Lecture 7

Structure of lecture

1. Limits and Continuity
2. Derivatives
3. Derivatives of power, exponential, logarithmic, and trigonometric functions
4. Differentiation rules: sums, differences, products and quotients.

What is Calculus?

Calculus is the mathematical study of continuous change.

For instance, Calculus is the mathematics of velocities, accelerations, tangent lines, areas, volumes, and a variety of many other concepts that have enabled scientists, engineers, economists to model real life situations.

What is Calculus?

The development of Calculus was stimulated in large part by two geometric problems:

1) Finding tangent lines to curves

   Differential Calculus: based on derivatives

2) Finding areas of plane regions

   Integral Calculus: based on integrals

Both these problems are closely related to a fundamental concept of calculus known as a '**Limit**'.

The Limit of a Function

- Suppose $f(x)$ is defined when x is near the number a. This means f is defined on some open interval that contains a, except possibly at a itself.

- We write

$$\lim_{x \to a} f(x) = L$$

  and say "the **limit** of $f(x)$, as x approaches a, equals L".

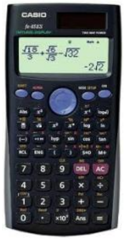
- We can make the values of $f(x)$ *arbitrarily close* to L, by restricting x to be *sufficiently close* to a, but not equal to a.

Estimating limits numerically

Example:

Estimate the limit numerically: $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

$\longrightarrow$ **2** $\longleftarrow$



x	1.9	1.99	1.999	2.001	2.01	2.1
Limit	3.9	3.99	3.999	4.001	4.01	4.1

$\longrightarrow$ **4** $\longleftarrow$

Thus, $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$

Estimating limits numerically

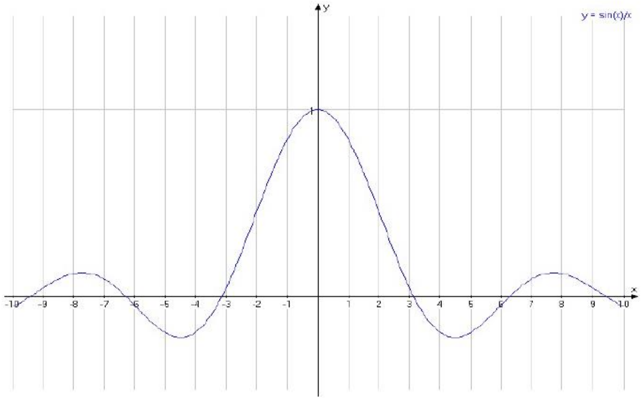
Example: Estimate the limit numerically.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	$f(x)$
-0.1	0.99833
-0.05	0.99958
-0.01	0.99998
-0.005	0.99999
0	?
0.005	0.99999
0.01	0.99998
0.05	0.99958
0.1	0.99833

From the table we see that the limit ≈ 1.

We can prove the limit is 1 later.

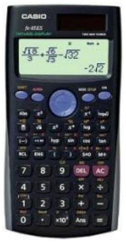


$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Estimating limits numerically

Example: Estimate the limit numerically.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$



$\longrightarrow \infty$

x	5	50	500	5000	50000	5000000
Limit	2.48832	2.691588029	2.715568521	2.71801005	2.718254646	2.718281557

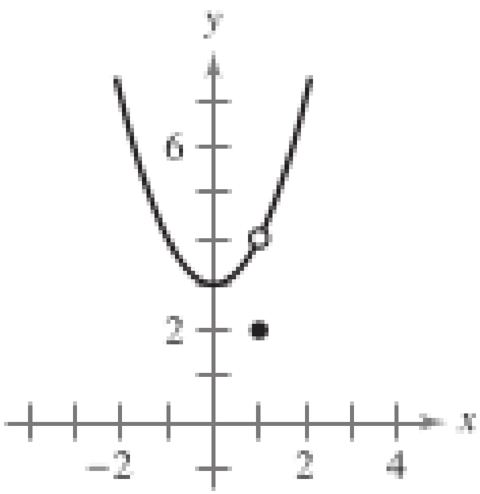
$\longrightarrow$ **?**

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx 2.71828$$

We can prove the limit is ***e*** later.

Estimating limits graphically

Example: Estimate the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ graphically.



$$\lim_{x\to 0} f(x) = 3$$

$$\lim_{x\to 1} f(x) = 4$$

$$f(1) = 2$$

Note: $f(x)$ is *continuous* at $x = 0$, and *discontinuous* at $x = 1$.

Continuity of a function

- A function $f(x)$ is **continuous at a point $\boldsymbol{x = a}$** if

$$\lim_{x \to a} f(x) = f(a)$$

- A function $f(x)$ is said to be **continuous over an interval** if it is continuous at very point on that interval.

- e.g. $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ and $(-\infty, 0)$, but not continuous on $(-\infty, \infty)$. Because $x = 0$ is a point of discontinuity of f.

Calculating limits algebraically

To find the limit: $\lim_{x \to a} f(x)$, we first try evaluating the function at a directly (direct substitution).

Case 1 If $f(a) = b$ and b is a real number,

then $\lim_{x \to a} f(x) = b$ (assuming f is continuous at a).

e.g. $\lim_{x \to 5} (x^2 + 1) = 5^2 + 1 = 26$

Case 2 If $f(a) = \frac{b}{0}$ and $b \neq 0$,

then $\lim_{x \to a} f(x) = \infty, -\infty$, or does not exist ($f$ probably has a vertical asymptote at $x = a$). *e.g.* $\lim_{x \to 2} \frac{1}{x-2}$ DNE.

Calculating limits algebraically

Case 3 If $f(a) = \frac{0}{0}$ (Indeterminate form):

Try rewrite $f(x)$ in equivalent form by factoring, multiplying by conjugates, using trigonometric identities, etc.

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3} \text{ (by factoring) } = \lim_{x \to -1} \frac{(x+1)(x-2)}{(x+1)(x-3)} = \lim_{x \to -1} \frac{x-2}{x-3} = \frac{3}{4}$$

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \text{ (multiply by conjugate) } = \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

$$\lim_{x \to 0} \frac{\sin x}{\sin 2x} \text{ (using a trig identity) } = \lim_{x \to 0} \frac{\sin x}{2 \sin x \cos x} = \lim_{x \to 0} \frac{1}{2 \cos x} = \frac{1}{2}$$

Calculating limits algebraically

Example: Find the limit: $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 1}$

Note: $x \to a$ means $(x - a)$ is one of the factor of $f(x)$.

$$= \lim_{x \to 1} \frac{\cancel{(x-1)} \cdot (x+2)}{\cancel{(x-1)} \cdot (x+1)}$$

$$= \lim_{x \to 1} \frac{(x+2)}{(x+1)} = \frac{3}{2}$$

Derivative (Slope, Instantaneous rate of change)

Many real-world phenomena involve changing quantities:

- In the speed of a rocket
- the inflation of currency
- the number of bacteria in a culture
- the voltage of an electrical signal

and so forth.

We introduce the concept of a 'derivative', (a mathematical tool for studying the rate at which one quantity changes relative to another); by noting that **the study of rates of change is closely related to the geometric concept of a tangent line to the curve**.

Slope of a line

Slope or **Gradient** of a line describes its steepness.

Slope (given two points on a line)

$$= \frac{\text{Difference of Y-coordinates}}{\text{Difference of X-coordinates}}$$

(taken in the same order)

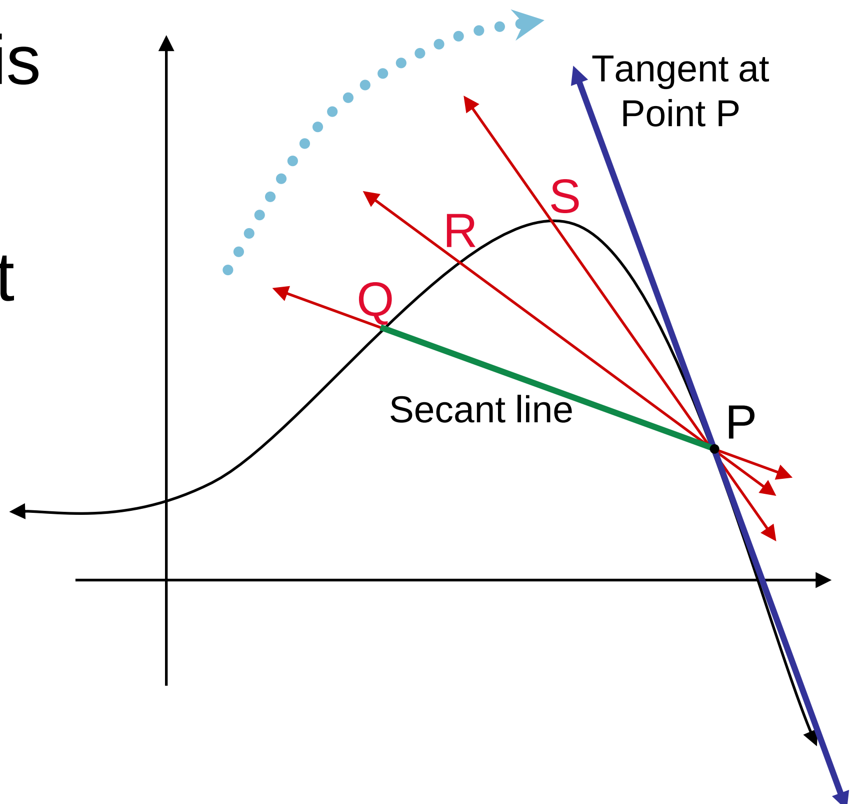
Tangent line

Tangent line (Tangent) to a curve at a given point is the straight line that *just touches* the curve at that point.

