Topic 1: Integration using substitution of $t = \tan\left(\frac{x}{2}\right)$

For integrals of the form:

$$\boxed{\int \frac{1}{a + b\cos x + c\sin x}\, dx}$$

substitution:

$$\text{Let } \tan\left(\frac{x}{2}\right) = t \;\Rightarrow\; dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$

Illustration: Evaluate $\displaystyle\int \frac{1}{1+2\cos x}\, dx$

$$I = \int \frac{1}{1+2\cos x}\, dx$$

$$\text{Let } \tan\left(\frac{x}{2}\right) = t \;\Rightarrow dx = \frac{2\ dt}{1+t^2} \qquad \text{Now, } \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \qquad I = \int \frac{1}{1 + 2\cdot\left(\frac{1-t^2}{1+t^2}\right)} \cdot \left(\frac{2dt}{1+t^2}\right)$$

$$\Rightarrow \qquad I = \int \frac{2}{3-t^2}\, dt$$

$$= 2\int \frac{1}{\left(\sqrt{3}\right)^2 - t^2}\, dt$$

$$\boxed{\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C}$$

$$= 2\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)+\sqrt{3}}{\tan\left(\frac{x}{2}\right)-\sqrt{3}}\right| + C$$

1. $\displaystyle\int \frac{1}{2+\sin x}\,dx$

Answer:

2. $\displaystyle\int \frac{1}{2-\cos x}\,dx$

Answer:

3. $\displaystyle\int \frac{1}{4\cos x+1}\,dx$

Answer:

4. $\displaystyle\int \frac{1}{3\cos x+4\sin x+5}\,dx$

Answer:

Topic 2: Integral of the form $\int \dfrac{1}{\mathrm{a}\cos^2\mathrm{x} + \mathrm{b}\sin^2\mathrm{x} + \mathrm{c}}\,\mathrm{dx}$

(i) Divide both the numerator and the denominator by $\cos^2 x$ and simplify

(ii) Substitute $\tan x$ by t $\left(\tan x = t\right)$, then $\sec^2 x\ dx = dt$

Illustration: Evaluate $\displaystyle\int \frac{1}{3\sin^2 x + 2}\ dx$

$$I = \int \frac{1}{3\sin^2 x + 2}\ dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{3\sin^2 x + 2}{\cos^2 x}}\ dx$$

$$= \int \frac{\sec^2 x}{3\tan^2 x + 2\sec^2 x}\ dx$$

$$= \int \frac{\sec^2 x}{5\tan^2 x + 2}\ dx$$

$$\text{Let} \quad \tan x = t \quad \Rightarrow \quad \sec^2 x\ dx = dt$$

$$\therefore\ I = \int \frac{1}{5t^2 + 2}\ dt = \frac{1}{5}\int \frac{1}{t^2 + \frac{2}{5}}\ dt$$

$$= \frac{1}{5}\int \frac{1}{t^2 + \left(\sqrt{\frac{2}{5}}\right)^2}\ dt$$

$$\boxed{\int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$= \frac{1}{5}\cdot\frac{1}{\sqrt{\frac{2}{5}}}\tan^{-1}\left(\frac{t}{\sqrt{\frac{2}{5}}}\right) + C$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{5}\tan x}{\sqrt{2}}\right) + C$$

1. $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx$

Answer:

2. $\displaystyle\int \frac{1}{4\cos^2 x+\sin^2 x}\,dx$

Answer:

Topic 3: Integration by Partial Fractions

Type 1: Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$
$$\Rightarrow \int \frac{p(x)}{(x+a)(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{x+b}\,dx$$

Type 2: Non-repeated quadratic factor

$$\frac{p(x)}{(x+a)(x^2+b)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b} = \frac{A}{x+a} + \frac{Bx}{x^2+b} + \frac{C}{x^2+b}$$
$$\Rightarrow \int \frac{p(x)}{(x+a)(x^2+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{Bx}{x^2+b}\,dx + \int \frac{C}{x^2+b}\,dx$$

Type 3: Repeated linear factor

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$
$$\Rightarrow \int \frac{p(x)}{(x+a)^2(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{(x+a)^2}\,dx + \int \frac{C}{x+b}\,dx$$

