# Seminar 03

## In this seminar you will study:

- The Remainder and Factor Theorem
- The method of synthetic division
  - Finding quotients and remainders
  - Solving polynomial equations
  - The method of synthetic division
- Partial Fractions

## The Remainder theorem and the Factor theorem

### The Remainder theorem

If a polynomial p(x) is divided by (x-c), then the remainder is p(c).

### The Factor theorem

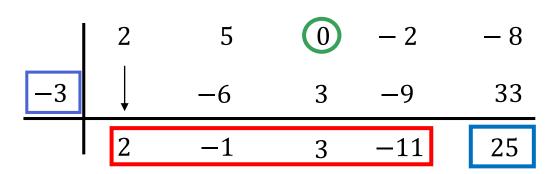
A polynomial p(x) has a factor (x-c) if and only if p(c)=0.

# The method of Synthetic division

**Example:** Use the method of synthetic division to find the quotient q(x) and the remainder r if  $p(x) = 2x^4 + 5x^3 - 2x - 8$  is divided by (x + 3).

 $+0x^{2}$ 

Here 
$$s(x) = x + 3 = x - c$$
  
 $\Rightarrow c = -3$ 



Quotient 
$$q(x)$$

Quotient 
$$q(x) = 2x^3 - x^2 + 3x - 11$$
 Remainder  $r =$ 

# Solving polynomial equations

### Result

Let  $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$  be a polynomial with integer coefficients (i.e.  $c_i \in \mathbb{Z}$ , for  $i = 0, 1, \dots, n$ ). If m is an integer zero of p(x), then m is a divisor of the constant term  $c_0$ .

For example, if  $p(x) = x^3 - 27$ , 3 is an integer zero of p(x), (i.e. m = 3), and 3 is a divisor of -27, (since  $c_0 = -27$ ).

# Solving polynomial equations

**Example:** Solve  $p(x) = x^3 - 2x^2 - 9x + 18 = 0$ 

**Solution:** 

Possible zeros are  $\pm 1, \pm 2 \pm 3, \pm 6, \pm 9, \pm 18$ .

Try  $\pm 1, \pm 2 \pm 3$ .

**Note:** exam question on solving cubic equation will have at least one of these integers as a zero of p(x).

$$p(1) = 8 \Rightarrow p(x) \neq 0$$
 : 1 is not a zero of  $p(x)$ 

$$p(-1) = 24 \Rightarrow p(x) \neq 0$$
 :  $-1$  is not a zero of  $p(x)$ 

$$p(2) = 0 \Rightarrow p(x) = 0$$
 : 2 is a zero of  $p(x)$ 

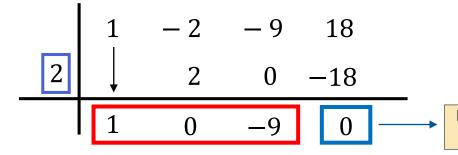
$$\Rightarrow (x-2)$$
 is one of the factors of  $p(x)$ 

Use the method of synthetic division to find the other factor.

## Solving polynomial equations

**Example:** Solve 
$$p(x) = x^3 - 2x^2 - 9x + 18 = 0$$

Here 
$$s(x) = x - 2 = x - c$$
  
 $\Rightarrow c = 2$ 



**Note:** remainder r = 0 : s(x) = x - 2 is a factor of  $p(x) = x^3 - 2x^2 - 9x + 18$ .

Thus, the other factor is  $x^2 - 9$ 

$$p(x) = (x-2) \cdot (x^2 - 9)$$

$$= (x-2) \cdot (x-3) \cdot (x+3)$$

$$p(x) = 0 \implies (x-2) \cdot (x-3) \cdot (x+3) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 3 \text{ or } x = -3$$

### The method of partial fractions

#### Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

#### Non-repeated quadratic factors

$$\frac{p(x)}{(x^2+a)(x+b)} = \frac{Ax+B}{x^2+a} + \frac{C}{x+b}$$

#### Repeated linear factors

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

In all these types, the constants A and B or A, B and C are to be determined.

## Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$
$$\Rightarrow p(x) = A(x+b) + B(x+a)$$

**Example:** Express  $\frac{14}{(x-9)(x+5)}$  as a sum of partial fractions.

$$\frac{14}{(x-9)(x+5)} = \frac{A}{x-9} + \frac{B}{x+5}$$

$$\therefore$$
 14 =  $A(x + 5) + B(x - 9)$ 

Let 
$$x = 9$$

$$\Rightarrow 14 = 14A$$

$$\Rightarrow A = 1$$
Let  $x = -5$ 

$$\Rightarrow 14 = -14B$$

$$\Rightarrow B = -1$$

$$\therefore \frac{14}{(x-9)(x+5)} = \frac{1}{x-9} - \frac{1}{x+5}$$



## Repeated linear factors

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{(x+b)} + \frac{B}{(x+a)} + \frac{C}{(x+a)^2}$$
$$\Rightarrow p(x) = A(x+a)^2 + B(x+a)(x+b) + C(x+b)$$

**Example:** Separate  $\frac{2x+5}{(x-1)(x+2)^2}$  into partial fractions.

**Solution:** 
$$\frac{2x+5}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\therefore 2x + 5 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

Let 
$$x = 1$$
 Let  $x = -2$  Let  $x = 0 \Rightarrow 5 = 4A - 2B - C$   
 $\Rightarrow 7 = 9A$   $\Rightarrow 1 = -3C$   $\therefore$  Substituting the values of  $A$  and  $B$   
 $\Rightarrow A = 7/9$   $\Rightarrow C = -1/3$   $\Rightarrow B = -7/9$ 

$$\therefore \frac{2x+5}{(x-1)(x+2)^2} = \frac{7}{9(x-1)} - \frac{7}{9(x+2)} - \frac{1}{3(x+2)^2}$$



## Non-repeated quadratic factors

$$\frac{p(x)}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$
  

$$\Rightarrow p(x) = (Ax+B)(x+b) + C(x^2+a)$$

**Example:** Express  $\frac{20}{(x-4)(x^2+4)}$  as a sum of partial fractions.

**Solution:** 

$$\frac{20}{(x-4)(x^2+4)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+4}$$

$$\therefore 20 = A(x^2 + 4) + (Bx + C)(x - 4) = (A + B)x^2 + (C - 4B)x + (4A - 4C)$$

Let 
$$x = 4$$
 Equating constant  

$$\Rightarrow A(4^2 + 4) = 20$$
 
$$\Rightarrow A = 1$$
 Equating constant  

$$\Rightarrow 4A - 4C = 20$$
 
$$\Rightarrow C = -4$$

Equating constant  

$$\Rightarrow 4A - 4C = 20$$

$$\Rightarrow C = -4$$

$$\therefore \frac{20}{(x-4)(x^2+4)} = \frac{1}{x-4} + \frac{-x-4}{x^2+4}$$

Equating coefficient of x

$$\Rightarrow C - 4B = 0$$

$$\Rightarrow B = -1$$