Homework_10

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Probability for Data Science

UC Berkeley, Fall 2019

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1 Homework 10

1.0.1 Instructions

Your homeworks have two components: a written portion and a portion that also involves code. Written work should be completed on paper, and coding questions should be done in the notebook. You are welcome to LaTeX your answers to the written portions, but staff will not be able to assist you with LaTeX related issues. It is your responsibility to ensure that both components of the homework are submitted completely and properly to Gradescope. Refer to the bottom of the notebook for submission instructions.

[1]:
```
# HIDDEN
from datascience import *
from prob140 import *
import numpy as np
import matplotlib.pyplot as plt
plt.style.use('fivethirtyeight')
%matplotlib inline
import math
from scipy import stats
```

1.0.2 1. Peter Meets Paul

Peter and Paul agree to meet at a restaurant at noon. Peter arrives at time normally distributed with mean 12:00 noon and SD 5 minutes. Paul arrives at a time normally distributed with mean 12:02 P.M. and SD 3 minutes.

Find the chances below assuming that the two arrival times are independent. First, write a formula for the chance in terms of the standard normal cdf Φ. Then use a code cell to find the numerical value. You do not have to turn in any coding work for this question.

a) P(Peter arrives before Paul)

b) P(both men arrive within 3 minutes of noon)

c) P(the two men arrive within 3 minutes of each other)

[2]:
```
# Calculation for 1a
1-stats.norm.cdf(0,2,34**0.5)
```

[2]: 0.6341997055200493

[3]:
```
# Calculation for 1b
peter= stats.norm.cdf(3,0,5)-stats.norm.cdf(-3,0,5)
#peter
paul= stats.norm.cdf(1,0,3)-stats.norm.cdf(-5,0,3)
#paul
peter*paul
```

[3]: 0.2631162570048904

[4]:
```
# Calculation for 1c
stats.norm.cdf(3,-2,34**0.5)-stats.norm.cdf(-3,-2,34**0.5)
```

[4]: 0.37249786716756705

1.0.3 2. Slices of a Normal Cake

Let X and Y be independent standard normal random variables.

a) Find $P(X > 0, Y > 0)$.

Yes, it's easy. But get a piece of paper and draw the event on the plane anyway. Imagine the joint density surface over the plane, and try to imagine the relevant volume under the joint density surface as a quadrant-shaped slice of a bell-shaped cake.

b) Find $P(X > 0, Y > X)$.

c) Find $P(X > 0, Y > \sqrt{3}X)$.

1.0.4 3. Distance Between Two Normal Points

Suppose two shots are fired at a target. Assume each shot hits with independent normally distributed coordinates, with the same means and equal unit variances. Let D be the distance between the point where the two shots strike.

a) Find $E(D)$. Your calculation will go faster if you remember that a normal $(0, \sigma^2)$ variable can be written as σZ where Z is standard normal.

b) Find $Var(D)$.

1.0.5 4. Min and Max of IID Uniforms

Let $U_1, U_2, \ldots, U_n$ be i.i.d. uniform on $(0, 1)$. Let $U_{(1)}$ and $U_{(n)}$ be the minimum and maximum of $U_1, U_2, \ldots, U_n$.

a) Find the joint density of $U_{(1)}$ and $U_{(n)}$.

b) Find the density of $U_{(1)}$.

c) Fix $x \in (0, 1)$ and find the conditional density of $U_{(n)}$ given $U_{(1)} = x$.

d) For fixed $x \in (0, 1)$, let $X_1, X_2, \ldots, X_{n-1}$ be $n - 1$ i.i.d. uniform $(x, 1)$ random variables. Find the density of $M = \max\{X_1, X_2, \ldots, X_{n-1}\}$ and compare it to your answer to Part **c**.

e) The random variable $R_n = U_{(n)} - U_{(1)}$ is called the *range* of the sample $U_1, U_2, \ldots, U_n$. Find $E(R_n)$.

1.0.6 5. Poisson MGF

Let X have Poisson(μ) distribution, and let Y independent of X have Poisson (λ) distribution.

a) Find the mgf of X.

b) Use the result of (a) to show that the distribution of $X + Y$ is Poisson.

1.1 Submission Instructions

Many assignments throughout the course will have a written portion and a code portion. Please follow the directions below to properly submit both portions.

1.1.1 Written Portion

- Scan all the pages into a PDF. You can use any scanner or a phone using an application. Please **DO NOT** simply take pictures using your phone.
- Please start a new page for each question. If you have already written multiple questions on the same page, you can crop the image or fold your page over (the old-fashioned way). This helps expedite grading.
- It is your responsibility to check that all the work on all the scanned pages is legible.

1.1.2 Code Portion

- Save your notebook using File > Save and Checkpoint.
- Generate a PDF file using File > Download as > PDF via LaTeX. This might take a few seconds and will automatically download a PDF version of this notebook.
  - If you have issues, please make a follow-up post on the general HW 10 Piazza thread.

1.1.3 Submitting

- Combine the PDFs from the written and code portions into one PDF. Here is a useful tool for doing so.
- Submit the assignment to Homework 10 on Gradescope.
- **Make sure to assign each page of your pdf to the correct question.**
- **It is your responsibility to verify that all of your work shows up in your final PDF submission.**

1.1.4 We will not grade assignments which do not have pages selected for each question.

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[ ]:
```

hw 10.

1. ⇒ porter normal$(12, 5^2)$

paul normal $(12{:}02, 3^2)$

a.

⇒ let the time peter arrive be x & paul's be y.

⇒ P (petere arrive before paul) = $P(Y > X)$

$= P(Y - X > 0)$

⇒ let $D = Y - X$ → $P(Y - X > 0) = P(D > 0)$

⇒ since X & Y are both normal dist

⇒ mean of $D = 12{:}02 - 12{:}00 = 2$ minute

variance of $D = 5^2 + 3^2 = 34$

SD of $D = \sqrt{34}$

⇒ D = normal $(2, \sqrt{34})$ distribution

⇒ $P(Y - X > 0) = P(D > 0) = 1 - \phi\left(\frac{0-2}{\sqrt{34}}\right)$

$= 0.634$

⇒ thus, P(peter arrives before paul) = 0.634.

b.

⇒ since paul & peter have the normal dist, set the center for peter & paul = 0 (peter = 12 & paul = 12:03), and the diff in time is represent by minute → same unit as SD.

⇒ since peter and paul arrives independently.

⇒ P(both men arrive ± 3 noon) = P(peter ± 3 noon) * P(paul ± 3 noon)

$$\Rightarrow P(\text{peter} \pm 3 \text{ noon}) = \phi\left(\frac{12{:}03-12}{5}\right) - \phi\left(\frac{11{:}57-12}{5}\right)$$

$$= \phi\left(\frac{3}{5}\right) - \phi\left(\frac{-3}{5}\right)$$

$$= 0.4514$$

$$\Rightarrow P(\text{paul} \pm 3 \text{ noon}) = \phi\left(\frac{12{:}03-12{:}02}{3}\right) - \phi\left(\frac{11{:}57-12{:}02}{3}\right)$$

$$= \phi\left(\frac{1}{3}\right) - \phi\left(\frac{-5}{3}\right)$$

$$= 0.583$$

⇒ P(both men arrive ±3 noon) = 0.4514 * 0.583

= 0.263

⇒ thus, P(both men arrive within 3 minutes of noon) = 0.263.

c.

⇒ let peter arrive be X & paul arrive be y.

⇒ P(arrive within 3 minutes apart) = $P(|X-Y|<3)$

$$= P(-3 < X - Y < 3)$$

$\Rightarrow$ let $D = X - Y \Rightarrow P(-3 < X - Y < 3) = P(-3 < D < 3)$

$\Rightarrow$ mean of $D = 12 - 12{:}02 = -2$ minute

variance of $D = 5^2 + 3^2 = 34$

SD of $D = \sqrt{34}$

$\Rightarrow P(-3 < X - Y < 3) = P(-3 < D < 3)$

$$= \phi\left(\frac{3-(-2)}{\sqrt{34}}\right) - \phi\left(\frac{-3-(-2)}{\sqrt{34}}\right)$$

$$= 0.372$$

$\Rightarrow$ thus, P(the two men arrive within 3 minute of each other) $= 0.372$.

2.

a.

$\Rightarrow P(X > 0, Y > 0) = P(X > 0) * P(Y > 0)$

$$= \int_0^\infty \int_0^\infty \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right) dy\, dx$$

$$= \int_0^\infty \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left[\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy\right] dx$$

