Robotic Navigation and Exploration

Week 2: Kinematic Model & Path Tracking Control

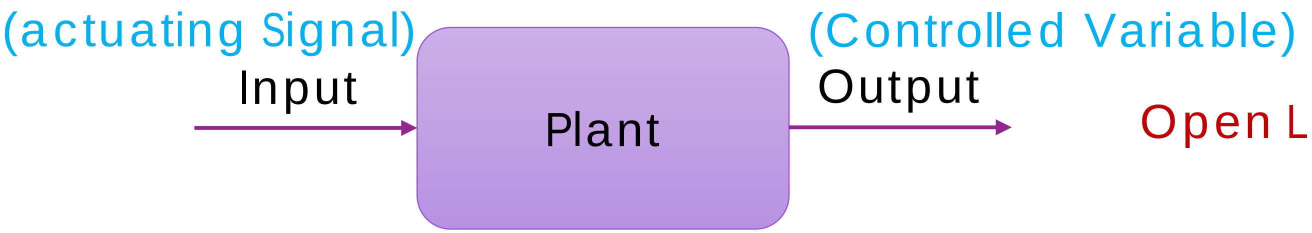
Min-Chun Hu anitahu@cs.nthu.edu.tw
CS, NTHU

Outline

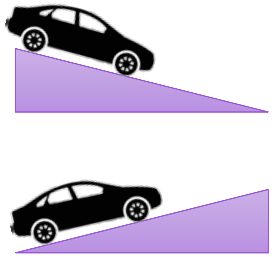
- Basics of Control System for Automobile
- Coordinate System
- Kinematic Model & Pure-Pursuit Control
- Differential Drive
- Bicycle Model
  - Pure Pursuit Control
  - Stanley Control (Path Coordinate and Control Stabilization)
  - Linear Quadratic Regulator (LQR)

Control Theory: Open Loop Control

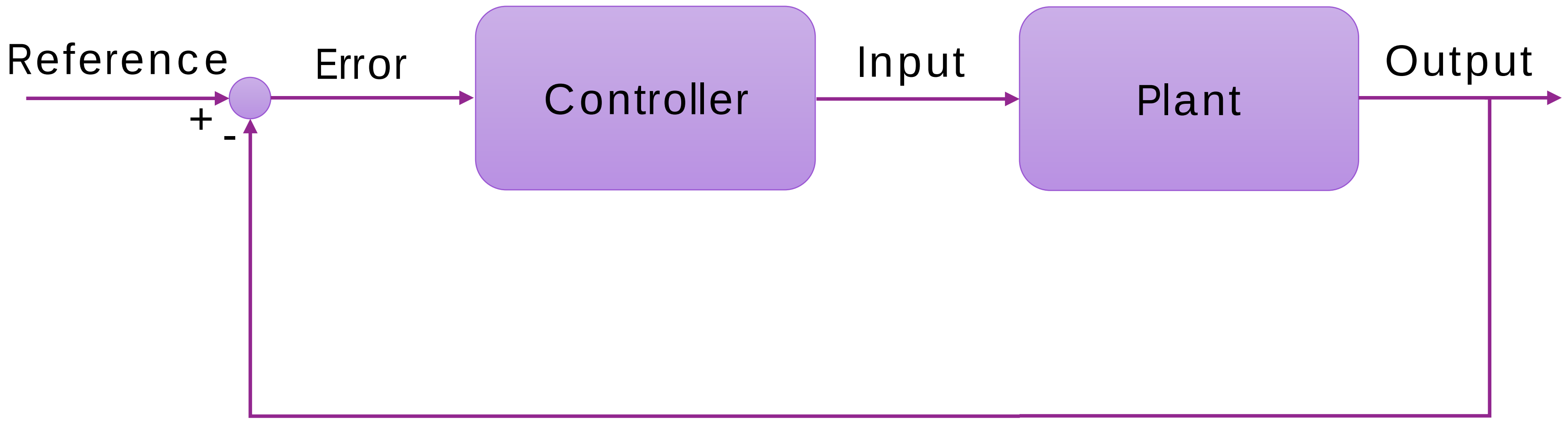
- Control System: the mechanism that affects the future state of a system
- Control Theory: a strategy to change input to desired output



Open Loop Control

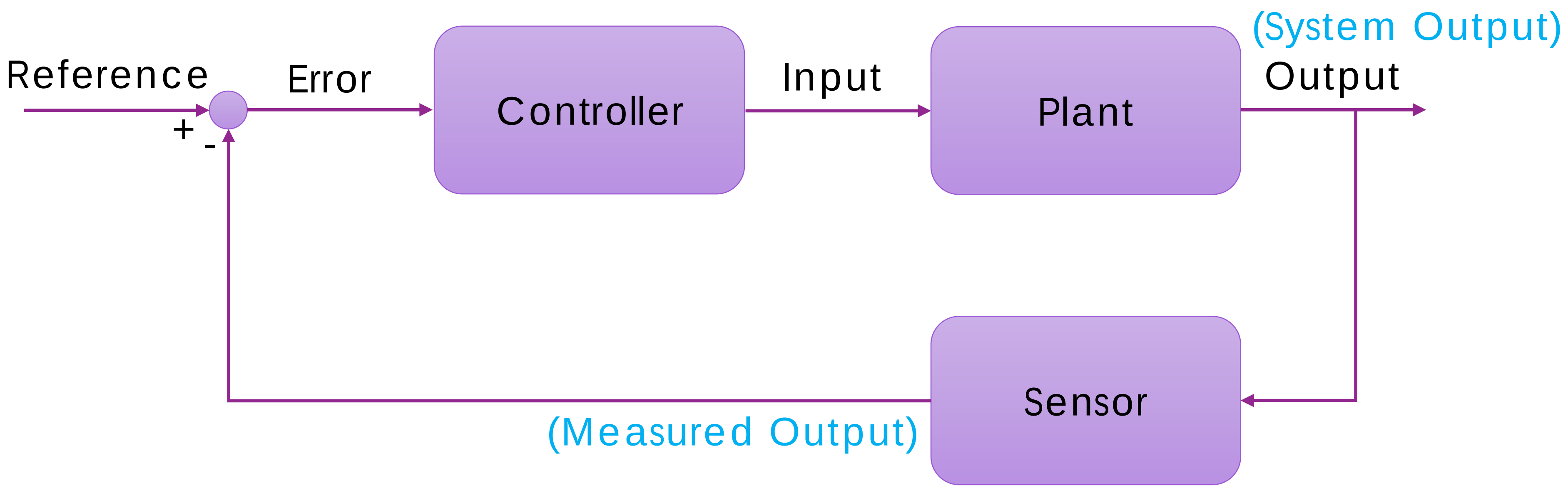


Control Theory: Close Loop Control



Close Loop Control
(Feedback Control)

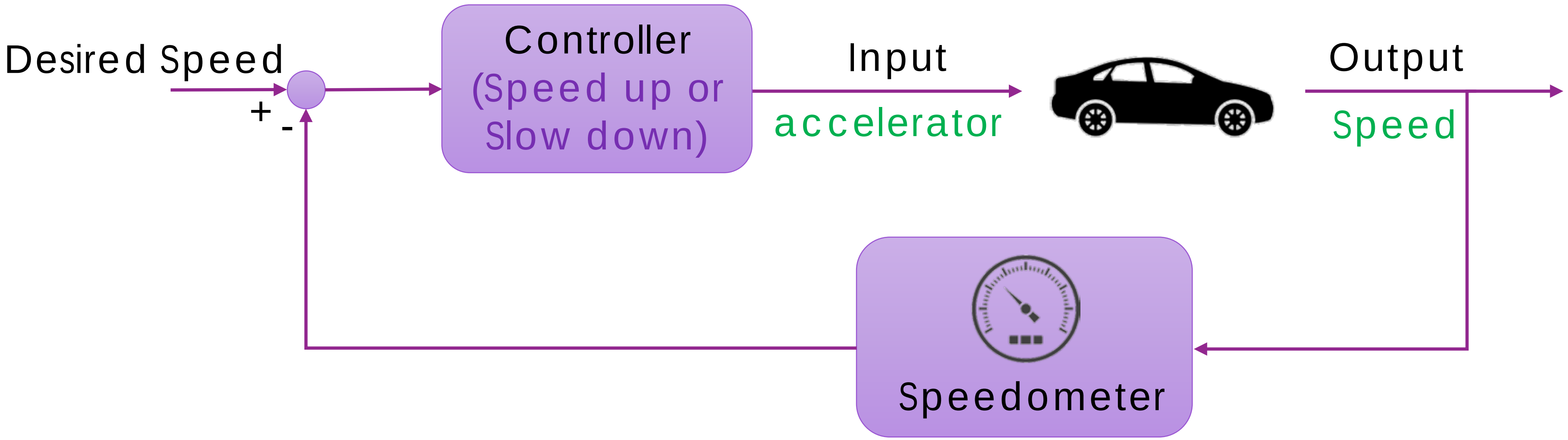
Control Theory: Close Loop Control



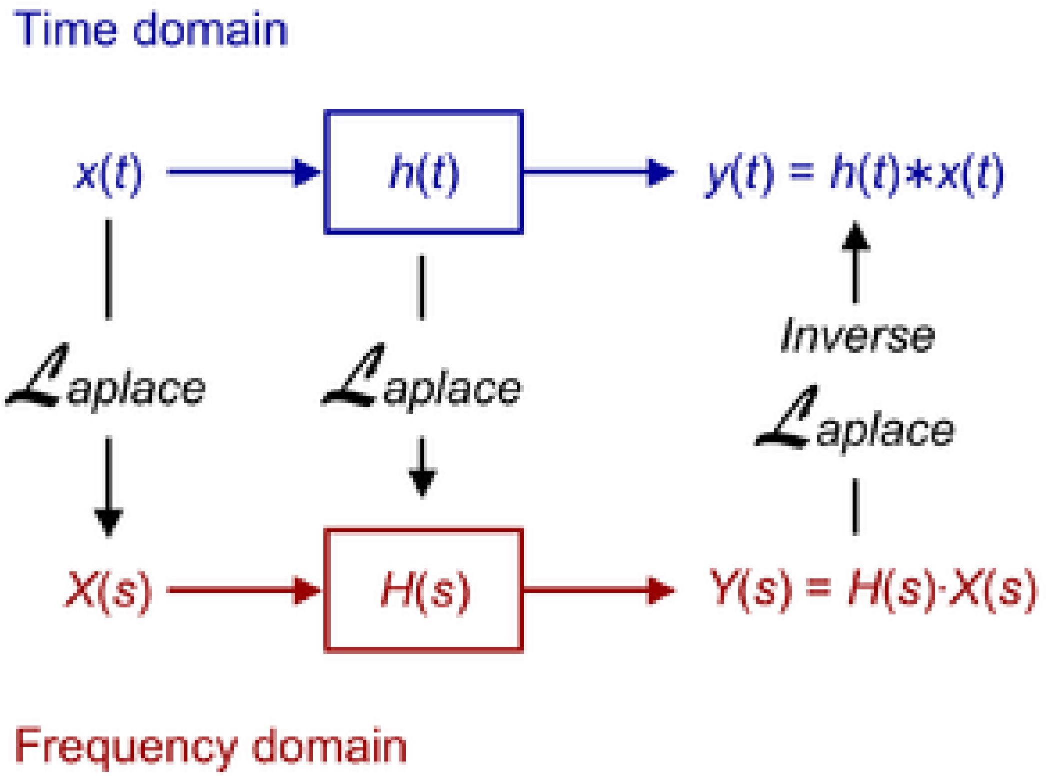
Close Loop Control

(Feedback Control)

Control Theory : Car Example



Linear Time Invariant System



Laplace transform

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

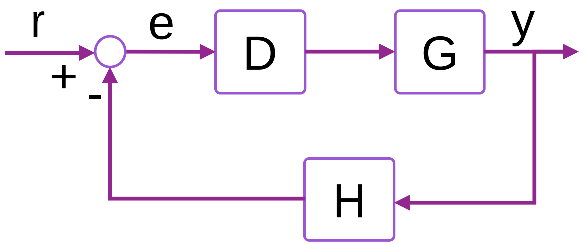
$$= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\, dt \quad \text{(by parts)}$$

$$= \left[-\frac{f(0^+)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},$$

Basic Laplace Transform Pairs

Signal or Function	f(t)	F(s)
Impulse	$\delta(t)$	1
Step	$u(t)=1, \quad t \geq 0$	$\frac{1}{s}$
Ramp	$r(t)=t, \quad t \geq 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t} \quad e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Damped Ramp	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2+\beta^2}$
Damped Sine	$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped Cosine	$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
Simple Complex Pole	see next pg	see next pg

Linear Time Invariant System



$$e = r - yH$$

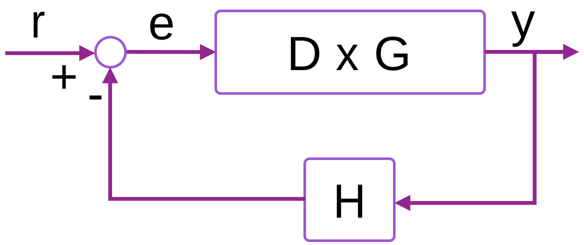
$$y = e \cdot D \cdot G$$

$$e = \frac{y}{DG}$$

$$r - yH = \frac{y}{DG}$$

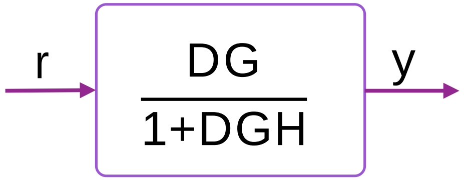
$$(DG)(r - yH) = y$$

$$DGr = y(1 + DGH) \quad \text{or} \quad y = \frac{DGr}{1 + DGH}$$



$\Longleftrightarrow$

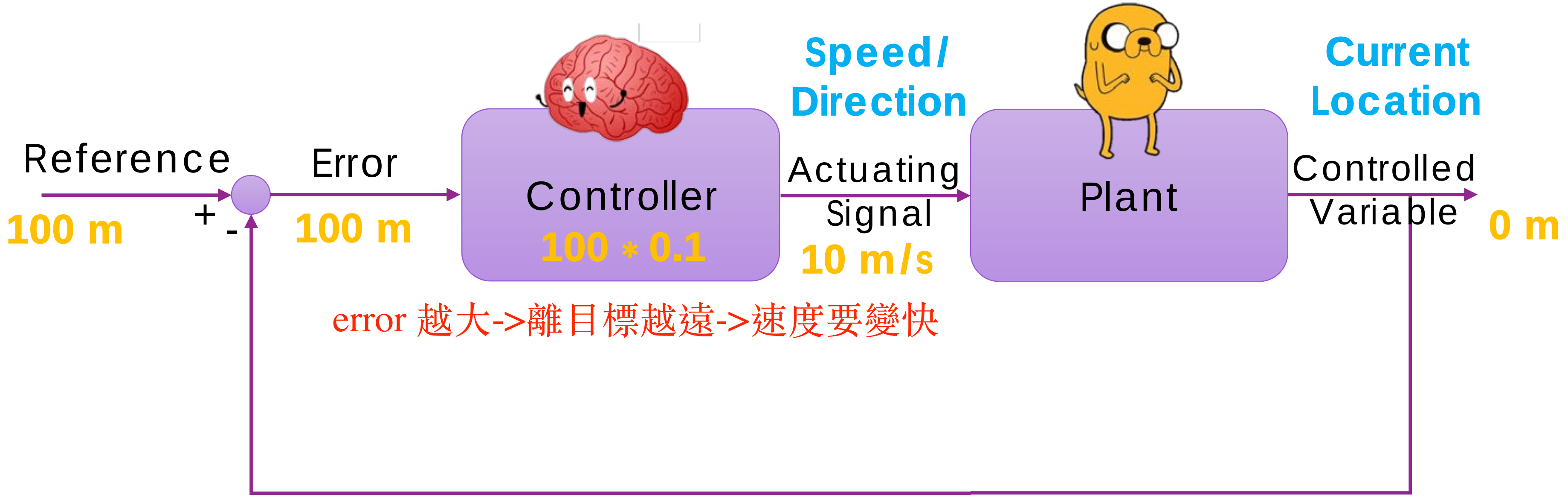
Equivalent



Plant

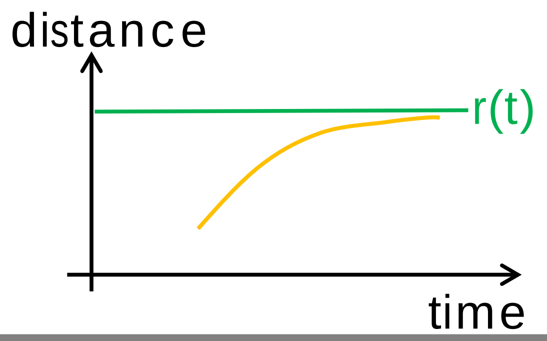
Open Loop

PID Control : Proportional Gain

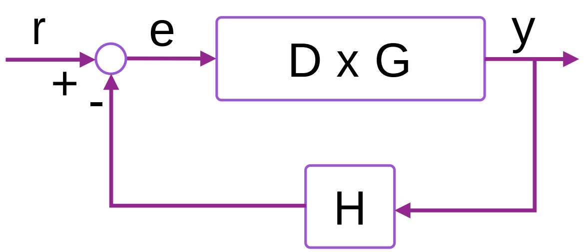


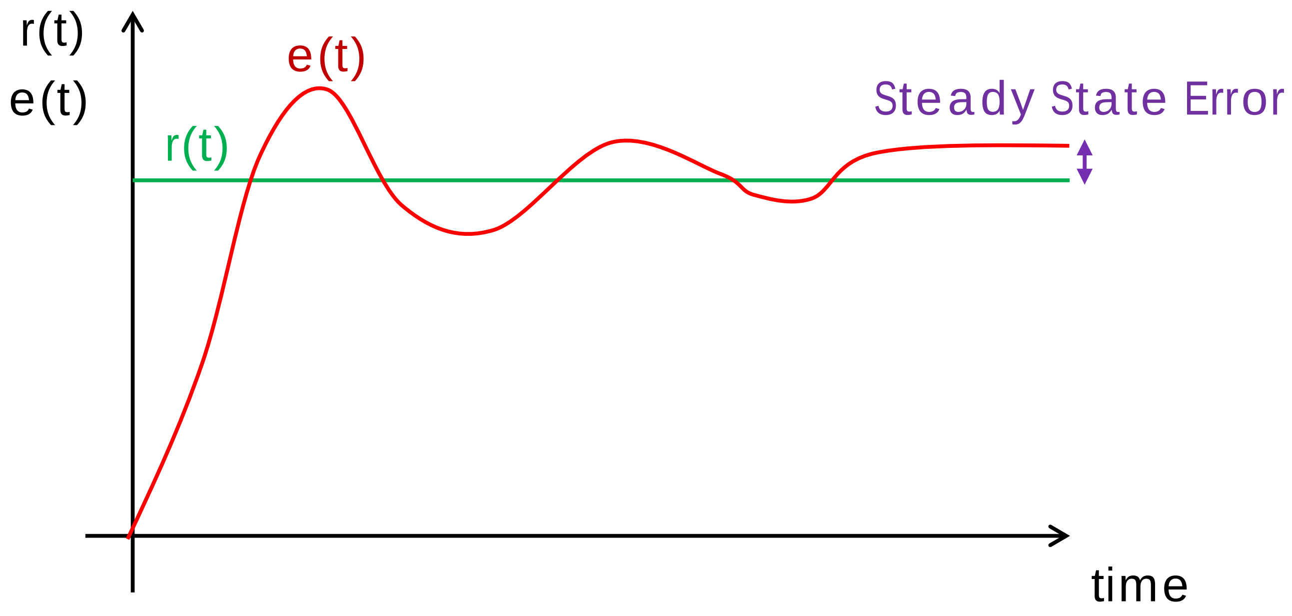
0 m

100 m (desired location)

