The University of Nottingham Ningbo China

Centre for English Language Education

Semester One

SAMPLE MID-SEMESTER EXAMINATION

FOUNDATION ALGEBRA FOR PHYSICAL SCIENCES & ENGINEERING

Time allowed: 60 minutes

Candidates may complete the information required on the front page of this booklet but must NOT write anything else until the start of the examination period is announced.

This paper comprises TWENTY questions. Answer all questions.

Answers must be written (with necessary steps) in this booklet.

Figures enclosed by square brackets, eg. [3], indicate marks for that question.

Only CELE approved calculator is allowed in this exam.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do not turn this page over until instructed to do so.

ADDITIONAL MATERIAL: Formula Sheet

INFORMATION FOR INVIGILATORS:

1. *Please give a 10 minutes warning before the end of exam.*
2. *Please collect this booklet at the end of the exam.*

Student ID: ______________

Seminar Group (*e.g.* A35): __________ **Marks (out of 100):** ______

1. Given $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = 2x+3$, find $x \in \mathbb{R}$ such that $(f \circ g)(x) = \dfrac{1}{2}$.

$(f \circ g)(x) = f(g(x))$

$\Rightarrow f(2x+3) = \dfrac{1}{(2x+3)^2+1}$

$\Rightarrow \dfrac{1}{(2x+3)^2+1} = \dfrac{1}{2}$

$\Rightarrow (2x+3)^2+1 = 2$

$\Rightarrow 2x+3 = \pm 1$

$\Rightarrow 2x = -4$ or $2x = -2$

$\therefore x = -2$ or $x = -1$

$x =$ -2 or -1

[3]

2. A function $f : [0, +\infty) \to [0, +\infty)$ is defined by $f(x) = \sqrt{x+1} - 1$.

(a) Find $f^{-1}(x)$ and its domain.

$y = f(x) = \sqrt{x+1} - 1$

$y+1 = \sqrt{x+1}$

$(y+1)^2 = x+1$

$x = (y+1)^2 - 1$

$\therefore f^{-1}(x) = (x+1)^2 - 1$

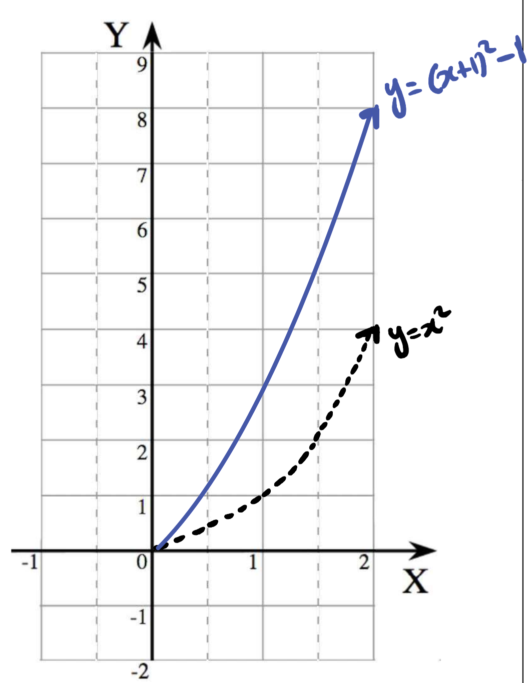
for $x \in [0, \infty)$

$\sqrt{x+1} \geq 1$

$\therefore \sqrt{x+1} - 1 \geq 0$

$\therefore$ domain of f^{-1} is $[0, \infty)$

(b) Sketch the graph of $y = f^{-1}(x)$.



$f^{-1}(x) =$ $(x+1)^2 - 1$

domain of $f^{-1}(x)$: $[0, \infty)$

[3]

3. Solve the modulus inequality $|x-3| \geq 2$ for $x \in \mathbb{R}$.

$\pm(x-3) \geqslant 2$

$\Rightarrow x-3 \geqslant 2$ or $-x+3 \geqslant 2$

$\Rightarrow x \geqslant 5$ or $x \leq 1$

or $x \in \mathbb{R}-(1,5)$

[2]

4. Use appropriate substitution to solve the exponential equation $e^x - 8e^{-x} = 2$ for $x \in \mathbb{R}$.

Let $e^x = t$

$\therefore e^x - 8e^{-x} - 2 = 0$

$\Rightarrow t - \frac{8}{t} - 2 = 0$

$\Rightarrow t^2 - 2t - 8 = 0$

$\Rightarrow (t-4)(t+2) = 0$

$\therefore t = 4$ or $t = -2$

But $e^x \neq -2$ (because $e^x > 0$)

$\therefore e^x = 4$

$\Rightarrow x = \ln 4$

$x =$ $\ln 4$

[4]

5. Solve the logarithmic equation $\log_2(4+2x) - \log_2(4-x) = 2$ for $x \in \mathbb{R}$.

$$\log_2(4+2x) - \log_2(4-x) = 2$$

$$\Rightarrow \log_2 \frac{4+2x}{4-x} = \log_2 2^2$$

$$\Rightarrow \frac{4+2x}{4-x} = 4$$

$$\Rightarrow 4+2x = 16-4x$$

$$\Rightarrow 6x = 12$$

$$\Rightarrow x = 2$$

$x =$ 2

[3]

6. Use the method of completing the square to find the range of the function

$$f(x) = 3x^2 - 12x + 7, \quad x \in \mathbb{R}.$$

$$f(x) = 3\left(x^2 - 4x + \frac{7}{3}\right)$$

$$= 3\left(\underline{x^2 - 4x + 2^2} + \frac{7}{3} - 2^2\right)$$

$$= 3\left(\underline{x^2 - 4x + 2^2} - \frac{5}{3}\right)$$

$$= 3\underline{(x-2)^2} - 3\left(\frac{5}{3}\right)$$

$$= 3(x-2)^2 - 5$$

But $3(x-2)^2 \geq 0$

$\therefore f(x) \geq -5$

Range of $f(x)$: $[-5, \infty)$

[3]

7. Given the function $f(x) = \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R}$,

(a) Express $f(x)$ in the form $R\cos(x-\theta)$, where $R > 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ are to be determined.

$\sin x + \sqrt{3}\cos x = R\cos x\cos\theta + R\sin x\sin\theta$

$\therefore \left.\begin{array}{l} R\cos\theta = \sqrt{3} \\ R\sin\theta = 1 \end{array}\right\} \Rightarrow R = \sqrt{1+\sqrt{3}^2} = 2$

$\therefore \cos\theta = \sqrt{3}/2$ & $\sin\theta = 1/2 \therefore \theta = \frac{\pi}{6}$

$R =$ 2 $\qquad \theta =$ $\pi/6$

(b) If $f(x)$ can also be expressed in the form $R\sin(x+\phi)$, where $\phi \in \left(0, \frac{\pi}{2}\right)$, find ϕ.

$\sin x + \sqrt{3}\cos x = 2\sin x\cos\phi + 2\cos x\sin\phi$

$\therefore \left.\begin{array}{l} 2\cos\phi = 1 \\ 2\sin\phi = \sqrt{3} \end{array}\right\} \Rightarrow \begin{array}{l} \cos\phi = 1/2 \\ \sin\phi = \sqrt{3}/2 \end{array}$

$\therefore \phi = \frac{\pi}{3}$

$\phi =$ $\pi/3$

[3]

8. Using the double-angle formulae and the half-angle formulae, simplify the following expression. Write your final result as an expression of $\frac{\alpha}{2}$.

$$\frac{2\sin\alpha - \sin 2\alpha}{2\sin\alpha + \sin 2\alpha}$$

$$= \frac{2\sin\alpha - 2\sin\alpha\cos\alpha}{2\sin\alpha + 2\sin\alpha\cos\alpha}$$

$$= \frac{2\sin\alpha(1-\cos\alpha)}{2\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$= \tan^2\frac{\alpha}{2}$$

