3.14pt

Wavelets

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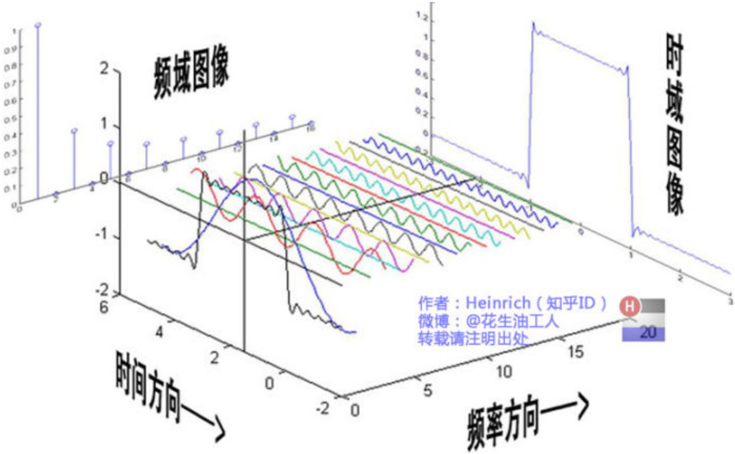
Outline of this presentation

3.14pt

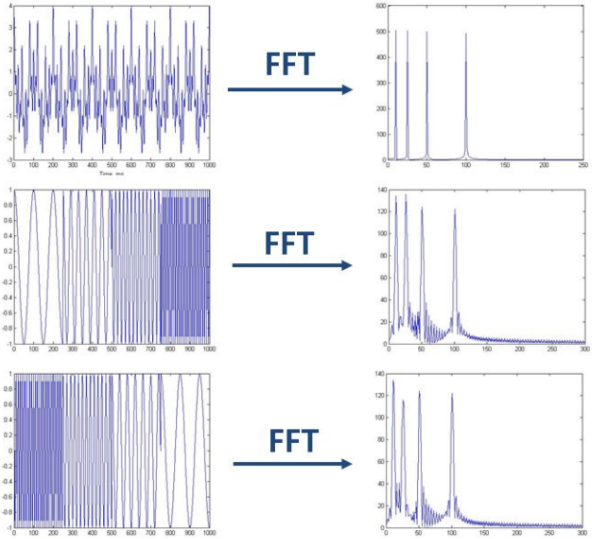
Outline

3.14pt

Fourier Transform

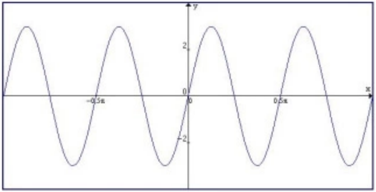


Fourier Transform

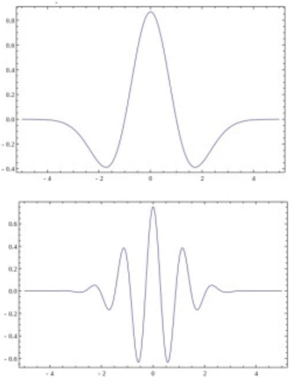


Difference between Fourier Transform and Wavelets

Sine wave (goes on forever)



Wavelets



Dilation

- A **dilation equation** is an equation where a function is defined in terms of a linear combination of scaled, shifted versions of itself. For instance,

$$f(x) = \sum_{k=0}^{d-1} c_k f(2x-k)$$

Lemma 11.1

If a dilation equation $f(x) = \sum_{k=0}^{d-1} c_k f(2x-k)$ has a solution, then $\sum_{k=0}^{d-1} c_k = 2$ or $\int_{-\infty}^{\infty} f(x)dx = 0$.

- Proof:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \sum_{k=0}^{d-1} c_k f(2x-k)dx \xlongequal{Funbini} \frac{1}{2} \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(x)dx$$

The Haar Wavelet

- Solve dilation equation $f(x) = f(2x) + f(2x-1)$, solution is

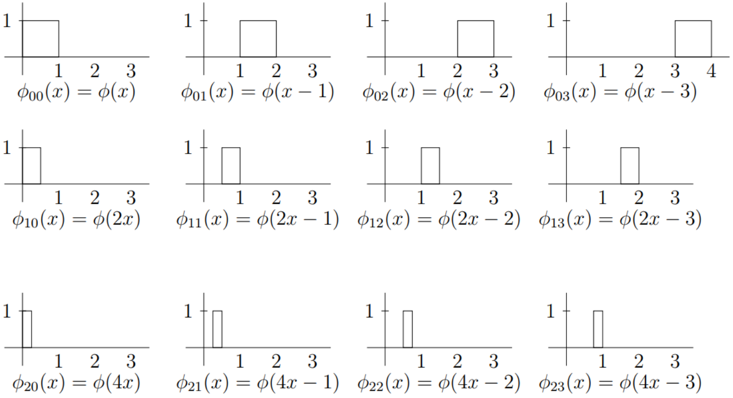
$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- The function ϕ is called a **scale function**.
- Scaling and shifting of the basic scale function ϕ gives a two dimensional set of scale functions

$$\phi_{jk}(x) = \phi\left(2^j x - k\right)$$

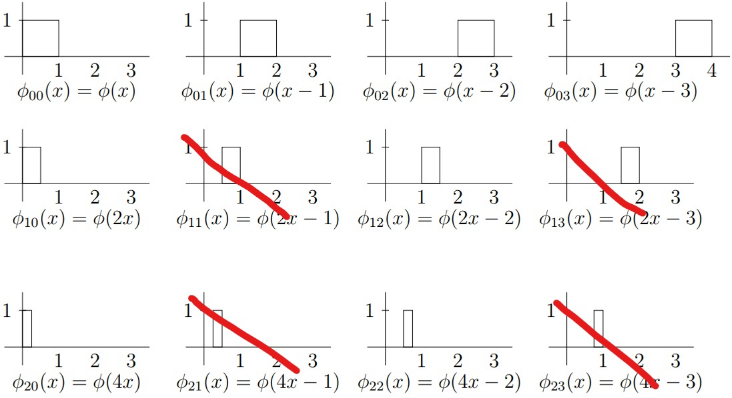
- For each j, the set of functions $\phi_{jk}, k = 0, 1, 2 \ldots$, form a basis for a vector space V_j and are orthogonal.
- But for different values of j, $\phi_{jk}, j = 0, 1, 2 \ldots$ are not orthogonal.

Haar wavelet scale functions



- Since $\phi_{jk}, \phi_{j+1,2k}$ and $\phi_{j+1,2k+1}$ are linearly dependent, for each value of j delete $\phi_{j+1,k}$ for odd values of k to get a linearly independent set of basis vectors.

Haar wavelet scale functions

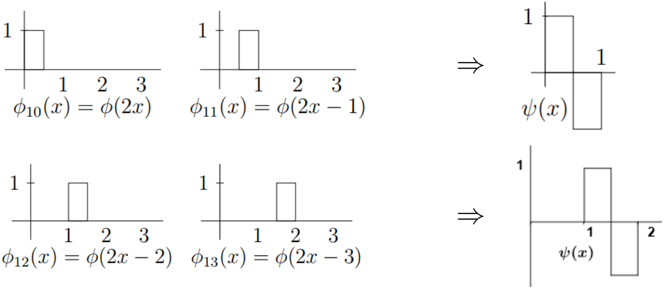


- To get an orthogonal set of basis vectors, define

$$\psi_{jk}(x) = \begin{cases} 1 & \frac{2k}{2^j} \le x < \frac{2k+1}{2^j} \\ -1 & \frac{2k+1}{2^j} \le x < \frac{2k+2}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

and replace $\phi_{j,2k}$ with $\psi_{j+1,2k}$.

- For instance,



The Haar Wavelet

- To approximate a function that has only finite support, select a scale vector $\phi(x)$.
- Next approximate the function by the set of scale functions $\phi\left(2^j x - k\right), k = 0, 1, \ldots$, for some fixed value of j.
- Once the value of j has been selected, the function is sampled at 2^j points, one in each interval of width 2^{-j}.
- Let the sample values be $s_0, s_1, \ldots$ The approximation to the function is $\sum_{k=0}^{2^j-1} s_k \phi\left(2^j x - k\right)$.
- Our goal is to represent the approximation to the function using the basis vectors rather than the nonorthogonal set of scale functions $\phi_{jk}(x)$.

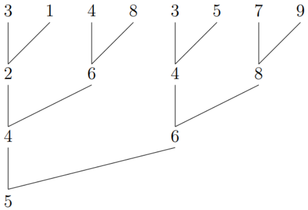
The Haar Wavelet Example

- To represent the function corresponding to vector $\begin{pmatrix} 3 & 1 & 4 & 8 & 3 & 5 & 7 & 9 \end{pmatrix}$, one needs to find the c_i such that

$$\begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

- The first column represents the scale function ϕ and subsequent columns the ψ's.

- Use tree methods to find the coefficients. Each vertex in the tree contains the average of the quantities of its two children.



- The result is

$$\begin{pmatrix}3\\1\\4\\8\\3\\5\\7\\9\end{pmatrix}=5\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}-1\begin{pmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{pmatrix}-2\begin{pmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{pmatrix}-2\begin{pmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{pmatrix}+1\begin{pmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix}-2\begin{pmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix}-1\begin{pmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{pmatrix}-1\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{pmatrix}$$

Wavelet Systems

- A wavelet system is built from a basic scaling function $\phi(x)$.
- A basic scale function $\phi(x)$ comes from a dilation equation.
- Scaling and shifting of the basic scale function gives a two dimensional set of scale functions

$$\phi_{jk}(x) = \phi\left(2^j x - k\right)$$

- For a fixed value of j, the ϕ_{jk} span a space V_j.
- If $\phi(x)$ satisfies a dilation equation

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k)$$

then ϕ_{jk} is a linear combination of the $\phi_{j+1,k}$'s and this implies that $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \cdots$.

Outline

3.14pt

Solving the Dilation Equation

- Consider solving a dilation equation

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k)$$

  to obtain the scale function for a wavelet system.
- The easiest way is to assume a solution and then calculate the scale function by successive approximation.

Example

- Example: solving

$$\phi(x) = \frac{1+\sqrt{3}}{4}\phi(2x) + \frac{3+\sqrt{3}}{4}\phi(2x-1) + \frac{3-\sqrt{3}}{4}\phi(2x-2) + \frac{1-\sqrt{3}}{4}\phi(2x-3)$$

- Begin with the coefficients of the dilation equation.

$$c_1 = \frac{1+\sqrt{3}}{4} \qquad c_2 = \frac{3+\sqrt{3}}{4} \qquad c_3 = \frac{3-\sqrt{3}}{4} \qquad c_4 = \frac{1-\sqrt{3}}{4}$$

- Execute the following loop until the values for $\phi(x)$ converge.
  1. Calculate $\phi(2x)$ by averaging successive values of $\phi(x)$ together. Fill out the remaining half of the vector representing $\phi(2x)$ with zeros.
  2. Calculate $\phi(2x-1)$, $\phi(2x-2)$, and $\phi(2x-3)$ by shifting the contents of $\phi(2x)$ the appropriate distance, discarding the zeros that move off the right end and adding zeros at the left end.
  3. Calculate the new approximation for $\phi(x)$ using the above values for $\phi(2x-1)$, $\phi(2x-2)$, and $\phi(2x-3)$ in the dilation equation for $\phi(2x)$.