$\Rightarrow \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 1 \Rightarrow$ bc it's area under density curve

$\Rightarrow$ since the standard normal curve is symmetric at 0.

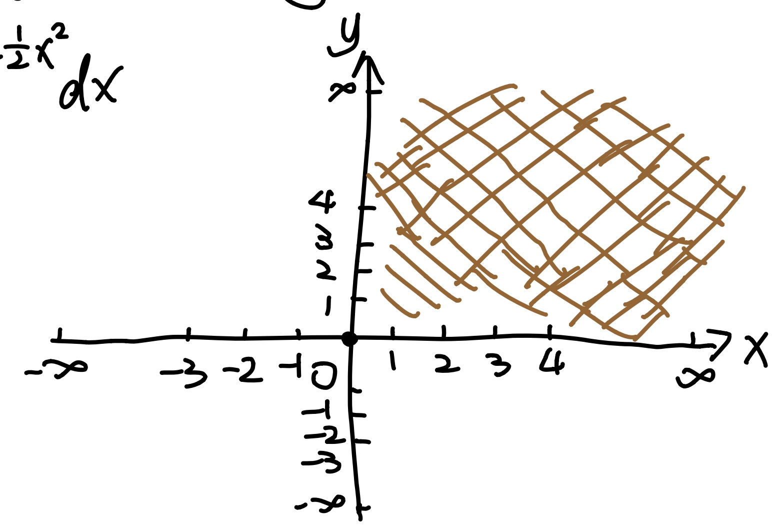
$$\Rightarrow \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = 1/2 = \frac{1}{2}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right) \left[\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy\right] dx$$

$$= \frac{1}{2} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{2} * \frac{1}{2}$$

$$= \frac{1}{4}$$

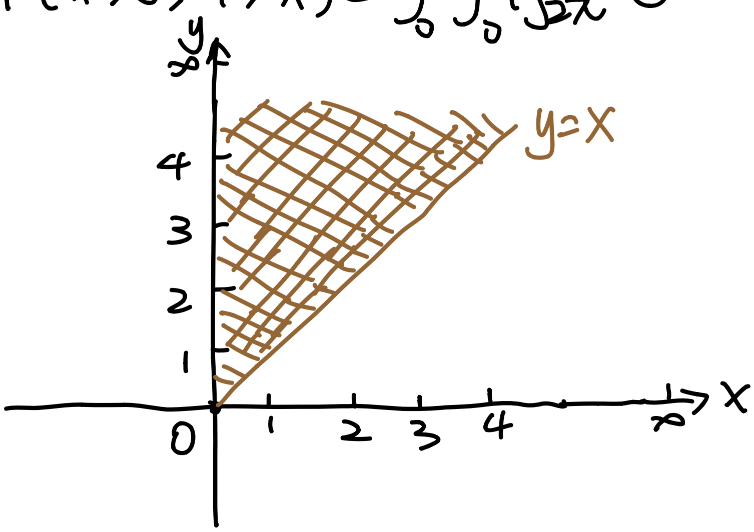


$\Rightarrow$ thus, $P(X>0, Y>0) = \frac{1}{4}$.

b.

$$\Rightarrow P(X>0, Y>X) = \int_0^{\infty}\int_0^{y} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right) dx\, dy.$$

$\Rightarrow$



$\Rightarrow$ for each same dxdy, the bottom surface area is cut in half compare to 2a.

$$\Rightarrow P(X>0, Y>0) = \frac{1}{2} * P(X>0, Y>0) = \frac{1}{2} * \frac{1}{4}$$

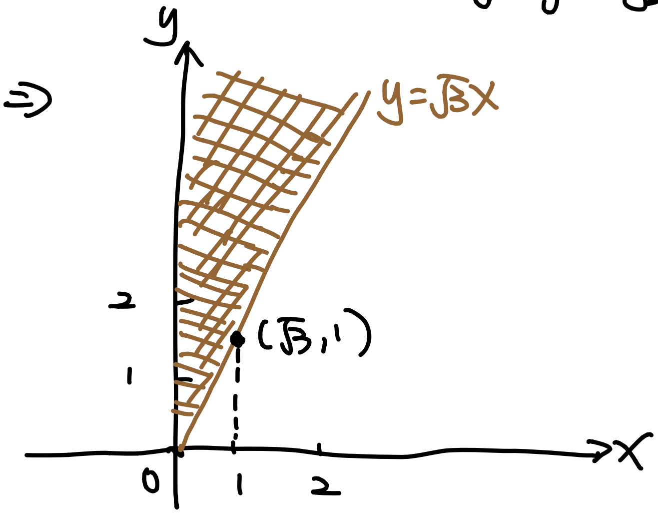
$$= \frac{1}{8}$$

$\Rightarrow$ thus, $P(X>0, Y>0) = \frac{1}{8}$.

c.

$$\Rightarrow P(X>0, Y>\sqrt{3}X) = \int_0^{\infty}\int_0^{y/\sqrt{3}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\right) dx\, dy.$$

$\Rightarrow$



$\Rightarrow$ bottom surface area

$\Rightarrow$ for $y = \sqrt{3}x$ $\rightarrow$ $x=1$ $y=\sqrt{3}$

$\Rightarrow \tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\Rightarrow \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} = 60^\circ$

$\Rightarrow$ actul θ occupy $= 90^\circ - 60^\circ = \frac{\pi}{2} - \frac{\pi}{3} = 30^\circ = \frac{1}{6}\pi$

$\Rightarrow$ from part 2a we know that $\frac{\pi}{2} = \frac{1}{4}$

$\rightarrow \frac{\pi}{2} \sim \frac{1}{4} \Rightarrow \pi = \frac{2}{4}$

$\rightarrow \frac{\pi}{6} = \frac{\frac{1}{2}}{6} = \frac{1}{12}$

$\Rightarrow$ thus, $P(X>0, Y>\sqrt{3}X) = \frac{1}{12}$.

3.

a.

$\Rightarrow$ let first shot $= (X_1, y_1)$ & second shot $= (X_2, y_2)$

$\Rightarrow D = \sqrt{(X_1 - X_2)^2 + (y_1 - y_2)^2}$

$\Rightarrow$ since $X_1 \overset{d}{=} X_2 \overset{d}{=} y_1 \overset{d}{=} y_2 = \text{norm}(\mu_i, \sigma_i^2) = \text{norm}(\mu_i, 1)$

$\Rightarrow X_1 - X_2 = \text{norm}(\mu_i - \mu_i, 1+1)$

$= \text{norm}(0, 2) = \sqrt{2} Z_x$

$\Rightarrow y_1 - y_2 = \text{norm}(\mu_i - \mu_i, 1+1)$