[3]

9. Given two polynomials $p(x) = 2x^4 - 3x^2 + x - 6$ and $s(x) = x - 3$,

(a) Apply the method of synthetic division to find the quotient and remainder of $\dfrac{p(x)}{s(x)}$.

$$\begin{array}{c|ccccc} & 2 & 0 & -3 & 1 & -6 \\ 3 & \downarrow & 6 & 18 & 45 & 138 \\ \hline & 2 & 6 & 15 & 46 & 132 \end{array}$$

$\therefore\ q(x) = 2x^3 + 6x^2 + 15x + 46$

$r = 132$

Quotient = $2x^3 + 6x^2 + 15x + 46$

Remainder = 132

(b) Apply the Remainder theorem to verify the value of the remainder you obtained above.

Here $s(x) \equiv x - c = x - 3$

$\therefore\ c = 3$

$\Rightarrow p(3) = r$

$\Rightarrow p(3) = 2(3)^4 - 3(3)^2 + 3 - 6$

$= 162 - 27 + 3 - 6$

$= 132$

[4]

10. If $(x + 5)$ is a factor of $p(x) = x^3 + 6x^2 + kx + 40$, use the Factor theorem to find the value of $k \in \mathbb{R}$.

Here $s(x) \equiv x - c = x - (-5)$

$\therefore\ c = -5$

$\Rightarrow p(-5) = 0$

$\Rightarrow (-5)^3 + 6(-5)^2 + k(-5) + 40 = 0$

$\Rightarrow -125 + 150 - 5k + 40 = 0$

$\Rightarrow -5k = -65$ $\therefore\ k = 13$

$k =$ 13

[2]

11. Given a polynomial $p(x) = x^3 - 2x^2 - 23x + 60$,

(a) Apply the Factor theorem to show that $(x-3)$ is a factor of $p(x)$.

Here $s(x) \equiv x - c = x - 3$
$\therefore c = 3$
$p(3) = 0$ if $x-3$ is a factor of $p(x)$
$= 3^3 - 2(3)^2 - 23(3) + 60$
$= 27 - 18 - 69 + 60 = 0$

(b) Use the method of Synthetic division to find the quotient of $\dfrac{p(x)}{x-3}$.

$$\begin{array}{c|cccc} & 1 & -2 & -23 & 60 \\ 3 & \downarrow & 3 & 3 & -60 \\ \hline & 1 & 1 & -20 & 0 \end{array}$$

Quotient = $x^2 + x - 20$

(c) Use the result from 11 (a) and (b) to factorise $p(x)$ completely into a product of linear factors. Hence, solve the equation $p(x) = 0$.

$\therefore p(x) = (x-3)(x^2+x-20)$
$= (x-3)(x-4)(x+5)$

$\therefore p(x) = 0$
$\Rightarrow (x-3)(x-4)(x+5) = 0$
$\Rightarrow x = 3, 4$ or -5.

[4]

12. Given a polynomial $p(x) = x^4 + ax^3 - 12x^2 + bx + 27$, find the value of $a, b \in \mathbb{R}$, if:

(i) $(x+3)$ is a factor of $p(x)$, and

(ii) the remainder of $p(x)$ divided by $(x-2)$ is equal to -25.

(i) Here $s(x) \equiv x - c = x - (-3)$
$\therefore c = -3$
$\Rightarrow p(-3) = 0$
$\Rightarrow 81 - 27a - 108 - 3b + 27 = 0$
$\Rightarrow 9a + b = 0$
(ii) Here $s(x) \equiv x - c = x - 2$
$\Rightarrow p(2) = -25$
$\Rightarrow 16 + 8a - 48 + 2b + 27 = -25$
$\Rightarrow 4a + b = -10$