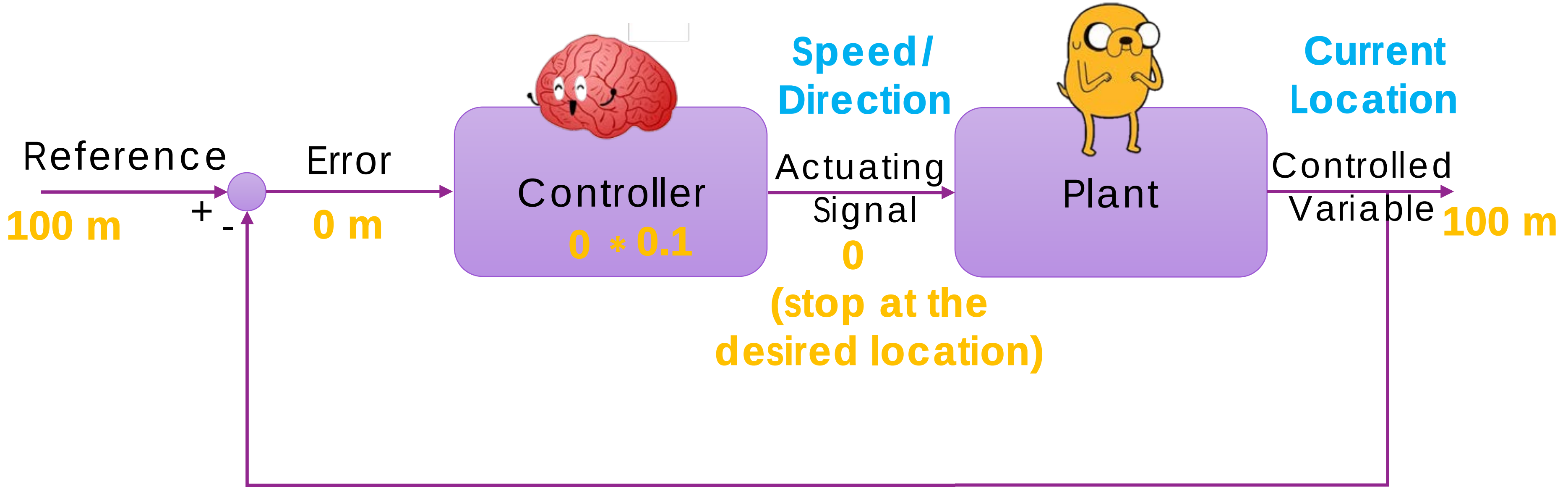


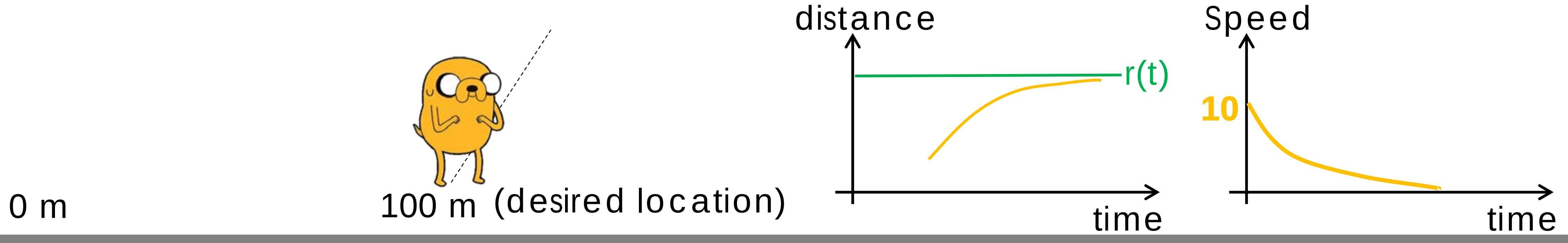
Steady State Error





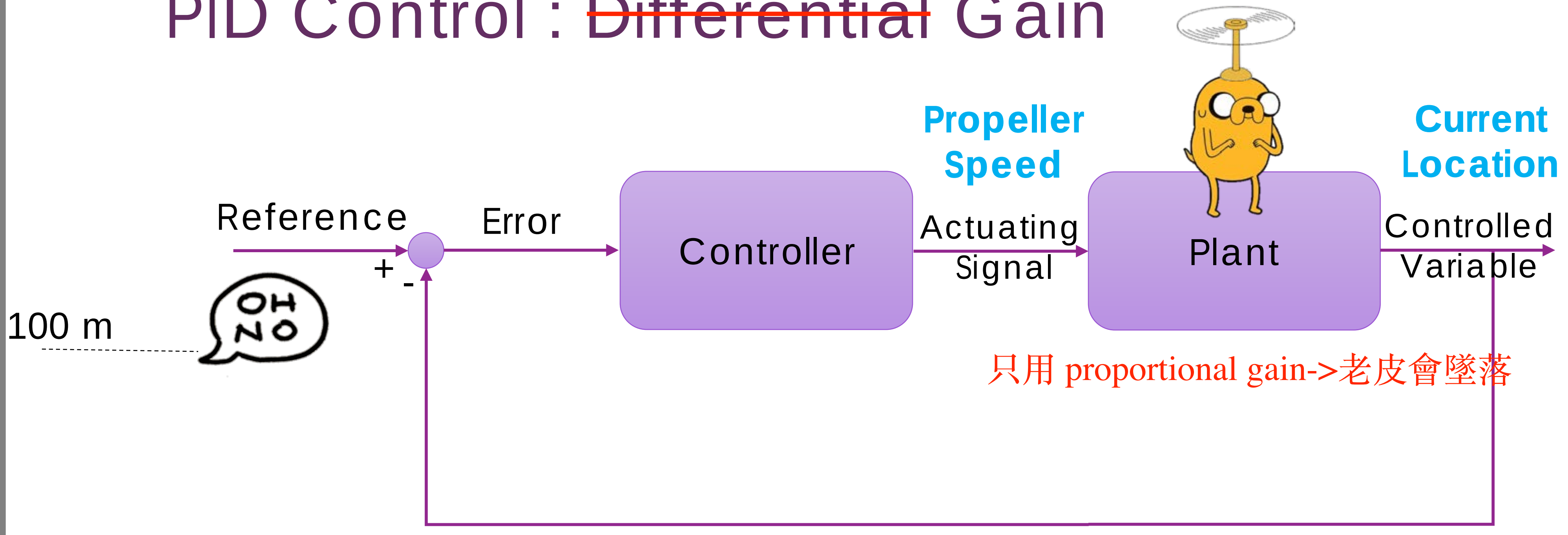
PID Control : Proportional Gain





Integral

PID Control : ~~Differential~~ Gain



只用 proportional gain->老皮會墜落

Error * Gain = Propeller Speed

Steady state error

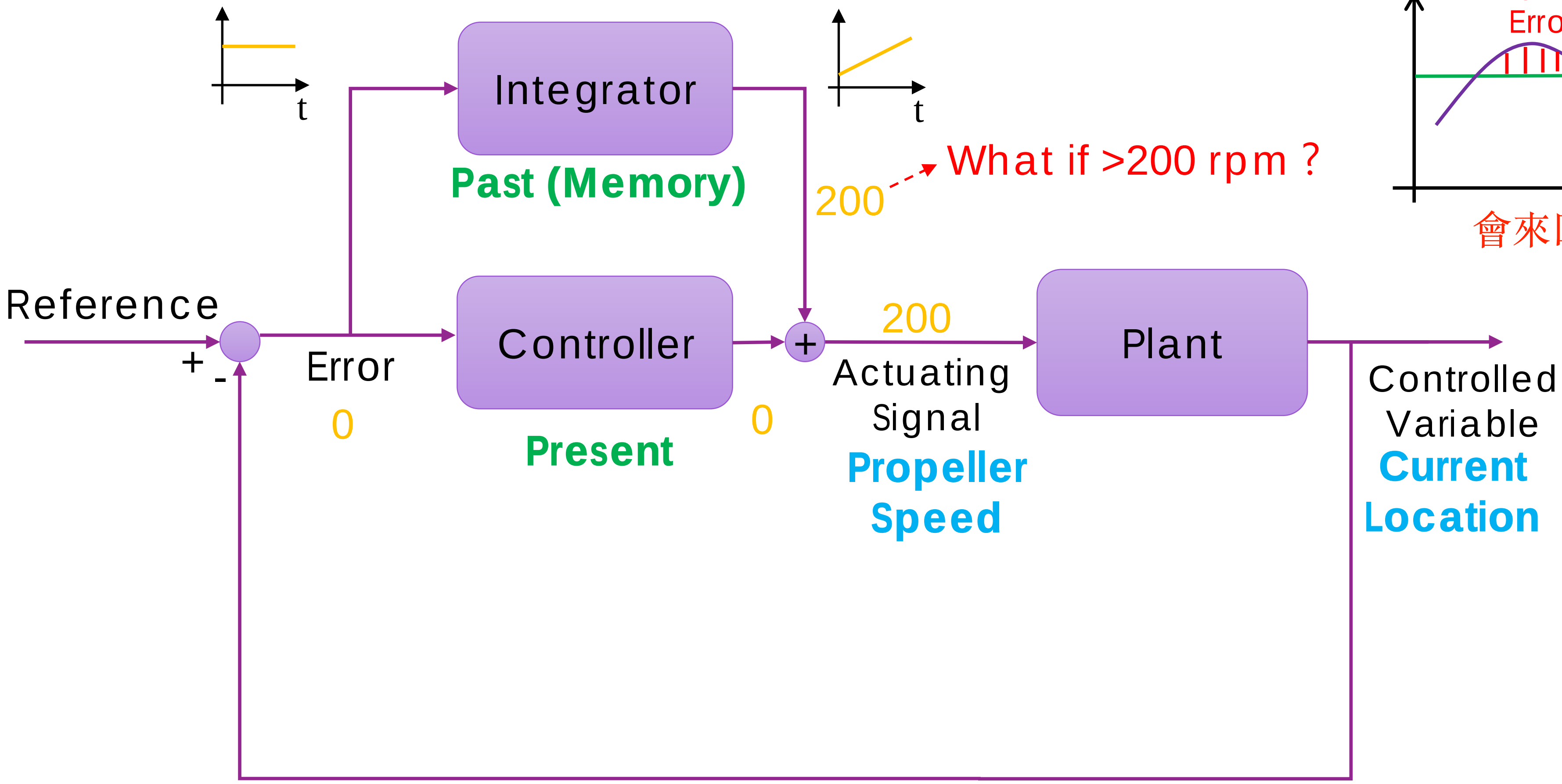
100	*	2	= 200 rpm
40	*	5	= 200 rpm
20	*	10	= 200 rpm
2	*	100	= 200 rpm

always get non-zero error

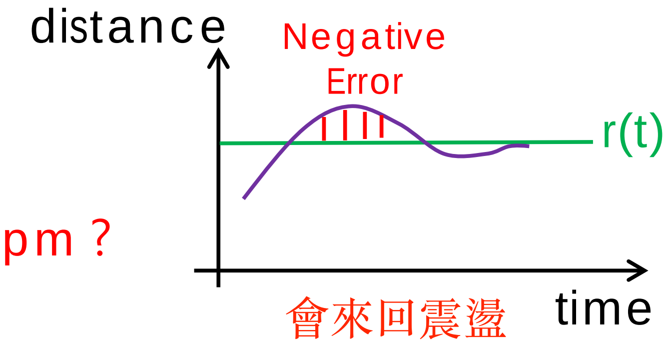
Idea: Consider past information ！

PID Control : Integral Gain

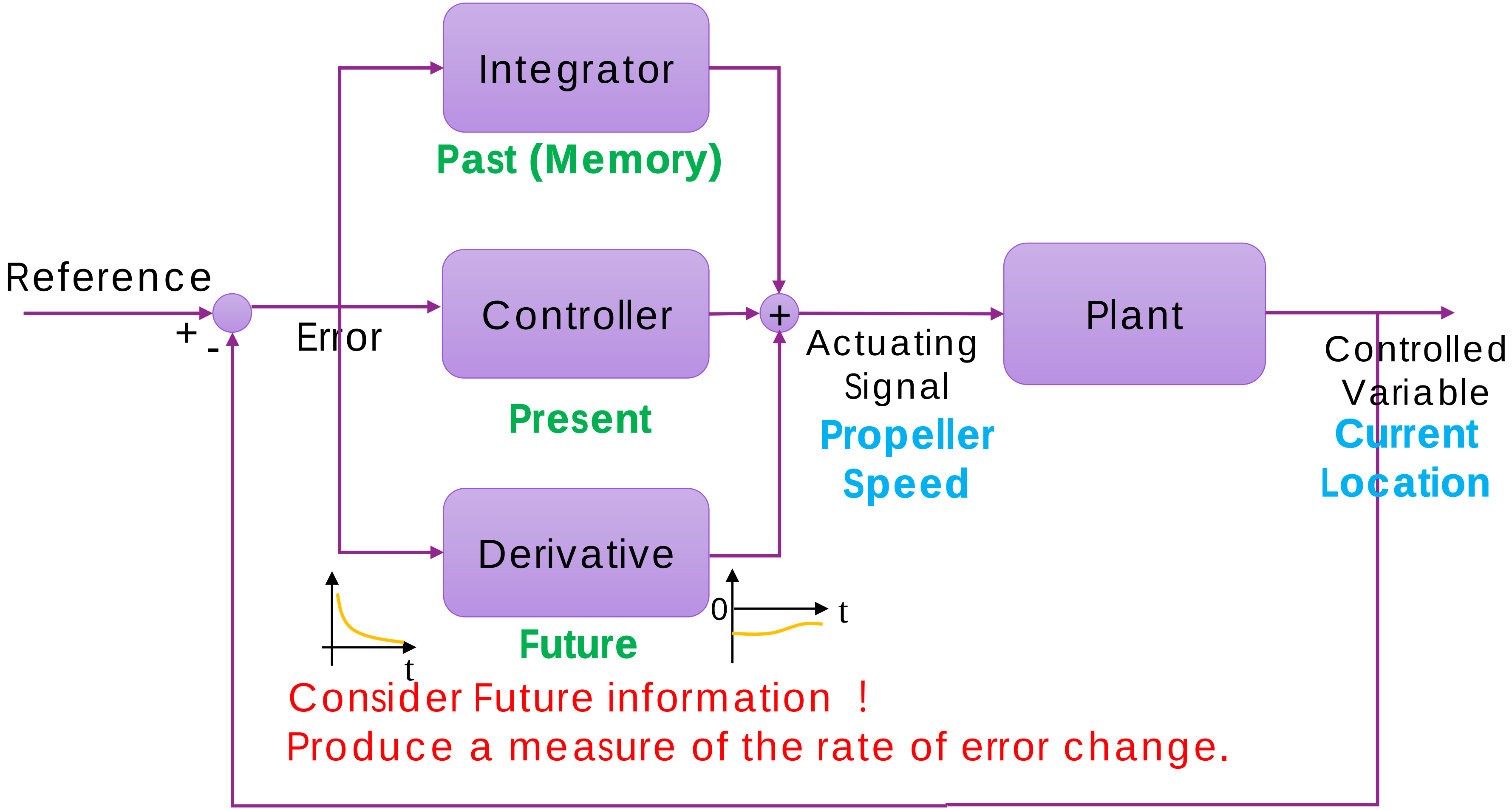
Consider past information ！
Sum up non-zero steady state error over time



Overshooting!



PID Control : Differential Gain



Consider Future information !

Produce a measure of the rate of error change.

PID Control

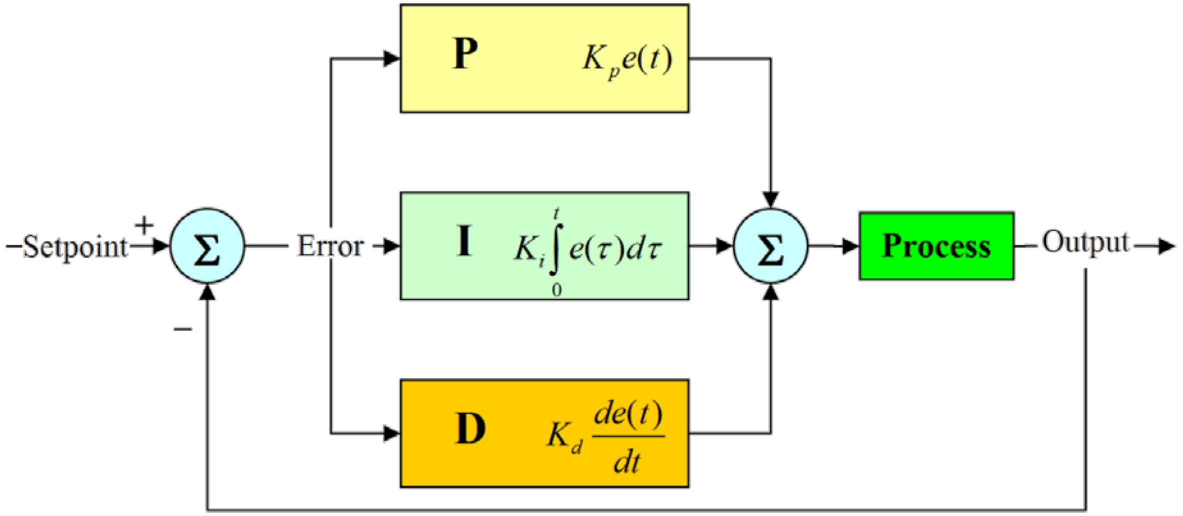
- **P**roportional / **I**ntegral / **D**ifferential Control

Continuous Form :

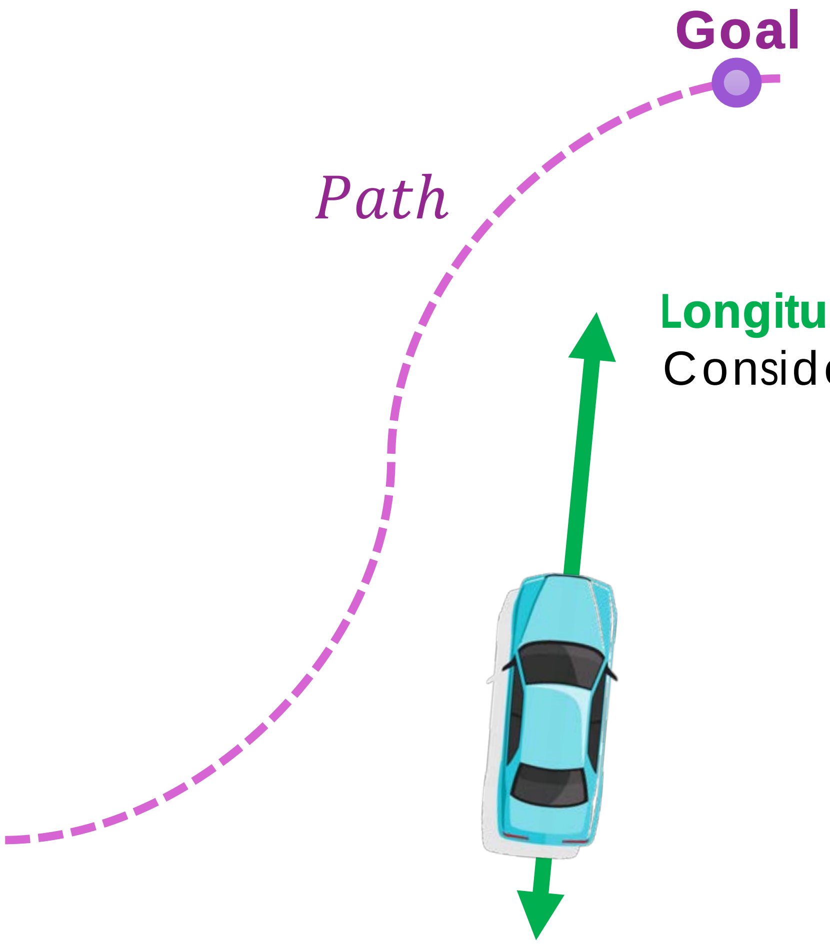
$$Output = K_p e(t) + K_i \int_0^t e_t \, dt + K_d \frac{de(t)}{dt}$$

Discrete Form :

