Lecture 6

Topics covered in this lecture session

1. Complex numbers - Introduction
   - Algebra of complex numbers.
   - Square root of a complex number.
2. Polar form of a complex number.
3. Algebraic operations on Argand diagram.

Mid-Semester Examination

Weighting	30%
Date	13 Nov. 2024 (Wednesday)
Time	15:00 – 16:00
Duration	1 hour
Type	20 short answer questions
Topics	Lecture and Seminar 1 - 5
Venue	Check email from CPSO

Bring your calculator! (Casio – fx82 family)

Complex Numbers - Introduction

In solving quadratic equations $ax^2 + bx + c = 0$ using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if the discriminant $\Delta < 0$, then no real root exist.

With Complex Numbers, we can explore further into the possibility of finding roots even when $\Delta = b^2 - 4ac < 0.$

For that, we first define *Imaginary Numbers*.

Complex Numbers - Introduction

An imaginary number is the one whose square is a negative real number.

e.g. $\sqrt{-1}, \quad \sqrt{-7}, \quad \sqrt{-8}, \quad \sqrt{-25}, \quad \sqrt{-1.21},$ etc.

are all imaginary numbers, because their squares

$-1, -7, -8, -25, -1.1$ are all negative real numbers.

We use the notation $\sqrt{-1} = i$ to represent imaginary numbers. e.g. $\sqrt{-7} = \sqrt{7}\,i, \;\; \sqrt{-25} = 5\,i,$ and so on.

Complex Numbers - Introduction

Note: $i = \sqrt{-1} \Rightarrow i^2 = -1.$

$\begin{aligned} i^3 &= i^2 \cdot i \\ &= (-1) \cdot i \\ &= -i \end{aligned}$	$\begin{aligned} i^4 &= i^2 \cdot i^2 \\ &= (-1) \cdot (-1) \\ &= 1 \end{aligned}$
$\begin{aligned} i^5 &= i^4 \cdot i \\ &= (1) \cdot i \\ &= i \end{aligned}$	$\begin{aligned} \frac{1}{i} = \frac{i}{i^2} &= \frac{i}{-1} \\ &= -i \end{aligned}$

Imaginary numbers

Using the notation $i = \sqrt{-1}$, it is now possible to solve quadratic equations with negative discriminants.

e.g. $x^2 - 2x + 2 = 0 \;\Rightarrow\; x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2}$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\,i}{2}$$

$$= 1 \pm i$$

$$= \boxed{(1) \pm i \cdot (1)}$$

Form: $a + i\,b$ where $a, b \in \mathbb{R}$ and $i^2 = -1.$

Complex Numbers

A Complex Number is of the form: $a + i\,b$ where $a, b \in \mathbb{R}$ and $i^2 = -1.$

a is called the Real part of the complex number z and is denoted by $Re(z).$

b is called the Imaginary part of the complex number z and is denoted by $Im(z).$

Thus, $\boxed{z = Re(z) + i\,Im(z).}$

Algebra of Complex Numbers

1. Equality

Two complex numbers are equal if and only if their real and imaginary parts are equal.

i.e. $z_1 = x_1 + i\, y_1 \quad \text{and} \quad z_2 = x_2 + i\, y_2 \quad \text{are equal}$

$$\Leftrightarrow \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

Algebra of Complex Numbers

2. Addition and Subtraction

For two complex numbers z_1 and z_2, the operations of addition and subtraction are defined by

<u>Addition</u>

$$\begin{aligned} z_1 + z_2 &= (x_1 + i\,y_1) + (x_2 + i\,y_2) \\ &= (x_1 + x_2) + i\,(y_1 + y_2) \end{aligned}$$

<u>Subtraction</u>

$$\begin{aligned} z_1 - z_2 &= (x_1 + i\,y_1) - (x_2 + i\,y_2) \\ &= (x_1 - x_2) + i\,(y_1 - y_2) \end{aligned}$$

Algebra of Complex Numbers

3. Multiplication

Multiplication of complex numbers is carried out in a similar way to expanding brackets, and then replacing i^2 by -1.

