University of Nottingham Ningbo China

CENTRE FOR ENGLISH LANGUAGE EDUCATION

PRELIMINARY YEAR, SEMESTER TWO, 2024-25

FOUNDATION CALCULUS AND MATHEMATICAL TECHNIQUES
MOCK END-OF-SEMESTER EXAM

Time allowed: ONE Hour THIRTY Minutes

Candidates must write their ID numbers on the Answer Booklet and fill-in their attendance card but must NOT write anything else until the start of the exam is announced.

This paper contains seven questions. The total number of points is 100.
Answer all questions with necessary steps.

Only general bilingual dictionaries are allowed. Subject-specific dictionaries are not permitted.

No electronic devices except for approved calculators (CASIO fx-82 series) can be used in this exam.

Do NOT open the examination paper until told to do so.

All answers must be written in the Answer Booklet.

ADDITIONAL MATERIAL: Formula Sheet

INFORMATION FOR INVIGILATORS:
1. A 15-minute warning should be given before the end of the exam.
2. Please collect the Question Paper, Answer Booklet, and Formula Sheet after the exam.
3. Please return the Answer Booklets in ID order.

1. (a) Given $y = \dfrac{1}{\sqrt{x-1}}$, use the definition of the derivative to find $\dfrac{dy}{dx}$.

   (b) Given $y = (3x^3 - 4x) \cdot e^x$, use the product rule of differentiation to find $\dfrac{dy}{dx}$.

   (c) Given $y = \dfrac{\sin^2 x}{e^{3x}}$, use the quotient rule of differentiation to find $\dfrac{dy}{dx}$.

   (d) Given $y = \ln\left[\sin\left(\sqrt{3x^3 - 4x + 1}\right)\right]$, use the chain rule of differentiation to find $\dfrac{dy}{dx}$.

   (e) Given $\tan(xy) = xy + xy^2$, use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

[10]

2. (a) Given $y = \dfrac{(x^2 - x + 3)^3}{\sqrt{x^3+2} \cdot (x-2)^4}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

   (b) The equation of a curve is given by the parametric equations $x = \sin t - \cos t$, and $y = \tan t$.

      (i) Find $\dfrac{dy}{dx}$.

      (ii) Find the equation of the tangent line to the curve at the point $t = \dfrac{\pi}{4}$.

   (c) The circumference of a circle is given by $C = 2\pi r$, and its area $A = \pi r^2$. If the circumference of the circle is increasing at a rate of 0.25 cm/s, find the rate at which its area is increasing when the radius is 2 cm.

   (d) Show that the function $f(x) = 3x^5 + 4x^3$ is always increasing for $x \in \mathbb{R}$.

[10]

3. (a) Given $f(x) = x^3 + 3x^2 - 9x + 1$.

   (i) Find the stationary points of $f(x)$.

   (ii) Use the second derivative tests to classify the stationary points obtained in $3(a)(i)$ as points of maximum or minimum values.

   (ii) Sketch the graph of $y = f(x)$.

   (b) Consider finding one root of the equation $x^3 - 4x + 1 = 0$. (3.1)

   (i) Show that the iterative formula obtained by applying the Newton-Raphson method to (3.1) is $x_{n+1} = \dfrac{2x_n^3 - 1}{3x_n^2 - 4}$.

   (ii) Starting with $x_0 = 2$, use the iterative formula (3.1) to obtain an approximate root of $f(x) = 0$ correct to 5 decimal places. Tabulate all the values of x_n until it converges.

[10]

4. (a) Given $f(x) = \ln(1 + x)$,

   (i) Obtain the Maclaurin series expansion of $f(x)$ up to the term with x^5.

   (ii) Use the result in 4(a)(i) to obtain the Maclaurin series expansion of $g(x) = \ln(1 - x)$.

   (iii) Show that $\ln\left(\dfrac{1+x}{1-x}\right) = 2\sum\limits_{k=1}^{\infty} \dfrac{x^{2k-1}}{2k-1}$.

   (b) Evaluate the following integrals

   (i) $\displaystyle\int e^x\left(e^x - 1\right)\, dx$.

   (ii) $\displaystyle\int \frac{\cos 2x}{\sin^2 x \, \cos^2 x}\, dx$.

   (iii) $\displaystyle\int \frac{\operatorname{cosec}^2 x}{4 - \cot^2 x}\, dx$ by using the substitution $\cot x = t$.

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5. (a) Evaluate the following integrals:

(i) $\int \sin^4 x \cdot \cos^5 x \, dx$, by using appropriate substitution.

(ii) $\int x^5 \sqrt{x^3 - 4} \, dx$, by using appropriate substitution.

(iii) $\int e^x \left(3x^3 + 9x^2\right) dx$ by using the result $\int e^x \left[f(x) + f'(x)\right] = e^x f(x) + C$.

(b) Evaluate $\int \frac{1}{x^2 + 2x + 2} \, dx$, by completing the square in the denominator.

(c) Use appropriate substitution to evaluate the integral $\int \frac{1}{2\sin^2 x + 3\cos^2 x} \, dx$.

[10]

6. (a) Apply the method of partial fractions to evaluate the integral $\int \frac{2x^2 + 3}{x(x-1)^2} \, dx$.

(b) Apply the method of integration by parts to evaluate the definite integral $\int_{-1}^{1} \ln(x + 2) \, dx$.

(c) Find the volume of solid of revolution formed when the region bounded by the curve $y = e^{2x}$ the lines $x = 0$, $x = 1$ and the $X-$axis is revolved about the $X-$axis.

(d) Apply the Simpson's rule to evaluate $\int_{0}^{1} \frac{4}{1 + x^2} \, dx$ to 5 decimal places by dividing $[0,\, 1]$ into 10 sub-intervals of equal width.

[10]

7. (a) Given $y = \dfrac{\sin x + a}{x} - \cos x$ where a is an arbitrary constant.

Show that $x\dfrac{dy}{dx} + y = x\sin x$.

(b) Solve the variable separable ordinary differential equation (ODE)

$$\frac{dy}{dx} = \frac{1+\sin x}{1-\cos y}.$$

(c) Solve the initial value problem of the variable separable ODE

$$x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0 \quad y(0) = 1.$$

(d) The differential equation model for the population of insects used in an experiment is defined by $\dfrac{dP}{dt} = kP$, where $k > 0$ is constant and t is time. (7.1)

(i) Show that the general solution for (7.1) is $P = P_0e^{kt}$, where $P = P(t)$ is the population of insects at time t, and $P_0 = P(0)$, is the initial population of the insects.

(ii) In 15 days the population of the insects increased from 1100 to 1450, how long will it take for the population of the insects to reach 1820.

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