$$Output = K_p e(t) + K_i \sum_0^t e_t + K_d(e(t) - e(t-1))$$

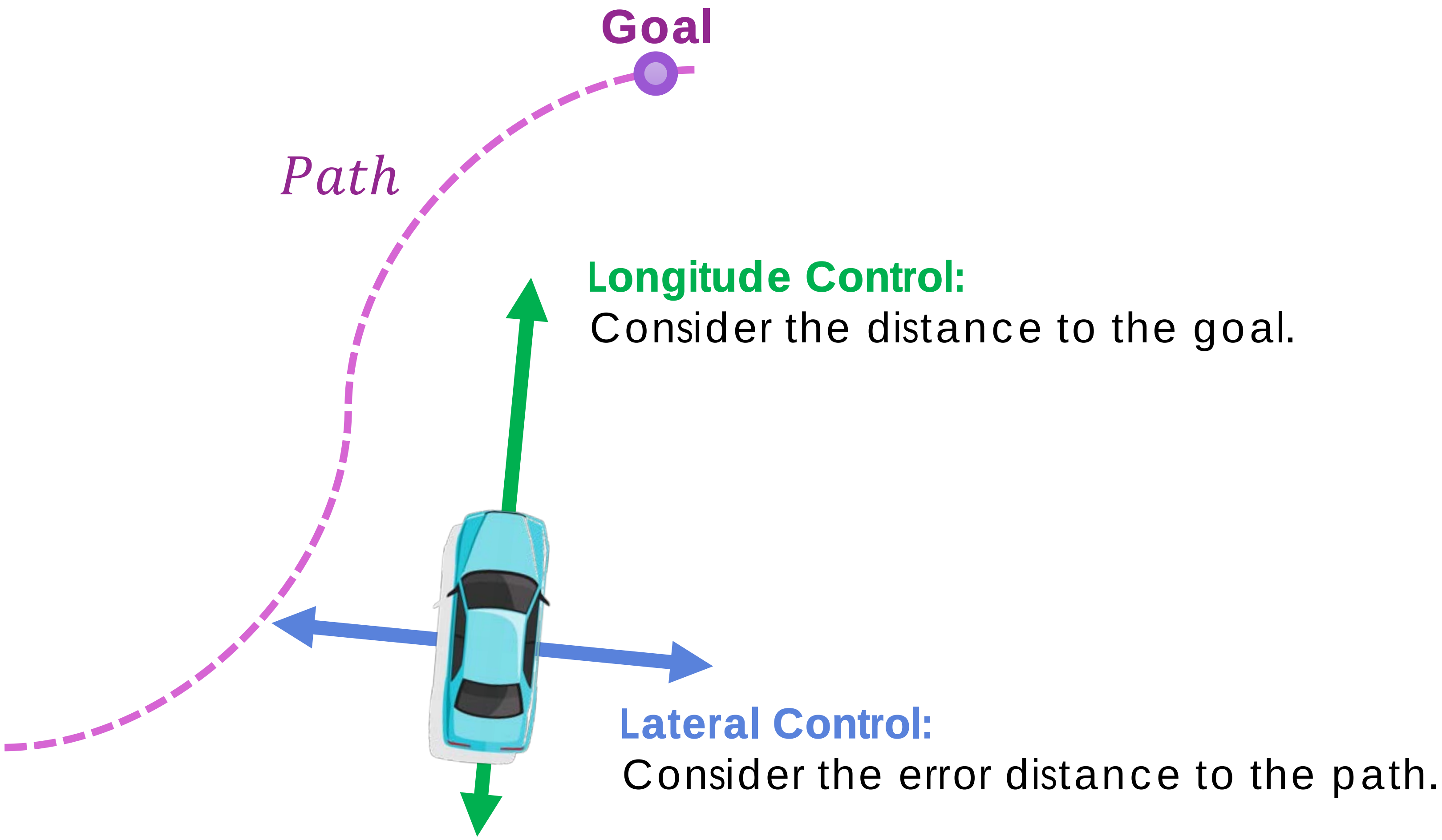


Path Tracking Problem



Longitude Control:
Consider the distance to the goal.

Path Tracking Problem



Basic Kinematic Model

低速->考慮幾何特性的運動模型
高速->摩擦力變低開始側向移動，需考慮動力學模型

State:

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Rotation Matrix:

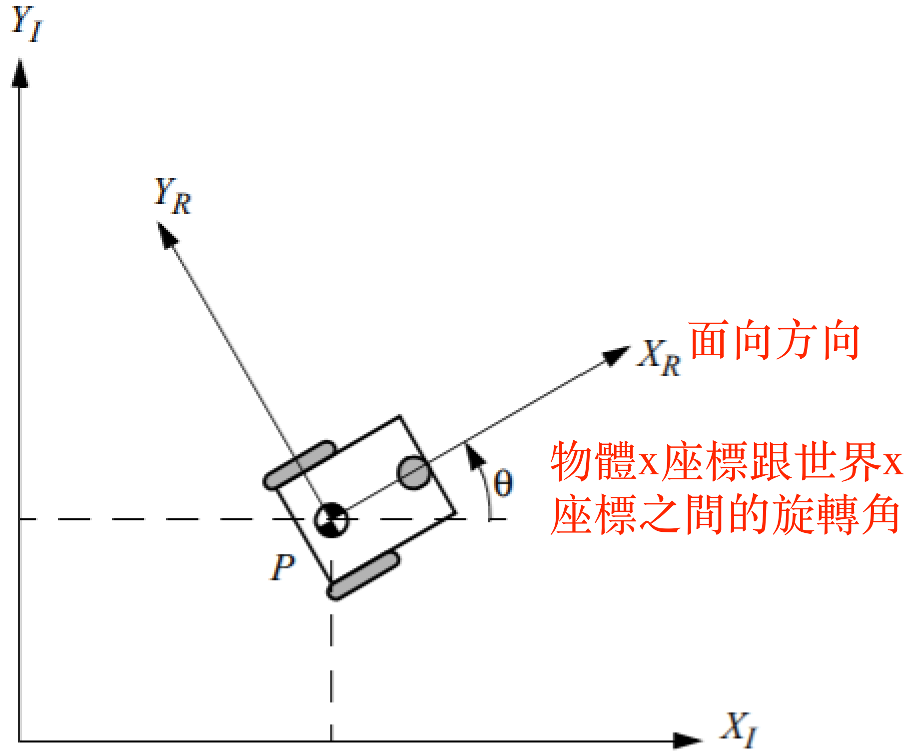
$$R(\theta) = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Kinematic Model:

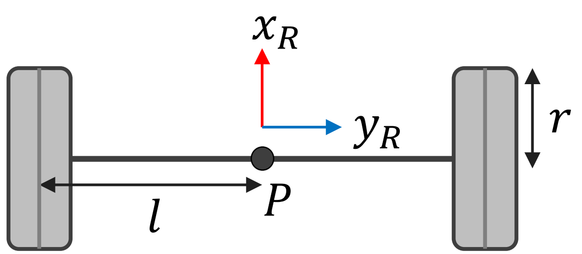
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix} \begin{matrix} \text{速度} \\ \\ \text{角速度} \end{matrix}$$

$$= \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ \omega \end{bmatrix}$$



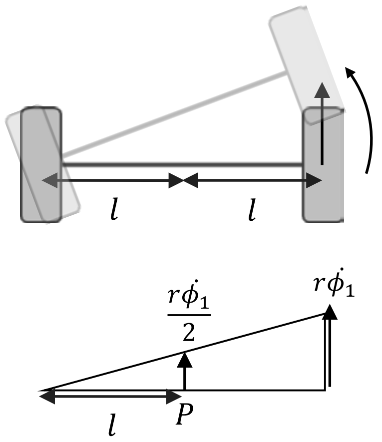
Differential Drive Vehicle (cont.)



Right Wheel:

$$\dot{x}_{R1} = \frac{r\boxed{\dot{\phi}_1}}{2}$$ 輪子的角速度

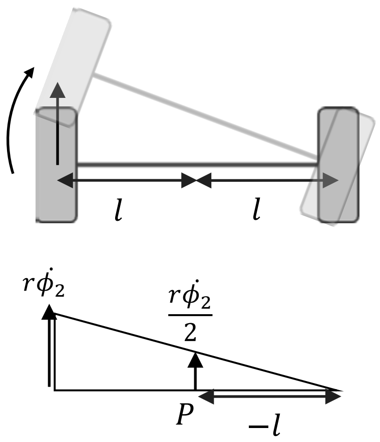
$$\omega_1 = \frac{r\dot{\phi}_1}{2l}$$



Left Wheel:

$$\dot{x}_{R2} = \frac{r\dot{\phi}_2}{2}$$

$$\omega_2 = \frac{-r\dot{\phi}_2}{2l}$$



Kinematic model for differential drive:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} - \frac{r\dot{\phi}_2}{2l} \end{bmatrix}$$

若時間很短，原點移動距離與速度是右/左輪的一半

Differential Drive Vehicle

- Given target velocity $\boldsymbol{v}$ and angular velocity $\boldsymbol{\omega}$

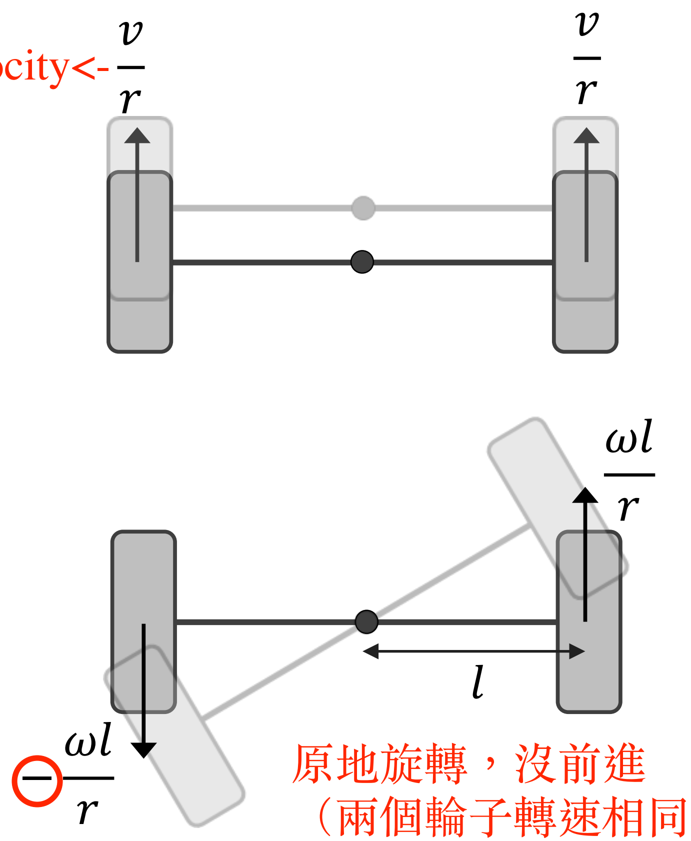
$$\begin{cases} v = \dfrac{r\dot{\phi}_1}{2} + \dfrac{r\dot{\phi}_2}{2} \\ \omega = \dfrac{r\dot{\phi}_1}{2l} - \dfrac{r\dot{\phi}_2}{2l} \end{cases}$$

$$\dot{\phi}_2 = \left(v - \frac{r\dot{\phi}_1}{2}\right)\frac{2}{r} = \frac{2v}{r} - \dot{\phi}_1$$

$$\omega = \frac{r\dot{\phi}_1}{2l} - \frac{r\left(\frac{2v}{r} - \dot{\phi}_1\right)}{2l} = \frac{r\dot{\phi}_1 - v}{l}$$

$$\dot{\phi}_1 = \frac{v}{r} + \frac{\omega l}{r}$$

$$\dot{\phi}_2 = \frac{v}{r} - \frac{\omega l}{r}$$



goal: 沿著弧線前進

Pure Pursuit Control

- Concept:
  - Modify the angular velocity to let the center achieve a point on path

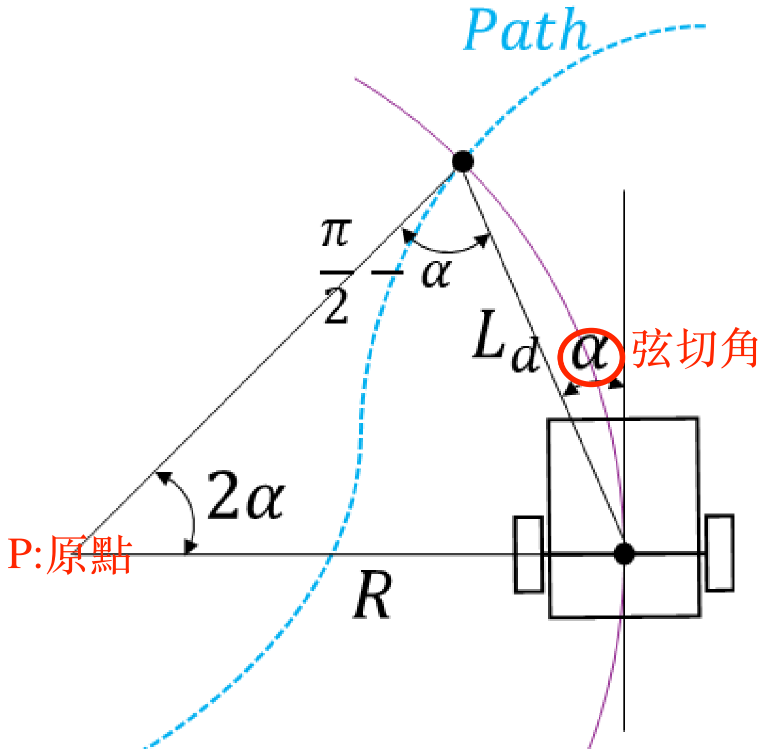
$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$\frac{L_d}{\sin(2\alpha)} = \frac{R}{\sin\left(\frac{\pi}{2} - \alpha\right)}$$ by 正弦定理

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

$$\omega = \frac{v}{R} = \frac{2\sin(\alpha)}{L_d}$$

Ld 距離越遠->速度越快

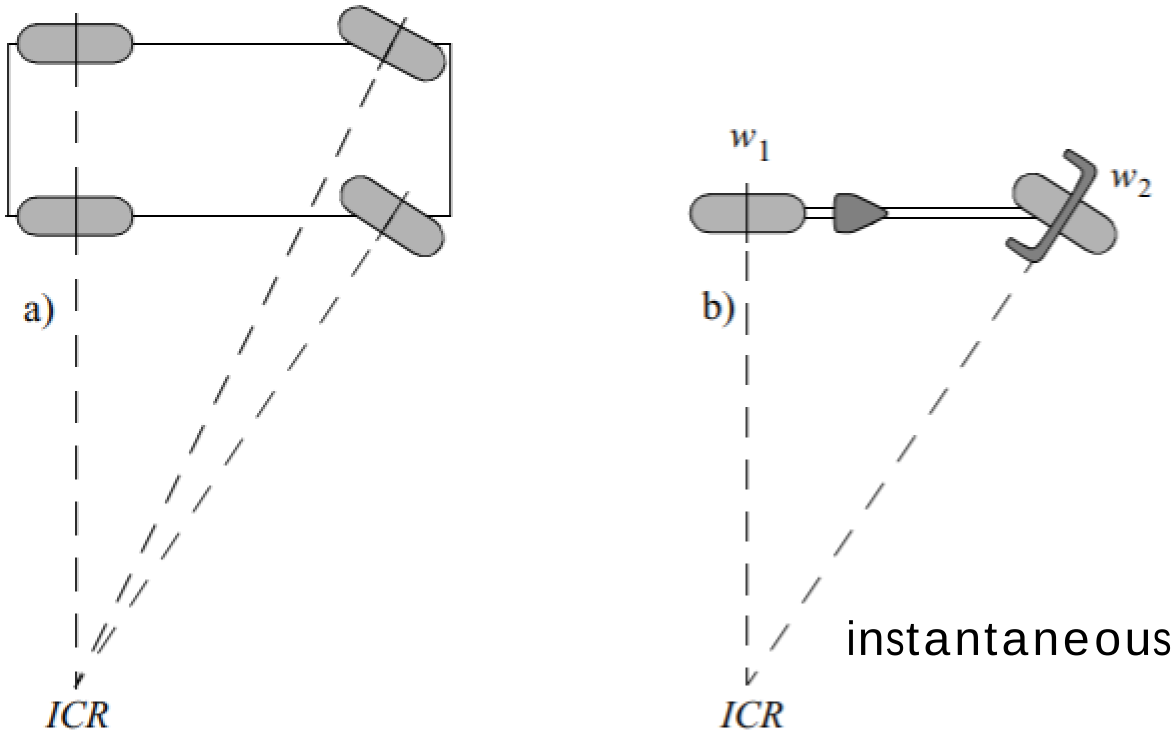


L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

idea: 四輪車想成二輪車

Kinematic Bicycle Model

- Speed and Steering Control



instantaneous center of rotation (ICR)

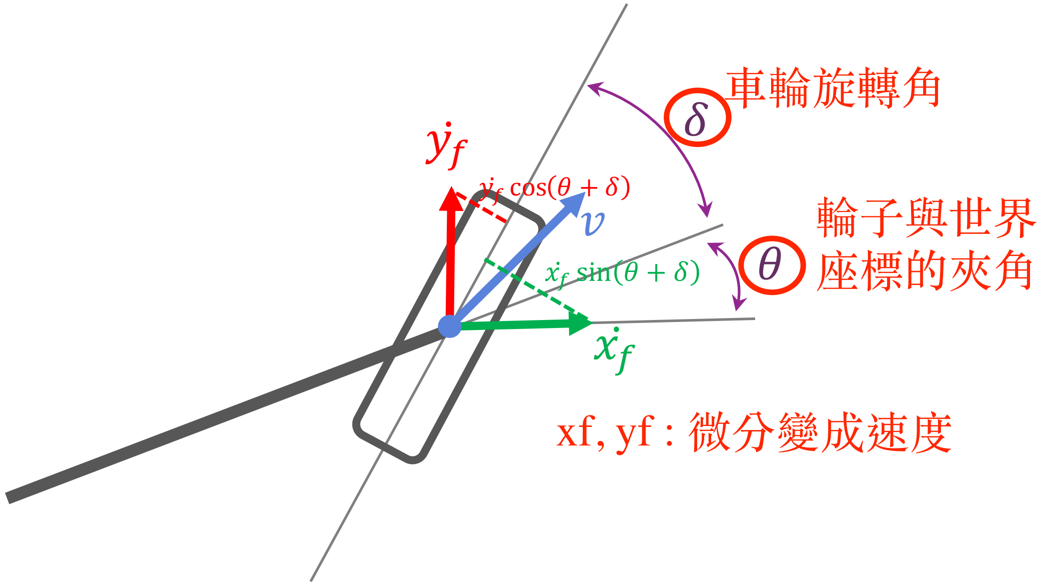
Figure 3.12
(a) Four-wheel with car-like Ackerman steering. (b) bicycle.

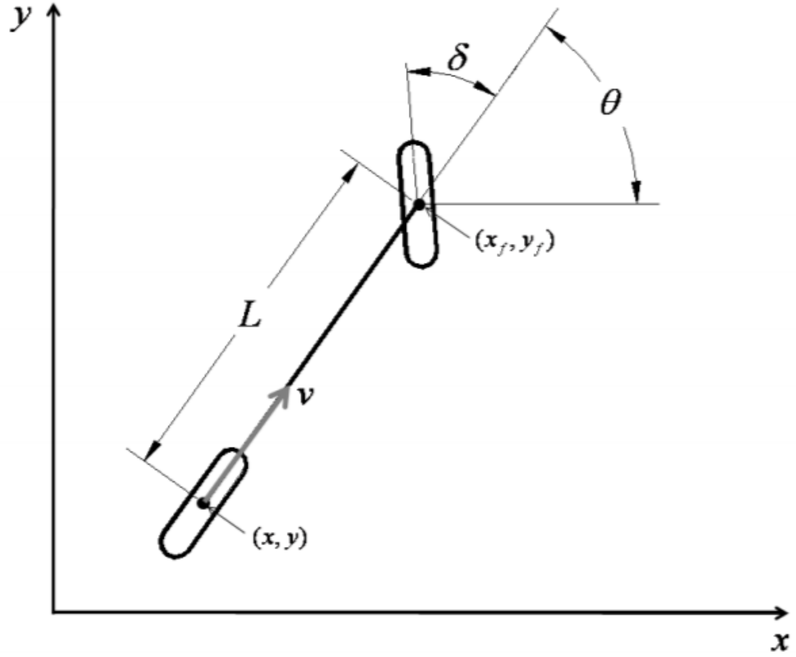
Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}_f \sin(\theta + \delta) - \dot{y}_f \cos(\theta + \delta) = 0 \quad (1)$$ Front Wheel

$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0 \quad (2)$$ Read Wheel



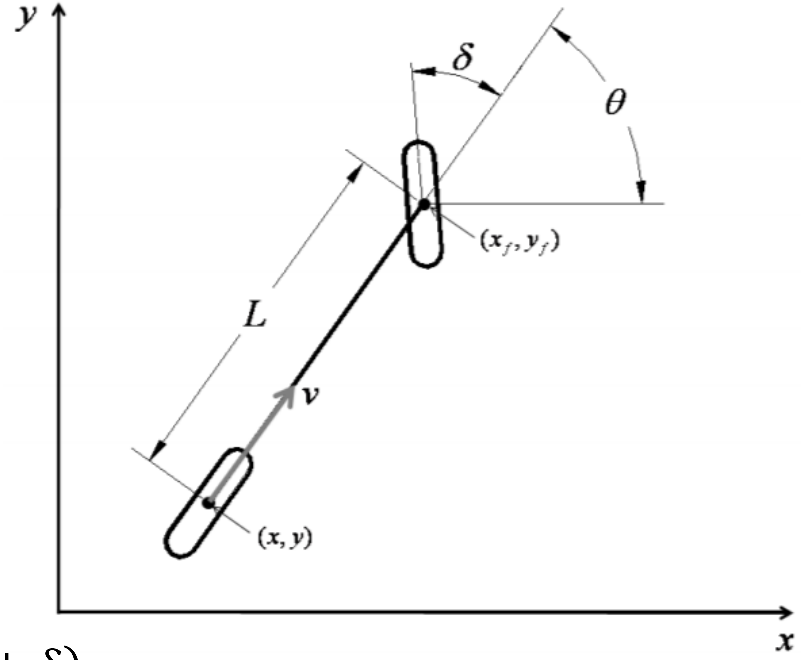


Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}_f \sin(\theta+\delta) - \dot{y}_f \cos(\theta+\delta) = 0 \qquad (1)$$ Front Wheel

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0 \qquad (2)$$ Rear Wheel



- Front Wheel Position

$$x_f = x + Lcos(\theta)$$
$$y_f = y + Lsin(\theta)$$

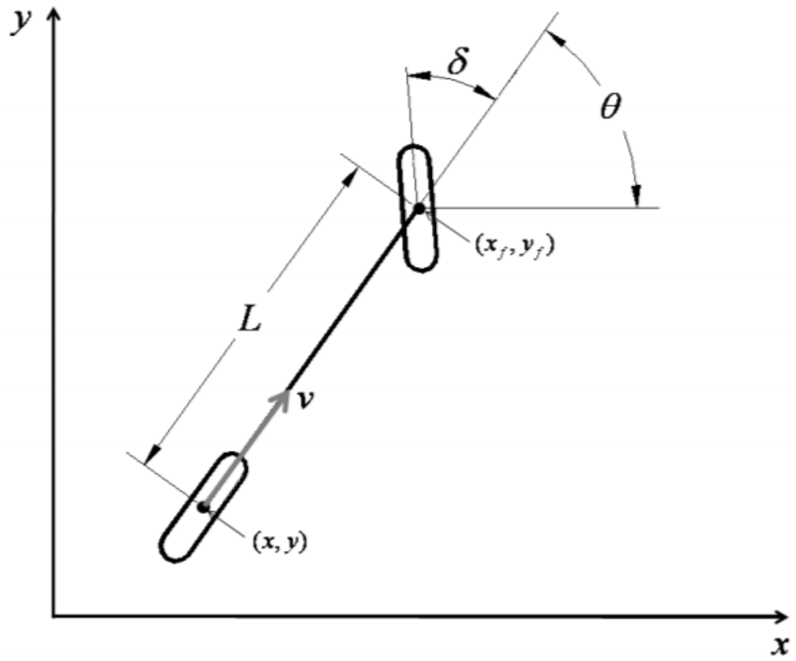
- Eliminating front wheel position from (1)

$$\begin{aligned} 0 &= \left(\dot{x} - \dot{\theta}Lsin(\theta)\right)\sin(\theta+\delta) - \left(\dot{y} + \dot{\theta}Lcos(\theta)\right)\cos(\theta+\delta) \\ &= \dot{x}\sin(\theta+\delta) - \dot{y}\cos(\theta+\delta) - \dot{\theta}Lsin(\theta)(\sin(\theta)\cos(\delta) + \cancel{\cos(\theta)\sin(\delta)}) \\ &\qquad - \dot{\theta}Lcos(\theta)(\cos(\theta)\cos(\delta) + \cancel{sin(\theta)\sin(\delta)}) \\ &= \dot{x}\sin(\theta+\delta) - \dot{y}\cos(\theta+\delta) - \dot{\theta}Lcos(\delta) \qquad (3) \end{aligned}$$

Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}\sin(\theta+\delta)-\dot{y}\cos(\theta+\delta)-\dot{\theta}Lcos(\delta)=0 \quad (3)$$
$$\dot{x}\sin(\theta)-\dot{y}\cos(\theta)=0 \quad (2)$$



- Rear wheel satisfied the constrain (2) when

$$\dot{x}=vcos(\theta) \quad (4)$$
$$\dot{y}=vsin(\theta) \quad (5)$$

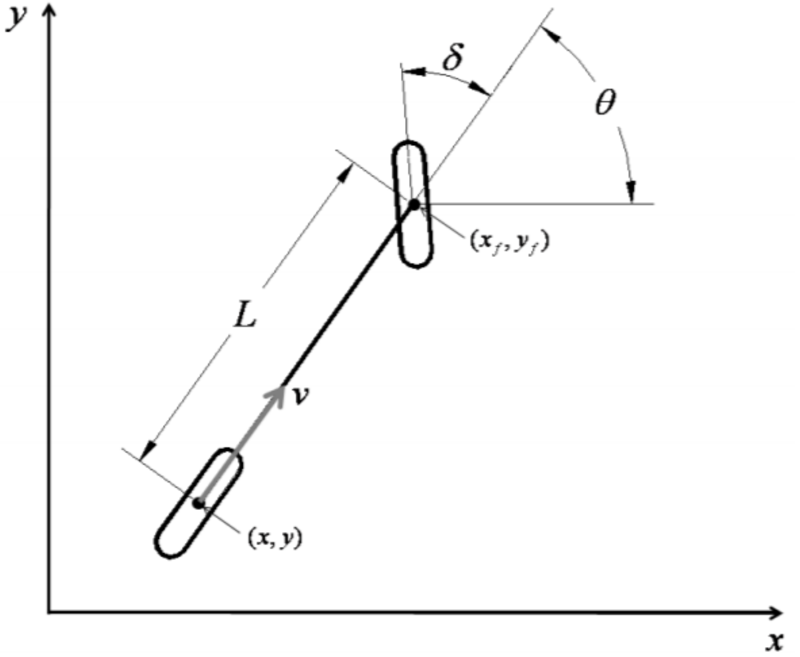
- Applying (4)(5) to (3)

$$\dot{\theta}=\frac{\dot{x}\sin(\theta+\delta)-\dot{y}\cos(\theta+\delta)}{Lcos(\delta)}$$
$$=\frac{vcos(\theta)(\cancel{\sin(\theta)\cos(\delta)}+\cos(\theta)\sin(\delta))-vsin(\theta)(\cancel{\cos(\theta)\cos(\delta)}+sin(\theta)\sin(\delta))}{Lcos(\delta)}$$
$$=\frac{v(\cos^2(\theta)+\sin^2(\theta))sin(\delta)}{Lcos(\delta)}=\frac{vtan(\delta)}{L}$$

Kinematic Bicycle Model

- Kinematic Model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \dfrac{\tan(\delta)}{L} \end{bmatrix} v$$

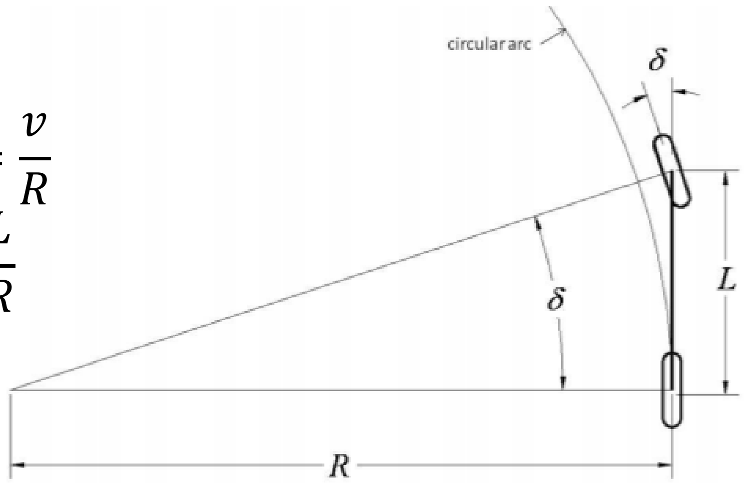


- Some Property

$$R\dot{\theta} = v$$

$$\frac{v\tan(\delta)}{L} = \frac{v}{R}$$

$$\tan(\delta) = \frac{L}{R}$$



goal: 找到一個目標點沿著圓弧並透過速度往前走

Pure Pursuit Control for Bicycle Model

- Concept:
  - Control the steer to let the rear wheel achieve a point on the path.

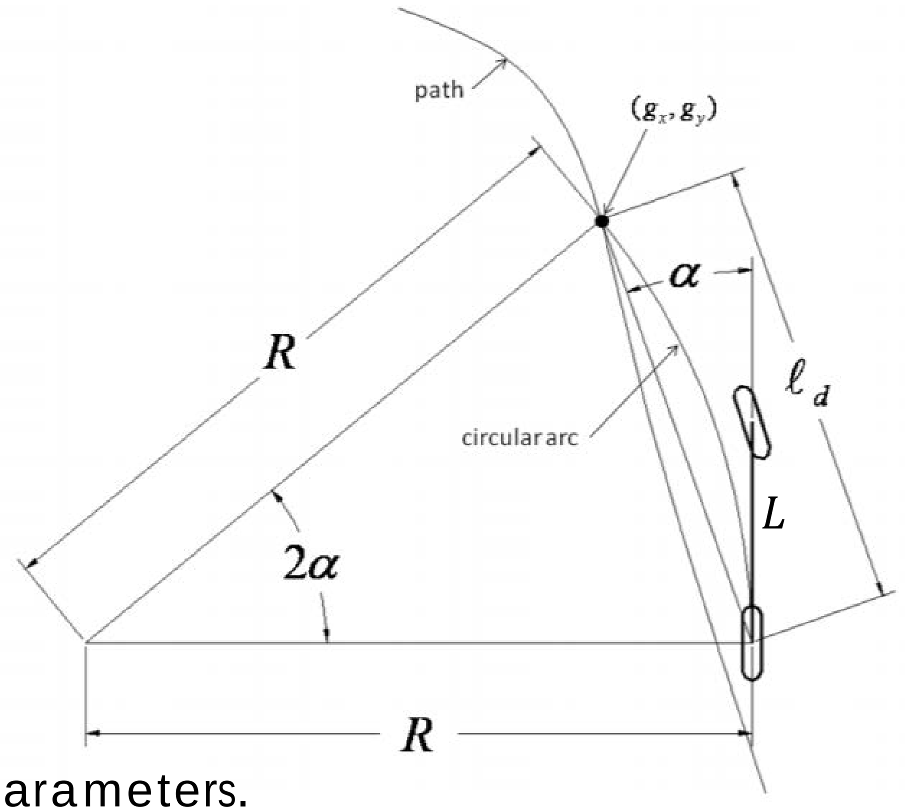
$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

$$\tan(\delta) = \frac{L}{R}$$

方向盤轉角

$$\delta = \arctan\left(\frac{L}{R}\right) = \arctan\left(\frac{2L\sin(\alpha)}{L_d}\right)$$



