K-SVDp: Image denoising via K-SVD with primal-dual active set algorithm

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Introduction

K-SVD Sparse Coding Stage:

$$\hat{x} = \arg\min \left\| y - Dx \right\|_2^2 \quad s.t. \ \| x \|_0 \leqslant T_0$$

- $D \in \mathbb{R}^{n\times K}$: dictionary with K prototype signal atom
- $y \in \mathbb{R}^{n\times 1}$: original signal
- $x \in \mathbb{R}^{K\times 1}$: sparse representation
- T_0 : sparsity level

Application:

- Image denoising via K-SVD

Benchmark

Popular Algorithms:

- ℓ_0 greedy algorithm: orthogonal matching pursuit (OMP)
- ℓ_1 optimization algorithm: basis pursuit denoising (BPDN, i.e. LASSO)

Challenges: accuracy/computational complexity

ℓ_0 greedy algorithm

- Computational efficiency
- Lack of Accuracy at high noise level

ℓ_1 optimization algorithm

- High Accuracy at high noise level
- Lack of computational efficiency

Goal: earn a tradeoff between accuracy and computational complexity

Algorithm Derivation

In order to find the coordinate-wise minimizer $x^\diamond = (x_1^\diamond, \ldots, x_K^\diamond)$, we have the following observations:

Observation 1:

$$\|y - Dx\|_2^2 = \sum_{i=1}^{n} (y_i - \sum_{q\neq j} D_{iq}x_q)^2 - 2\sum_{i=1}^{n}(y_i - \sum_{q\neq j} D_{iq}x_q)D_{ij}x_j + x_j^2$$

Observation 2:

Define: $l_j(t) = \sum_{i=1}^{n} (y_i - \sum_{q\neq j} D_{iq}x_q^\diamond)^2 - 2\sum_{i=1}^{n}(y_i - \sum_{q\neq j} D_{iq}x_q^\diamond)D_{ij}t + t^2, \quad j = 1,\ldots K$

Then $t_j^* = \arg\min l_j(t) \iff t_j^* = x_j^\diamond + g_j^\diamond$

$$g_j^\diamond = (y - Dx^\diamond)^\diamond{}' d_j \quad \forall j = 1,\ldots,K \qquad (1)$$

d_j : the j-th column of D

Through derivation, we get following equations:

Equation 1: (KKT condition)

$$x_j^\diamond = \begin{cases} x_j^\diamond + g_j^\diamond, & \text{if } h_j \geq h_{[T_0]} \\ 0 & \text{else} \end{cases} \quad \text{for } j = 1,\ldots,K$$

$h_{[1]} \geq h_{[2]} \geq \ldots, \geq h_{[K]}$: the decreasing permutation of h.

Equation 2:

$$\begin{cases} x_{\mathcal{F}} = 0, \ g_{\mathcal{A}} = 0, \\ x_{\mathcal{A}} = \left(D_{\mathcal{A}}' D_{\mathcal{A}}\right)^{-1} D_{\mathcal{A}}' y, \\ g_{\mathcal{F}} = (y - Dx)^T D_{\mathcal{F}}, \\ h_{\mathcal{F}} = \frac{1}{2}(g_{\mathcal{F}})^2, \\ h_{\mathcal{A}} = \frac{1}{2}(x_{\mathcal{A}})^2 \end{cases} \qquad (2)$$

$\mathcal{A}$: the indicator set of nonzero elements in x

$\mathcal{F}$: $\mathcal{A}^C$

Equation 3:

$$\mathcal{A} = \left\{ j : h_j \geq h_{[k]} \right\}, \quad \mathcal{F} = \left\{ j : h_j < h_{[k]} \right\}$$

Algorithm:

Sparse coding algorithm in K-SVDp

Input: Signal y, fixed dictionary $D \in \mathbb{R}^{n\times K}$, the maximum number of iterations R and ℓ_0-$norm$ constraint T_0

Output: Sparse representation x

Initialization: Randomly set $\mathcal{A}^0$ be a T_0 subset of $\{1,\ldots,K\}$ and $\mathcal{F}^0 = \left(\mathcal{A}^0\right)^c$;

For $r \in \{0,1\ldots,R\}$

- Compute $x_{\mathcal{F}}^r, x_{\mathcal{A}}^r, g_{\mathcal{F}}^r, g_{\mathcal{A}}^r, h_{\mathcal{F}}^r, h_{\mathcal{A}}^r$ by equation (2) where $\mathcal{A} = \mathcal{A}^r$;
- Sort h_j by $h_{[1]}^r \geq h_{[2]}^r \geq \ldots, \geq h_{[K]}^r$;
- Update the active and inactive sets by
$$\mathcal{A}^{r+1} = \left\{ j : h_j^r \geq h_{[T_0]}^r \right\}, \quad \mathcal{F}^{r+1} = \left\{ j : h_j^r < h_{[T_0]}^r \right\}$$
- If $\mathcal{A}^{r+1} = \mathcal{A}^r$ then stop; else $r = r + 1$ and return to steps above.

End

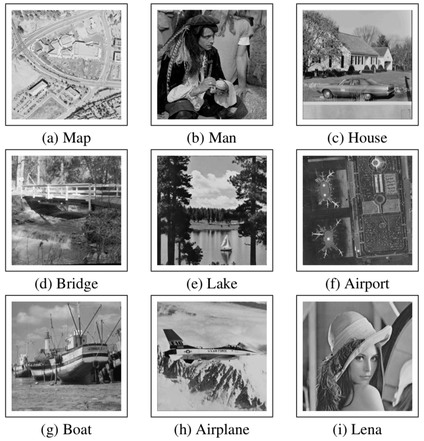
Numerical Experiments

Experiment Design:

- 9 images from classic USC-SIPI image database
- 6 noise levels $\sigma = 15,20,25,50,75,100$. (The first three are low noise levels, and the last three are the high noise levels)

Assessment:

- Reconstruction Time: average reconstruction time among different images since the time expense is stable when images change;
- Reconstruction Quality: peak signal-to-noise ratio (PSNR), structural similarity index (SSIM) and visual comparison.



(a) Map (b) Man (c) House (d) Bridge (e) Lake (f) Airport (g) Boat (h) Airplane (i) Lena

Figure1: Chosen images from USC-SIPI Image Database

Results

Running Time:

σ	14	20	25	50	75	100
BPDN	1178.91	1177.59	1180.31	-	-	-
OMP	78.04	80.82	83.20	78.76	79.33	78.37
K-SVDp	93.29	96.19	84.28	58.93	59.47	61.56

Table1: Reconstruction time(s) with $\sigma = 15,20,25,50,75,100$

$\sigma = \mathbf{15, 20, 25}$: Matlab® R2017b environment on the Macbook with 2.9 GHz Intel® Core™ i5 processor and 8G memory

$\sigma = \mathbf{50, 75, 100}$: Matlab® R2019a online environment

Reconstruction quality:

Peak signal-to-noise ratio (PSNR)

Figure	Map		Man		House		Bridge		Lake	
$\sigma/PSNR$	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp
50/14.15	19.75	**21.76**	19.92	**23.74**	19.87	**23.26**	19.68	**22.26**	18.88	**23.35**
75/10.63	16.40	**20.67**	16.48	**22.30**	16.46	**21.85**	16.39	**21.29**	16.47	**21.87**
100/8.13	13.97	**19.83**	14.02	**21.17**	14.00	**20.78**	13.99	**20.26**	14.04	**20.81**
Figure	Airport		Boat		Airplane		Lena		Average	
$\sigma/PSNR$	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp
50/14.15	19.90	**23.85**	19.96	**24.13**	19.96	**24.03**	19.99	**25.67**	19.88	**23.56**
75/10.63	16.48	**22.48**	16.48	**22.61**	16.49	**22.26**	16.54	**23.87**	16.47	**22.13**
100/8.13	14.03	**21.34**	14.02	**21.36**	14.04	**21.11**	14.06	**22.41**	14.02	**21.01**

