COMS W4701: Artificial Intelligence, Spring 2024

Homework 4

Instructions: Compile all written solutions for this assignment in a single, typed PDF file. Coding solutions may be directly implemented in the provided Python file(s). **Do not modify any filenames or code outside of the indicated sections.** Submit all files on Gradescope in the appropriate assignment bins, and make sure to **tag all pages for written problems**. Please be mindful of the deadline and late policy, as well as our policies on citations and academic honesty.

Problem 1: Recommendation Model (20 points)

A certain app is deciding whether to show you a personalized **ad**, represented by a binary random variable A. It can possibly consider two factors: whether you browse the **virtual** marketplace and whether you make a trip to the **physical** store location (it can track your location!). These decisions are represented by binary random variables V and P. The joint distribution over all three variables is as follows:

V	P	A	$\Pr(V, P, A)$
$+v$	$+p$	$+a$	0.12
$+v$	$+p$	$-a$	0.08
$+v$	$-p$	$+a$	0.18
$+v$	$-p$	$-a$	0.12
$-v$	$+p$	$+a$	0.06
$-v$	$+p$	$-a$	0.14
$-v$	$-p$	$+a$	0.09
$-v$	$-p$	$-a$	0.21

1. Find the marginal distributions of each of the three random variables.
2. Find the joint distributions of each of the three different *pairs* of the random variables.
3. Using the distributions you found above, show whether each pair of random variables is independent.
4. Find the following conditional distributions: $\Pr(V|+a)$, $\Pr(P|+a)$, $\Pr(V, P|+a)$.
5. Can we conclude that V and P are conditionally independent given A without further calculations? If not, what additional calculations do we need to verify?
6. Explain whether it is possible to recover $\Pr(V, P, A)$ using only the three marginal distributions you computed in part 1. Consider the same question if we were to only use three joint distributions you computed in part 2. Of the six distributions, what is the *minimal* set needed (i.e., minimum number of table entries) to recover $\Pr(V, P, A)$?

Problem 2: Wandering Robot (18 points)

A robot is wandering around a room with some obstacles, labeled as # in the grid below. It can occupy any of the free cells labeled with a letter, but we are uncertain about its true location and so we keep a belief distribution over its current location. At each timestep, it may either stay in its current cell, or move to an adjacent cardinal cell, all with uniform probability. For example, from A the robot can move to A, B, or D each with probability $\frac{1}{3}$, while from C it can move to B or C, each with probability $\frac{1}{2}$.

A	B	C
D	E	#
#	F	#

The robot also makes an observation after each transition, returning what it sees in a random cardinal direction. Possibilities include observing #, "wall", or "empty" (for a free cell). For example, in D the robot observes "wall", # (each with probability $\frac{1}{4}$), or "empty" (probability $\frac{1}{2}$).

We highly recommend that you use Python or an equivalent computing library to complete this problem. You can screenshot all of the relevant code that you wrote or attach a notebook file in place of showing work. Please still include the final answer for each part in your writeup.

1. Suppose the robot wanders around for a long time without making any observations. What is the stationary distribution π over the robot's predicted location?[1]

2. The robot's sensors just started working. Take the stationary distribution that you found above to be $\Pr(X_0)$. Starting from this belief state, the robot makes one transition and observes $e_1 =$ "wall". What is the updated belief distribution $\Pr(X_1 \mid e_1)$?

3. The robot makes a second transition and observes $e_2 = \#$. What is the updated belief distribution $\Pr(X_2 \mid e_1, e_2)$?

4. Compute the joint distribution $\Pr(X_1, X_2 \mid e_1, e_2)$. You can either write the result as a matrix A, where $A_{ij} = \Pr(X_1 = i, X_2 = j \mid e_1, e_2)$, or just list the joint probability values that are nonzero.

5. Using your result above, compute the distribution $Pr(X_1|e_1, e_2)$.

Problem 3: Robot Learning (16 points)

The robot suspects that the transition and observation models described above are no longer accurate. We will try to update them by running through an iteration of the Baum-Welch algorithm, starting with the same initial distribution and models that you used in the last problem.
As with the last problem, we recommend completing this in Python.

[1] `numpy.linalg.eig` in Python returns a 1D array of eigenvalues and a 2D array where each column is a corresponding eigenvector. Remember that eigenvectors may not sum to 1 by default.

1. Compute (in order) the probability arrays β_2, β_1, and β_0.

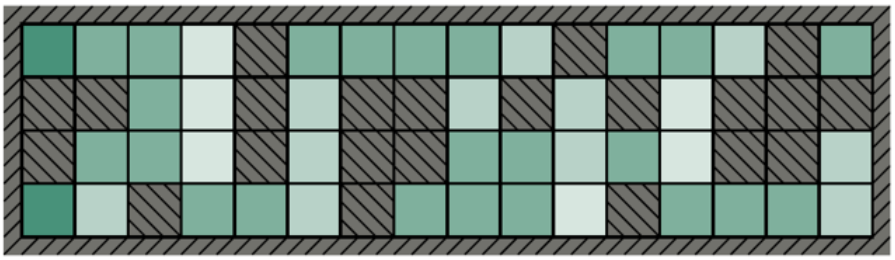
2. Compute the distribution of state occurrences in each timestep: γ_0, γ_1, γ_2. What is the new initial state distribution $\Pr(\hat{X}_0)$?

3. Compute the new set of observation probabilities $\Pr(\hat{e}|X)$ for each of the three observation values. Would you say that the parameters for $e =$ "empty" are accurate?

4. Compute the two sets of "expected" transition matrices ξ_0 and ξ_1.

5. Compute the updated transition matrix $\Pr(\hat{X_t}|X_{t-1})$ using your results above.

Problem 4: Grid World Localization (46 points)



You will be implementing grid world localization and HMM learning (see Section 14.3.2 in AIMA for a similar example). An agent is moving around and gathering observations on a grid, where each cell may be either passable or not passable (e.g., a wall). We do not know exactly where the agent is, so we will have a belief state over each cell of the grid. For simplicity, the belief state will also include the blocked cells, which will always have probability 0. You will be completing the `Gridworld_HMM` class to implement the environment model, as well as filtering, smoothing, and learning algorithms.

4.1: Transition Probabilities (8 points)

The transition model is such that the agent may either stay in the current cell or move to an adjacent free cell, all with uniform probability. Thus, if the set S includes all adjacent free neighbors of state x in addition to x itself, then $\Pr(x'|x) = \frac{1}{|S|}$ for all $x' \in S$.

Write the function `initT()`, which should return a $n \times n$ NumPy array T, where n is equal to `grid.size` and $T_{ij} = \Pr(x_j|x_i)$. We recommend that you populate this array one row (state) at a time. You may use the `neighbors()` helper function, which returns a list of all adjacent free cells as well as the given cell. Be sure to check that the rows of T sum to 1.

4.2: Observation Probabilities (8 points)

From any state x, the agent can make an observation e, which is an integer value between 0 and 15 (inclusive). The 4-bit representation of e indicates whether each of the four adjacent neighbors is a blocked (1) or free (0) cell. The order of the bits is north, east, south, west (NESW). For example, a correct observation for the lower-right corner cell in the figure would be the value 6, with bit representation 0110 indicating free cells north and west and blocked cells east and south.

However, the observations are also *noisy*; there is a ϵ probability that each bit may be independently wrong. If d is the bitwise discrepancy (XOR operation) between a possible observation e and the correct observation, then $\Pr(e|x) = (1-\epsilon)^{4-d}\epsilon^d$. Following the example above, the probability of $e = 6$ is $(1-\epsilon)^4$, while the probabilities of $e = 0$ and $e = 15$ are both $(1-\epsilon)^2\epsilon^2$.

Write the function `initO()`, which should return a $16 \times N$ array O where $O_{ij} = \Pr(e = i|x_j)$. Again, it may be easiest to populate the array one row (state) at a time. First find the "correct" observation value for each state. Then for each possible observation sequence with decimal values between 0 and 15, compute the discrepancy and insert the observation probability into the array[2].

4.3: Filtering and Smoothing (16 points)

Now that we have the model defined, we can start performing inference. We split these tasks into four functions. `forward()` and `backward()` perform a single step of each respective algorithm, given either the `alpha` or `beta` probability message and a single `observation`. Both can be easily done in one or two lines[3]. Then return the updated α or β vectors.

We can then use these two methods to implement `filtering()` and `smoothing()`, each given a list of `observations` and initial belief state `init`. Each method should return two $T \times N$ NumPy arrays, where T is the number of observations and N is equal to `grid.size`. `filtering()` returns an array of α vectors and the normalized $\Pr(X_t|e_{1:t})$ belief states; `smoothing()` returns an array of β vectors and the $\Pr(X_t|e_{1:T})$ belief states.

For `filtering()`, it should be straightforward to populate the array one row at a time, each one computed by calling `forward()` on the previous row. For `smoothing()`, you will want to actually call `filtering()` first; the returned array contains all the α vectors. Then populate a β array in the same format, but going from the last row to the first and calling `backward()` for each one. The smoothed belief states can then be computed by taking the elementwise product of α with β and then normalizing each row.

4.4: Learning Observation Probabilities (8 points)

For this last part, we will no longer assume a known observation model. The agent receives a sequence of measurements, which we will use to estimate a new set of observation probabilities using the Baum-Welch algorithm in the `baum_welch()` function.

In each iteration of Baum-Welch, first call `smoothing` to get all the `gamma` distributions. Use these to compute a new observation probability array, and update `self.obs` in place. Separately, compute the *log likelihood* $\log \Pr(e_{1:T})$ in each iteration. To do so, you can compute $\log(\sum_{x_t} \Pr(x_t, e_{1:T})) = \log(\sum_{x_t} \alpha_t * \beta_t)$ for any t. We recommend that you use $t = 1$ since you already have β_1 from `smoothing`, and you will just need one `forward` call to obtain α_1.

Store the computed log likelihoods in a list, and stop running Baum-Welch when the difference in log likelihoods between two successive iterations is 10^{-3} or smaller. Return both the new matrix of learned observation probabilities, as well as the list of log likelihoods.

[2] `^` in Python implements bitwise XOR.

[3] `@` in Python implements matrix multiplication, while `*` implements elementwise array multiplication.

4.5: Analysis (6 points)

When you are finished, you can test your implementation by invoking `python main.py`. It can run under three "modes", specified by the argument `-m`, as described below:

- Mode 0 (default): Run both filtering and smoothing for `-t` timesteps (default 50), `-n` episodes (default 500), and a pre-specified set of ϵ values. When finished, plots showing the average localization error[4] for each algorithm will be shown.
- Mode 1: Run one filtering episode for `-t` timesteps on an environment with ϵ equal to `-e` (default 0). An animation will pop up showing the agent's true location and the estimated belief distribution colored on the grid over time. Bright yellow corresponds to higher probabilities.
- Mode 2: Run Baum-Welch after generating a sequence of `-t` observations using the given `-e` value. The learned observation probabilities are then visualized using color intensity on a 16×64 grid, and the observation log likelihood is plotted as well.

After verifying that each mode above produces sensible results with your implementation, briefly address the following questions.

1. Inspect the results of mode 0 with no additional parameters, and do the same with a similar set of ϵ values for mode 1 (e.g., 0.0, 0.1, and 0.4). How does the value of ϵ affect the performance of each inference algorithm? Explain how the animations of mode 1 verify your hypotheses.
2. Looking at mode 0, which inference algorithm generally has better localization error over time and why? Also explain the differences in the shapes of the error curves over time. Please include the two localization plots in your writeup.
3. Run mode 2 for a few different values of ϵ (e.g., 0.0, 0.1, and 0.4) and take a look at the log likelihood curves of the Baum-Welch learning experiment. Give an intuitive explanation for what this measures (do not just repeat the definition $\log \Pr(e_{1:T})$), and explain whether the shape of the curve indicates whether the learning algorithm is successful. Please include one of the log likelihood plots in your writeup.

Coding Submission

Please submit the completed `gridworld_hmm.py` file under the HW4 Coding bin on Gradescope.

[4] This is computed as the total Manhattan distance between the agent's belief distribution and its true location (a distribution with a value of 1 in its true location and 0 elsewhere).