CELEN037
Foundation Calculus and Mathematical Techniques

Lecture 6

Main content

- Integrals of the form $\int f(x)g(x)\,dx$ where $f(x) = \sin(rx)$ or $f(x) = \cos(rx)$ and $g(x) = \sin(sx)$ or $g(x) = \cos(sx)$
- Integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$
- Some useful results
- Integration by completing the square
- Integration of improper rational functions
- Trigonometric substitution

Integrals of the form $\int f(x)g(x)\,dx$ where $f(x) = \sin(rx)$ or $f(x) = \cos(rx)$ and $g(x) = \sin(sx)$ or $g(x) = \cos(sx)$

Suppose that we want to evaluate an integral of the form

$$\int f(x)g(x)\,dx$$

where

$$f(x) = \sin(rx) \text{ or } f(x) = \cos(rx)$$

and

$$g(x) = \sin(sx) \text{ or } g(x) = \cos(sx)$$

with r and s being real numbers.

Then use (the appropriate) one of the trigonometric identities:

- $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
- $\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$
- $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$
- $\sin(A)\sin(B) = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$

Problem 1

Evaluate $\int \cos(4x)\cos(2x)\,dx$.

$$\begin{aligned}
\int \cos(4x)\cos(2x)\,dx &= \frac{1}{2}\int (\cos(4x+2x)+\cos(4x-2x))\,dx \\
&= \frac{1}{2}\int \cos(6x)\,dx + \frac{1}{2}\int \cos(2x)\,dx \\
&= \frac{1}{12}\sin(6x) + \frac{1}{4}\sin(2x) + C
\end{aligned}$$

Problem 2

Evaluate $\int \sin(\pi x)\cos(2\pi x)\,dx$.

$$\begin{aligned}
\int \sin(\pi x)\cos(2\pi x)\,dx &= \frac{1}{2}\int (\sin(\pi x + 2\pi x) + \sin(\pi - 2\pi x))\,dx \\
&= \frac{1}{2}\int \sin(3\pi x)\,dx - \frac{1}{2}\int \sin(\pi x)\,dx \\
&= -\frac{1}{6\pi}\cos(3\pi x) + \frac{1}{2\pi}\cos(\pi x) + C
\end{aligned}$$

Problem 3

Evaluate $\int \sin(x)\sin(2x)\sin(4x)\,dx$.

$$
\begin{aligned}
& \int \sin(x)\sin(2x)\sin(4x)\,dx \\
= \;& \frac{1}{2}\int \sin(x)(\cos(-2x)-\cos(6x))\,dx \\
= \;& \frac{1}{2}\int \sin(x)\cos(2x)\,dx - \frac{1}{2}\int \sin(x)\cos(6x)\,dx \\
= \;& \frac{1}{4}\int (\sin(3x)+\sin(-x))\,dx - \frac{1}{4}\int (\sin(7x)+\sin(-5x))\,dx \\
= \;& \frac{1}{4}\int \sin(3x)\,dx - \frac{1}{4}\int \sin(x)\,dx - \frac{1}{4}\int \sin(7x)\,dx + \frac{1}{4}\int \sin(5x)\,dx \\
= \;& -\frac{1}{12}\cos(3x) + \frac{1}{4}\cos(x) + \frac{1}{28}\cos(7x) - \frac{1}{20}\cos(5x) + C
\end{aligned}
$$

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Integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$

Suppose that we want to evaluate an integral of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

where m and n are nonnegative integers.

If m and n are odd then:

- Use the substitution $t = \sin(x)$ and the trigonometric identity $\cos^2(x) = 1 - \sin^2(x)$ if $\sin(x)$ is being raised to a higher power than $\cos(x)$ ($m > n$).
- Use the substitution $t = \cos(x)$ and the trigonometric identity $\sin^2(x) = 1 - \cos^2(x)$ if $\cos(x)$ is being raised to a higher power than $\sin(x)$ ($n > m$).
- Use either of the above two approaches if $m = n$.

If only one of m and n is odd (in which case the other is even) then:

- Use the substitution $t = \sin(x)$ and the trigonometric identity $\cos^2(x) = 1 - \sin^2(x)$ if $\sin(x)$ is being raised to the even power (m is even).
- Use the substitution $t = \cos(x)$ and the trigonometric identity $\sin^2(x) = 1 - \cos^2(x)$ if $\cos(x)$ is being raised to the even power (n is even).

If m and n are even then:

- Use the trigonometric identities $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$ and $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.

Problem 4

Evaluate $\int \cos^3(x)\sin^7(x)\,dx$.

Let $t = \sin(x)$.

Then $\dfrac{dt}{dx} = \cos(x)$ and $dt = \cos(x)\,dx$.

$$\begin{aligned}
\therefore \int \cos^3(x)\sin^7(x)\,dx &= \int \left(1-\sin^2(x)\right)\sin^7(x)\cos(x)\,dx \\
&= \int \left(1-t^2\right)t^7\,dt \\
&= \int \left(t^7-t^9\right)dt \\
&= \frac{t^8}{8}-\frac{t^{10}}{10}+C \\
&= \frac{\sin^8(x)}{8}-\frac{\sin^{10}(x)}{10}+C
\end{aligned}$$

Problem 5

Evaluate $\int \sin^5(x)\cos^4(x)\,dx$.

Let $t = \cos(x)$. Then $\dfrac{dt}{dx} = -\sin(x)$ and $-dt = \sin(x)\,dx$.

$$\begin{aligned}
\therefore \int \sin^5(x)\cos^4(x)\,dx &= \int \left(1-\cos^2(x)\right)^2 \cos^4(x)\sin(x)\,dx \\
&= -\int \left(1-t^2\right)^2 t^4\,dt \\
&= -\int \left(t^4 - 2t^6 + t^8\right) dt \\
&= -\left(\frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9}\right) + C \\
&= -\frac{\cos^5(x)}{5} + \frac{2\cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C
\end{aligned}$$

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Problem 6

Use an appropriate substitution to evaluate $\displaystyle\int \cos^3(x)\,dx$.

Let $t = \sin(x)$. Then $\dfrac{dt}{dx} = \cos(x)$ and $dt = \cos(x)\,dx$.

$$\begin{aligned}
\therefore \int \cos^3(x)\,dx &= \int \left(1 - \sin^2(x)\right)\cos(x)\,dx \\
&= \int \left(1 - t^2\right)dt \\
&= t - \frac{t^3}{3} + C \\
&= \sin(x) - \frac{\sin^3(x)}{3} + C
\end{aligned}$$

Problem 7

Use an appropriate substitution to evaluate $\displaystyle\int \sin^3(x)\,dx$.

Let $t = \cos(x)$. Then $\dfrac{dt}{dx} = -\sin(x)$ and $dt = -\sin(x)\,dx$.

$$
\begin{aligned}
\therefore \int \sin^3(x)\,dx &= \int \left(1-\cos^2(x)\right)\sin(x)\,dx \\
&= -\int \left(1-t^2\right)dt \\
&= -t + \frac{t^3}{3} + C \\
&= \frac{\cos^3(x)}{3} - \cos(x) + C
\end{aligned}
$$

Problem 8

Evaluate $\int \sin^2(x)\cos^2(x)\,dx$.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{(1-\cos(2x))(1+\cos(2x))}{2^2}\,dx \\
&= \frac{1}{4}\int \left(1-\cos^2(2x)\right)dx \\
&= \frac{1}{4}\int \left(1-\frac{1+\cos(4x)}{2}\right)dx \\
&= \frac{1}{8}\int (1-\cos(4x))\,dx \\
&= \frac{x}{8}-\frac{\sin(4x)}{32}+C
\end{aligned}$$

Problem 9

Evaluate $\int \sin^4(x)\cos^2(x)\,dx$.

$$\begin{aligned}
&\int \sin^4(x)\cos^2(x)\,dx \\
= \;& \int \left(\frac{1-\cos(2x)}{2}\right)^2 \frac{1+\cos(2x)}{2}\,dx \\
= \;& \frac{1}{8}\int (1-\cos(2x))\big(1-\cos^2(2x)\big)\,dx \\
= \;& \frac{1}{8}\int \big(1-\cos(2x)-\cos^2(2x)+\cos^3(2x)\big)\,dx \\
= \;& \frac{1}{8}\int \left(1-\cos(2x)-\frac{1+\cos(4x)}{2}+\frac{\cos(6x)+3\cos(2x)}{4}\right)dx \\
= \;& \frac{1}{8}\int \left(\frac{1}{2}-\frac{\cos(2x)}{4}-\frac{\cos(4x)}{2}+\frac{\cos(6x)}{4}\right)dx \\
= \;& \frac{x}{16}-\frac{\sin(2x)}{64}-\frac{\sin(4x)}{64}+\frac{\sin(6x)}{192}+C
\end{aligned}$$

Some useful results

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad \left(\text{since } \frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)}\right).$$

For real numbers $n \neq -1$,

$$\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C \qquad \left(\text{since } \frac{d}{dx}\left(\frac{(f(x))^{n+1}}{n+1}\right) = (f(x))^n f'(x)\right).$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C \qquad \left(\text{since } \frac{d}{dx}\left(2\sqrt{f(x)}\right) = \frac{f'(x)}{\sqrt{f(x)}}\right).$$

$$\int e^x(f(x) + f'(x))\,dx = e^x f(x) + C \qquad \left(\text{since } \frac{d}{dx}(e^x f(x)) = e^x(f(x) + f'(x))\right).$$

Problem 10

Evaluate $\displaystyle\int \frac{\cos(x)}{1+\sin(x)}\,dx$.

Let $f(x) = 1 + \sin(x)$.
Then $f'(x) = \cos(x)$.

$$\therefore \int \frac{\cos(x)}{1+\sin(x)}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|1+\sin(x)| + C$$

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Problem 11

Evaluate $\displaystyle\int \frac{1}{x(1+\ln(x))}\,dx$.

Let $f(x) = 1 + \ln(x)$.
Then $f'(x) = \dfrac{1}{x}$.

$$\therefore \int \frac{1}{x(1+\ln(x))}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|1+\ln(x)| + C$$

Problem 12

Evaluate $\displaystyle\int \frac{x^2 + 2x}{x^3 + 3x^2 - 5}\, dx$.

Let $f(x) = x^3 + 3x^2 - 5$.
Then $f'(x) = 3x^2 + 6x = 3(x^2 + 2x)$.

$$\therefore \int \frac{x^2 + 2x}{x^3 + 3x^2 - 5}\, dx = \frac{1}{3}\int \frac{f'(x)}{f(x)}\, dx = \frac{1}{3}\ln\left|x^3 + 3x^2 - 5\right| + C$$

Problem 13

Evaluate $\displaystyle\int \frac{e^{2x}+1}{e^{2x}-1}\,dx$.

We first note that

$$\frac{e^{2x}+1}{e^{2x}-1} = \frac{e^{-x}\left(e^{2x}+1\right)}{e^{-x}(e^{2x}-1)} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}.$$

Let $f(x) = e^{x} - e^{-x}$.
Then $f'(x) = e^{x} + e^{-x}$.

$$\therefore \int \frac{e^{2x}+1}{e^{2x}-1}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln\left|e^{x}-e^{-x}\right| + C$$

Problem 14

Evaluate $\int \cot(x)\,dx$.

We first note that

$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

Let $f(x) = \sin(x)$.
Then $f'(x) = \cos(x)$.

$$\therefore \int \cot(x)\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|\sin(x)| + C$$

Problem 15

Evaluate $\displaystyle\int \tan(x)\,dx$.

We first note that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Let $f(x) = \cos(x)$.
Then $f'(x) = -\sin(x)$.

$$\therefore \int \tan(x)\,dx = -\int \frac{f'(x)}{f(x)}\,dx = -\ln|\cos(x)| + C$$

Problem 16

Evaluate $\int \sec(x)\,dx$.

We first note that

$$\sec(x) = \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}.$$

Let $f(x) = \sec(x) + \tan(x)$.
Then $f'(x) = \sec(x)\tan(x) + \sec^2(x)$.

$$\therefore \int \sec(x)\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|\sec(x)+\tan(x)| + C$$

Problem 17

Evaluate $\displaystyle\int \operatorname{cosec}(x)\,dx$.

We first note that

$$\operatorname{cosec}(x) = \frac{\operatorname{cosec}(x)(\operatorname{cosec}(x) - \cot(x))}{\operatorname{cosec}(x) - \cot(x)} = \frac{\operatorname{cosec}^2(x) - \operatorname{cosec}(x)\cot(x)}{\operatorname{cosec}(x) - \cot(x)}.$$

Let $f(x) = \operatorname{cosec}(x) - \cot(x)$.
Then $f'(x) = -\operatorname{cosec}(x)\cot(x) + \operatorname{cosec}^2(x)$.

$$\therefore \int \operatorname{cosec}(x)\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|\operatorname{cosec}(x) - \cot(x)| + C$$

Problem 18

Evaluate $\displaystyle\int 2x\left(x^2+1\right)^{50}\,dx$.

Let $f(x) = x^2 + 1$.
Then $f'(x) = 2x$.

$$\therefore \int 2x\left(x^2+1\right)^{50}\,dx = \int (f(x))^{50} f'(x)\,dx = \frac{\left(x^2+1\right)^{51}}{51} + C$$

Problem 19

Evaluate $\int \tan^3(x)\,dx$.

We first note that

$$\tan^3(x) = \tan(x)\big(\sec^2(x) - 1\big) = \tan(x)\sec^2(x) - \frac{\sin(x)}{\cos(x)}.$$

Let $f(x) = \tan(x)$ and $g(x) = \cos(x)$.
Then $f'(x) = \sec^2(x)$ and $g'(x) = -\sin(x)$.

$$\therefore \int \tan^3(x)\,dx = \int f(x)f'(x)\,dx + \int \frac{g'(x)}{g(x)}\,dx = \frac{\tan^2(x)}{2} + \ln|\cos(x)| + C$$

Problem 20

Evaluate $\displaystyle\int \frac{4-2x}{\sqrt{5-x^2+4x}}\,dx$.

Let $f(x) = 5 - x^2 + 4x$.
Then $f'(x) = -2x + 4$.

$$\therefore \int \frac{4-2x}{\sqrt{5-x^2+4x}}\,dx = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{5-x^2+4x} + C$$

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Problem 21

Evaluate $\int \sec^2(x)(3+\tan(x))^{-1/2}\,dx$.

Let $f(x) = 3 + \tan(x)$.
Then $f'(x) = \sec^2(x)$.

$$\therefore \int \sec^2(x)(3+\tan(x))^{-1/2}\,dx = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{3+\tan(x)} + C$$

Problem 22

Evaluate $\int e^x(\ln(\sec(x)) + \tan(x))\,dx$.

Let $f(x) = \ln(\sec(x))$.

Then $f'(x) = \dfrac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$.

$$\therefore \int e^x(\ln(\sec(x)) + \tan(x))\,dx = \int e^x\big(f(x) + f'(x)\big)\,dx = e^x\ln(\sec(x)) + C$$

Problem 23

Evaluate $\displaystyle\int \frac{xe^x}{(1+x)^2}\,dx$.

We first note that

$$\frac{xe^x}{(1+x)^2} = \frac{(1+x-1)e^x}{(1+x)^2} = e^x\left(\frac{1}{1+x} - \frac{1}{(1+x)^2}\right).$$

Let $f(x) = \dfrac{1}{1+x}$.
Then $f'(x) = -\dfrac{1}{(1+x)^2}$.

$$\therefore \int \frac{xe^x}{(1+x)^2}\,dx = \int e^x\big(f(x) + f'(x)\big)\,dx = \frac{e^x}{1+x} + C$$

Integration by completing the square

Suppose that we want to evaluate an integral of the form

$$\int \frac{1}{q(x)}\,dx$$

or

$$\int \frac{1}{\sqrt{q(x)}}\,dx$$

where q is a quadratic polynomial.
We can complete the square to write $q(x)$ in a form from which we can proceed to evaluate the integral.
The results that can be used to evaluate such integrals of the form $\displaystyle\int \frac{1}{q(x)}\,dx$ include the results on the next two pages.

For nonzero real numbers a,

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C,$$

since

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

and so

$$\begin{aligned}
\int \frac{1}{x^2 - a^2}\, dx &= \frac{1}{2a}\left(\int \frac{1}{x-a}\, dx - \int \frac{1}{x+a}\, dx\right) \\
&= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C \\
&= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.
\end{aligned}$$

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For nonzero real numbers a,

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C,$$

since

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{x+a} - \frac{1}{x-a}\right)$$

and so

$$\begin{aligned}
\int \frac{1}{a^2 - x^2}\, dx &= \frac{1}{2a}\left(\int \frac{1}{x+a}\, dx - \int \frac{1}{x-a}\, dx\right) \\
&= \frac{1}{2a}(\ln|x+a| - \ln|x-a|) + C \\
&= \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C.
\end{aligned}$$

Problem 24

Evaluate $\displaystyle\int \frac{1}{x^2+2x-3}\,dx$.

$$\begin{aligned}
\int \frac{1}{x^2+2x-3}\,dx &= \int \frac{1}{(x+1)^2-1-3}\,dx \\
&= \int \frac{1}{(x+1)^2-4}\,dx \\
&= \int \frac{1}{(x+1)^2-2^2}\,dx \\
&= \frac{1}{4}\ln\left|\frac{x+1-2}{x+1+2}\right| + C \\
&= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C
\end{aligned}$$

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Problem 25

Evaluate $\int \frac{1}{\sqrt{5-x^2+4x}}\,dx$.

$$\begin{aligned}
\int \frac{1}{\sqrt{5-x^2+4x}}\,dx &= \int \frac{1}{\sqrt{5-(x-2)^2+4}}\,dx \\
&= \int \frac{1}{\sqrt{9-(x-2)^2}}\,dx \\
&= \int \frac{1}{\sqrt{3^2-(x-2)^2}}\,dx \\
&= \sin^{-1}\left(\frac{x-2}{3}\right)+C
\end{aligned}$$

Problem 26

Evaluate $\int \frac{1}{x^2+2x+3}\,dx$.

$$\begin{aligned}\int \frac{1}{x^2+2x+3}\,dx &= \int \frac{1}{(x+1)^2-1+3}\,dx \\ &= \int \frac{1}{(x+1)^2+2}\,dx \\ &= \int \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}\,dx \\ &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C\end{aligned}$$

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Class activity

$$\int \frac{3}{4x^2 - 25}\, dx = \frac{3}{20} \ln\left|\frac{6x - 15}{2x + 5}\right| + C \tag{1}$$

$$\int \frac{3}{4x^2 - 25}\, dx = \frac{3}{20} \ln\left|\frac{2x - 5}{2x + 5}\right| + C \tag{2}$$

$$\int \frac{3}{4x^2 - 25}\, dx = \frac{1}{20} \ln\left|\frac{6x - 15}{2x + 5}\right| + C \tag{3}$$

Which, if any, of the above are correct?

Integration of improper rational functions

Suppose that we want to evaluate an integral of the form

$$\int \frac{p(x)}{q(x)}\,dx$$

where p is a polynomial of degree m and q is a polynomial of degree n where $m \geq n \geq 1$.
There exists a polynomial s of degree $m - n$ and a polynomial r such that

$$p(x) = q(x)s(x) + r(x)$$

where $r = 0$ or the degree of r is less than n.
Hence,

$$\int \frac{p(x)}{q(x)}\,dx = \int s(x)\,dx + \int \frac{r(x)}{q(x)}\,dx.$$

Problem 27

Evaluate $\int \frac{x^2+3}{x^2-3}\,dx$.

$$\begin{aligned}
\int \frac{x^2+3}{x^2-3}\,dx &= \int \frac{x^2-3+6}{x^2-3}\,dx \\
&= \int \left(\frac{x^2-3}{x^2-3} + \frac{6}{x^2-3}\right) dx \\
&= \int 1\,dx + 6\int \frac{1}{x^2-\left(\sqrt{3}\right)^2}\,dx \\
&= x + \frac{6}{2\sqrt{3}} \ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C \\
&= x + \sqrt{3}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C
\end{aligned}$$

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Problem 28

Evaluate $\displaystyle\int \frac{x^2+4}{x-5}\,dx$.

$$\begin{aligned}
\int \frac{x^2+4}{x-5}\,dx &= \int \frac{(x-5)(x+5)+25+4}{x-5}\,dx \\
&= \int \left(x+5+\frac{29}{x-5}\right)dx \\
&= \frac{x^2}{2}+5x+29\ln|x-5|+C
\end{aligned}$$

Extended topic: Trigonometric substitution

Integrand contains	Substitution to try
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$x = a\sin(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$
$\dfrac{1}{x^2 + a^2}$ or $\dfrac{1}{\sqrt{x^2 + a^2}}$	$x = a\tan(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$x = \begin{cases} a\sec(t) \text{ with } 0 < t < \dfrac{\pi}{2} \text{ if } \dfrac{x}{a} > 1 \\ a\sec(t) \text{ with } \dfrac{\pi}{2} < t < \pi \text{ if } \dfrac{x}{a} < -1 \end{cases}$

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Problem 29

Show that, for all positive real numbers a,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

if $x > a$.

Let $x = a\sec(t)$ with $0 < t < \dfrac{\pi}{2}$.

Then, $\dfrac{dx}{dt} = a\sec(t)\tan(t)$ and $dx = a\sec(t)\tan(t)\,dt$.

Now,

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - a^2}}\, dx &= \int \frac{a\sec(t)\tan(t)}{\sqrt{a^2\sec^2(t) - a^2}}\, dt \\ &= \int \frac{a\sec(t)\tan(t)}{\sqrt{a^2(\sec^2(t) - 1)}}\, dt \\ &= \int \frac{a\sec(x)\tan(t)}{\sqrt{a^2}\sqrt{\tan^2(t)}}\, dt \\ &= \int \frac{a\sec(x)\tan(t)}{a\tan(t)}\, dt\end{aligned}$$

since $a > 0$ and $\tan(t) > 0$ as $0 < t < \dfrac{\pi}{2}$.

$$
\begin{aligned}
\therefore \int \frac{1}{\sqrt{x^2 - a^2}}\, dx &= \int \sec(t)\, dt \\
&= \ln\left|\sec(t) + \tan(t)\right| + c \\
&= \ln\left|\sec(t) + \sqrt{\sec^2(t) - 1}\right| + c \\
&= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c \\
&= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2 - a^2}{a^2}}\right| + c \\
&= \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c \\
&= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln(a) + c \\
&= \ln\left|x + \sqrt{x^2 - a^2}\right| + C
\end{aligned}
$$

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Problem 30

Show that, for all positive real numbers a,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

if $x < -a$.

Let $x = a\sec(t)$ with $\frac{\pi}{2} < t < \pi$.

Then, $\frac{dx}{dt} = a\sec(t)\tan(t)$ and $dx = a\sec(t)\tan(t)\,dt$.

Now,

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2 - a^2}}\,dx &= \int \frac{a\sec(t)\tan(t)}{\sqrt{a^2\sec^2(t) - a^2}}\,dt \\
&= \int \frac{a\sec(t)\tan(t)}{\sqrt{a^2(\sec^2(t) - 1)}}\,dt \\
&= \int \frac{a\sec(x)\tan(t)}{\sqrt{a^2}\sqrt{\tan^2(t)}}\,dt \\
&= \int \frac{a\sec(x)\tan(t)}{-a\tan(t)}\,dt
\end{aligned}
$$

since $a > 0$ and $\tan(t) < 0$ as $\frac{\pi}{2} < t < \pi$.

So,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\,dx &= -\int \sec(t)\,dt \\
&= -\ln|\sec(t)+\tan(t)|+c \\
&= -\ln\left|\sec(t)-\sqrt{\sec^2(t)-1}\right|+c \\
&= -\ln\left|\frac{x}{a}-\sqrt{\frac{x^2}{a^2}-1}\right|+c \\
&= -\ln\left|\frac{x}{a}-\sqrt{\frac{x^2-a^2}{a^2}}\right|+c \\
&= -\ln\left|\frac{x-\sqrt{x^2-a^2}}{a}\right|+c \\
&= \ln\left|\frac{a}{x-\sqrt{x^2-a^2}}\right|+c
\end{aligned}$$

$$
\begin{aligned}
\therefore \int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \ln\left|\frac{a\left(x+\sqrt{x^2-a^2}\right)}{\left(x-\sqrt{x^2-a^2}\right)\left(x+\sqrt{x^2-a^2}\right)}\right| + c \\
&= \ln\left|\frac{a\left(x+\sqrt{x^2-a^2}\right)}{x^2-(x^2-a^2)}\right| + c \\
&= \ln\left|\frac{a\left(x+\sqrt{x^2-a^2}\right)}{a^2}\right| + c \\
&= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| - \ln(a) + c \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| + C
\end{aligned}
$$

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Problem 31

Show that, for all positive real numbers a,

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C.$$

Let $x = a\tan(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Then $\dfrac{dx}{dt} = a\sec^2(t)$ and $dx = a\sec^2(t)\,dt$.

Now,

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{a\sec^2(t)}{\sqrt{a^2\tan^2(t)+a^2}}\,dt \\
&= \int \frac{a\sec^2(t)}{\sqrt{a^2(\tan^2(t)+1)}}\,dt \\
&= \int \frac{a\sec^2(t)}{\sqrt{a^2}\sqrt{\sec^2(t)}}\,dt \\
&= \int \frac{a\sec^2(t)}{a\sec(t)}\,dt
\end{aligned}
$$

since $a > 0$ and $\sec(t) > 0$ as $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

$$
\begin{aligned}
\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \sec(t)\,dt \\
&= \ln\left|\tan(t)+\sec(t)\right| + c \\
&= \ln\left|\tan(t)+\sqrt{\tan^2(t)+1}\right| + c \\
&= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right| + c \\
&= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2+a^2}{a^2}}\right| + c \\
&= \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c \\
&= \ln\left|x+\sqrt{x^2+a^2}\right| - \ln(a) + c \\
&= \ln\left|x+\sqrt{x^2+a^2}\right| + C
\end{aligned}
$$