Another approach

- Example: soving

$$\phi(x) = \frac{1}{2}f(2x) + f(2x-1) + \frac{1}{2}f(2x-2)$$

- Consider continuous solutions with support in $0 \le x < 2$:

$$\begin{array}{ll} \phi(0) = \frac{1}{2}\phi(0) + \phi(-1) + \phi(-2) = \frac{1}{2}\phi(0) + 0 + 0 & \phi(0) = 0 \\ \phi(2) = \frac{1}{2}\phi(4) + \phi(3) + \phi(2) = \frac{1}{2}\phi(2) + 0 + 0 & \phi(2) = 0 \\ \phi(1) = \frac{1}{2}\phi(2) + \phi(1) + \phi(0) = 0 + \phi(1) + 0 & \phi(1) \quad \text{arbitrary} \end{array}$$

- Set $\phi(1) = 1$. Then

$$\begin{aligned} \phi\left(\tfrac{1}{2}\right) &= \tfrac{1}{2}\phi(1) + \phi(0) + \tfrac{1}{2}\phi(-1) = \tfrac{1}{2} \\ \phi\left(\tfrac{3}{2}\right) &= \tfrac{1}{2}\phi(3) + \phi(2) + \tfrac{1}{2}\phi(1) = \tfrac{1}{2} \\ \phi\left(\tfrac{1}{4}\right) &= \tfrac{1}{2}\phi\left(\tfrac{1}{2}\right) + \phi\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\phi\left(-\tfrac{3}{2}\right) = \tfrac{1}{4} \end{aligned}$$

- One can continue this process and compute $\phi\left(\frac{i}{2^j}\right)$ for larger values of j until ϕ is approximated to a desired accuracy.

Conditions on the Dilation Equation

Lemma 11.2

Let

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k)$$

If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k \neq 0$ and $\phi(x)$ has been normalized so that $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$, then $\sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k)$.

Lemma 11.3

If $0 \leq x < d$ is the support of $\phi(x)$, and the set of integer shifts, $\{\phi(x-k) | k \geq 0\}$, are linearly independent, then $c_k = 0$ unless $0 \leq k \leq d-1$.

Outline

3.14pt

Derivation of the Wavelets from the Scaling Function

Lemma 1

(Orthogonality of $\psi(x)$ and $\psi(x-k)$)
Let $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$. If $\psi(x)$ and $\psi(x-k)$ are orthogonal for $k \neq 0$ and $\psi(x)$ has been normalized so that $\int_{-\infty}^{\infty} \psi(x)\psi(x-k)dx = \delta(k)$, then

$$\sum_{i=0}^{d-1} (-1)^k b_i b_{i-2k} = 2\delta(k)$$

Lemma 2

(Orthogonality of $\phi(x)$ and $\psi(x-k)$)
Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$ *and* $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$. *If* $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$ *and* $\int_{-\infty}^{\infty} \phi(x)\psi(x-k)dx = 0$ *for all* k, *then for all* k

$$\sum_{i=0}^{d-1} c_i b_{i-2k} = 2\delta(k)$$

Derivation of the Wavelets from the Scaling Function

Proof.

$$
\begin{aligned}
\int_{x=-\infty}^{\infty} \phi(x)\psi(x-k)dx &= \int_{x=-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=1}^{d-1} b_j \phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \int_{x=-\infty}^{\infty} \phi(2x-i)\phi(2x-2k-j)dx \\
&= \frac{1}{2} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \int_{y=-\infty}^{\infty} \phi(y)\phi(y-2k-j+i)dy \\
&= \frac{1}{2} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \delta(2k+j-i) \\
&= \frac{1}{2} \sum_{i=0}^{d-1} c_i b_{i-2k} = 0
\end{aligned}
$$

$\square$

Derivation of the Wavelets from the Scaling Function

Lemma 3

Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$ *and* $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$*. If* $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$ *and* $\int_{-\infty}^{\infty} \phi(x)\psi(x-k)dx = 0$ *for all* k*, then for all* k

$$b_k = (-1)^k c_{d-1-k}$$

Derivation of the Wavelets from the Scaling Function I

Proof.

By last lemma, $\sum_{j=0}^{d-1} c_j b_{j-2k} = 0$ for all k, which can be written as

$$\sum_{j=0}^{\frac{d}{2}-1} c_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} b_{2j+1-2k} = 0$$

Similarly, By Lemmas 11.2 and 11.4, we can get

$$\sum_{j=0}^{\frac{d}{2}-1} c_{2j} c_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} c_{2j+1-2k} = 2\delta(k)$$

and

$$\sum_{j=0}^{\frac{d}{2}-1} b_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} (-1)^j b_{2j+1} b_{2j+1-2k} = 2\delta(k)$$

$\square$

Derivation of the Wavelets from the Scaling Function II

Proof.

$$
\begin{aligned}
c_0b_0 + c_2b_2 + c_4b_4 + \cdots + c_1b_1 + c_3b_3 + c_5b_5 + \cdots &= 0 \quad k=0\\
c_2b_0 + c_4b_2 + \cdots \quad + c_3b_1 + c_5b_3 + \cdots &= 0 \quad k=1\\
c_4b_0 + \cdots \quad + c_5b_1 + \cdots &= 0 \quad k=2\\
c_0c_0 + c_2c_2 + c_4c_4 + \cdots + c_1c_1 + c_3c_3 + c_5c_5 + \cdots &= 0 \quad k=0\\
c_2c_0 + c_4c_2 + \cdots \quad + c_3c_1 + c_5c_3 + \cdots &= 0 \quad k=1\\
c_4c_0 + \cdots \quad + c_5c_1 + \cdots &= 0 \quad k=2\\
b_0b_0 + b_2b_2 + b_4b_4 + \cdots + b_1b_1 + b_3b_3 + b_5b_5 + \cdots &= 0 \quad k=0\\
b_2b_0 + b_4b_2 + \cdots \quad + b_3b_1 + b_5b_3 + \cdots &= 0 \quad k=1\\
b_4b_0 + \cdots \quad + b_5b_1 + \cdots &= 0 \quad k=2
\end{aligned}
$$

Let $C_e = (c_0, c_2, \ldots, c_{d-2}), C_o = (c_1, c_3, \ldots, c_{d-1}), B_e = (b_0, b_2, \ldots, b_{d-2})$. Equation 12.1, 12.2 and 11.3 can be expressed as convolutions of these sequences. We can get the matrix format as

$$
\begin{pmatrix} C_e & C_o \\ B_e & B_o \end{pmatrix} * \begin{pmatrix} C_e^R & B_e^R \\ C_o^R & B_o^R \end{pmatrix} = \begin{pmatrix} 2\delta & 0 \\ 0 & 2\delta \end{pmatrix}
$$

Taking the Fourier of $z-$transform yields

$$
\begin{pmatrix} F(C_e) & F(C_o) \\ F(B_e) & F(B_o) \end{pmatrix} \begin{pmatrix} F(C_e^R) & F(B_e^R) \\ F(C_o^R) & F(B_o^R) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
$$

Derivation of the Wavelets from the Scaling Function I

Proof.

Taking the determinant yields
$(F(C_e)F(B_o) - F(B_e)F(C_o))(F(C_e)F(B_o) - F(C_o)F(B_e)) = 4$.
Thus, $F(C_e)F(B_o) - F(C_o)F(B_e) = 2$ and the inverse transform yields
$C_e * B_o - C_o * B_e = 2\delta(k)$. Convolution by C_e^R yields

$$\begin{aligned} C_e^R * C_e * B_o + C_o^R * B_o * C_o &= 2C_e^R * \delta(k) \\ \left(C_e^R * C_e + C_o^R * C_o\right) * B_o &= 2C_e^R * \delta(k) \\ 2\delta(k) * B_o &= 2C_e^R * \delta(k) \\ C_e &= B_o^R \end{aligned}$$

Thus, $c_i = b_{d-1-i}$ for even i. Similarly, we can get $c_i = -b_{d-1-i}$ for all odd i and hence $c_i = (-1)^i b_{d-1-i}$ for all i. □

Outline

3.14pt

Sufficient Conditions for the Wavelets to be Orthogonal

Wavelets system:

- Wavelets, $psi_j(2^j x - k)$, at all scales and shifts to be orthogonal to the scale function $phi(x)$
- All wavelets to be orthogonal. That is

$$\int_{-\infty}^{\infty} \psi_j\left(2^j x - k\right) \psi_l\left(2^l x - m\right) dx = \delta(j-l)\delta(k-m)$$

- $\phi(x)$ and $\psi_{jk}, j \leq l$, and all k, to span V_l, the space spanned by $\phi(2^j x - k)$ for all k.

Sufficient Conditions for the Wavelets to be Orthogonal

Lemma 4

If $b_k = (-1)^k c_{d-1-k}$, *then* $\int_{-\infty}^{\infty} \phi(x)\psi\left(2^j x - l\right) dx = 0$ *for all* j *and* l.

Proof.

We first show that $\phi(x)$ and $\psi(x-k)$ are orthogonal for all values of k. Then we modify the proof to show that $\phi(x)$ and $\psi(2^j x - k)$ are orthogonal for all j and k. □

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\psi(x-k) &= \int_{i=0}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i (-1)^j c_{d-1-j} \int_{-\infty}^{\infty} \phi(2x-i)\phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} (-1)^j c_i c_{d-1-j} \delta(i-2k-j) \\
&= \sum_{j=0}^{d-1} (-1)^j c_{2k+j} c_{d-1-j} \\
&= c_{2k}c_{d-1} - c_{2k+1}c_{d-2} + \cdots + c_{d-2}c_{2k-1} - c_{d-1}c_{2k} \\
&= 0
\end{aligned}
$$

The last step requires that d be even which we have assumed for all scale functions. □

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

For the case where the wavelet is $\psi(2^j - l)$, first express $\phi(x)$ as a linear combination of $\phi(2^{j-1}x - n)$. Now for each these terms

$$\int_{-\infty}^{\infty} \phi\left(2^j x - m\right) \psi\left(2^j x - k\right) dx = \frac{1}{2^{j-1}} \int_{-\infty}^{\infty} \phi(y - m)\psi(2y - k)dy = 0$$

□

Sufficient Conditions for the Wavelets to be Orthogonal

Lemma 5

If $b_k = (-1)^k c_{d-1-k}$, *then*

$$\int_{-\infty}^{\infty} \frac{1}{2^j}\psi_j\left(2^j x - k\right)\frac{1}{2^k}\psi_l\left(2^l x - m\right)dx = \delta(j-l)\delta(k-m)$$

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

This first level wavelets are orthogonal.

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(x)\psi(x-k)dx &= \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} b_i \phi(2x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1} b_i \sum_{j=0}^{d-1} b_j \int_{-\infty}^{\infty} \phi(2x-i)\phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} b_i b_j \delta(i-2k-j) \\
&= \sum_{i=0}^{d-1} bib_{i-2k} \\
&= \sum_{i=0}^{d-1} (-1)^i c_{d-1-i} (-1)^{i-2k} c_{d-1-i+2k} \\
&= \sum_{i=0}^{d-1} (-1)^{2i-2k} c_{d-1-i} c_{d-1-i+2k}
\end{aligned}$$

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

Substituting j for $d-l-i$ yields

$$\sum_{j=0}^{d-1} c_j c_{j+2k} = 2\delta(k)$$

Example of orthogonality when wavelets are of different scale.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(2x)\psi(x-k)dx &= \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} b_i \phi(4x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)dx \\ &= \sum_{i=0}^{d-1} \sum_{i=0}^{d-1} b_i b_j \int_{-\infty}^{\infty} \phi(4x-i)\phi(2x-2k-j)dx\end{aligned}$$

□

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

Since $\phi(2x-2k-j)=\sum_{l=0}^{d-1} c_l\phi(4x-4k-2j-l)$

$$\begin{aligned}
\int_{-\infty}^{\infty}\psi(2x)\psi(x-k)dx &= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{l=0}^{d-1} b_i b_j c_l \int_{-\infty}^{\infty}\psi(4x-i)\phi(4x-4k-2j-l)dx \\
&= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{l=0}^{d-1} b_i b_j c_l \delta(i-4k-2j-l) \\
&= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} b_i b_j c_{i-4k-2j}
\end{aligned}$$

Since $\sum_{j=0}^{d-1} c_j b_{j-2k}=0, \sum_{i=0}^{d-1} b_i c_{i-4k-2j}=\delta(j-2k)$ Thus

$$\int_{-\infty}^{\infty}\psi(2x)\psi(x-k)dx=\sum_{j=0}^{d-1} b_j\delta(j-2k)=0$$

□

Sufficient Conditions for the Wavelets to be Orthogonal

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\psi(2x-k)dx &= \int_{-\infty}^{\infty} \sum_{j=0}^{d-1} c_j \phi(2x-j)\psi(2x-k)dx \\
&= \sum_{j=0}^{d-1} c_j \int_{-\infty}^{\infty} \phi(2x-j)\psi(2x-k)dx \\
&= \frac{1}{2} \sum_{j=0}^{d-1} c_j \int_{-\infty}^{\infty} \phi(y-j)\psi(y-k)dy \\
&= 0
\end{aligned}$$

If ψ was of scale 2^j, ϕ would be expanded as a linear combination of ϕ of scale 2^j all of which would be orthogonal to ψ. □

Outline

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Expressing a Function in Terms of Wavelets

- Aim: Express a function $f(x)$ in terms of an orthogonal basis of the wavelet system using given wavelet system with scale function ϕ and mother wavelet ψ.
- Let $f(x) = \sum_{k=0}^{\infty} a_{jk}\phi_j(x-k)$ where the a_{jk} are the coefficients in the expansion of $f(x)$ using level j scale functions. Since the $\phi_j(x-k)$ are orthogonal

$$a_{jk} = \int_{x=-\infty}^{\infty} f(x)\phi_j(x-k)dx$$

Expanding ϕ_j in terms of ϕ_{j+1} yields

$$\begin{aligned} a_{jk} &= \int_{x=-\infty}^{\infty} f(x) \sum_{m=0}^{d-1} c_m \phi_{j+1}(2x-2k-m)dx \\ &= \sum_{m=0}^{d-1} c_m \int_{x=-\infty}^{\infty} f(x)\phi_{j+1}(2x-2k-m)dx \\ &= \sum_{m=0}^{d-1} c_m a_{j+1,2k+m} \\ &= \sum_{n=2k}^{d-1} c_{n-2k} a_{j+1,n} \end{aligned}$$

Outline

3.14pt

Designing a Wavelet System

- If one uses d terms in the dilation equation, one defree of freedom can be used to satisfy

$$\sum_{i=0}^{d-1} c_i = 2$$

which insures the existence of a solution with a nonzero mean. Another $\frac{d}{2}$ degrees of freedom are used to satisfy

$$\sum_{i=0}^{d-1} c_i c_{i-2k} = \delta(k)$$

which insures the orthogonal properties. The remaining $\frac{d}{2} - 1$ degrees of freedom can be used to obtain some desirable properites such as smoothness.

The Haar Wavelet

- Use scal function to generate the two dimensional family of functions $\phi_{jk}(x) = \phi\left(2^j x - k\right)$.
- For a given value of j, the shifted versions, $\{\phi_{jk} | k \geq 0\}$, span a space V_j.
- Since $\phi(x)$ is the solution of a dilation equation, for any fixed j, ϕ_{jk} is a linear combination of the $\left\{\phi_{j+1,k'} | k' \geq 0\right\}$. So $V_j \subseteq V_{j+1}$.
- For each j, the set of functions $\phi_{jk}, k = 0, 1, 2 \ldots$, form a basis for a vector space V_j and are orthogonal. But for different values of j are not orthogonal.
- Since $\phi_{jk}, \phi_{j+1,2k}$ and $\phi_{j+1,2k+1}$ are linearly dependent, for each value of j delete $\phi_{j+1,k}$ for odd values of k to get a linearly independent set of basis vectors.
- To get an orthogonal set of basis vectors, define

$$\psi_{jk}(x) = \begin{cases} 1 & \frac{2k}{2^j} \leq x < \frac{2k+1}{2^j} \\ -1 & \frac{2k+1}{2^j} \leq x < \frac{2k+2}{2^j} \\ 0 & \text{otherwise} \end{cases}$$