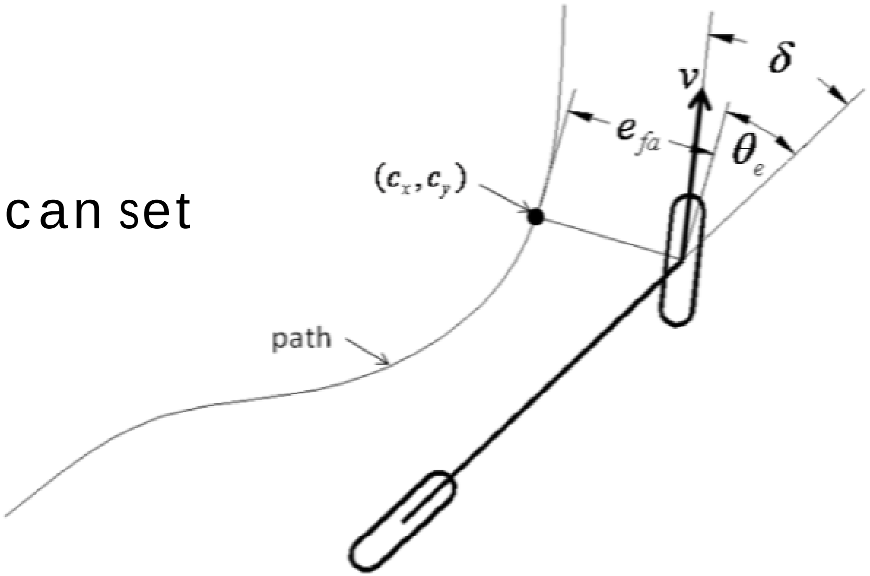
L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

goal: 考慮行徑平穩性
idea: 前輪為原點，找到路徑上的最近點得到目標點，找到切線跟法線

Stanley Control

- Concept:
  - Exponential stability for front wheel feedback
- Differential of error distance 法線微分方程式

$$\dot{e} = v_f sin(\delta - \theta_e)$$ 路徑方向的法線角度



- To achieve exponential stability to path, we can set

$\dot{e} = -ke$, where k>0

$$-ke = v_f sin(\delta - \theta_e)$$

$$\delta = \arcsin\left(-\frac{ke}{v_f}\right) + \theta_e$$

- It is not defined when **|–ke/vf|>1**. We can modify the control law to

$$\delta = \arctan\left(-\frac{ke}{v_f}\right) + \theta_e$$, which satisfy the **local exponential stability (LES).**

theta 趨近於 0 -> arcsin 可近似成 arctan

LQR Control optimization problem

- If we use the motion model with more complex form (e.g. dynamic model), it is hard to directly analyze the error function.
- **L**inear **Q**uadratic **R**egulator (LQR) introduce the concept of **cost function**, and try to solve the optimization problem when the motion model is **linear form** and the cost function is **quadratic form**.
- The formulation of LQR problem:
  - Define state **x** and control **u,** the motion model is $\dot{x} = Ax + Bu.$ x, u are vector A, B are matrix
  - The cost function is setting to the quadratic form $c = \underbrace{x^T Q x}_{\text{State Error}} + \underbrace{u^T R u}_{\text{Minimum Control}}$

    x 包含距離誤差 e、轉角誤差 theta
    u 包含方向盤角度 delta、加速度

    , in which **Q** is the state weighting matrix and **R** is the control weighting matrix.
  - The total objective function of an episode $J = \int_0^T [x(t)^T Q x(t) + u(t)^T R u(t)] dt + x^T(T) S x(T)$

LQR Control (cont.)

- The goal is to find the optimal control **u*** which minimize the total object function: $\min_{u} J = \min_{\mathrm{u}} \int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt + x(T)^T S x(T)$

- To solve this problem, we first introduce the concept of optimal principle. If we have a optimal control sequence $[u_t^*, u_{t+1}^*, u_{t+2}^*, \dots, u_T^*]$, then the subsequence $[u_{t+1}^*, u_{t+2}^*, \dots, u_T^*]$ is also an optimal control sequence.

  T is terminal

- Follow the concept, we can apply **dynamic programming** to recursively solve the optimal control from terminal state to current time.

LQR Control (cont.)

- However, we do not know the terminal time or even the terminal time is infinite in most time. In this case, we can solve the LQR using the recursive relation of value function.

negative reward, 越小越好

- Introduce the value function **V(x)**, which is the summing of the future cost. We can write down the recursive form of the discrete time value function:

$$V(x_t) = \min_{\mathrm{u}}\{ x_t^T Q x_t + u_t R u_t + V(x_{t+1})\}$$

- We can guess the value function to be quadratic form $V(x_t) = x_t^T P_t x_t$ (which P is symmetric positive-definite), and apply the linear motion model $Ax_t + Bu_t$ to value function:

$$V(x_t) = \min_{\mathrm{u}}\{ x_t^T Q x_t + u_t R u_t + x_{t+1}^T P_{t+1} x_{t+1}\}$$
$$= \min_{\mathrm{u}}\{ x_t^T Q x_t + u_t R u_t + (Ax_t + Bu_t)^T P_{t+1}(Ax_t + Bu_t)\}$$
$$= \min_{\mathrm{u}}\{x_t^T \big(Q + A^T P_{t+1} A\big) x_t + 2x^T A^T P B u + u_t^T \big(R + B^T P_{t+1} B\big) u_t\}$$

LQR Control (cont.)

- Solve the minimum equation

$$V(x_t) = x_t^T P_t x_t = \min_{\mathrm{u}}\{x_t^T\left(Q + A^T P_{t+1} A\right)x_t + 2x^T A^T PBu + u_t^T\left(R + B^T P_{t+1} B\right)u_t\}$$

$$\frac{\partial}{\partial u}\left[x_t^T\left(Q + A^T P_{t+1} A\right)x_t + 2x^T A^T PBu_t^* + u_t^{*T}\left(R + B^T P_{t+1} B\right)u_t^*\right] = 0$$

$$2\left(x^T A^T P_{t+1} B\right)^T + 2\left(R + B^T P_{t+1} B\right)u_t^* = 0$$

$$u_t^* = -\left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A x_t$$ the optimal control equation

- Apply u* to the value function, and get the equation of P

$$x_t^T P_t x_t = x_t^T\left(Q + A^T P_{t+1} A - A^T P_{t+1} B\left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A\right)x_t$$

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B\left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A$$

Discrete **R**iccati **A**lgebra **E**quation (DARE)

Remark: In continuous case, $\dot{P} = -PA - A^T P + PBR^{-1}P - Q$ is the **C**ontinuous **R**iccati **A**lgebra **E**quation (CARE)

LQR Control (cont.)

- Given discrete Riccati algebra equation

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B \left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A$$

- Suppose the value function is time-invariant, then

$$P = Q + A^T P A - A^T P B \left(R + B^T P B\right)^{-1} B^T P A$$

- In practice, we can first initialize $P^{(0)} = Q$, then iteratively apply the Riccati equation on until converge :

```
INITIALIZE: P ← Q
REPEAT
    P_next ← Q + AᵀPA − AᵀPB(R + BᵀPB)⁻¹BᵀPA
    ε ← ||P_next − P||
    IF ε < threshold THEN
        return P_next
    ENDIF
    P ← P_next
END
```

INITIALIZE: $P \leftarrow Q$
REPEAT
  $P_{next} \leftarrow Q + A^T P A - A^T P B \left(R + B^T P B\right)^{-1} B^T P A$->DARE
  $\epsilon \leftarrow ||P_{next} - P||$
  IF $\epsilon < threshold$ THEN
    return P_{next}
  ENDIF
  $P \leftarrow P_{next}$
END

LQR Control for Kinematic Model

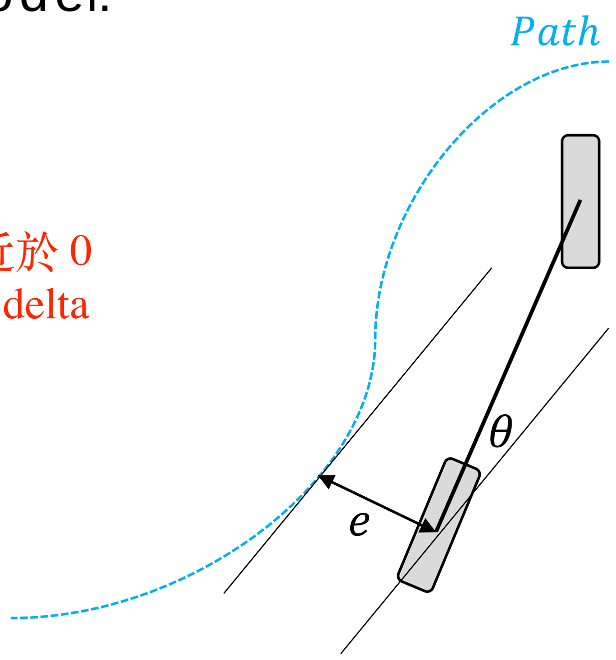
- Take an example to solve the LQR optimal control of the kinematic model.
- Define State: $x = [e, \dot{e}, \theta, \dot{\theta}]$,

the linear approximate of kinematic motion model:

- $$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{v\tan(\delta)}{L} \end{bmatrix}$$

當 delta 趨近於 0
tan(delta) =: delta

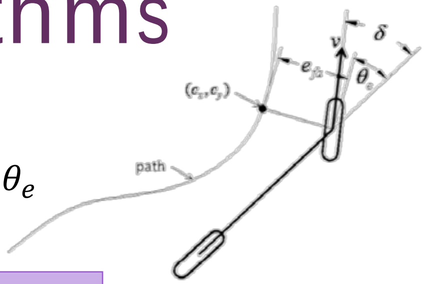
- $$\approx \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{v}{L} \end{bmatrix}\delta = Ax + Bu$$

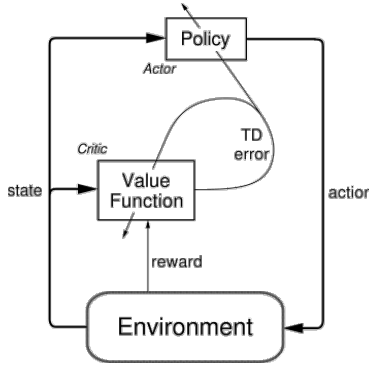


Review of Control Algorithms

$$\delta = K_p e(t) + K_i \sum_{0}^{t} e_t + K_d(e(t) - e(t-1))$$

$$\delta = \arctan\left(-\frac{ke}{v_f}\right) + \theta_e$$





PID Control

Stanley Control

Model-free Reinforcement Learning

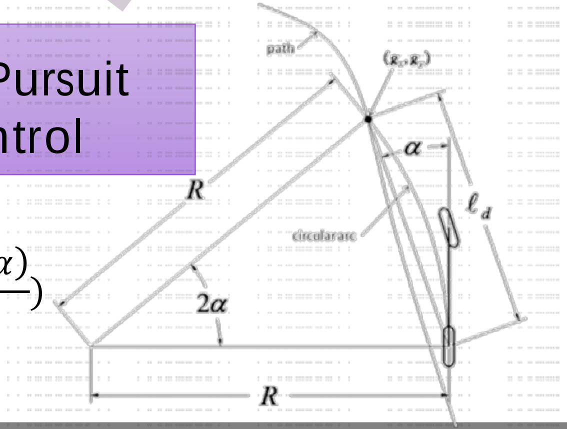
Apply the kinematic property.

Consider the progressive stability.

More complex motion model.

Don’t need model. Non-linear case.

Pure-Pursuit Control



$$\delta = \arctan\left(\frac{2L\sin(\alpha)}{L_d}\right)$$

LQR Control

DARE:

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B \left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A$$

Next Week

- Path Planning