

**Topic 1: Integration using product formula e.g. $\int \sin(Ax) \cos(Bx) dx$**

Illustration: Evaluate $\int \sin 3x \cos 2x dx$.

$$\begin{aligned} I &= \int \sin 3x \cos 2x dx \\ &= \frac{1}{2} \int 2 \sin 3x \cos 2x dx \end{aligned}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$\begin{aligned} &= \frac{1}{2} \int [\sin(3x + 2x) + \sin(3x - 2x)] dx \\ &= \frac{1}{2} \int \sin 5x dx + \frac{1}{2} \int \sin x dx \\ &= -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C \end{aligned}$$

1. $\int \cos 4x \sin 2x dx$

Answer:

2. $\int \cos 4x \cos 3x dx$

Answer:



1. $\int \sin 4x \sin x \, dx$

Answer:

2. $\int \cos 5x \sin 4x \, dx$

Answer:



Topic 2: Integral of the form $\int \sin^m x \cdot \cos^n x \, dx$

Given $\int \sin^m x \cdot \cos^n x \, dx$, where $m, n \in \mathbb{N}$.

(i) If m and n are both odd, let $\sin x = t$ if $m > n$, and $\cos x = t$ if $m < n$

(ii) If m is even and n is odd, let $\sin x = t$

If m is odd and n is even, let $\cos x = t$

(iii) If m and n are both even, then transform the integrand using:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Illustration: Evaluate $\int \sin^3 x \cdot \cos^7 x \, dx$

$$I = \int \sin^3 x \cdot \cos^7 x \, dx$$

Both m and n are odd, $m < n$.

Let $\cos x = t$, then $-\sin x \, dx = dt \Rightarrow \sin x \, dx = -dt$

$$\therefore I = \int \sin^2 x \cdot \cos^7 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cdot \cos^7 x \cdot \sin x \, dx$$

$$= \int (1 - t^2) \cdot t^7 (-dt)$$

$$= - \int (t^7 - t^9) \, dt$$

$$= -\frac{t^8}{8} + \frac{t^{10}}{10} + C = -\frac{\cos^8 x}{8} + \frac{\cos^{10} x}{10} + C$$



1. $\int \sin^3 x \cdot \cos^2 x \, dx$

Answer:

2. $\int \sin^2 x \cdot \cos^5 x \, dx$

Answer:



Topic 3: Integral of the form $\int [f(x)]^n \cdot f'(x) dx$

$$\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

Illustration: Evaluate $\int (8x^3 + 5x^2 - 1)^6 \cdot (24x^2 + 10x) dx$

$$I = \int (8x^3 + 5x^2 - 1)^6 \cdot (24x^2 + 10x) dx$$

Let $f(x) = 8x^3 + 5x^2 - 1 \Rightarrow f'(x) = 24x^2 + 10x \therefore I = \frac{(8x^3 + 5x^2 - 1)^7}{7} + C$

<div>1. $\int \tan^5 x \sec^2 x dx$</div> <div>Answer:</div>	<div>2. $\int (e^{9x-6}) \cdot (e^{3x-2}) dx$</div> <div>Answer:</div>
<div>3. $\int (\ln x + 3x)^{10} \cdot \left(\frac{1+3x}{x}\right) dx$</div> <div>Answer:</div>	<div>4. $\int (3x^2 - 4x + 1)^{\frac{5}{2}} \cdot (3x - 2) dx$</div> <div>Answer:</div>

**Topic 4: Integral of the form** $\int \frac{f'(x)}{f(x)} dx$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

Illustration: Evaluate $\int \frac{x^2}{1+x^3} dx$

$$I = \int \frac{x^2}{1+x^3} dx$$

Let $f(x) = 1 + x^3 \Rightarrow f'(x) = 3x^2$

$$\therefore I = \frac{1}{\color{red}{3}} \int \frac{\color{red}{3}x^2}{1+x^3} dx$$

$$= \frac{1}{3} \ln |1+x^3| + C$$

1. $\int \frac{2^{2x} - 1}{2^{2x} + 1} dx$

Answer:

2. $\int \frac{e^{3x}}{e^{3x} + 1} dx$

Answer:



1. $\int \frac{\sin 5x}{1 + \cos 5x} dx$

Answer:

2. $\int \operatorname{cosec} x dx$
 $= \int \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

