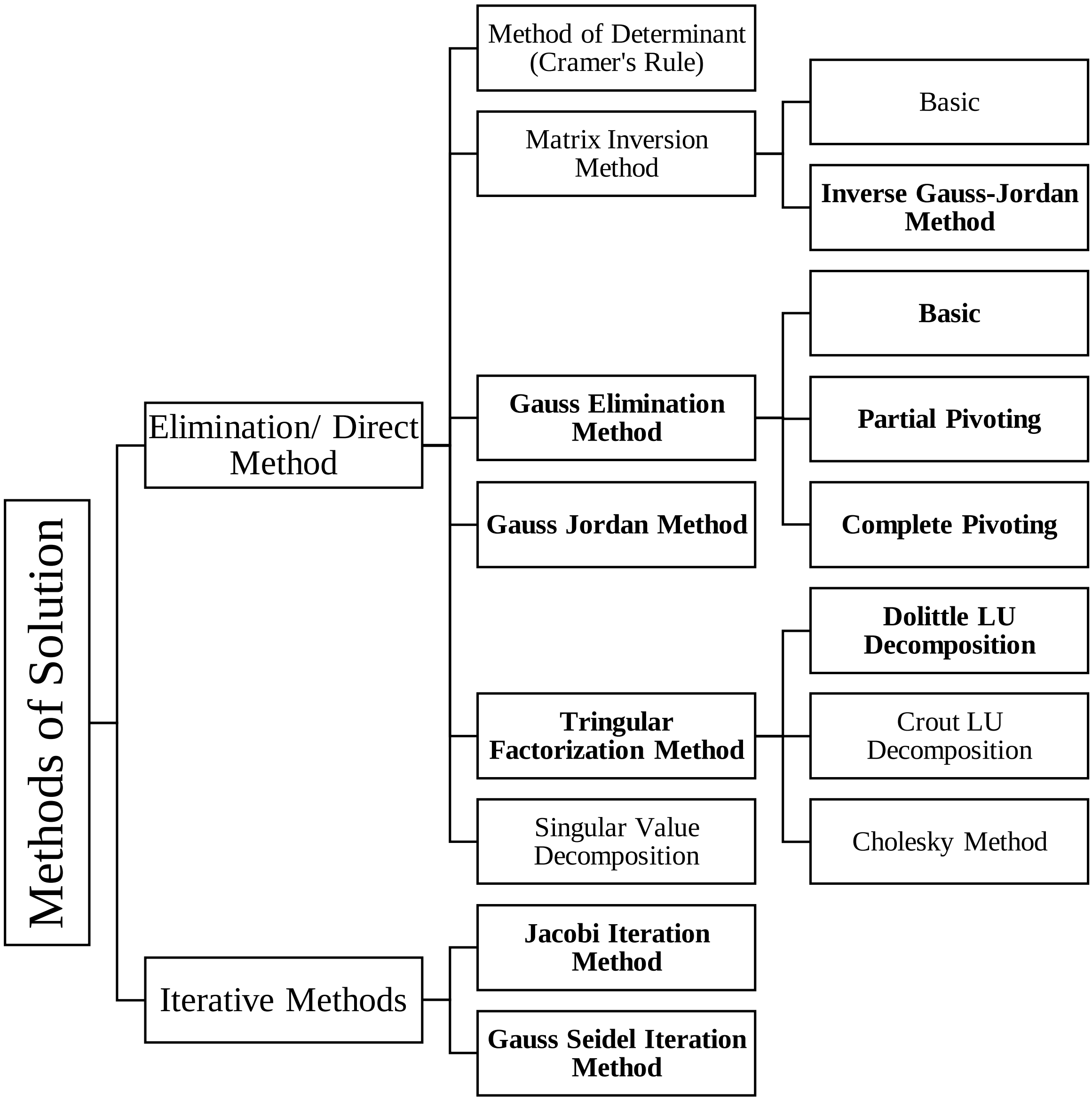
Chapter 3

Solution of System of Linear Algebraic Equations

Solutions of System of Linear Equations



Q. Using Gauss-Jordan Method, find the inverse of the matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{-4} \end{bmatrix}$$

Solution:

The augmented matrix is

$$\left\{\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right\}$$

Applying R2 - R1 => R2, R3 + 2R1 => R3

$$\left\{\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right\}$$

Applying R2/2 => R2

$$\left\{\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right\}$$

Applying R1 - R2 => R1, R3 + 2R2 => R3

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right\}$$

Applying R3 * (-1/4) => R3

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right\}$$

Applying R1 - 6R3 => R1, R2 + 3R3 => R2

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right\}$$

Conclusion: Hence, the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Q. Using Gauss-Jordan Method, find the inverse of the matrix.

$$\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \end{bmatrix}$$

Solution:

The augmented matrix is

$$\left\{\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right\}$$

Applying R1/3 => R1

$$\left\{\begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right\}$$

Applying R2 - R1 => R2, R3 – 2R1 => R3

$$\left\{\begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 0 & 7/3 & 7/3 & -1/3 & 1 & 0 \\ 0 & 11/3 & 11/3 & -2/3 & 0 & 1 \end{array}\right\}$$

Applying R2 * 3/7 => R2

$$\left\{\begin{array}{ccc|ccc} 1 & -1/3 & 2/3 & 1/3 & 0 & 0 \\ 1 & 1 & 1 & -1/7 & 3/7 & 0 \\ 2 & 11/3 & 11/3 & -2/3 & 0 & 1 \end{array}\right\}$$

Applying R1 + R2/3 => R1, R3 – 11R2/3 => R3

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 1 & 2/7 & 1/7 & 0 \\ 0 & 1 & 1 & -1/7 & 3/7 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1/7} & \mathbf{-11/7} & \mathbf{1} \end{array}\right\} \longleftarrow \textit{Zero Row}$$

Conclusion: The matrix is not invertible.

Triangular Factorization Method

Since, the system of linear equation can be expressed in the matrix form as: **AX = B**. So, here in the triangular factorization method, the coefficient matrix A of a system of linear equations can be factorized or decomposed into two triangular matrices L and U such that:

$$A = LU \text{ .... (i).}$$

$$\text{Where, } L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & l_{44} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ l_{n1} & l_{n2} & \cdots & \cdots & l_{n(n-1)} & l_{nn} \end{bmatrix} \text{, known as lower triangular matrix.}$$

$$\& \ U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & u_{24} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & u_{34} & \cdots & u_{3n} \\ \cdots & \cdots & \cdots & u_{44} & u_{45} \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & \cdots & u_{n(n-1)} & u_{nn} \end{bmatrix} \text{, known as upper triangular matrix.}$$

Once, A is factorized into L and U, the system of equations AX = B can be expressed by: **(LU)X = B**

$$\text{i.e. } L(UX) = B \text{ ... (ii)}$$

If we assume, **UX = Y**, where Y is an unknown vector. Then:

$$LY = B \text{ .... (iii)}$$

Now, we can solve AX = B in two stages:

a) Solving the equation: LY = B for Y by forward substitution and
b) Solving the equation UX = Y for X using Y by backward substitution.

The elements of L and U can be determined by comparing the elements of the product of L and U with those of A. This is done by assuming the diagonal elements of L or U to be unity.

- The decomposition with L having unit diagonal values is called the Doolittle LU Decomposition.
- The decomposition with U having unit diagonal elements is called the Crout LU Decomposition

Example: Solve the system of three simultaneous linear equations by using Do-little LU Decomposition Method.

$$\begin{aligned} 3x + 2y + z &= 10 \\ 2x + 3y + 2z &= 14 \\ x + 2y + 3z &= 14 \end{aligned}$$

Solution:

Using Do-little LU Decomposition, we have:

$$[A] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{12} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

On comparison, we will have the following relations:

$u_{11} = 3$, $u_{12} = 2$, $u_{13} = 1$, $l_{21}u_{11} = 2 : l_{21} = 2/3$,
$l_{21}u_{12} + u_{22} = 3: u_{22} = 3 - (2/3)2 = 5/3$,
$l_{21}u_{12} + u_{23} = 2: u_{23} = 2 - (2/3)1 = 4/3$,
$l_{31}u_{11} = 1: l_{31} = 1/3$, $l_{31}u12 + l_{32}u_{22} = 2: l_{32} = (2 - (1/3)2)/(5/3) = 4/5$
$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3: u_{33} = (3 - (4/5)(4/3) - (1/3)(1) = 24/15$

Thus we have: $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{24}{15} \end{bmatrix}$

(i) Forward Substitution: Solving: $\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$, we will get:

$y_1 = 10$, $y_2 = 22/3$ and $y_3 = 72/15$.

(ii) Backward Substitution: Solving: $\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{24}{15} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{72}{15} \end{bmatrix}$, we will get:

$z = 3$, $y = 2$ and $x = 1$.