Lecture 3

Topics covered in this lecture session

1. Remainder and Factor Theorems.
2. Polynomial Division.
3. Polynomial Factorisation.
4. Partial Fraction

Division Process (for numbers)

Example $26 \div 3$

$$\frac{26}{3} \Rightarrow \begin{array}{r|l} & 8 \\ \hline 3 & 26 \\ & 24 \\ \hline & 2 \end{array}$$

8 → Quotient

3 → Divisor

2 → Remainder

$$\boxed{\therefore \frac{26}{3} = 8 + \frac{2}{3} = Quotient + \frac{Remainder}{Divisor}}$$

Division of polynomials (Analogous result)

$$\text{e.g.}\quad p(x) \div s(x) \text{ where } s(x) \neq 0$$

$$\frac{p(x)}{s(x)} \Rightarrow s(x)\overline{\smash{)}\,p(x)\,}$$

$q(x)$ → Quotient

Poly 2

$r(x)$ → Remainder

$s(x)$ → Divisor

$r(x)$ is either a constant

OR

$\deg(r(x)) < \deg(s(x))$

Division of polynomials

Thus, $\dfrac{p(x)}{s(x)} = q(x) + \dfrac{r(x)}{s(x)} \quad \Rightarrow \quad p(x) = s(x)\, q(x) + r(x)$

where, $q(x)$ is the quotient, and

$r(x)$ is the remainder - which is either a constant (r)
or $\deg(r(x)) < \deg(s(x))$.

In particular, when $p(x)$ is divided by $(x - c)$, the remainder must be some constant r.

Remainder Theorem

$$\text{i.e. } \frac{p(x)}{(x-c)} = q(x) + \frac{r}{(x-c)}$$

$$\Rightarrow \quad p(x) = (x-c)\,q(x) + r$$

$$\Rightarrow \quad p(c) = r$$

Remainder Theorem

If a polynomial $p(x)$ is divided by $(x-c)$, then the remainder is $p(c)$.

Example

If $x^2 - 7x + k$ has a remainder 1 when divided by $(x+1)$, find k.

Solution: $(x+1) \equiv (x-c) \Rightarrow c = -1$

By Remainder Theorem, $p(c) = r$

$$\Rightarrow \; p(-1) = 1$$

$$\Rightarrow \; (-1)^2 - 7(-1) + k = 1$$

$$\Rightarrow \; k + 8 = 1 \quad \Rightarrow \quad k = -7.$$

Factor Theorem

Factorising a polynomial $p(x)$ means to write it as a product of lower-degree polynomials - called factors of $p(x)$.

For $s(x)$ to be a factor of $p(x)$, there must be no remainder when $p(x)$ is divided by $s(x)$.

i.e. $\dfrac{p(x)}{s(x)} = q(x) + \dfrac{0}{s(x)}$ **or** $p(x) = s(x)q(x) + 0$

Factor Theorem

In particular, when $(x-c)$ is a factor of the polynomial $p(x)$, $p(x)$ can be expressed as

$$p(x) = (x-c)\,q(x) \quad \text{i.e.} \quad p(c) = 0.$$

Factor Theorem

A polynomial $p(x)$ has a factor $(x-c)$, if any only if $p(c) = 0$.

Note: $p(c) = r$ is the Remainder Theorem

$p(c) = 0$ is the Factor Theorem

Example

If $(x-2)$ is a factor of $ax^2 - 12x + 4,$ find $a.$

Solution: Here, $(x-c) = (x-2) \Rightarrow c = 2$

By Factor theorem, $p(c) = 0.$

$$\Rightarrow p(2) = 0 \Rightarrow a\,(2)^2 - 12\,(2) + 4 = 0$$

$$\Rightarrow 4a - 24 + 4 = 0$$

$$\Rightarrow 4a = 20. \quad \Rightarrow a = 5.$$

What is the remainder when $x^2 + 5x - 6$ is divided by $(x + 1)$

A. 0

B. $\boxed{-10}$

C. 5

Given that $(\mathrm{x} - \mathrm{a})$ is a factor of $\frac{1}{a}(x^2) + 2x - 9$, find a.

