=\mu}$ by Sommerfeld expansion

$\because \int_0^\mu d\varepsilon\, N(\varepsilon) = \frac{2}{3}\mu N(\mu)$, $N'(\varepsilon) = \frac{1}{2}\frac{N(\mu)}{\mu}$

$\therefore n \approx \frac{2}{3}\mu N(\mu)\left(1 + \frac{3}{2}\frac{\pi^2}{12}\left(\frac{k_BT}{\mu}\right)^2\right)$

$\mu = \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)$