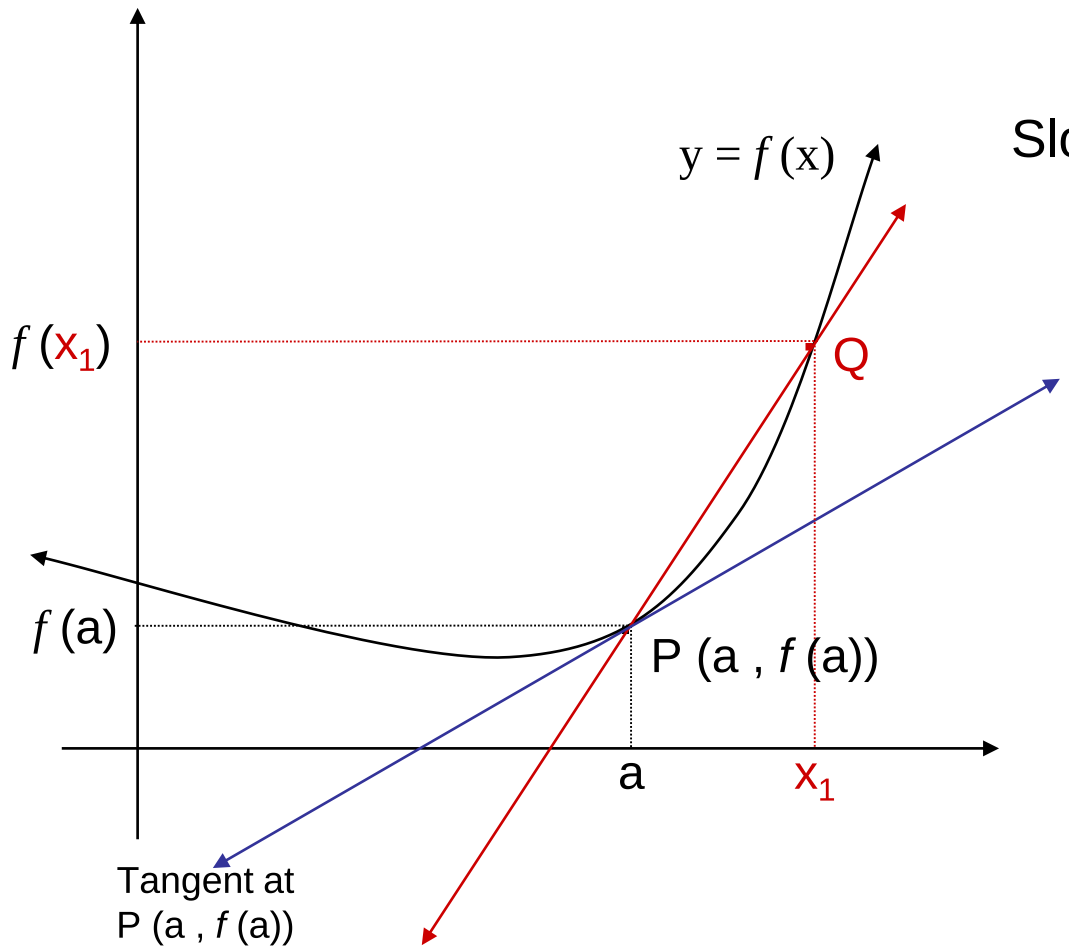
when $Q \to P$

Slope of tangent

$\approx$ Slope of secant line



Definition of derivative



Slope of tangent

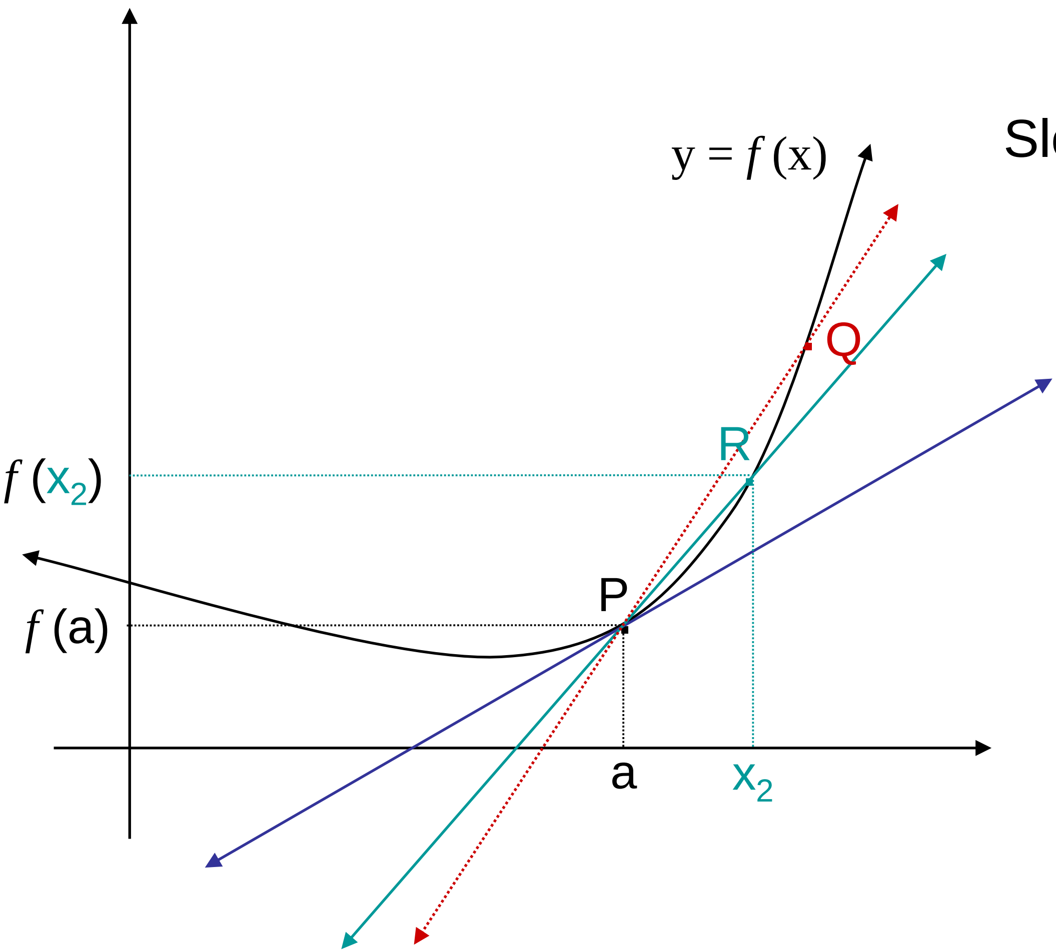
$\approx$ Slope of secant line

$$= \frac{\text{Difference of Y-coordinates}}{\text{Difference of X-coordinates}}$$

