1. $H_U = \frac{U}{2}\sum n_i(n_i-1) + V\sum_{\langle ij\rangle} n_i n_j - \mu\sum_i n_i$

$= \frac{U}{2}\sum (x_i+1)x_i + V\sum_{\langle ij\rangle}(x_i+1)(x_j+1) - \mu\sum_i(x_i+1)$

$x_i \to -x_i$

$\therefore H_U \to \frac{U}{2}\sum_i(-x \quad \cdots$

$= H_U - U\sum_i x_i - 2V\sum_{\langle ij\rangle}(x_i+x_j) + 2\mu\sum_i x_i$

$= H_U + (2\mu - U - 4Vz)\sum x_i$

$\therefore$ H preserves if $\mu = \frac{1}{2}(U+4Vz)$

2. $\begin{cases} c_{i\sigma}^{(0)} = c_{i\sigma} \\ c_{i\sigma}^{(n+1)} = [c_{i\sigma}^{(n)}, H_U] \end{cases}$

ansatz $c_{i\sigma}^{(n)} = \alpha_n \eta_{i\sigma} + \beta_n c_{i\sigma}$

$\alpha_{n+1}\eta + \beta_{n+1}C = \alpha_{n+1}\eta + \beta_{n+1}(\xi+\eta) = -[H_U - \mu N, \alpha_n\eta + \beta_n(\xi+\eta)]$

$= (U\alpha_n + U\beta_n - \mu\alpha_n)\eta - \mu\beta_n C$

$\therefore \begin{cases} \alpha_{n+1} = (U-\mu)\alpha_n + \beta_n \\ \beta_{n+1} = -\mu\beta_n \end{cases}$

$\therefore \beta_n = (-\mu)^n, \Rightarrow \alpha_{n+1} = (U-\mu)\alpha_n + (-\mu)^n, \quad \beta_n = (-\mu)^n$

3. $|b\pm\rangle = \frac{1}{\sqrt{2}}(c_{1\uparrow}^{+} c_{2\downarrow}^{+} \pm c_{2\uparrow}^{+} c_{1\downarrow}^{+})|0\rangle$

$|d\pm\rangle = \frac{1}{\sqrt{2}}(c_{1\uparrow}^{+} c_{1\downarrow}^{+} \pm c_{2\uparrow}^{+} c_{2\downarrow}^{+})|0\rangle$

$$\begin{cases} H|d+\rangle = U|d+\rangle \\ H|d-\rangle = U|d-\rangle + 2t|b-\rangle \\ H|b+\rangle = 0 \\ H|b-\rangle = 2t|d-\rangle \end{cases}$$

$\therefore \begin{pmatrix} U & 2t \\ 2t & 0 \end{pmatrix}\begin{pmatrix} d \\ b \end{pmatrix} = E\begin{pmatrix} d \\ b \end{pmatrix}$

$\therefore H = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} \frac{U}{2}+\varepsilon & 0 \\ 0 & \frac{U}{2}-\varepsilon \end{pmatrix}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$

where $\varepsilon\cos\theta = \frac{U}{2}$, $\varepsilon\sin\theta = 2t$, $\varepsilon = \sqrt{(\frac{U}{2})^2+(2t)^2}$

$\therefore |\psi_+\rangle = \sqrt{\frac{1}{2}\left(1+\frac{U/2}{\sqrt{(U/2)^2+(2t)^2}}\right)}|d-\rangle - \sqrt{\frac{1}{2}\left(1-\frac{U/2}{\sqrt{(U/2)^2+(2t)^2}}\right)}|b-\rangle$

$E_+ = \frac{U}{2} + \sqrt{(\frac{U}{2})^2+(2t)^2}$

$|\psi_-\rangle = \sqrt{\frac{1}{2}\left(1-\frac{U/2}{\sqrt{(\frac{U}{2})^2+(2t)^2}}\right)}|d-\rangle + \sqrt{\quad \cdots \quad}|b-\rangle$

$E_- = \frac{U}{2} - \sqrt{(\frac{U}{2})^2+(2t)^2}$

4. $S = S^{(1)} + S^{(2)} + O(U^{-3})$

$$\begin{cases} S^{(1)} = \frac{1}{U}(\hat{T}_1 - \hat{T}_{-1}) \\ S^{(2)} = \frac{1}{U^2}[\hat{T}_1 + \hat{T}_{-1}, T_0] \end{cases}$$

$$O = e^S \hat{O} e^{-S} = \hat{O} + [S, \hat{O}] + \frac{1}{2!}[S, [S, \hat{O}]] + O(U^{-3})$$

$$= \hat{O} + \frac{1}{U}[\hat{T}_1 - \hat{T}_{-1}, \hat{O}] + \frac{1}{U^2}[[\hat{T}_1 + \hat{T}_{-1}, \hat{T}_0], \hat{O}]$$

$$+ \frac{1}{2U^2}[\hat{T}_1 - \hat{T}_{-1}, [\hat{T}_1 - \hat{T}_{-1}, \hat{O}]] + O(U^{-3})$$

$\because A \hat{O}_n U^b \subset mU$, $A \subset aU$, $b \subset bU$

$\therefore n = m - a - b$

$\therefore O_n U = \cdots$

as required.