Multiplication

$$
\begin{aligned}
z_1 \cdot z_2 &= (x_1 + i\,y_1)\cdot(x_2 + i\,y_2) \\
&= x_1\,x_2 + i\,x_1\,y_2 + i\,x_2\,y_1 + i^2\,y_1\,y_2 \\
&= (x_1\,x_2 - y_1\,y_2) + i\,(x_1\,y_2 + x_2\,y_1)
\end{aligned}
$$

Algebra of Complex Numbers

4. Division

To define division of complex numbers, we first need to define the conjugate complex number.

Conjugate Complex Number

For $z = a + i\,b$, the conjugate complex number, denoted by $\overline{z}$, is defined by $\overline{z} = a - i\,b$.

Clearly, $\overline{\overline{z}} = \overline{a - i\,b} = a + i\,b = z \;\Rightarrow$

z and $\overline{z}$ are conjugates of each other.

Algebra of Complex Numbers

Division

$$\frac{z_1}{z_2} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2} = \left(\frac{x_1 + i\,y_1}{x_2 + i\,y_2}\right) \cdot \left(\frac{x_2 - i\,y_2}{x_2 - i\,y_2}\right)$$

(Multiply and Divide by the Conjugate of the Denominator)

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 - i^2\,y_2^2}$$

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 + y_2^2} \quad (\because\ i^2 = -1)$$

$$= \left(\frac{x_1\,x_2 + y_1\,y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2\,y_1 - x_1\,y_2}{x_2^2 + y_2^2}\right)$$

Square root of a complex number

Example: Find $\sqrt{5-12\,i}$

Suppose $\sqrt{5-12\,i} = a + i\,b$

$$\Rightarrow 5 - 12\,i = a^2 + 2\,i\,ab + i^2\,b^2$$

$$\Rightarrow 5 - 12\,i = (a^2 - b^2) + i\,(2ab)$$

Equating real and imaginary parts $\Rightarrow a^2 - b^2 = 5 \text{ and } 2ab = -12$

which upon solving gives: $a = 3,\ b = -2 \quad \textbf{or} \quad a = -3,\ b = 2$

Thus, $\sqrt{5-12\,i} \ = \ 3 - 2\,i \qquad \textbf{or} \qquad -3 + 2\,i$

Argand diagram

A complex number can be represented on the Argand diagram; where

- Real numbers are represented on the X-axis (called real axis);

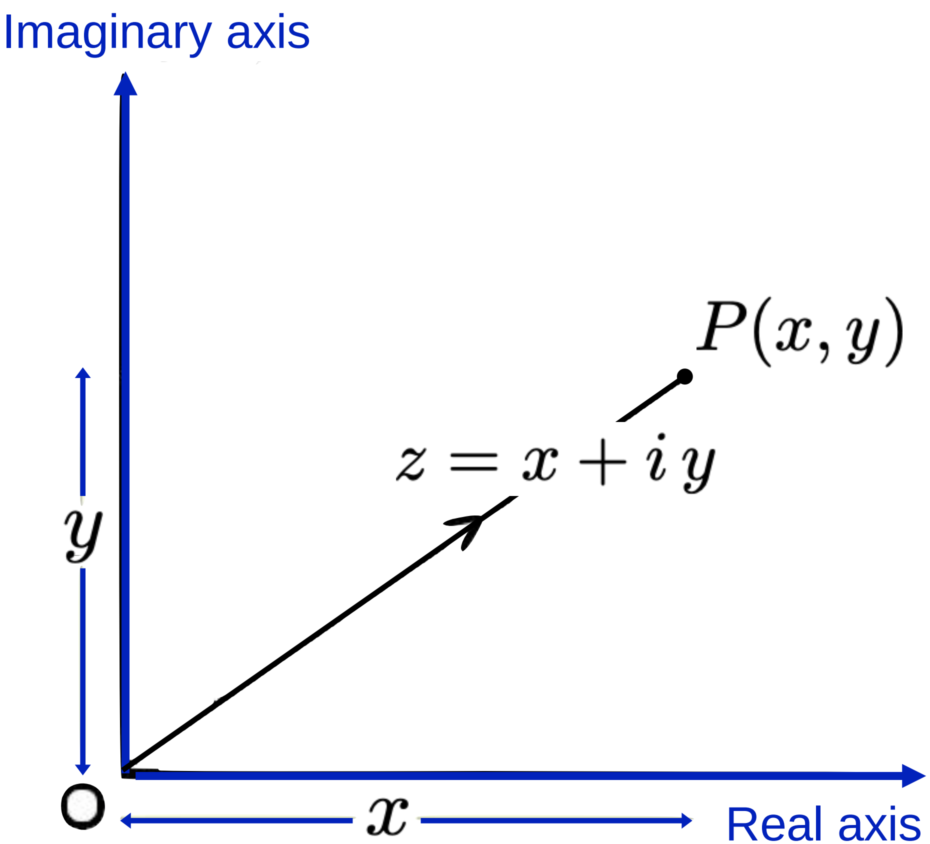
and

- Imaginary numbers are represented on the Y–axis (called imaginary axis).

Argand diagram

Thus, a general complex number $z = x + i\,y$ is represented by the vector $\overrightarrow{OP}$ where $P(x, y)$ is the point (x, y) in the XY-plane

(called the Argand plane or complex plane).

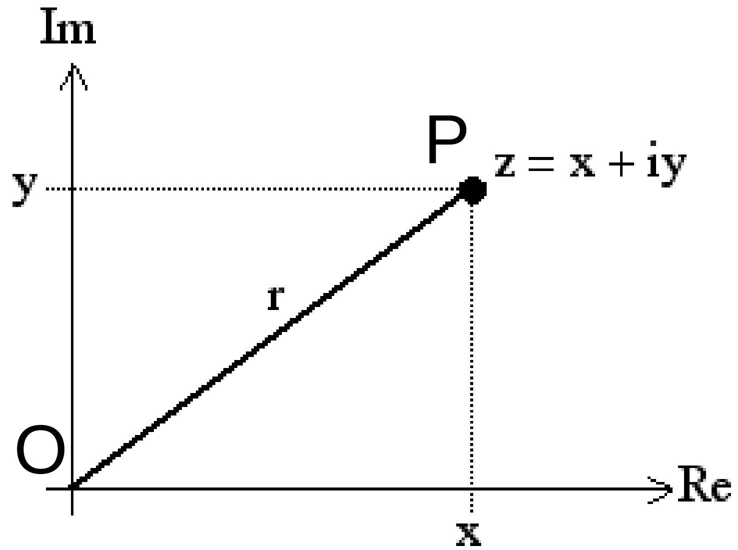


Modulus of a complex number

The length of $\overline{OP}$ is called the modulus of the complex number $z = x + i\,y$ and is denoted by:

$$r = |\, z \,| = |\, x + i\,y \,|$$

$$= \sqrt{x^2 + y^2}$$



i.e. $|\, z \,| = \sqrt{[Re(z)]^2 + [Im(z)]^2}$

Properties of Modulus

1) $$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

2) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

3) $$|z_1 + z_2| \leq |z_1| + |z_2|$$

4) $$|z_1 - z_2| \geq |z_1| \sim |z_2|$$ ($\sim$ denotes positive difference)

Worked Examples

1) Find $|-4+7\,i\,|$

$$|-4+7\,i\,| = \sqrt{(-4)^2+(7)^2} = \sqrt{65}$$

2) Find $\left|\dfrac{2-3\,i}{4+\sqrt{2}\,i}\right|$

$$\left|\frac{2-3\,i}{4+\sqrt{2}\,i}\right| = \frac{|\,2-3\,i\,|}{|\,4+\sqrt{2}\,i\,|} = \frac{\sqrt{2^2+(-3)^2}}{\sqrt{4^2+(\sqrt{2})^2}} = \sqrt{\frac{13}{18}}$$

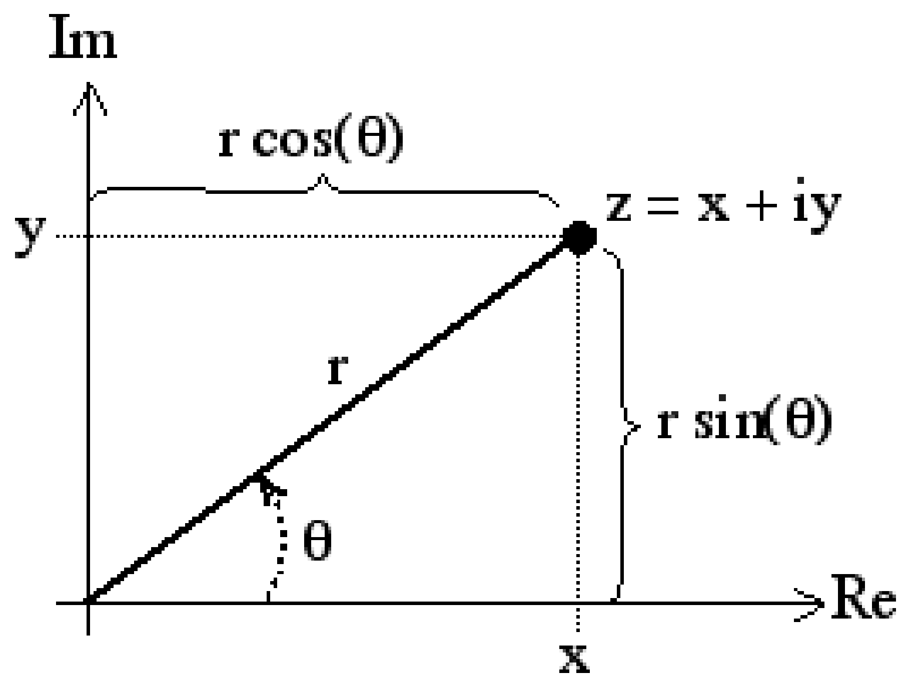
Polar form of a complex number

There is another way of representing a complex number using Polar coordinates (r, θ), and is called the Polar form of a complex number.

Suppose, the complex number $z = x + i\,y$ is represented in Cartesian form on the Argand diagram, by the point $P(x, y)$.

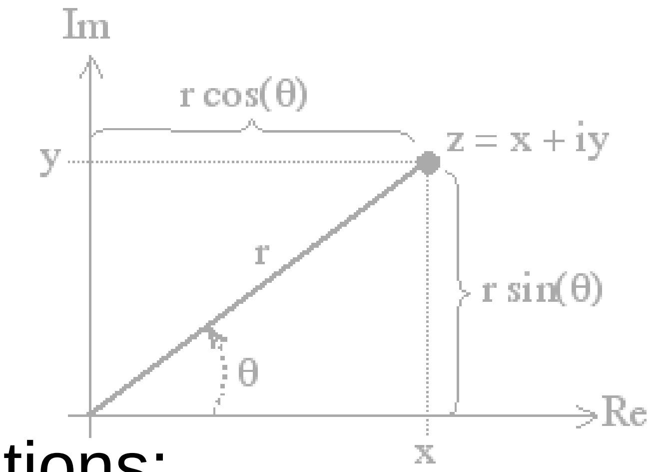
Polar form of a complex number

The same point P can be located by using its distance r from the origin O, and the angle θ made by the line $\overrightarrow{OP}$ with the real axis (X-axis).