$$= \frac{f(x_1) - f(a)}{x_1 - a}$$

First approximation

Definition of derivative



Slope of tangent

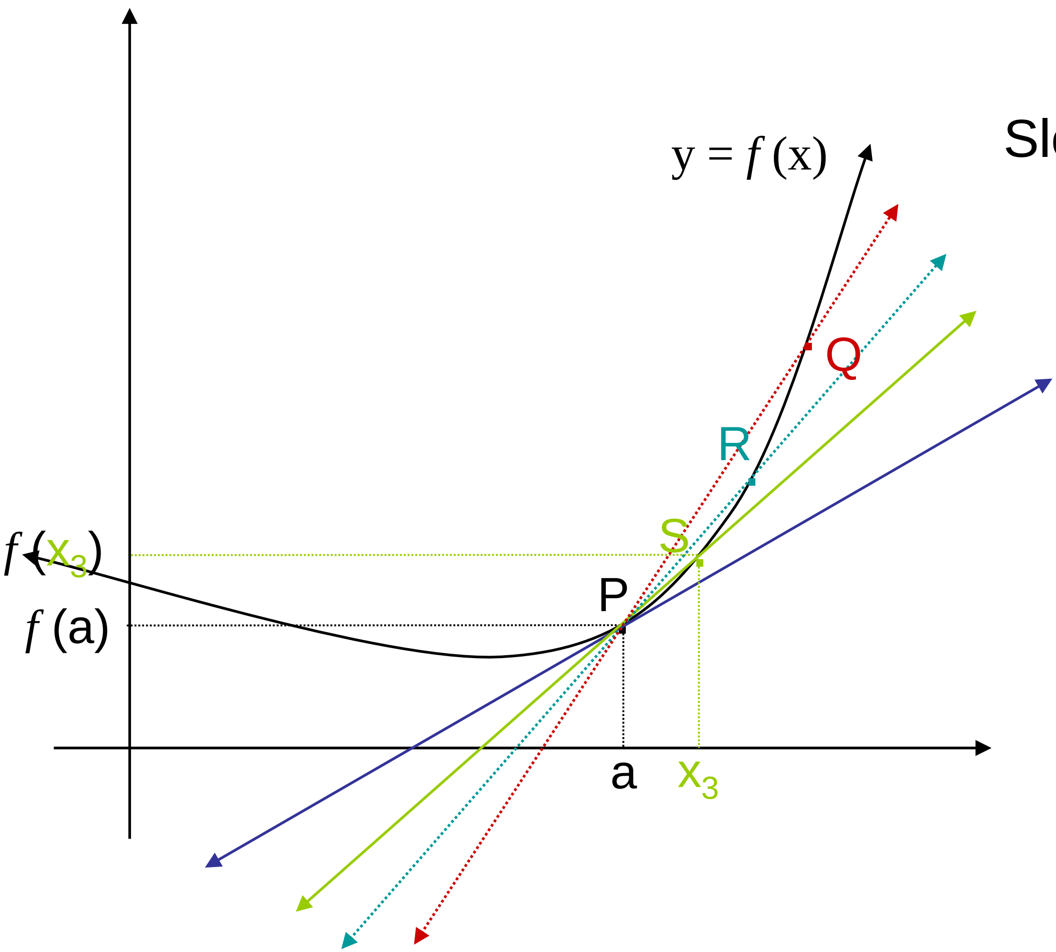
$\approx$ Slope of secant line

$= \dfrac{\text{Difference of Y-coordinates}}{\text{Difference of X-coordinates}}$