Illustration 1: Evaluate $\displaystyle\int \frac{13}{(3x-2)(2x+3)}dx$

Let $\displaystyle\frac{13}{(3x-2)(2x+3)} = \frac{A}{3x-2} + \frac{B}{2x+3}$ Then $13 = A(2x+3) + B(3x-2)$

When $x = \dfrac{2}{3} \Rightarrow 13 = A\left(2\times\dfrac{2}{3}+3\right) \Rightarrow A = 3$

$x = -\dfrac{3}{2} \Rightarrow 13 = B\left[3\times\left(-\dfrac{3}{2}\right)-2\right] \Rightarrow B = -2$

$$\therefore \frac{13}{(3x-2)(2x+3)} = \frac{3}{3x-2} + \frac{-2}{2x+3}$$

$$\therefore \quad I = \int \frac{13}{(3x-2)(2x+3)}\,dx = \int \frac{3}{3x-2}\,dx - \int \frac{2}{2x+3}\,dx$$
$$= \ln|3x-2| - \ln|2x+3| + C$$
$$= \ln\left|\frac{3x-2}{2x+3}\right| + C$$

1. $\displaystyle\int \frac{3x}{(x-1)(x+2)}\,dx$ **Answer:**	2. $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$ **Answer:**
3. $\displaystyle\int \frac{5x-8}{(x+4)(x-3)}\,dx$ **Answer:**	4. $\displaystyle\int \frac{3x-1}{x^2+x-12}\,dx$ **Answer:**

Topic 3: Integration by Partial Fractions

Illustration 2: Evaluate $\displaystyle\int \frac{3}{(x+1)(x^2+2)}dx$

Let $\displaystyle \frac{3}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} \Rightarrow 3 = A(x^2+2) + (Bx+C)(x+1)$

$$\begin{array}{llll} \text{When} & x=-1 & \Rightarrow & A=1 \\ & x=0 & \Rightarrow & C=1 \\ & x=1 & \Rightarrow & B=-1 \end{array}$$

$$\begin{aligned} \therefore \quad I &= \int \frac{3}{(x+1)(x^2+2)}\,dx \\ &= \int \frac{1}{x+1}\,dx - \int \frac{x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx \\ &= \ln|x+1| - \frac{1}{2}\ln\left(x^2+2\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \end{aligned}$$

1. $\displaystyle\int \frac{17}{(x-4)(x^2+1)}\,dx$ **Answer:**	2. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$ **Answer:**

Topic 3: Integration by Partial Fractions

Illustration 3: Evaluate $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

Let $\displaystyle \frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}$

Then $1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$

When $x = 1 \Rightarrow C = \dfrac{1}{36}$

$x = -5 \Rightarrow B = -\dfrac{1}{6}$

$x = 0 \Rightarrow A = -\dfrac{1}{36}$

$$\therefore \quad I = \int \frac{1}{(x+5)^2(x-1)}\,dx = -\frac{1}{36}\int \frac{1}{x+5}\,dx - \frac{1}{6}\int \frac{1}{(x+5)^2}\,dx + \frac{1}{36}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

1. $\displaystyle\int \frac{25}{(x-3)^2(x+2)}\,dx$ **Answer:**	2. $\displaystyle\int \frac{9}{(x+1)(x-2)^2}\,dx$ **Answer:**

Topic 4: Integration by Parts

$$\boxed{\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx}$$

LIATE Rule: Choose the function that appears first in the following list as $\boldsymbol{u}$ and the other as $\boldsymbol{\frac{dv}{dx}}$.

L: **L**ogarithmic functions ($\ln x$, $\log_a x$, etc.)
I : **I**nverse functions ($\sin^{-1} x$, $\tan^{-1} x$, etc.)
A: **A**lgebraic functions (x^2, x^n, etc.)
T: **T**rigonometric functions ($\sin x$, $\cos x$, $\tan x$, etc.)
E: **E**xponential functions (e^x, a^x, etc.)

Illustration 1: Evaluate $\displaystyle\int x^2 \ln x \; dx$.

$L \rightarrow \ln x$
I
$A \rightarrow x^2$
T
E

$$I = \int x^2 \ln x \; dx. \quad \text{Let} \quad u = \ln x, \quad \frac{dv}{dx} = x^2$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad \int dv = \int x^2\, dx \Rightarrow v = \frac{x^3}{3}$$

$$\boxed{\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx}$$

$$\therefore \quad I = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx$$

$$\Rightarrow \quad I = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3}\, dx$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$

1. $\displaystyle\int x^2 \sin^{-1} x \, dx$ **Answer:**	2. $\displaystyle\int \cos^{-1} x \, dx$ **Answer:**
3. $\displaystyle\int x \tan^{-1} x \, dx$ **Answer:**	4. $\displaystyle\int x \sec^2 x \, dx$ **Answer:**

Topic 4: Integration by Parts

Sometimes, we need to apply the method of integration by parts for multiple times.

Illustration 2: Evaluate $\displaystyle\int e^x \cos x \; dx$

Let $u = \cos x$

and $\dfrac{dv}{dx} = e^x$ $\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = -\sin x \\ v = \displaystyle\int e^x \; dx = e^x \end{array} \right.$

L

I

A

$T \longrightarrow \cos x$

$E \longrightarrow e^x$

$$\boxed{\int u \cdot \frac{dv}{dx} \; dx = u \cdot v - \int v \cdot \frac{du}{dx} \; dx}$$

$$\Rightarrow I = \cos x \cdot e^x - \int e^x \cdot (-\sin x) \; dx$$

$$\therefore \; I = e^x \cdot \cos x + \int e^x \cdot \sin x \; dx$$

$$\therefore \; I = e^x \cdot \cos x + \int e^x \cdot \sin x \; dx$$

Again integrating by parts (in integral on the righ

Let $u = \sin x$

and $\dfrac{dv}{dx} = e^x$ $\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = \cos x \\ v = \displaystyle\int e^x \; dx = e^x \end{array} \right.$

$$\therefore \; I = e^x \cdot \cos x + \sin x \cdot e^x - \int e^x \cdot \cos x \; dx$$

$$\therefore \; I = e^x \cdot \cos x + e^x \cdot \sin x - I$$

i.e. $\quad 2\,I = e^x \cdot (\cos x + \sin x)$

$$\therefore \; I = \int e^x \cos x \; dx = \frac{e^x}{2} \cdot (\cos x + \sin x) + C$$

Topic 4: Integration by Parts

> Sometimes, we need to apply the method of integration by parts for multiple times.

Illustration 3: Evaluate $\displaystyle\int \sin(\ln x)\ dx$.

$$I = \int \sin(\ln x)\ dx.$$

Let $\ln x = t$. Then $\boxed{x = e^t} \Rightarrow dx = e^t\ dt$.

L

I

A

$T \longrightarrow \sin t$

$E \longrightarrow e^t$

$$\therefore\quad I = \int e^t \sin t\ dt$$

Let $u = \sin t$, $\dfrac{dv}{dt} = e^t$

$$\Rightarrow \frac{du}{dt} = \cos t \text{ and } \int dv = \int e^t\,dt \Rightarrow v = e^t$$

$$\boxed{\int u\cdot\frac{dv}{dt}\ dt = u\cdot v - \int v\cdot\frac{du}{dt}\ dt}$$

$$\therefore\quad I = e^t \sin t - \int e^t \cos t\ dt$$

$$I = e^t \sin t - \int e^t \cos t\ dt$$

For $\displaystyle\int e^t \cos t\ dt$, use integration by parts again.

L

I

A

$T \longrightarrow \cos t$

$E \longrightarrow e^t$

Let $u = \cos t$, $\dfrac{dv}{dt} = e^t$

$$\Rightarrow \frac{du}{dt} = -\sin t \text{ and } dv = \int e^t\,dt \Rightarrow v = e^t$$

$$\therefore\quad \int e^t \cos t\ dt = e^t \cos t + \int e^t \sin t\ dt$$

$$\Rightarrow I = e^t \sin t - e^t \cos t - I \qquad \boxed{\because\quad I = \int e^t \sin t\ dt}$$

$$\Rightarrow 2I = \left(e^t \sin t - e^t \cos t\right)$$

$$\Rightarrow I = \frac{1}{2}\left(e^t \sin t - e^t \cos t\right)$$

$$\therefore I = \int \sin(\ln x)\ dx = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

1. $\displaystyle\int x^3 e^{x^2}\, dx$

Answer:

2. $\displaystyle\int \frac{\ln x}{\sqrt{x}}\, dx$

Answer: