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Homework 3

1. Let X, Y, Z *and* W be independent random variables with $E(X) = 1$, $E(Y) = 0$, $E(Z) = 2$, $E(W) = 1$, $E(W^2) = 4$; $Var(X) = 1$, $Var(Y) = 2$, $Var(Z) = 2$. Evaluate:

(15 p) (a) $E(3X - Y + Z)$ and $Var(3X - Y + Z)$

$\Rightarrow \mathbb{E}(3X - Y + Z)$
$= \mathbb{E}(3X) - \mathbb{E}(Y) + \mathbb{E}(Z)$
$= 3\mathbb{E}(X) - \mathbb{E}(Y) + \mathbb{E}(Z)$
$= 3 - 0 + 2 = 5$

$\Rightarrow \mathbb{E}(3X - Y + Z) = 5$

$Var(3X - Y + Z)$
$= Var(3X) + Var(-Y) + Var(Z)$ ← iid. X, Y & Z
$= 9Var(X) + Var(Y) + Var(Z)$
$= 9 + 2 + 2 = 13$

$\Rightarrow Var(3X - Y + Z) = 13$

(10p) (b) $Var(4X-W)$

$\Rightarrow Var(4X - W)$
$= Var(4X) + Var(-W)$
$= 16Var(X) + Var(W)$ ← iid. X & W
$= 16 + 3 = 19$

$\Rightarrow Var(W) = \mathbb{E}(W^2) - (\mathbb{E}(W))^2$
$= 4 - 1 = 3$

$\Rightarrow Var(4X - W) = 19.$

(10p) (c) $E(1+X^2)$

$\Rightarrow \mathbb{E}(1 + X^2)$
$= \mathbb{E}(1) + \mathbb{E}(X^2)$
$= 1 + \mathbb{E}(X^2)$
$= 1 + 0 = 1$

$\Rightarrow Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
$1 = \mathbb{E}(X^2) - 1$
$\mathbb{E}(X^2) = 0$

$\Rightarrow \mathbb{E}(1 + X^2) = 0.$

2. Let $x_1, \ldots, x_n$ be independent random variables, all having the same distribution with expected value μ and variance σ^2.

(65 points) The random variable defined as the arithmetic average of these variables, is called the *sample mean.* That is, the sample mean is given by $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

(24p.) (a) Derive $E()$ and $Var()$ as functions of μ and σ^2 respectively.

$\Rightarrow \mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\mathbb{E}(X_1 + X_2 \ldots\ldots + X_n)$
$= \frac{1}{n}[\mathbb{E}(X_1) + \mathbb{E}(X_2) + \ldots\ldots + \mathbb{E}(X_n)]$ ← in total of n^{th} term.
$= \frac{1}{n} * (n * \mu)$
$= \mu$

$\Rightarrow \mathbb{E}(\bar{X}) = \mu.$

$$\Rightarrow Var(\bar{X}) = Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)$$

$= Var\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \dots\dots + \frac{1}{n}X_n\right) \leftarrow$ total of n^{th} term.

$= Var\left(\frac{1}{n}X_1\right) + Var\left(\frac{1}{n}X_2\right) + \dots\dots Var\left(\frac{1}{n}X_n\right) \leftarrow X_1, \dots\dots, X_n$ are independent variable.

$= \frac{1}{n^2}Var(X_1) + \frac{1}{n^2}Var(X_2) + \dots\dots + \frac{1}{n^2}Var(X_n)$

$= \frac{1}{n^2} * [n * \sigma^2) = \sigma^2/n$

$\Rightarrow \boxed{Var(\bar{X}) = \sigma^2/n.}$

(10p) (b) Is consistent? Explain.

$\Rightarrow$ ① $\mathbb{E}(\bar{X}) = \mu \Rightarrow$ thus $\bar{X}$ is an unbaised estimator

$\Rightarrow$ ② $Var(\bar{X}) = \sigma^2/n \Rightarrow$ as $n \to \infty$, $Var(\bar{X})$ approach to 0 (a single point).

$\Rightarrow$ thus, $\bar{X}$ is consistent.

(10p) (c) Consider an alternative linear estimator $\widetilde{X} = \sum_{i=1}^{n} a_i x_i$ where a_i are constant. What restrictions on a_i are needed for this alternative estimator to be unbiased?

$\Rightarrow \mathbb{E}(\tilde{X}) = \mathbb{E}(\sum_i^n a_i X_i) = \mathbb{E}(a_1 X_1 + a_2 X_2 \dots\dots + a_n X_n)$

$= a_1 * \mu + \dots\dots a_n * \mu) = \mu * (a_1 + a_2 + \dots\dots a_n)$

$\Rightarrow \mathbb{E}(\tilde{X}) == \mu$ if unbaised $\Rightarrow$ thus, for unbaised $\tilde{X}$, $\sum_{i=1}^{n} a_i = 1$.

(11p) (d) Derive Var(.

$\Rightarrow Var(\tilde{X}) = Var(\sum_{i=1}^{n} a_i X_i) = Var(a_1 X_1 + a_2 X_2 + \dots\dots X_n)$

$= Var(a_1 X_1) + Var(a_2 X_2) + \dots\dots + Var(a_n X_n)$

$= a_1^2 Var(X_1) + \dots\dots a_n^2 Var(X_n) = a_1^2 \sigma^2 + \dots\dots + a_n^2 \sigma^2$

$= \sigma^2 * \sum_{i=1}^{n} a_i^2$

$\Rightarrow \boxed{Var(\tilde{X}) = \sigma^2 \sum_{i=1}^{n} a_i^2}$

(10p) (e) Which linear estimator is more efficient? (Hint: use the inequality $\frac{(\sum_{i=1}^{n} a_i)^2}{n} \le \sum_{i=1}^{n} a_i^2$)

$\Rightarrow Var(\bar{X}) = \sigma^2/n$ & $Var(\tilde{X}) = \sigma^2 \sum_{i=1}^{n} a_i^2$

$\Rightarrow$ for $\tilde{X}$ to be unbaised (only then we can compare)

$\Rightarrow \sum_{i=1}^{n} a_i = 1 \leftarrow$ derive in 2c

$\Rightarrow$ we know $\frac{(\sum_{i=1}^{n} a_i)^2}{n} \le \sum_{i=1}^{n} a_i^2$

$\Rightarrow \frac{1}{n} \leq \sum_{i=1}^{n} a_i^2$ $\leftarrow \sum_{i=1}^{n} a_i = 1$

$\Rightarrow$ for $Var(\bar{x})$ & $Var(\hat{x})$, the diff in constant define the diff in those two variable

$\Rightarrow$ since $\frac{1}{n} \leq \sum_{i=1}^{n} a_i^2$

$\Rightarrow Var(\bar{x}) \leq Var(\hat{x})$

$\Rightarrow$ thus, for linear estimator $\bar{x}$ is more efficient than $\hat{x}$.