FCMT SAMPLE EXAM

SOLUTIONS

1. (a) $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x-1} - \sqrt{x+h-1}}{h(\sqrt{x+h-1})(\sqrt{x-1})}$$

$$= \lim_{h \to 0} \frac{\sqrt{x-1} - \sqrt{x+h-1}}{h(\sqrt{x+h-1})(\sqrt{x-1})} \cdot \left(\frac{\sqrt{x-1} + \sqrt{x+h-1}}{\sqrt{x-1} + \sqrt{x+h-1}}\right)$$

$$= \lim_{h \to 0} \frac{-h}{h[\sqrt{x+h-1} \cdot \sqrt{x-1}]\{\sqrt{x-1} + \sqrt{x+h-1}\}}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x+h-1} \cdot \sqrt{x-1}(\sqrt{x-1} + \sqrt{x+h-1})}$$

$$= \frac{-1}{2(x-1) \cdot \sqrt{x-1}}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}(x-1)^{-3/2}$$

(b) $$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(3x^3 - 4x) + (3x^3 - 4x) \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot (9x^2 - 4) + (3x^3 - 4x) \cdot e^x$$

$$= e^x (3x^3 + 9x^2 - 4x - 4)$$

(c)

$$\frac{dy}{dx} = \frac{e^{3x}\cdot\frac{d}{dx}(\sin^2 x) - \sin^2 x\cdot\frac{d}{dx}(e^{3x})}{(e^{3x})^2}$$

$$= \frac{e^{3x}(2\sin x\cos x) - 3e^{3x}\cdot\sin^2 x}{(e^{3x})^2}$$

$$= \frac{\sin 2x - 3\sin^2 x}{e^{3x}}$$

(d)

$$\frac{dy}{dx} = \frac{1}{\sin(\sqrt{3x^3-4x+1})}\cdot\frac{d}{dx}\left[\sin(\sqrt{3x^3-4x+1}\right]$$

$$= \frac{1}{\sin(\sqrt{3x^3-4x+1})}\cdot\cos(\sqrt{3x^3-4x+1})\cdot\frac{d}{dx}\sqrt{3x^3-4x+1}$$

$$= \cot(\sqrt{3x^3-4x+1})\cdot\frac{1}{2\sqrt{3x^3-4x+1}}\cdot\frac{d}{dx}(3x^3-4x+1)$$

$$= \frac{\cot(\sqrt{3x^3-4x+1})}{2\sqrt{3x^3-4x+1}}\cdot(9x^2-4)$$

$$= \frac{(9x^2-4)\cdot\cot(\sqrt{3x^3-4x+1})}{2\sqrt{3x^3-4x+1}}$$

(e)

$$\frac{d}{dx}\{\tan(xy)\} = \frac{d}{dx}(xy+xy^2)$$

$$\sec^2(xy)\left[x\frac{dy}{dx}+y\right] = \left[x\frac{dy}{dx}+y+2xy\frac{dy}{dx}+y^2\right]$$

$$\frac{dy}{dx}\left[x\cdot\sec^2(xy)-x-2xy\right] = y^2+y-y\sec^2(xy)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(y-\sec^2(xy)+1)}{x(\sec^2(xy)-2y-1)}$$

2 (a)

$$\ln y = \ln(x^2-x+3)^3 - \ln(x^3+2)^{1/2} - \ln(x-2)^4$$

$$\ln y = 3\ln(x^2-x+3) - \frac{1}{2}\ln(x^3+2) - 4\ln(x-2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2-x+3}\cdot(2x-1) - \frac{1}{2}\frac{1}{(x^3+2)}\cdot(3x^2) - \frac{4}{x-2}$$

$$= \frac{3(2x-1)}{x^2-x+3} - \frac{3x^2}{2(x^3+2)} - \frac{4}{x-2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^2-x+3)}{\sqrt{x^3+2}\cdot(x-2)^4}\left[\frac{3(2x-1)}{x^2-x+3} - \frac{3x^2}{2(x^3+2)} - \frac{4}{x-2}\right]$$

(b) (i)

$$\frac{dx}{dt} = \cos t + \sin t$$

$$\frac{dy}{dt} = \sec^2 t$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\sec^2 t}{\cos t + \sin t}$$

(ii) at $t = \frac{\pi}{4}$: $x = 0$ and $y = 1$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec^2(\pi/4)}{\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

$$\therefore (y-1) = \sqrt{2}(x-0)$$

$\Rightarrow \quad y - \sqrt{2}x - 1 = 0$: equation of tangent line.

(c)

$$C = 2\pi r$$

$$\frac{dC}{dt} = \frac{d}{dr}(2\pi r) \cdot \frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{dC}{dt} \Big/ 2\pi = \frac{1}{4} \Big/ 2\pi$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{8\pi}$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot \left(\frac{dr}{dt}\right)$$

$$\left.\frac{dA}{dt}\right|_{r=2} = 2\pi(2) \cdot \frac{1}{8\pi}$$

$$= \frac{1}{2} \text{ cm}^2/\text{s}$$

(d)

$f'(x) = 15x^4 + 12x^2$

for all $x \in \mathbb{R}$ $f'(x) \geq 0$

$\therefore$ $f(x)$ is always increasing

3 (a) (i) stationary points are at $f'(x) = 0$

$$3x^2 + 6x - 9 = 0 \Rightarrow (x+3)(x-1) = 0$$

$\therefore$ $x = -3$ or $x = 1$

Now, $x = -3 \Rightarrow f(-3) = 28$

$x = 1 \Rightarrow f(1) = -4$

$\therefore$ $(-3, 28)$ & $(1, -4)$ are stationary points

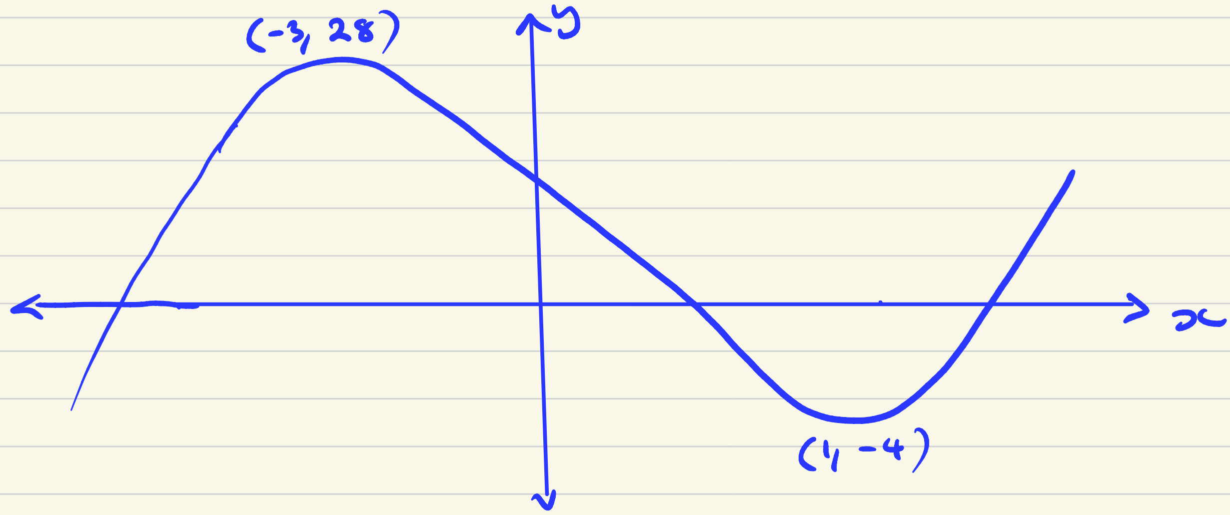
(ii) $f''(x) = 6x + 6$

$f''(x)\big|_{x=-3} = -12 < 0$

$\Rightarrow$ $(-3, 28)$ is a point of maximum value

$f''(x)\big|_{x=1} = 12 > 0$

$\Rightarrow$ $(1, -4)$ is a point of minimum value.



(b) $f'(x) = 3x^2 - 4$

$$\therefore\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n - 4x_n + 1}{3x_n^2 - 4}\right)$$

$$\Rightarrow \quad x_{n+1} = \frac{3x_n^3 - 4x_n - x_n^3 + 4x_n - 1}{3x_n^2 - 4}$$

$$= \frac{2x_n^3 - 1}{3x_n^2 - 4}$$

n	x_n
0	2
1	1.87500
2	1.86098
3	1.86081
4	1.86081

$\therefore \quad x_{root} = 1.86081$

4 (a) (i)

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{(1+x)} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{(4)}(0) = -6$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5} \Rightarrow f^{(5)}(0) = 24$$

$$\therefore f(x) = \ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \cdots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$g(x) = \ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} + \cdots$$

$$\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\}$$

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$$

(b) (i) $$I = \int (e^{2x} - e^{x})\,dx$$

$$= \frac{1}{2}e^{2x} - e^{x} + c$$

(ii) $$I = \int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx$$

$$= \int \frac{\cos 2x}{(\frac{1}{2}\sin 2x)^2}\,dx$$

$$= 4\int \frac{\cos 2x}{\sin^2 2x}\,dx$$

$$= 4\int \frac{\cos 2x}{\sin 2x}\cdot\frac{1}{\sin 2x}\,dx$$

$$= 4\int \cot 2x \cdot \operatorname{cosec} 2x\,dx$$

$$= 4\cdot\frac{1}{2}(-\operatorname{cosec} 2x) + c$$

$$= -2\operatorname{cosec} 2x + c$$ {or any other equivalent form}

(iii) $$I = \int \frac{\operatorname{cosec}^2 x}{4 - \cot^2 x}\,dx$$

let $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\,dx = dt$

$$\therefore\ I = -\int \frac{(-\operatorname{cosec}^2 x)}{4 - \cot^2 x}\,dx$$

$$= -\int \frac{dt}{4 - t^2}$$

$$= -\frac{1}{4}\ln\left|\frac{t+2}{t-2}\right| + c$$

$$\Rightarrow I = -\frac{1}{4}\ln\left|\frac{\cot x + 2}{\cot x - 2}\right| + c$$

$$= \frac{1}{4}\ln\left|\frac{\cot x - 2}{\cot x + 2}\right| + c$$

5(a)(i) $I = \int \sin^4 x \cdot \cos^5 x \, dx$

$= \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx$

let $\sin x = t \Rightarrow \cos x \, dx = dt$

$I = \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$

$= \int t^4 (1 - t^2)^2 \, dt$

$= \int t^4 (1 - 2t^2 + t^4) \, dt$

$= \int (t^4 - 2t^6 + t^8) \, dt$

$= \frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9} + C$

$\therefore I = \frac{(\sin x)^9}{9} - \frac{2(\sin x)^7}{7} + \frac{(\sin x)^5}{5} + C$

(ii)

$I = \int x^5 \sqrt{x^3 - 4} \, dx$

let $x^3 - 4 = t^2$

$\Rightarrow x^3 = t^2 + 4$

$\Rightarrow 3x^2 \, dx = 2t \, dt$

$\Rightarrow x^2 \, dx = \frac{2}{3} t \, dt$

$I = \int x^3 \cdot \sqrt{x^3 - 4} \cdot x^2 \, dx$

$= \int (t^2 + 4) \cdot t \cdot \frac{2}{3} t \, dt$

$$= \frac{2}{3}\int (t^2+4)\,t^2\,dt$$

$$= \frac{2}{3}\int (t^4+4t^2)\,dt$$

$$= \frac{2}{3}\left[\frac{t^5}{5}+\frac{4t^3}{3}\right]+C$$

$$= \frac{2}{3}\left[\frac{(x^3-4)^{5/2}}{5}+\frac{4\,(x^3-4)^{3/2}}{3}\right]+C$$

(iii) $I = \int e^x (3x^3 + 9x^2)\, dx$

Here $f(x) = 3x^3 \Rightarrow f'(x) = 9x^2$

$\therefore I = e^x (3x^3) + C$

$= 3e^x \cdot x^3 + C$

(b) $I = \int \frac{1}{x^2+2x+2}\, dx$

$= \int \frac{1}{(x^2+2x+1)+1}\, dx$

$= \int \frac{1}{(x+1)^2+1}\, dx$

$= \tan^{-1}(x+1) + C$

(c) $I = \int \frac{1}{2\sin^2 x + 3\cos^2 x}\, dx$

$= \int \frac{\left(\frac{1}{\cos^2 x}\right)}{2\left(\frac{\sin^2 x}{\cos^2 x}\right) + 3\left(\frac{\cos^2 x}{\cos^2 x}\right)}\, dx$

$= \int \frac{\sec^2 x}{2\tan^2 x + 3}\, dx$

let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore I = \int \frac{dt}{2t^2+3}$

$= \frac{1}{2} \int \frac{dt}{t^2 + \sqrt{\frac{3}{2}}^{\,2}}$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left[\sqrt{\frac{2}{3}}\, t\right] + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left[\sqrt{\frac{2}{3}} \cdot \tan x\right] + C$$

6 (a) $$I = \int \frac{2x^2+3}{x(x-1)^2}\,dx$$

$$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$2x^2+3 = A(x-1)^2 + Bx[x-1] + Cx \qquad \text{-----} \; (1)$$

put $x = 1 \Rightarrow C = 5$

put $x = 0 \Rightarrow A = 3$

Equate coefficients of x^2 from (1)

$\therefore \quad 2 = A + B \Rightarrow B = -1$

$$\Rightarrow \frac{2x^2+3}{x(x-1)^2} = \frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}$$

$$\Rightarrow I = \int \left(\frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2} \right) dx$$

$$= 3\ln|x| - \ln|x-1| - \frac{5}{(x-1)} + C$$

(b) $$I = \int_{-1}^{1} \ln(x+2)\,dx$$

let $x+2 = t \Rightarrow dx = dt$

change limits of integration:

x	-1	1
t	1	3

$$\therefore \quad I = \int_1^3 \ln(t)\,dt$$

$$= \int_1^3 (1)\cdot \ln(t)\,dt$$

$$I = \int_a^b u \cdot \frac{dv}{dt}\,dt = \left[u \cdot v\right]_a^b - \int_a^b v \cdot \frac{du}{dt}\,dt$$

let $u = \ln(t) \Rightarrow \frac{du}{dt} = \frac{1}{t}$

let $\frac{dv}{dt} = 1 \Rightarrow v = t$

$$\therefore I = \left[\ln(t) \cdot t\right]_1^3 - \int_1^3 1\,dt$$

$$= \left[3\ln(3) - 0\right] - \left[t\right]_1^3$$

$$= 3\ln(3) - (3 - 1)$$

$$= 3\ln(3) - 2$$

(c) $$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$= \pi \int_0^1 \left(e^{2x}\right)^2 dx$$

$$= \pi \int_0^1 e^{4x}\,dx$$

$$= \pi \left[\frac{e^{4x}}{4}\right]_0^1$$

$$= \pi \left[\frac{e^4}{4} - \frac{1}{4}\right]$$

$$= \frac{\pi}{4}\left(e^4 - 1\right)$$

(d) h (width of sub-interval) $= \dfrac{b-a}{n}$

$$\Rightarrow h = \frac{1-0}{10} = 0.1$$

Here, $f(x) = \dfrac{4}{1+x^2}$

x_n	f_n	$f(x_n)$
0	f_0	4.00000
0.1	f_1	3.96040
0.2	f_2	3.84615
0.3	f_3	3.66972
0.4	f_4	3.44828
0.5	f_5	3.20000
0.6	f_6	2.94118
0.7	f_7	2.68456
0.8	f_8	2.43902
0.9	f_9	2.20994
1.0	f_{10}	2.00000

$$\int_a^b f(x) \approx \frac{h}{3}\left[f_0 + 2(f_2+f_4+f_6+f_8) + 4(f_1+f_3+f_5+f_7+f_9) + f_{10}\right]$$

$$\therefore I = \int_0^1 \frac{4}{1+x^2}dx \approx \frac{1}{30}\left[4 + 2(3.84615 + 3.44828 + 2.94118 + 2.43902) + 4(3.96040 + 3.66972 + 2.68456 + 2.20994) + 2\right]$$

$$\Rightarrow I \approx \frac{1}{30}\left[4 + 25.34926 + 62.89848 + 2\right]$$

$$\approx \frac{1}{30}(94.24774)$$

$$\approx 3.14159$$

7 (a) $$y = \frac{\sin x + a}{x} - \cos x$$

$$\frac{dy}{dx} = \frac{x\cos x - (\sin x + a)}{x^2} + \sin x$$

Now $$x\frac{dy}{dx} + y = x\sin x$$

$$LHS = x \cdot \left[\frac{x\cos x - \sin x - a}{x^2} + \sin x\right]$$

$$+ \frac{\sin x + a}{x} - \cos x$$

$$\Rightarrow \cancel{\cos x} - \cancel{\frac{(\sin x + a)}{x}} + x\sin x + \cancel{\frac{(\sin x + a)}{x}} \cancel{- \cos x}$$

$$= x\sin x = RHS$$

(b) $$(1 - \cos y)\,dy = (1 + \sin x)\,dx$$

$$\int (1 - \cos y)\,dy = \int (1 + \sin x)\,dx$$

$$y - \sin y = x - \cos x + C$$

(c) $$x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} = 0 \Rightarrow 2y\,dy = \frac{-x}{\sqrt{x^2+1}}\,dx$$

$$\Rightarrow 2\int y\,dy = -\int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$\Rightarrow 2\int y\,dy = -\frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx$$

$$\Rightarrow y^2 = -\sqrt{x^2+1} + C$$ (general solution)

Now, $x = 0 \Rightarrow y = 1$

$$\therefore\ 1 = -\sqrt{0+1} + C \Rightarrow -1 + C = 1$$

$\Rightarrow \quad c = 2$

$\Rightarrow y = 2 - \sqrt{x^2+1}$ (particular solution)

(d) (i) $\dfrac{dP}{dt} = kP$

$\Rightarrow \int \dfrac{dP}{P} = k\int dt$

$\Rightarrow \ln|P| = kt + c$ (general solution)

Now $t=0 \Rightarrow P(0) = P_0$

$\therefore \ln|P_0| = k(0) + c$

$\Rightarrow c = \ln|P_0|$

$\therefore \ln|P| = kt + \ln|P_0|$ (particular solution)

$\ln\left|\dfrac{P}{P_0}\right| = kt \Rightarrow \dfrac{P}{P_0} = e^{kt}$

$\Rightarrow P = P_0 e^{kt}$

(ii) $P_0 = 1100$

$\therefore P = 1100 e^{kt}$

t	15	?
P	1450	1820

$\Rightarrow 1450 = 1100 e^{15k}$

$\therefore k = \dfrac{1}{15}\ln\left(\dfrac{29}{22}\right)$

$$\therefore \quad P = 1100 e^{\left[\frac{1}{15}\ln\left(\frac{29}{22}\right)\right]\cdot t}$$

when $P = 1820$

$$\frac{1820}{1100} = e^{\frac{1}{15}\ln\left(\frac{29}{22}\right)\cdot t}$$

$$\Rightarrow t = 15 \cdot \frac{\ln(1820/1100)}{\ln(29/22)}$$

$$\Rightarrow t \simeq 27.34 \text{ days}$$