$= \text{norm}(0, 2) = \sqrt{2} Z_y$

$\Rightarrow D = \sqrt{(X_1 - X_2)^2 + (y_1 - y_2)^2} = \sqrt{2Z_x^2 + 2Z_y^2}$

$= \sqrt{2} * \sqrt{Z_1^2 + Z_2^2}$

$\Rightarrow \sqrt{Z_1^2 + Z_2^2}$ is a rayheight dist

$\rightarrow$ Expectation of rayheight dist $= E(\sqrt{T}) = \frac{\sqrt{2\pi}}{2}$.

$\hookrightarrow$ external source: statisticshowto.datasciencecentral.com.

$\Rightarrow D = \sqrt{2} * \frac{\sqrt{2\pi}}{2} = \sqrt{\pi}$

b.

$\Rightarrow \text{Var}(D) = E(D^2) - [E(D)]^2$

$\Rightarrow D^2 = (X_1 - X_2)^2 + (y_1 - y_2)^2$

$\Rightarrow z^2 = \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$\Rightarrow (X_1 - X_2)^2 = 2 z_x^2 = \text{gamma}(\frac{1}{2}, \frac{\frac{1}{2}}{2}) = \text{gamma}(\frac{1}{2}, \frac{1}{4})$

$\Rightarrow (y_1 - y_2)^2 = 2 z_y^2 = \text{gamma}(\frac{1}{2}, \frac{\frac{1}{2}}{2}) = \text{gamma}(\frac{1}{2}, \frac{1}{4})$

$\Rightarrow D^2 = \text{gamma}(\frac{1}{2} + \frac{1}{2}, \frac{1}{4}) = \text{gamma}(1, \frac{1}{4})$

$\Rightarrow$ Expetation of gamma $(r, \lambda) = \frac{r}{\lambda} = \frac{1}{\frac{1}{4}} = 4$

$\Rightarrow E(D^2) = 4$

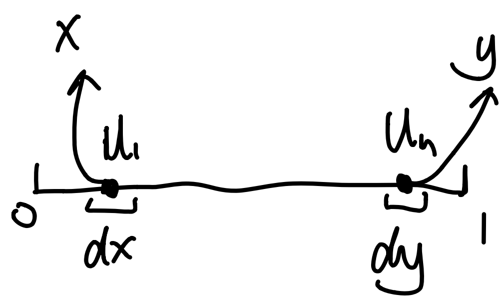
$\Rightarrow Var(D) = E(D^2) - [E(D)]^2 = 4 - (\sqrt{\pi})^2$

$= 4 - \pi$

$\Rightarrow$ thus, $Var(D) = 4 - \pi$.

4.

a.



$\Rightarrow P(U_{(1)} \in dx, U_{(n)} \in dy)$

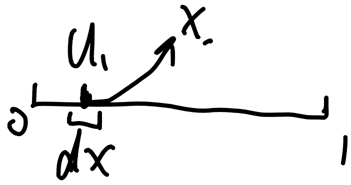
$= \frac{n!}{1!(n-2)!1!} (1dx)^1 (y-x)^{n-2} (1dy)^1$

$\Rightarrow$ joint density $f(x, y) = \frac{n!}{(n-2)!} (y-x)^{n-2}$

$\Rightarrow$ thus. the joint density is $\frac{n!}{(n-2)!} (y-x)^{n-2}$.

b.

$$\Rightarrow f_{U(k)}(X) = \frac{n!}{(k-1)!(n-k)!} X^{k-1} (1-X)^{n-k}$$



$$\Rightarrow f_{U(1)} = \frac{n!}{(n-1)!} (1-X)^{n-1}$$

$$= n * (1-X)^{n-1}$$

$\Rightarrow$ thus, the density of $U_{(1)} = n * (1-X)^{n-1}$

c.

$$\Rightarrow f_{Y|X=x} = \frac{f(x,y)}{f_X(x)} \text{ for all } Y.$$

$$\Rightarrow f_{U_n|X_1=x} = \frac{f(U_1, U_n)}{f_{U_1}(x)} = \frac{\cancel{n} * (n-1) * (y-x)^{n-2}}{\cancel{n} * (1-X)^{n-1}}$$

$$= \frac{(n-1)(y-x)^{n-2}}{(1-X)^{n-1}}$$

$$\Rightarrow \text{thus, } f(U_n | X_1 = x) = \frac{(n-1)(y-x)^{n-2}}{(1-x)^{n-1}}$$

d.

$\Rightarrow$

$$\Rightarrow f_M = f_{U_{n-1}} = \frac{(n-1)!}{(n-2)!} * \left(\frac{r-x}{1-x}\right)^{n-2}$$

$$= (n-1) * \left(\frac{r-x}{1-x}\right)^{n-2}$$

$$= \frac{(n-1) * (r-x)^{n-2}}{(1-X)^{n-2}}$$