Polar form of a complex number

Thus, $P(x, y)$ becomes the point



$$P(r, \theta) \equiv (r\cos\theta, r\sin\theta)$$

where, $r = \sqrt{x^2 + y^2}$

and θ is found from the set of equations:

$$\cos\theta = \frac{x}{r} \quad ; \quad \sin\theta = \frac{y}{r}.$$

Thus, $z = x + i\,y = r\cos\theta + i\,r\sin\theta$

Argument of a complex number

The angle θ is called the argument of the complex number

$$z = x + i\,y = r\cos\theta + i\,r\sin\theta$$

It is written as $Arg(x + i\,y)$, and obtained from the set of equations: $\cos\theta = \dfrac{x}{r}\quad;\quad \sin\theta = \dfrac{y}{r}$.

As there are infinite number of angles that satisfy the above set of equations, the definition needs to be tightened so that everyone gets the same answer.

(Principal) Argument of a complex number

We denote the principal value of the argument by

$$\arg(z) = \theta \qquad \text{if} \quad \underline{-\pi < \theta \leq \pi}.$$

Example

Express the following complex numbers in polar form and show them on the Argand diagram:

$(i) \quad z_1 = 1 + i$

$(ii) \quad z_2 = -1 + i$

$(iii) \quad z_3 = -1 - i$

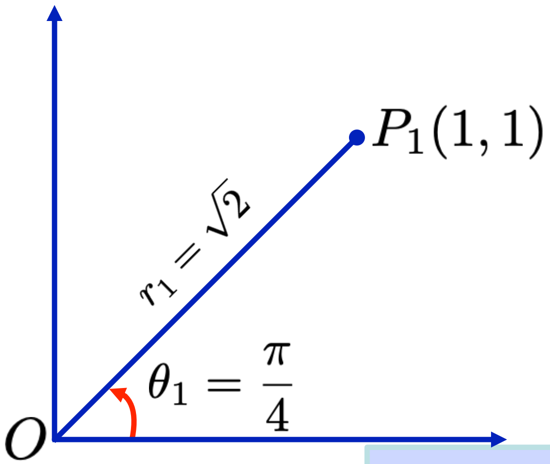
$(iv) \quad z_4 = 1 - i$

(Principal) Argument of a complex number

$$(i) \quad z_1 = 1 + i \;\equiv\; x + i\,y \;\Rightarrow\; x = 1,\; y = 1$$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow\; \theta = \frac{\pi}{4}$$



$$-\pi < \theta \leq \pi$$

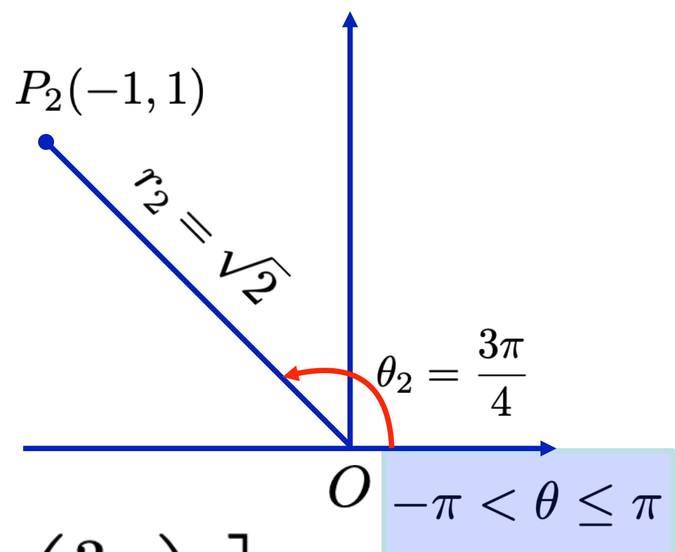
Thus, $z_1 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(ii) \quad z_2 = -1 + i \equiv x + i\,y \Rightarrow x = -1,\ y = 1$$

$$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{-1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = \frac{3\pi}{4}$$



