Early Module Feedback

Topic 1: Higher order derivatives

Available from 09:00 Monday 10 March - 17:00 Sunday 16 March 2025

The derivative of $\dfrac{dy}{dx}$ w.r.t. x is called the **second** order derivative of $y = f(x)$ and is denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$. Thus,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) = \frac{d^2y}{dx^2}$$

Similarly, successive derivatives are: $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \cdots, \dfrac{d^ny}{dx^n}$ or $f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)$

Illustration: Given $y = e^{3x}$, show that $\dfrac{d^3y}{dx^3} - 27y = 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(e^{3x}) = 3e^{3x}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3e^{3x}) = 9e^{3x}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(9e^{3x}) = 27e^{3x} \quad \Rightarrow \frac{d^3y}{dx^3} = 27y \qquad \therefore \frac{d^3y}{dx^3} - 27y = 0$$

1. Given $y = 2x^2 + \ln x$, $x > 0$. Show that $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2 = 0$.

Answer:

2. Given $y = \ln\left(x + \sqrt{1 + x^2}\right)$. Show that $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

Answer:

1. Given $y = \tan x$, find $\dfrac{d^2y}{dx^2}$ **Answer:**	2. Given $y = e^x \sin x$, find $\dfrac{d^2y}{dx^2}$ **Answer:**
3. Given $y = \sin^{-1} x\,;\ \lvert\, x \,\rvert < 1$, find $\left.\dfrac{d^2y}{dx^2}\right\rvert_{x=\frac{1}{3}}$ **Answer:**	4. Given $y = (1 + x^2)\tan^{-1} x$, find $\dfrac{d^2y}{dx^2}$ **Answer:**

Topic 2: Parametric Differentiation

Illustration: Given $x = a(t - \sin t)$ and $y = a(1 - \cos t)$ (where $a \neq 0$ is a constant), find $\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{2}}$

$$\frac{dx}{dt} = a(1 - \cos t)$$

$$\frac{dy}{dt} = a \sin t$$

Note: $\dfrac{dy}{dt}$ is in the numerator

$\dfrac{dx}{dt} \neq 0$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{\sin \frac{\pi}{2}}{1 - \cos \frac{\pi}{2}} = 1$$

1. Given $x = \cos t$, $y = \sin^2 t$. Use the method of parametric differentiation to find $\left.\dfrac{dy}{dx}\right|_{t=\pi/3}$

Answer:

1. Given $x = a\sec\theta\,,\ y = b\tan\theta$ use the method of parametric differentiation to find $\left.\frac{dy}{dx}\right|_{\theta=\pi/4}$

Answer:

2. Given $x = a\cos\theta\,,\ y = b\sin\theta$. Use the method of parametric differentiation to find $\left.\frac{dy}{dx}\right|_{\theta=\pi/4}$

Answer:

3. Given $x = \cos^3\theta\,,\ y = \sin^3\theta$. Use the method of parametric differentiation to find $\left.\frac{dy}{dx}\right|_{\theta=3\pi/4}$

Answer:

Topic 3: Maclaurin's Series

Let $f(x)$ be a continuously differentiable function then the Maclaurin's series expansion of $f(x)$ is given by:

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2\,!} \cdot x^2 + \frac{f'''(0)}{3\,!} \cdot x^3 + \cdots$$

Illustration: Given $f(x) = \dfrac{1}{1-x}$; $-1 < x < 1$, obtain the Maclaurin's series expansion of $f(x)$ up to the terms in x^4.

$$f(x) = \frac{1}{1-x} \qquad \Rightarrow f(0) = 1$$

$$f'(x) = \frac{-1}{(1-x)^2} \cdot (-1) = \frac{1}{(1-x)^2} \qquad \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-2}{(1-x)^3} \cdot (-1) = \frac{2}{(1-x)^3} \qquad \Rightarrow f''(0) = 2$$

$$f'''(x) = \frac{-6}{(1-x)^4} \cdot (-1) = \frac{6}{(1-x)^4} \qquad \Rightarrow f'''(0) = 6$$

$$f^{(4)}(x) = \frac{-24}{(1-x)^5} \cdot (-1) = \frac{24}{(1-x)^5} \qquad \Rightarrow f^{(4)}(0) = 24$$

$$\therefore \frac{1}{1-x} = 1 + 1 \cdot x + \frac{2}{2!} \cdot x^2 + \frac{6}{3!} \cdot x^3 + \frac{24}{4!} \cdot x^4 + \cdots$$

$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

1. Given $f(x) = \ln(1+x)\,;\ \lvert x \rvert < 1$, find the Maclaurin's series expansion of $f(x)$ up to terms with x^4 **Answer:**	2. Given $f(x) = \tan^{-1} x$, find the Maclaurin's series expansion of $f(x)$ up to terms with x^3 **Answer:**
3. Given $f(x) = \tan x$, find the Maclaurin's series expansion of $f(x)$ up to terms with x^3 **Answer:**	4. Given $f(x) = \dfrac{1}{1-x^2}\,;\ \lvert x \rvert < 1$, find the Maclaurin's series expansion of $f(x)$ up to terms with x^2 **Answer:**

1. Obtain Maclaurin's series expansion of $f(x) = e^x$, and show that:

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Answer:

2. Obtain Maclaurin's series expansion of $f(x) = \ln(1+x)$, and show that:

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)}$$

Answer:

Topic 3: Maclaurin's Series (additional illustration)

Let $f(x)$ be a continuously differentiable function then the Maclaurin's series expansion of $f(x)$ is given by:

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2\,!} \cdot x^2 + \frac{f'''(0)}{3\,!} \cdot x^3 + \cdots$$

Illustration: Given $f(x) = \cos^2 x$, find the Maclaurin's series expansion of $f(x)$ up to the terms in x^4. Hint: use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Let $g(x) = \cos 2x \quad \Rightarrow \quad g(0) = 1$

$g'(x) = -2\sin 2x \quad \Rightarrow \quad g'(0) = 0$

$g''(x) = -4\cos 2x \quad \Rightarrow \quad g''(0) = -4$

$g'''(x) = 8\sin 2x \quad \Rightarrow \quad g'''(0) = 0$

$g^{(4)}(x) = 16\cos 2x \quad \Rightarrow \quad g^{(4)}(0) = 16$

$$g(x) = g(0) + g'(0) \cdot x + \frac{g''(0)}{2\,!} \cdot x^2 + \frac{g'''(0)}{3\,!} \cdot x^3 + \frac{g^{(4)}(0)}{4\,!} \cdot x^4 + \cdots$$

$$= 1 + 0 \cdot x + \frac{-4}{2\,!} \cdot x^2 + \frac{0}{3\,!} \cdot x^3 + \frac{16}{4\,!} \cdot x^4 + \cdots = 1 - 2x^2 + \frac{2}{3}x^4 + \cdots$$

$$f(x) = \cos^2 x = \frac{1}{2}(1 + g(x)) = 1 - x^2 + \frac{x^4}{3} + \cdots$$

1. Given $f(x) = x^2 \sin 2x$, find the Maclaurin's series expansion of $f(x)$, show the first 3 non-zero terms.

   Hint: find the Maclaurin's series of $g(x) = \sin 2x$, then $f(x) = x^2 \cdot g(x)$

Answer:

2. Given $f(x) = x \sin^2 2x$, find the Maclaurin's series expansion of $f(x)$, show the first 3 non-zero terms. Hint: use $\sin^2 2x = \dfrac{1}{2}(1 - \cos 4x)$

Answer:

Topic 4: Equations of Tangent and Normal Lines

The equation of a tangent line to the curve $y = f(x)$ at the point (x_1, y_1) is given by

$$y - y_1 = m\,(x - x_1)$$

where $m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$

The equation of a normal line to the curve $y = f(x)$ at the point (x_1, y_1) is given by

$$y - y_1 = n\,(x - x_1)$$

where $n = -\frac{1}{m}$

Illustration: The equation of a curve is given by $y = \frac{1}{x}$. Find the equation of the tangent line and the normal line to the curve at point (1,1).

$$\frac{dy}{dx} = -\frac{1}{x^2} \implies m = \left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{(1)^2} = -1$$

Equation of tangent line: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 1 = -1(x - 1)$$

$\therefore \quad y + x - 2 = 0$ Equation of the tangent line at $(1, 1)$

Equation of normal line: $y - y_1 = -\frac{1}{m}(x - x_1)$

$$\Rightarrow y - 1 = 1(x - 1)$$

$\therefore \quad y = x$ Equation of the normal line at $(1, 1)$

1. Find the equations of the tangent and the normal lines to the curve $x^2y + 3xy^2 = 2$ at point $(2, -1)$. **Answer:**	2. Find the equations of the tangent and the normal lines to the curve $y^2e^x + x^2 = 9$ at point $(0, 3)$. **Answer:**
3. Find the equation of the tangent to the curve $y = x^4 + 4x$ which is parallel to the line $y = 36x + 47$. **Answer:**	4. Find the equation of the tangent to the curve $y = x^2 + 4$ which is perpendicular to the line $3x - y + 1 = 0$. **Answer:**

Topic 5: Related Rates

Illustration: The area and circumference of a circle with radius r are given by πr^2 and $2\pi r$ respectively. Given that the area of the circle is decreasing at a rate of $0.5\,\text{cm}^2/\text{s}$, find the rate at which the circumference is decreasing when the radius is 2 cm.

Given information: Area of circle $A = \pi r^2$,

Circumference of circle $C = 2\pi r$,

$$\frac{dA}{dt} = -0.5\text{cm}^2/\text{s}.$$

To find: $\left.\dfrac{dC}{dt}\right|_{r=2\,\text{cm}} = ?$

$$C = 2\pi r \quad \Rightarrow \quad \frac{dC}{dt} = \frac{dC}{dr}\cdot\frac{dr}{dt} = 2\pi\frac{dr}{dt}$$

$$A = \pi r^2 \quad \Rightarrow \quad \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = \pi(2r)\cdot\frac{dr}{dt} \quad \Rightarrow \quad -0.5 = \pi(2r)\cdot\frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = -\frac{0.5}{2\pi r} = -\frac{1}{4\pi r}$$

$$\therefore \quad \frac{dC}{dt} = 2\pi\left(-\frac{1}{4\pi r}\right) = -\frac{1}{2r}$$

$$\left.\frac{dC}{dt}\right|_{r=2\,\text{cm}} = -\frac{1}{2(2)} = -\frac{1}{4}\,\text{cm}/\text{s}$$

1. The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone. If the height of the cone is increasing at the rate of $3\ \mathrm{cm/s}$ and radius of the base is not changing with time, find the rate at which its volume is increasing if the radius of the base is 5 cm.

Answer:

2. A spherical balloon is inflated by a machine which pumps-in air at a rate of $10\ \mathrm{cm}^3/\mathrm{s}$. Find the rate at which its radius is increasing when its radius is 10 cm.

Answer: