

1.2.1 Secondary goals

- Learn proof techniques
- Learn some mathematics
- Have fun: Algorithms can be beautiful and ever poetic.

1.3 Typical Functions

- **polynomials** – $n, n^{\frac{1}{2}}, n^2, n^{10}, n^k$, etc (grows fast)
- **exponentials** – $2^n, 3^n, e^n, 11^{2n}$, etc (grows very fast)
- **logarithms** – $\log n = \log_2 n, \ln n, \log_{10} n$, etc (grows slowly)
- **poly logarithms** – $(\log n)^2$ (grows slowly)
- **log-logarithmic** – $\log \log n$ (grows slowly)
- **$\log^* n$** – the number of times you have to take the log of a number before you get something less than 1. (grows very slowly)

1.3.1 Illustrating the Growth of $\log \log n$ and $\log^* n$

- **Given:** Let N be the number of particles in the Universe. It is estimated that there are over 10^{80} particles in the universe.
 1. **Question:** What is the $\log \log$ of that number?
 - **Answer:** We know that $10^3 \approx 2^{10}$. This means that 10^{80} can be represented as $10^{3(27)}$, which is about 2^{270} . The $\log 2^{270} \approx 270$ and the $\log \log 2^{270}$ is about 8.1.
 2. **Question:** What is the \log^* of that number?
 - **Answer:** $5 \leq \log^*(10^{80}) \leq 6$

Moral: $\log \log n$ and $\log^* n$ are very slowly growing functions.

1.4 Review of logs

Here are some basic properties of logs:

- $\log_a x = y \iff x = a^y$
- $\log_a x = y$
- $\log_a (x \cdot y) = \log_a x + \log_a y$
- $\log_a x^y = y \cdot \log_a x$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- Note that ‘a’ is the base of the log. If omitted, assume base of 2.

1.4.1 Examples of logs

1. **Claim:** $2^{\log n} = n$.

- **Proof:** Take the log of both sides:
 $\Rightarrow \log 2^{\log n} = \log n$

2. **Claim:** $n^{\frac{1}{\log n}} = 2$.

- **Proof:** Let $n = 2^{\log n}$.
 $\Rightarrow (2^{\log n})^{\frac{1}{\log n}} = 2^1 = 2$.

3. **Claim:** $3^{\log n} = n^{\log 3}$.

- **Proof:** Let $3 = 2^{\log 3}$.
 $\Rightarrow (2^{\log 3})^{\log n} = 2^{\log 3 \cdot \log n} = (2^{\log n})^{\log 3} = n^{\log 3}$

4. **Question:** How to convert $\log_a n$ to $\log_b n$?

- **Answer:** $\frac{\log_b n}{\log_a n} = \log_a b$.

1.5 Definitions of: $O(f(n)), \Omega(f(n)), \Theta(f(n)), o(f(n)), \omega(f(n))$

1.5.1 Definitions of $O(f(n))$ (“Big OH”) (“Order”)

We use O -notation to give an upper bound on a function, to within a constant factor. Figure 1.1 shows the intuition behind O -notation.

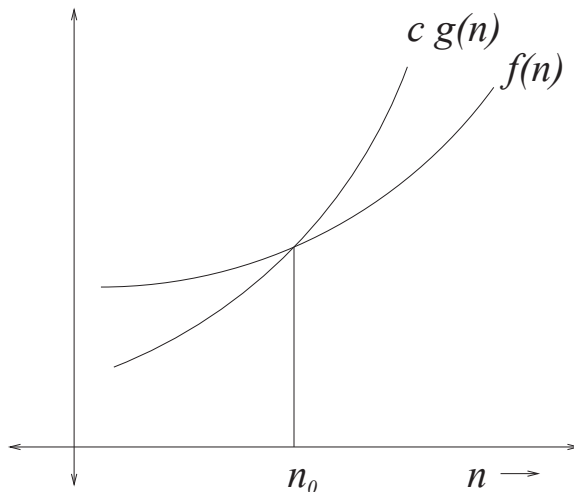


Figure 1.1: $f(n) = O(g(n))$

The following are three equivalent definitions of $O(f(n))$:

Definition 1. $f(n) \in O(g(n))$ means \exists constant $c > 0$ and some number n_0 such that for $n > n_0$, $f(n) \leq c \cdot g(n)$

Definition 2. $f(n) \in O(g(n))$ if $\exists c > 0$ such that for sufficiently large n , $c \cdot g(n) \geq f(n)$

Definition 3. $f(n) \in O(g(n))$ if $\exists c > 0$ such that,

$$\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) \leq c.$$

Examples:

- $n = O(n)$
- $n = O(n^2)$
- $\log n = O(\log n^2)$

We are interested in positive monotonically increasing functions and positive or non negative values of n .

1.5.2 Definitions of $\Omega(f(n))$ (“Omega”)

Just as O -notation provides an asymptotic *upper* bound on a function, Ω -notation provides an **asymptotic lower bound**.

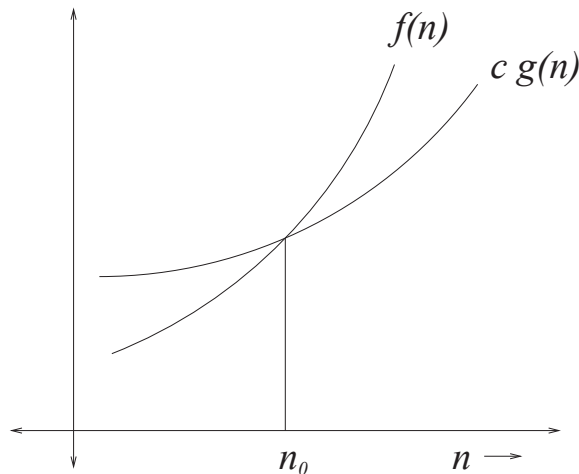


Figure 1.2: $f(n) = \Omega(g(n))$

The following are three equivalent definitions of $\Omega(f(n))$:

Definition 4. $f(n) \in \Omega(g(n))$ if $\exists c > 0$ and $n_0 > 0$ such that $\forall n > n_0$, $f(n) \geq c \cdot g(n)$.

Definition 5. $f(n) \in \Omega(g(n))$ if $\exists c > 0$ such that for sufficiently large n , $f(n) \geq c \cdot g(n)$.

Definition 6. $f(n) \in \Omega(g(n))$ if $\exists c > 0$ such that,

$$\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) > c.$$

Examples:

- $n^2 = \Omega(n)$.
- $n^2 = \Omega(n^2)$.
- $n^c = \Omega(\log n)$ for any constant c .

1.5.3 Definitions of $\Theta(f(n))$ (“Theta”)

Θ -notation is used to provide **asymptotic tight bound** on a given function.

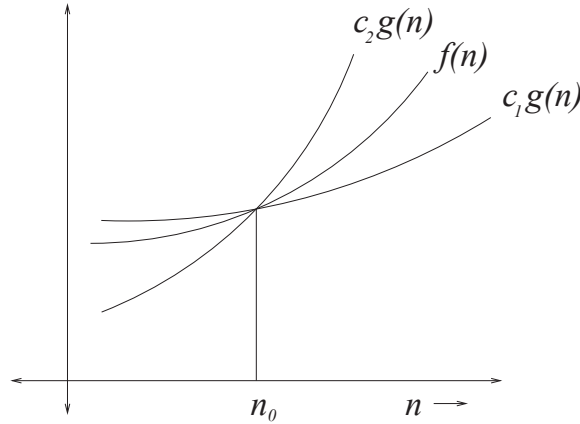


Figure 1.3: $f(n) = \Theta(g(n))$

The following are three definitions of $\Theta(f(n))$:

Definition 7. $f(n) \in \Theta(g(n))$ means that $\exists c_1, c_2 > 0$ and $n_0 > 0$ such that $\forall n > n_0$
 $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ where $c_1 \leq c \leq c_2$

Definition 8. $f(n) \in \Theta(g(n))$ if $\exists c$ such that

$$\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) = c.$$

Definition 9. $f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

Counter Example

Shows that the second definition and third definition of $\Theta(g(n))$ are not equivalent.

- Let $f(n) = 3^n$, if n is even, $f(n) = 2 * 3^n$, if n is odd,
and $g(n) = 3^n$.
- Then $f(n)$ is still strictly increasing,
but $\frac{f(n)}{g(n)} = 1$, if n is even,
and $\frac{f(n)}{g(n)} = 2$, if n is odd.
- So the limit of $\frac{f(n)}{g(n)}$ does not exist.
- Therefore, the third definition is correct.

1.5.4 Definition of $o(f(n))$ (“little oh”)

Definition 10. $f(n) \in o(g(n))$ means

$$\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) = 0.$$

Also read “ $g(n)$ dominates $f(n)$ ”

Examples:

- **Given:** Suppose $a > b$.

1. **Question:** Does n^a dominate n^b ?

– **Answer:** Yes. Because,

$$\lim_{n \rightarrow \infty} \left(\frac{n^b}{n^a} \right) = 0.$$

2. **Question:** What about $\log_a n$ and $\log_b n$. Does $\log_a n$ dominate $\log_b n$?

– **Answer:** No. Because,

$$\lim_{n \rightarrow \infty} \left(\frac{\log_b n}{\log_a n} \right) = \log_a b \neq 0.$$

1.5.5 Definition of $\omega(f(n))$ (“little omega”)

Definition 11. If $f(n)$ is $\Omega(g(n))$, but not $\Theta(g(n))$, then $f(n)$ is said to be $\omega(g(n))$.

Definition 12. $f(n) \in \omega(g(n)) \iff g(n) \in o(f(n))$

1.6 Examples:

1.6.1 True & False Questions:

1. Claim: $\log_2 n \in o(\log_{10} n)$

- Answer: FALSE.
- Counter Example: Let $n = 1000$,
 $\Rightarrow \log_2 1000 \approx 10$ and $\log_{10} 1000 = 3$.
 $10 > 3$ therefore, $\log_2 n$ is not $\in o(\log_{10} n)$.

2. Claim: $\log_2 n \in \Theta(\log_2 n)$.

- Answer: TRUE.
- Proof:

$$\lim_{n \rightarrow \infty} \left(\frac{\log_2 n}{c \cdot \log_2 n} \right) = c \Rightarrow \log_2 n \in \Theta(\log_2 n).$$

3. Claim: $\log_{10} n \in \Theta(\log_2 n)$.

- Answer: TRUE.
- Proof: Let $\log_{10} n = \frac{\log_2 n}{\log_2 10}$,

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{\log_2 n}{\log_2 10}}{c \cdot \log_2 n} \right) = \frac{1}{c \cdot \log_2 10} \Rightarrow \log_{10} n \in \Theta(\log_2 n).$$

4. Claim: $3^n \in O(2^n)$.

- Answer: FALSE.
- Counter Example:

$$\lim_{n \rightarrow \infty} \left(\frac{2^n}{3^n} \right) = 0 \Rightarrow 2^n \in o(3^n).$$