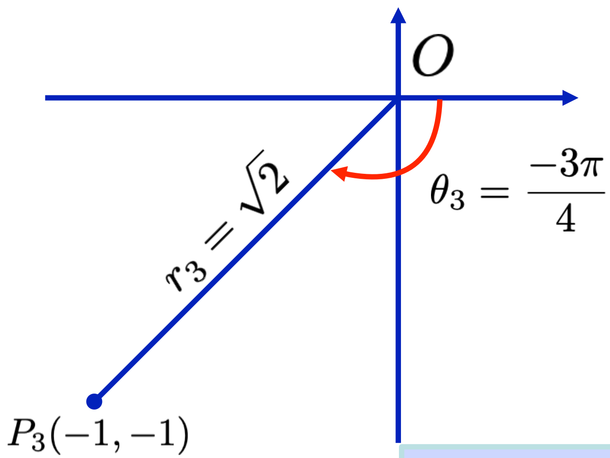
Thus, $z_2 = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(iii) \quad z_3 = -1 - i \equiv x + i\,y \Rightarrow x = -1,\ y = -1$$

$$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{-1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{-1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = \frac{-3\pi}{4}$$



$-\pi < \theta \leq \pi$

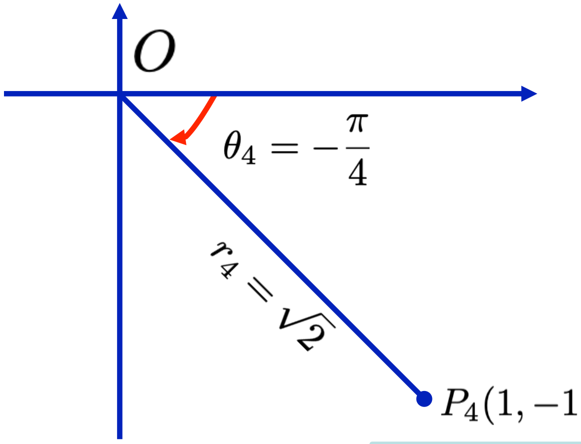
Thus, $z_3 = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\,\sin\left(\frac{-3\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(iv) \quad z_4 = 1 - i \equiv x + i\,y \Rightarrow x = 1,\ y = -1$$

$$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{-1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = -\frac{\pi}{4}$$



$$\boxed{-\pi < \theta \le \pi}$$

Thus, $z_4 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

Finding arg(z) using a calculator

Quadrant	First	Second	Third	Fourth
Interval	$\left(0, \frac{\pi}{2}\right)$	$(0, \pi)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Signs of x **and** y	$x > 0,\ y > 0$	$x < 0,\ y > 0$	$x < 0,\ y < 0$	$x > 0,\ y < 0$
Principal argument $\theta = \arg(z)$	$\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$\pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$

Example

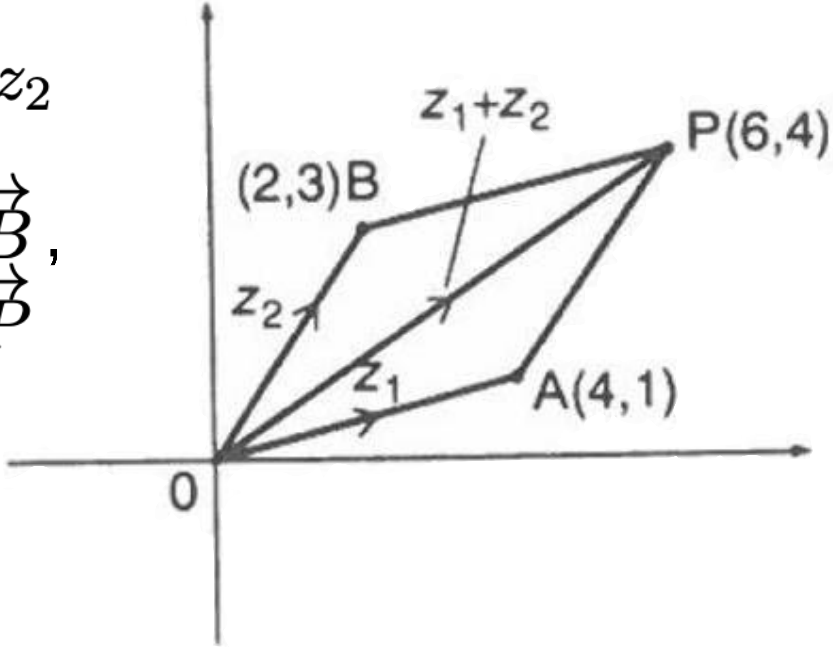
Express the complex number $z = -2 + 5i$ in polar form

$z = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Algebraic operations on Argand diagram

1. Addition

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 + z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPB$.



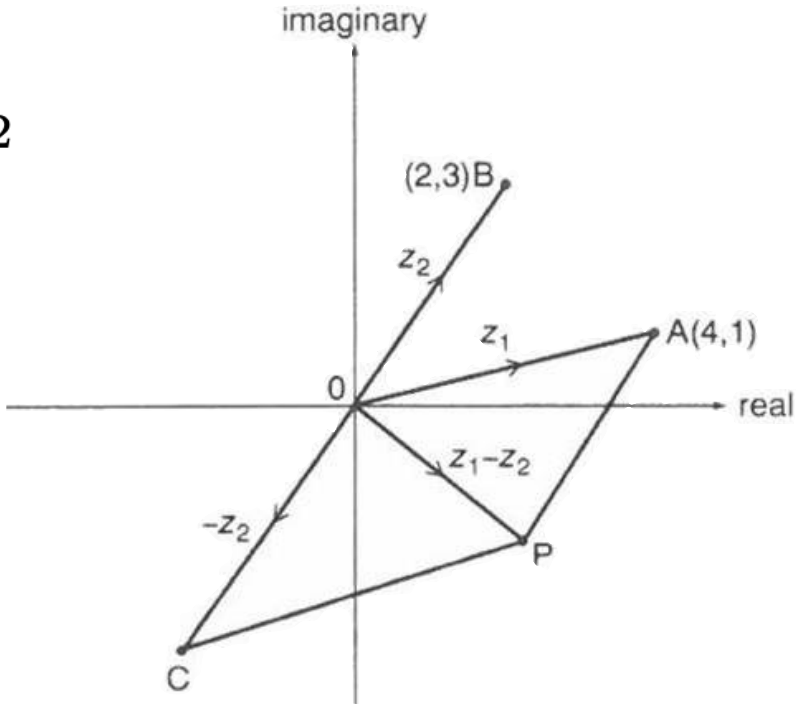
e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3i$

then, $z = z_1 + z_2 = 6 + 4i$ is the point $P(6,4)$.

Algebraic operations on Argand diagram

2. Subtraction

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 - z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPC$.



e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3\,i$

then, $z = z_1 - z_2 = 2 - 2\,i$ is the point $P(2, -2)$.

Algebraic operations on Argand diagram

3. Multiplication

To show the product of complex numbers on the Argand plane, it is useful to first represent them in polar form.

Let $z_1 = r_1\,(\cos\theta_1 + i\,\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

be two complex numbers in polar form.

Then, $z_1 \cdot z_2 = r_1\,(\cos\theta_1 + i\,\sin\theta_1) \cdot r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

Algebraic operations on Argand diagram

$$\therefore\ z_1 \cdot z_2 = r_1 \cdot r_2 \left(\cos\theta_1\ \cos\theta_2 + i\ \cos\theta_1\ \sin\theta_2 + i\ \sin\theta_1\ \cos\theta_2 + i^2\ \sin\theta_1\ \sin\theta_2\right)$$

$$= r_1 \cdot r_2 \left(\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2 + i\left(\sin\theta_1\ \cos\theta_2 + \cos\theta_1\ \sin\theta_2\right)\right)$$

$$= r_1 \cdot r_2\ \left[\cos\left(\theta_1 + \theta_2\right) + i\ \sin\left(\theta_1 + \theta_2\right)\right]$$

Thus,

$z_1 \cdot z_2 \equiv R\ \left(\cos\theta + i\ \sin\theta\right)$ where $R = r_1 \cdot r_2$, $\theta = \theta_1 + \theta_2$

Algebraic operations on Argand diagram

e.g. $z_1 = 2 + 2\,i$

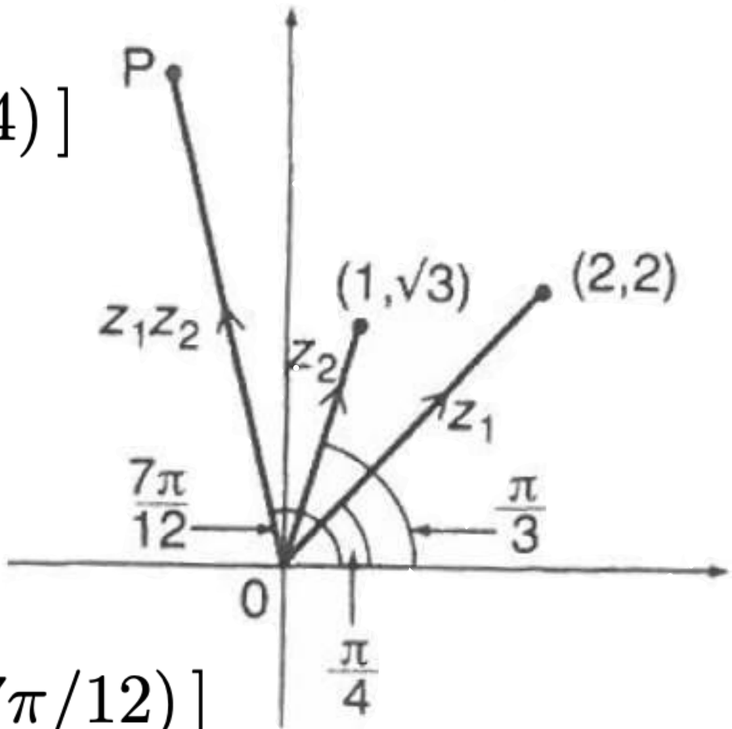
$$= 2\sqrt{2}\,\left[\,\cos(\pi/4) + i\,\sin(\pi/4)\,\right]$$

$$z_2 = 1 + \sqrt{3}\,i$$

$$= 2\,\left[\,\cos(\pi/3) + i\,\sin(\pi/3)\,\right]$$

then, $z = z_1 \cdot z_2$

$$= 4\sqrt{2}\,\left[\,\cos(7\pi/12) + i\,\sin(7\pi/12)\,\right]$$



Algebraic operations on Argand diagram

4. Division

In a similar way, it can be shown that:

$$\frac{z_1}{z_2} = \frac{r_1\,(\cos\theta_1 + i\,\sin\theta_1)}{r_2\,(\cos\theta_2 + i\,\sin\theta_2)} = \frac{r_1}{r_2}\,[\cos(\theta_1-\theta_2) + i\,\sin(\theta_1-\theta_2)]$$

Thus,

$$\boxed{\frac{z_1}{z_2} \equiv R\,(\cos\theta + i\,\sin\theta) \text{ where } R = \frac{r_1}{r_2} \text{ and } \theta = \theta_1 - \theta_2.}$$

Independent Learning Week (ILW)

Week 9 w/c 11 - Nov. 2024

CELE has named the week commencing on Monday 11th November as your Independent Learning Week.

You will have **no lectures and seminars** throughout the Independent Learning Week, but you will be given some learning activities to engage in.

The goal is for you to use this week to reflect on what learning independently outside the classroom means to you, sharpen your study skills, and reinforce your preparation moving toward the end of the semester. To help you:

1) Instructions will be posted on Moodle on Monday 11th November.

2) Office hours will remain at the usual times in case you need supports from tutors.

THANKS FOR YOUR ATTENTION