$$\Rightarrow \text{thus, density } f_M = \frac{(n-1) * (r-x)^{n-2}}{(1-X)^{n-2}}$$

it's pretty similar to 4c expect there is one more 1-x in the denominator.

e.

$\Rightarrow f_{U_1} = n * (1-x)^{n-1}$

$\Rightarrow f_{U_n} = n * x^{n-1}$

$\Rightarrow E(R_n) = E(U_n - U_1) = E(U_n) - E(U_1)$

$= \int_0^1 x * n * x^{n-1} dx - \int_0^1 x * n * (1-x)^{n-1} dx$

$= n \int_0^1 x^n dx - n \int_0^1 x(1-x)^{n-1} dx$

$\Rightarrow n \int_0^1 x(1-x)^{n-1} dx \Leftarrow$ sustitute $1-x=u$ $du=-dx$ & $x=1-u$

$= + \frac{n}{n(n+1)}$

$\Rightarrow E(R_n) = n * \frac{x^{n+1}}{n+1} \Big|_0^1 - \frac{n}{n(n+1)}$

$= n * \left(\frac{1}{n+1} - 0\right) - \frac{\cancel{n}\, 1}{\cancel{n}(n+1)}$

$= \frac{n}{n+1} - \frac{1}{n+1}$

$= \frac{n-1}{n+1}$

$\Rightarrow$ thus, $E(R_n) = \frac{n-1}{n+1}$

J.

a.

$\Rightarrow$ Posisson $(\mu) \Rightarrow P(X=k) = e^{-\mu} \frac{\mu^k}{k!}$

$$\Rightarrow M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} * e^{-\mu} * \frac{\mu^x}{x!} dx.$$

$$= \sum_{x=0}^{\infty} e^{tx-\mu} * \frac{\mu^x}{x!} dx$$

$$= e^{-\mu} \sum_{x=0}^{\infty} e^{tx} * \frac{\mu^x}{x!} dx$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(e^t * \mu)^x}{x!} dx$$

$$\Rightarrow e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} \cdots\cdots$$ (used in hw 4 before)

$$\Rightarrow \sum_{x=0}^{\infty} \frac{(e^t \mu)^x}{x!} = e^{e^t * \mu}$$

$$\Rightarrow E(e^{tx}) = e^{-\mu} * e^{e^t * \mu} = e^{e^t * \mu - \mu}$$

$$= e^{\mu(e^t - 1)}$$

$\Rightarrow$ thus, the mgf of x, $M_X(t) = e^{\mu(e^t-1)}$ for all t.

b.

$\Rightarrow$ let $S = X + Y$

$$\Rightarrow M_S(t) = M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} * e^{tY})$$

$$= M_X(t) * M_Y(t)$$

$\Rightarrow$ since X = possion (μ_X) dist & Y = possion (λ_Y) dist

$$\Rightarrow M_X(t) * M_Y(t) = e^{\mu(e^t-1)} * e^{\lambda(e^t-1)}$$

$$= e^{(e^t-1)*(\mu+\lambda)}$$

$$\Rightarrow M_{X+Y}(t) = e^{(\mu+\lambda)*(e^t-1)}$$

$\Rightarrow$ let a variable G have poission $(\lambda + \mu)$ dist

$\Rightarrow M_G(t) = e^{(\mu+\lambda) * (e^t - 1)}$ $\rightarrow$ proved in 1a.

$\Rightarrow M_{X+Y}(t) == M_G(t) \rightarrow X+Y \overset{d}{=} G \overset{d}{=}$ poission $(\lambda+\mu, t)$

$\Rightarrow$ thus, since mgf of $X+Y$ is equal to mgf of poisson $(\lambda+\mu)$ dist,

the result show that $X+Y$ is possion dist, with parameter $\lambda+\mu$.