$$= \frac{f(x_2) - f(a)}{x_2 - a}$$

Second approximation

Definition of derivative



Slope of tangent

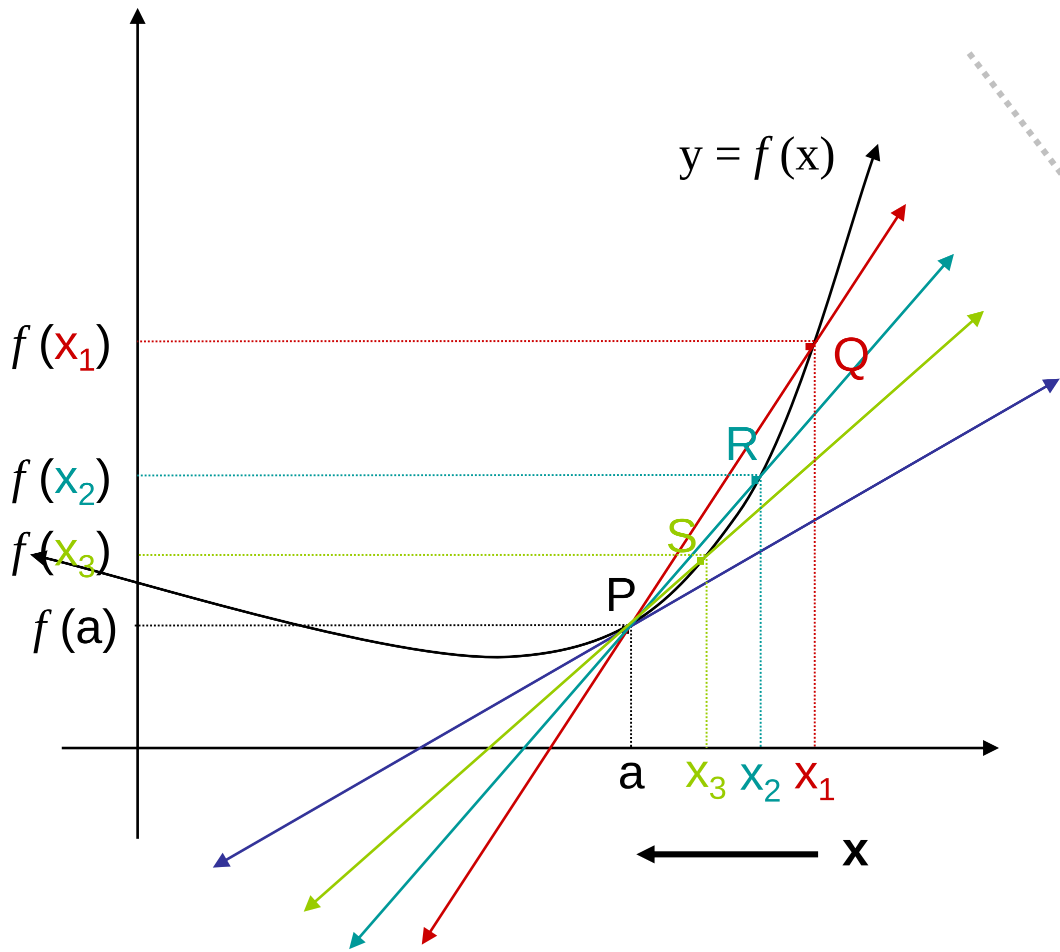
$\approx$ Slope of secant line

$= \dfrac{\text{Difference of Y-coordinates}}{\text{Difference of X-coordinates}}$

$$= \frac{f(x_3) - f(a)}{x_3 - a}$$

Third approximation

Definition of derivative



Observation

Secant lines $\rightarrow$ Tangent line

as

$x \rightarrow a$

$$\boxed{\frac{f(x_1) - f(a)}{x_1 - a}}$$

$$\boxed{\frac{f(x_2) - f(a)}{x_2 - a}}$$

$$\boxed{\frac{f(x_3) - f(a)}{x_3 - a}}$$

Better approxs. for Slope

Definition of derivative (at a point)

As $x \to a$, Secant line $\to$ Tangent line.

$$\therefore \text{ Slope of tangent line} \approx \text{Slope of secant line}$$

$$\therefore \text{ Slope of tangent line} = \lim_{x\to a} (\text{Slope of secant line})$$

$$i.e. \text{ Slope of tangent line } = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Definition of derivative (at a point)

The slope (gradient) of the tangent at point $P(a)$ on the curve $y = f(x)$ is defined as the derivative of y with respect to x (at point a).

$$\boxed{\left.\frac{dy}{dx}\right|_{(\text{at point } x=a)} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Definition of derivative function

Alternatively, we can define the derivative of $f(x)$ at point $x = a$ to be:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

And, the **derivative function** of $f(x)$ for any x in the domain of f is defined as:

$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Derivatives of power functions

1. Given $y = f(x) = x^2$, find $\dfrac{dy}{dx}$ by using the definition of derivative (**first principles**).

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}x + \cancel{h}^2}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

Derivatives of power functions

2. Given $y = f(x) = \dfrac{1}{x}$, find $\dfrac{dy}{dx}$ by using the definition of derivative.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cancel{x} - \cancel{x} - h}{x \cdot (x-h) \cdot h} \right)$$

$$= \lim_{h\to 0} \left(\frac{-\cancel{h}}{x \cdot (x-h) \cdot \cancel{h}} \right)$$

$$= -\frac{1}{x^2}$$

Derivatives of power functions

3. Given $y = f(x) = \sqrt{x}$, find $\dfrac{dy}{dx}$ by using the definition of derivative.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Power rule

The derivative of a power function: $y = f(x) = x^n$, where n is any real number, is given by the power rule:

$$\frac{d}{dx}(x^n) = f'(x) = nx^{n-1}$$

Derivatives of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{2 \cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right)}{h}$$

Derivatives of trigonometric functions

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right)\cdot \boxed{\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}$$

$$\boxed{\therefore \frac{d}{dx}(\sin x) = \cos x}$$

Derivatives of trigonometric functions

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x} \; ; \quad x \neq (2k+1)\frac{\pi}{2} \; ; \; k \in \mathbb{Z}$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x} \; ; \quad x \neq (2k+1)\frac{\pi}{2} \; ; \; k \in \mathbb{Z}$$

$$\boxed{\frac{d}{dx}(\mathrm{cosec} x) = -\mathrm{cosec} x \cot x} \; ; \quad x \neq k\pi \; ; \; k \in \mathbb{Z}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x} \; ; \quad x \neq k\pi \; ; \; k \in \mathbb{Z}$$

Derivatives of exponential functions

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h}$$

$$= e^x \cdot \lim_{h\to 0}\left(\frac{e^h-1}{h}\right)$$

$$\therefore \frac{d}{dx}(e^x) = e^x \cdot (1)$$

Thus, $\boxed{\frac{d}{dx}(e^x) = e^x}$

Estimate limit numerically to obtain 1, or use one of the definitions of e to find the limit algebraically.

Derivatives of exponential functions

Derivative of the exponential functions:

$$\frac{d}{dx}(e^x) = e^x$$

and

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x$$

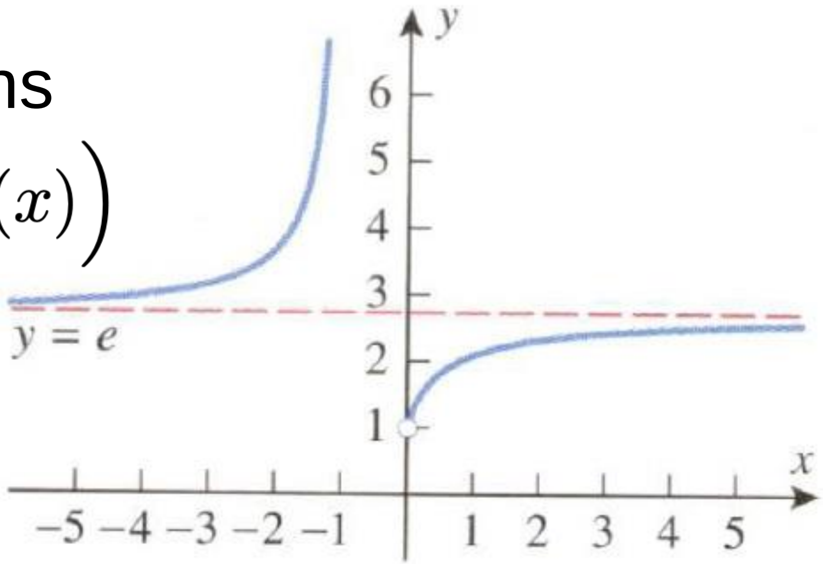
Derivative of logarithmic functions

Here is one way to find the formula of the derivative of $y = \log_e x = \ln x$. Firstly, we need to know:

(1) $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e \quad \textbf{or} \quad \lim_{x\to 0}\,(1+x)^{1/x} = e$$

(2) Limit of composite functions

$$\lim_{x\to a} f\left(g(x)\right) = f\left(\lim_{x\to a} g(x)\right)$$



Derivative of logarithmic functions

$$\begin{aligned}
\frac{d}{dx}(\log_e x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_e(x+h)-\log_e x}{h} \\
&= \lim_{h\to 0} \frac{\log_e\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{\log_e\left(1+\frac{h}{x}\right)}{h}
\end{aligned}$$

Derivative of logarithmic functions

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[\log_e \left(1 + \frac{h}{x} \right) \right]$$

$$= \lim_{h \to 0} \log_e \left(1 + \frac{h}{x} \right)^{1/h}$$

$$= \log_e \left[\lim_{h \to 0} \left(1 + \frac{h}{x} \right)^{x \, 1/h} \right]^{\frac{1}{x}} \text{ using (2)}$$

$$\frac{h}{x} \to 0$$

Derivative of Logarithmic functions

$$= \frac{1}{x} \cdot \log_e e \qquad \text{using (1)}$$

$$= \frac{1}{x}$$

$$\boxed{\therefore \frac{d}{dx}(\log_e x) = \frac{1}{x} \quad ; \quad x \in \mathbb{R}^+}$$

Derivatives of logarithmic functions

Derivative of the logarithmic functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

and

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a \cdot x}$$

Alternatively, we can use *implicit differentiation* (w2) to derive the formula.

Rules of differentiation

Differentiation is the process of finding the derivatives.

The SUM Rule

If $u = f(x)$ and $v = g(x)$ are differentiable functions of x, then

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}}$$

The DIFFERENCE Rule

$$\boxed{\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}}$$

Rules of differentiation

Examples:

1 Given $y = x^3 + \sin x - e^x + \ln x$ find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\sin x\right) - \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(\ln x\right).$$

$$\therefore \frac{dy}{dx} = 3\,x^2 + \cos x - e^x + \frac{1}{x}.$$

Rules of differentiation

Examples:

2 Given $y = (x^2 - 1) \cdot (x^2 + 1)$ find $\dfrac{dy}{dx}$.

$$\text{Here } \; y = x^4 - 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}(1)$$

$$\Rightarrow\; \frac{dy}{dx} = 4x^3 - 0 \;= 4x^3$$

Rules of differentiation

The PRODUCT Rule

If u and v are differentiable functions of x then,

$$\boxed{\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}$$

Example: Given $y = x \cdot \sin x$ find $\dfrac{dy}{dx}$.

$$y = x \cdot \sin x \;\Rightarrow\; \frac{dy}{dx} = x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x)$$

$$= x \cos x + \sin x \cdot (1) = x \cos x + \sin x$$

Rules of differentiation

Extension of the Product Rule

If $u = f(x), v = g(x),$ and $w = h(x)$ are differentiable functions of $x,$ then

$$\frac{d}{dx}(u \cdot v \cdot w) = \boxed{u\, v \cdot \frac{dw}{dx}} + \boxed{v\, w \cdot \frac{du}{dx}} + \boxed{u\, w \cdot \frac{dv}{dx}}$$

Keep 2 functions fixed, take the derivative of the 3rd

Rules of differentiation

Example: Find $\dfrac{d}{dx}\left(x\,e^x\ \cot x\right)$.

$$\frac{d}{dx}\left(x\,e^x\ \cot x\right)$$

$$= \boxed{x\,e^x \cdot \frac{d}{dx}(\cot x)} + \boxed{e^x\ \cot x \cdot \frac{d}{dx}(x)} + \boxed{x\ \cot x \cdot \frac{d}{dx}(e^x)}$$

Keep 2 functions fixed, take the derivative of the 3rd

$$= x\,e^x\left(-\mathrm{cosec}^2 x\right) + e^x\ \cot x\,(1) + x\ \cot x\,(e^x)$$

$$= e^x\ \left(-\,x\,\mathrm{cosec}^2 x + \cot x + x\ \cot x\right)$$

Rules of differentiation

The QUOTIENT Rule

If u and v are differentiable functions of x then,

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}$$

Example: Given $y = \dfrac{x}{\sin x}$ find $\dfrac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{\sin x \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{\sin x - x \cdot \cos x}{\sin^2 x}$$

Rules of differentiation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Example: Find $\dfrac{d}{dx}\left(\dfrac{x^2-1}{x^2+1}\right)$.

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\boxed{(x^2+1)\cdot\frac{d}{dx}(x^2-1)} - \boxed{(x^2-1)\cdot\frac{d}{dx}(x^2+1)}}{\boxed{(x^2+1)^2}}$$

$$= \frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{2x\left(x^2+1-x^2+1\right)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

Thank You!