A. $\boxed{3}$

B. -3

C. -3

Polynomial Division

1. Method of Long Division (or actual division)

The process of long division for dividing polynomials is similar to that of division of numbers.

Suppose, we want to determine

$$\frac{27x^3+9x^2-3x-10}{3x-2}$$

$$\begin{array}{r|l}
 & 9x^2 + 9x + 5 \\ \hline
3x-2 & 27x^3 + 9x^2 - 3x - 10 \\
 & 27x^3 - 18x^2 \\ \hline
 & \qquad 27x^2 - 3x \\
 & \qquad 27x^2 - 18x \\ \hline
 & \qquad\qquad 15x - 10 \\
 & \qquad\qquad 15x - 10 \\ \hline
 & \qquad\qquad\qquad 0
\end{array}$$

$$\text{Thus, } \frac{27x^3+9x^2-3x-10}{3x-2} = 9x^2+9x+5$$

Polynomial Division

2. Method of Synthetic Division

The method of Synthetic Division is a powerful alternative to the Method of Long Division.

We study this method only for linear divisors of the form $(x-c)$.

To understand the method, let us consider the example:

Example: If $\dfrac{x^3-9x^2-20}{(x-3)} = q(x) + \dfrac{r(x)}{(x-3)}$, find $q(x)$ and $r(x)$.

Method of Synthetic Division

Step 1

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ & & & & \\ \hline & & & & \end{array}$$

Write the coefficients of the polynomial to be divided at the top. Put zero as coefficient for unseen power(s) of x.

Step 2

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & & & & \\ \hline & & & & \end{array}$$

$$\frac{x^3-9x^2-20}{x-3}$$

Negate the constant term in the divisor, and write-in on the left side, that is, if $(x-a)$ is the divisor, write a on the left side.

Method of Synthetic Division

Step 3

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Drop the first coefficient after the bar to the last row.

Step 4

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & & \\ \hline & 1 & & & \end{array}$$

Multiply the dropped number with the number before the bar, and place it in the next column.

Method of Synthetic Division

Step 5

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & & \\ \hline & 1 & -6 & & \end{array}$$

Perform addition in the next column.

Repeat the previous two steps to obtain the following.

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & -18 & -54 \\ \hline & 1 & -6 & -18 & \mathbf{-74} \end{array}$$

Thus,

$$\frac{x^3 - 9x^2 - 20}{(x-3)} = (x^2 - 6x - 18) + \frac{-74}{(x-3)}$$

Factorising Polynomials

(with at least one integer zero)

Result:

Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$ be a polynomial with integer coefficients. Then, r is an integer zero of $p(x)$, if r is a divisor of the constant term c_0.

Examples:

Factorise $p(x) = x^3 - 3x^2 - 13x + 15$ completely into linear factors.

Factorising Polynomials
(with at least one integer zero)

Example: Solve the cubic equation: $x^3 + 3x^2 - 7x - 21 = 0.$

Possible zeros are $\pm 1, \pm 3, \pm 7, \pm 21$.

$p(1) = -24 \Rightarrow p(x) \neq 0 \ \therefore\ 1$ is not a zero of $p(x)$

$p(-1) = -12 \Rightarrow p(x) \neq 0 \ \therefore\ -1$ is not a zero of $p(x)$

$p(3) = 12 \Rightarrow p(x) \neq 0 \ \therefore\ 3$ is not a zero of $p(x)$

$p(-3) = 0 \Rightarrow p(x) \neq 0 \ \therefore\ -3$ is a zero of $p(x)$

$\Rightarrow (x - (-3)) = (x + 3)$ is one of the factors of $p(x)$

Use the method of synthetic division to find the other factor.

Factorising Polynomials (with at least one integer zero)

Example: Solve the cubic equation: $x^3 + 3x^2 - 7x - 21 = 0.$

Here s$(x) = x + 3 = x - c \Rightarrow c = \boxed{-3}$

$$\begin{array}{c|cccc} & 1 & 3 & -7 & -21 \\ \boxed{-3} & \downarrow & -3 & 0 & 21 \\ \hline & \boxed{1 \quad 0 \quad -7} & & & \boxed{0} \end{array}$$

Thus, the other factor is $\boxed{x^2 - 7}$

$$\therefore\ p(x) = (x+3)\cdot(x^2-7) = (x+3)\cdot(x+\sqrt{7})\cdot(x-\sqrt{7})$$

$$\therefore\ p(x) = 0\ \Rightarrow (x+3)\cdot(x^2-7) = (x+3)\cdot(x+\sqrt{7})\cdot(x-\sqrt{7}) = 0$$

$\therefore\ x = -3,\ x = \sqrt{7},$ or $x = -\sqrt{7}$

Partial Fractions

Process: Simplifying algebraic fractions

$$\frac{1}{(x+1)} - \frac{1}{(x+2)}$$

$$= \frac{(x+2)-(x+1)}{(x+1)(x+2)}$$

$$= \frac{1}{(x^2+3x+2)}$$

Process: Finding partial fractions for a given expression

Partial Fractions

Thus, in the method of partial fraction, we decompose

a rational fraction

$$f(x) = \frac{p(x)}{q(x)} \quad ; \quad q(x) \neq 0,$$

where $p(x)$ and $q(x)$ are polynomials,

as a sum of several fractions with a

simpler denominator.

$$\frac{1}{(x^2+3x+2)}$$

$$\Downarrow$$

$$\frac{1}{(x+1)} - \frac{1}{(x+2)}$$

Partial Fractions

The method is applicable when the following conditions are satisfied.

a) $\deg[\, p(x)\,] < \deg[\, q(x)\,]$,

b) The expression in the denominator is factorable.

e.g. $$\frac{2x+3}{x^2+3x+2} \qquad \therefore \deg[p(x)] = 1 < 2 = \deg[q(x)]$$

$$\frac{3x+1}{(x-1)^2\,(x+2)} \qquad \therefore \deg[p(x)] = 1 < 3 = \deg[q(x)]$$

Forms of Partial Fractions

1. Non-repeated Linear Factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

e.g. $$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Forms of Partial Fractions

2. Non-repeated Quadratic Factors

$$\frac{1}{(x+a)(x^2+b)} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+b)}$$

$$\frac{1}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

e.g. $$\frac{13}{(x^2+1)\,(2x+3)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(2x+3)}$$

Forms of Partial Fractions

3. Repeated Linear Factors

$$\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$$

$$\frac{1}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

e.g. $$\frac{x}{(x-3)^2(2x+1)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(2x+1)}$$

Partial Fractions - The method

Non repeated linear factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\Rightarrow \; A\,(x+b) + B\,(x+a) = 1$$

Put $x = -a$ to find the value of A

and then

put $x = -b$ to find the value of B.

Non repeated quadratic factor

$$\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$

$$\Rightarrow \; (Ax+B)\,(x+b) + C\,(x^2+a) = 1$$

Put $x = -b$ to find the value of C

and then

equate the terms in x^2 or x

or constants, to find A and B.

Partial Fractions - The method

Step 1:

Express the given rational function of the form $\frac{p(x)}{q(x)}$ as a sum of partial fractions with constants A and B (and C).

Step 2:

Find the constants A and B (and C) as explained earlier.

Step 3:

Finally, write the given expression as a sum of partial fractions with obtained values of constants A and B (and C).

Examples: Partial Fractions

Express the following as a sum of partial fractions.

1. $\dfrac{2x}{(x-1)(x-3)}$

2. $\dfrac{1}{(x^2+1)(x-1)}$

3. $\dfrac{2x}{(x-1)(x+2)^2}$

THANKS FOR YOUR ATTENTION