Answer:

3. $\int \cot x dx$

Answer:

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Answer:



Topic 5: Integral of the form $\int \frac{f'(x)}{\sqrt{f(x)}} dx$

Illustration: Evaluate $\int \frac{3x^2 - 4x}{\sqrt{x^3 - 2x^2 + 1}} dx$
 $I = \int \frac{3x^2 - 4x}{\sqrt{x^3 - 2x^2 + 1}} dx$

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$$

Let $f(x) = x^3 - 2x^2 + 1 \Rightarrow f'(x) = 3x^2 - 4x \therefore I = 2\sqrt{x^3 - 2x^2 + 1} + C$

<div>1. $\int \frac{1 - \cos x}{\sqrt{x - \sin x}} dx$</div> <div>Answer:</div>	<div>2. $\int \frac{\sin 2x}{\sqrt{1 + \sin^2 x}} dx$</div> <div>Answer:</div>
<div>3. $\int \frac{e^{-x}(1 - x)}{\sqrt{x}e^{-x}} dx$</div> <div>Answer:</div>	<div>4. $\int \frac{1}{x\sqrt{1 + \ln x}} dx$</div> <div>Answer:</div>



Topic 6: Integral of the form $\int e^x [f(x) + f'(x)] dx$

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

Illustration: Evaluate $\int e^x \left[\frac{x}{(x+1)^2} \right] dx$

$$\int e^x \left[\frac{x}{(x+1)^2} \right] dx = \int e^x \left[\frac{x+1-1}{(x+1)^2} \right] dx$$

$$= \int e^x \left[\frac{1}{x+1} - \frac{1}{(x+1)^2} \right] dx$$

$$\int e^x [f(x) + f'(x)] dx \text{ where } f(x) = \frac{1}{x+1}, f'(x) = -\frac{1}{(x+1)^2}$$

$$= e^x f(x) + C = e^x \frac{1}{x+1} + C$$

1. $\int e^x (\sin x + \cos x) dx$

Answer:

2. $\int e^x (1 - x^2 - 2x) dx$

Answer:



1. $\int e^x(x)dx$

Answer:

2. $\int e^x \left(\frac{x-2}{x^3} \right) dx$

Answer:

**Topic 7: Integration by Completing the Square in the Denominator**

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left| x + \sqrt{x^2 + a^2} \right| + C$$

Illustration 1:

Evaluate $\int \frac{1}{x^2 + 4x + 9} dx$

$$\begin{aligned} I &= \int \frac{1}{x^2 + 4x + 9} dx \\ &= \int \frac{1}{(x^2 + 4x + 4) + 9 - 4} dx \\ &= \int \frac{1}{(x + 2)^2 + 5} dx \\ &= \int \frac{1}{(x + 2)^2 + (\sqrt{5})^2} dx \end{aligned}$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{x + 2}{\sqrt{5}} \right) + C$$

Illustration 2:

Evaluate $\int \frac{1}{9 - x^2 - 4x} dx$

$$\begin{aligned} I &= \int \frac{1}{9 - x^2 - 4x} dx \\ &= \int \frac{1}{-(x^2 + 4x + 4) + 9 + 4} dx \\ &= \int \frac{1}{13 - (x + 2)^2} dx \\ &= \int \frac{1}{(\sqrt{13})^2 - (x + 2)^2} dx \end{aligned}$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + C$$

$$\therefore I = \frac{1}{2\sqrt{13}} \ln \left| \frac{x + 2 + \sqrt{13}}{x + 2 - \sqrt{13}} \right| + C$$



1. $\int \frac{1}{5 - 4x - x^2} dx$

Answer:

2. $\int \frac{1}{x^2 + 2x + 10} dx$

Answer:

3. $\int \frac{1}{\sqrt{5 - x^2 + 4x}} dx$

Answer:

4. $\int \frac{1}{x^2 - x + 2} dx$

Answer:



1. $\int \frac{1}{\sqrt{4x^2 + 4x + 3}} dx$

Answer:

2. $\int \frac{1}{8 - x^2 + 2x} dx$

Answer:

3. $\int \frac{1}{x^2 + 4x - 5} dx$

Answer:

4. $\int \frac{1}{\sqrt{x^2 + 4x + 5}} dx$

Answer: