Seminar 8

In this seminar you will study:

- The Binomial Theorem
- Applications of the Binomial Theorem in:
    - Approximation
    - Error Analysis

The Binomial Theorem

The Binomial Theorem Formula 1: $x \in \mathbb{R}, \ n \in \mathbb{N}$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots\cdots + x^n$$

Example: Expand $\left(1 + \frac{x}{2}\right)^4$ using the Binomial theorem.

Solution:

$$\left(1+\frac{x}{2}\right)^4 = 1 + \binom{4}{1} \cdot \frac{x}{2} + \binom{4}{2} \cdot \left(\frac{x}{2}\right)^2 + \binom{4}{3} \cdot \left(\frac{x}{2}\right)^3 + \binom{4}{4} \cdot \left(\frac{x}{2}\right)^4$$

$$= 1 + 4 \cdot \frac{x}{2} + 6 \cdot \left(\frac{x}{2}\right)^2 + 4 \cdot \left(\frac{x}{2}\right)^3 + 1 \cdot \left(\frac{x}{2}\right)^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}$$

Note: final result of the expansion is a **polynomial**

The Binomial Theorem

The Binomial Theorem Formula 2: $a,\ b \in \mathbb{R},\ \ n \in \mathbb{N}$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}\, b + \binom{n}{2} a^{n-2}\, b^2 + \binom{n}{3} a^{n-3}\, b^3 + \cdots + b^n \qquad ;\ T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

Example: Expand $\left(3 + \frac{2}{x}\right)^4$ using the Binomial theorem.

Solution: Here, $a = 3,\ b = \frac{2}{x},\ n = 4.$

$$\therefore \left(3 + \frac{2}{x}\right)^4 = 3^4 + \binom{4}{1} \cdot 3^3 \cdot \frac{2}{x} + \binom{4}{2} \cdot 3^2 \cdot \left(\frac{2}{x}\right)^2 + \binom{4}{3} \cdot 3 \cdot \left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4$$

$$= 81 + 4 \cdot 27 \cdot \frac{2}{x} + 6 \cdot 9 \cdot \frac{4}{x^2} + 4 \cdot 3 \cdot \frac{8}{x^3} + \frac{16}{x^4}$$

$$= 81 + \frac{216}{x} + \frac{216}{x^2} + \frac{96}{x^3} + \frac{16}{x^4}$$

Note: final result of the expansion is **finite**

The Binomial Theorem: Finding the coefficient of x^n

Example: Find the coefficient of x^3 in the expansion of $\left(3-\dfrac{2x}{5}\right)^5$.

Solution:

$$\left(3-\frac{2x}{5}\right)^5 = 3^5 + \binom{5}{1}\cdot 3^4\cdot\frac{-2x}{5} + \binom{5}{2}\cdot 3^3\cdot\left(\frac{-2x}{5}\right)^2 + \boxed{\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2x}{5}\right)^3} + \binom{5}{4}\cdot 3\cdot\left(\frac{-2x}{5}\right)^4 + \left(\frac{-2x}{5}\right)^5$$

Thus, the term in x^3 is $\boxed{\binom{5}{3}}\cdot\boxed{3^2}\cdot\boxed{\left(\frac{-2}{5}\right)^3}x^3$

$\therefore$ The coefficient of x^3 is $\boxed{10}\cdot\boxed{9}\cdot\boxed{\dfrac{-8}{125}} = -\dfrac{144}{25}$

The Generalised Binomial Theorem

The Generalised Binomial Theorem : $x,\, n \in \mathbb{R},\ |\,x\,| < 1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}\cdot x^2 + \frac{n(n-1)(n-2)}{3\,!}\cdot x^3 + \cdots$$

Example: Expand $(1+x)^{-3}$ up to the term with x^3, $(|\,x\,| < 1)$, using the Generalised Binomial Theorem.

Solution:

$$(1+x)^{-3} = 1 + (-3)\cdot x + \frac{(-3)(-3-1)}{2\,!}\cdot x^2 + \frac{(-3)(-3-1)(-3-2)}{3\,!}\cdot x^3 + \cdots$$

$$= 1 - 3x + \frac{(-3)(-4)}{2\cdot 1}\cdot x^2 + \frac{(-3)(-4)(-5)}{3\cdot 2\cdot 1}\cdot x^3 + \cdots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \cdots$$

Note: final result of the expansion is an **infinite series**

Approximation using the Binomial Theorem

Approximation using the Binomial Theorem :

Given $(1+x)^n$, where $x,\ n \in \mathbb{R},\ |\,x\,| < 1,$ apply the Generalised Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}\cdot x^2 + \frac{n(n-1)(n-2)}{3!}\cdot x^3 + \cdots$$

Example 1: Use the first three terms of the Generalised Binomial Theorem to find an approximate value of $\dfrac{1}{1.05}$.

Solution: $\dfrac{1}{1.05} = (1.05)^{-1}$

$= (1+0.05)^{-1}$. Here $n = -1$, and $x = 0.05 \Rightarrow |\,x\,| < 1$.

$$\therefore (1+0.05)^{-1} = 1 + (-1)\times 0.05 + \frac{(-1)\times(-1-1)}{2\,!}\times 0.05^2 + \cdots$$

$$= 1 - 0.05 + 0.05^2 + \cdots$$

$$\approx 0.9525$$

Approximation using the Binomial Theorem

Approximation using the Binomial Theorem :

Given $(1+x)^n$, where $x,\, n \in \mathbb{R},\ |\,x\,| < 1,$ apply the Generalised Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \cdots$$

Example 2: Use the first three terms of the Generalised Binomial Theorem to find an approximate value of $\sqrt[3]{0.99}$.

Solution: $\sqrt[3]{0.99} = (1-0.01)^{\frac{1}{3}}$

$$= [1 + (-0.01)]^{\frac{1}{3}}.$$ Here $n = \frac{1}{3}$, and $x = -0.01 \Rightarrow |\,x\,| < 1.$

$$\therefore (1-0.01)^{\frac{1}{3}} = [1+(-0.01)]^{\frac{1}{3}} = 1 + \frac{1}{3} \times (-0.01) + \frac{\frac{1}{3} \times (\frac{1}{3}-1)}{2!} \times (-0.01)^2 + \cdots$$

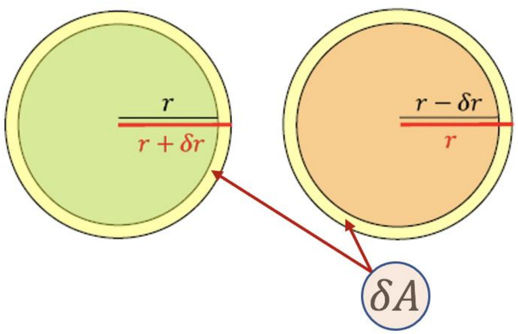
$$= 1 - \frac{0.01}{3} - \frac{0.0001}{9} + \cdots$$

$$\approx 0.9967$$

Application of the Binomial Theorem in Error Analysis

Example: The radius r of a circle is measured with an error $\delta r = 1.25\%$ of r.
Use the approximation $(1+x)^n \approx 1 + nx$ to find the resulting error δA in the the calculated area. Area of a circle: $A = \pi\, r^2$.

Solution:



$$\delta r = 1.25\%\, r \;\Rightarrow\; \delta r = 0.0125\, r$$

$$\Rightarrow \cancel{A} + \delta A = \pi(r + \delta r)^2$$

$$= \pi(r + 0.0125\, r)^2$$

$$= \pi r^2 (1 + 0.0125)^2$$

$$\approx A(1 + 2 \times 0.0125)$$

$$= \cancel{A} + 0.025 A$$

$$\Rightarrow \delta A = 0.025 A$$

$\Rightarrow$ The error in the area is 2.5% of A