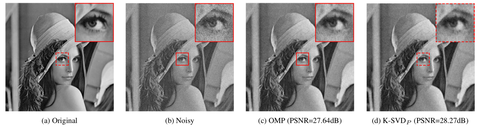
Table2: Accuracy of the reconstruction in terms of the PSNR (in dB) with $\sigma = 50,75,100$ (The best are highlighted in bold)

Structural similarity index (SSIM)

Figure	Map		Man		House		Bridge		Lake	
$\sigma/PSNR$	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp
15/0.829	**0.928**	0.921	**0.910**	0.907	**0.892**	0.891	**0.931**	0.923	**0.896**	**0.896**
20/0.753	**0.892**	0.887	0.866	**0.867**	0.846	**0.847**	**0.898**	0.893	0.846	**0.851**
25/0.683	**0.854**	0.848	0.819	**0.835**	0.795	**0.816**	**0.864**	0.852	0.797	**0.828**
50/0.443	0.678	**0.679**	0.619	**0.707**	0.596	**0.714**	**0.683**	0.677	0.596	**0.743**
75/0.307	0.534	**0.599**	0.475	**0.620**	0.459	**0.626**	0.536	**0.614**	0.464	**0.649**
100/0.223	0.430	**0.544**	0.372	**0.556**	0.360	**0.551**	0.425	**0.557**	0.370	**0.569**
Figure	Airport		Boat		Airplane		Lena		Average	
$\sigma/PSNR$	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp	OMP	K-SVDp
15/0.829	**0.897**	0.893	**0.897**	0.895	0.874	**0.878**	0.881	**0.889**	**0.901**	0.899
20/0.753	**0.850**	0.848	**0.847**	**0.847**	0.818	**0.828**	0.827	**0.839**	0.854	**0.856**
25/0.683	0.800	**0.810**	0.795	**0.819**	0.759	**0.800**	0.769	**0.821**	0.806	**0.825**
50/0.443	0.585	**0.664**	0.579	**0.711**	0.544	**0.743**	0.541	**0.754**	0.602	**0.710**
75/0.307	0.434	**0.574**	0.432	**0.621**	0.414	**0.635**	0.395	**0.660**	0.460	**0.622**
100/0.223	0.333	**0.509**	0.333	**0.540**	0.331	**0.554**	0.304	**0.575**	0.362	**0.551**

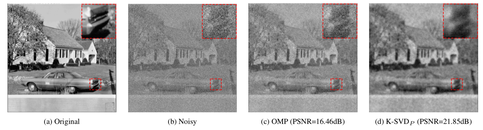
Table3: Accuracy of the reconstruction in terms of the SSIM with $\sigma = 15,20,25,50,75,100$ (The best are highlighted in bold)

Visual Comparison



(a) Original (b) Noisy (c) OMP (PSNR=27.64dB) (d) K-SVDp (PSNR=28.27dB)

Figure2: Denoising results in Lenna, $\sigma = 20$



(a) Original (b) Noisy (c) OMP (PSNR=16.46dB) (d) K-SVDp (PSNR=21.85dB)

Figure3: Denoising results in House, $\sigma = 75$

Conclusion

- **Low Noise:** K-SVDp is superior to BPDN in time complexity and clarity;
- **High Noise:** K-SVDp is better than the OMP algorithm and reduces the time complexity compared to BPDN which makes it possible to utilize;
- **SSIM & visual comparisons:** KSVDp performs better on local patterns.

Future work

- Improve the restoration performance at low noise levels;
- Decrease the time complexity further;
- Extend to situations where the noise level is unknown.

Link to the video:https://youtu.be/bF0dtJ9mhT8