$\therefore 9a + b = 0$ ----- ①
$4a + b = -10$ ----②
$\Rightarrow 5a = 10$ ---[① − ②]
$\Rightarrow a = 2$
$\Rightarrow 9(2) + b = 0$ ---[put a in ①]
$\Rightarrow b = -18$

$a =$ 2 $\quad b =$ −18

[2]

13. Given $f(x) = 5\,\sin(3x + \pi) - 2, \quad x \in \mathbb{R}$,

(a) Find the range and period of $f(x)$.

$$\Rightarrow -1 \le \sin(3x+\pi) \le 1$$
$$\Rightarrow -5 \le 5\sin(3x+\pi) \le 5$$
$$\Rightarrow -7 \le 5\sin(3x+\pi) - 2 \le 3$$
$$\Rightarrow -7 \le f(x) \le 3$$

Range = $\{-7, 3\}$ period = $2\pi/3$

(b) If $g(x) = |f(x)|$, find the range of $g(x)$.

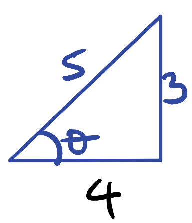
$$g(x) = |5\sin(3x+\pi) - 2|$$
$$\therefore\ 0 \le g(x) \le 7$$

Range = $[0, 7]$

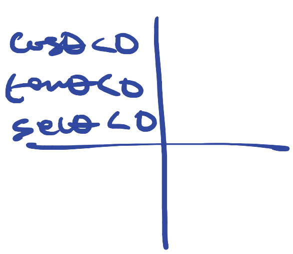
[4]

14. Given that $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $\sin\theta = \frac{3}{5}$, find the value of the expression:

$\cos\theta + \tan\theta + \sec\theta$



$$\therefore\ \cos\theta = -\frac{4}{5}$$
$$\tan\theta = \frac{-3}{4}$$
$$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4}$$



$$\Rightarrow \cos\theta + \tan\theta + \sec\theta$$
$$= -\frac{4}{5} - \frac{3}{4} - \frac{5}{4}$$
$$= -14/5$$

[2]

15. Use appropriate trigonometric identities to verify the given equalities:

$$(a) \quad \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1-\tan\alpha\cdot\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

$$LHS = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$= \frac{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{1 + \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta} = RHS$$

$$(b) \quad \sin x + \sin 3x = 4\sin x \cdot \cos^2 x$$

$$LHS = 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)$$

$$= 2\sin 2x \cos x$$

$$= 2(2\sin x\cos x)\cos x$$

$$= 4\sin x\cos^2 x = RHS$$

You may use any equivalent form

[4]

16. Use appropriate trigonometric identities to find the value of

$$\frac{\sin 70^\circ + \cos 40^\circ}{\cos 70^\circ + \sin 40^\circ}$$

$$\frac{\sin 70^\circ + \cos 40^\circ}{\cos 70^\circ + \sin 40^\circ} = \frac{\sin 70^\circ + \sin 50^\circ}{\cos 70^\circ + \cos 50^\circ}$$

$$= \frac{\cancel{2}\sin 60^\circ \cancel{\cos 10^\circ}}{\cancel{2}\cos 60^\circ \cancel{\cos 10^\circ}}$$

$$= \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

[2]

17. Solve the following equation for $\theta \in \left(\frac{\pi}{2}, \pi\right)$:

$$\tan^2\theta + \sec\theta - 1 = 0$$

$\tan^2\theta + \sec\theta - 1 = 0$

$\sec^2\theta - 1 + \sec\theta - 1 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

let $m = \sec\theta$

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow (m-1)(m+2) = 0$

$\Rightarrow m = 1$ or $m = -2$

$\Rightarrow \sec\theta = 1$ or $\sec\theta = -2$

$\Rightarrow \cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$

But $\theta \in (\pi/2, \pi)$ $\therefore \theta = 2\pi/3$

$\theta =$ $2\pi/3$

[4]

18. Find the value of the following expression:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) + \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$$

$\Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$\Rightarrow \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) + \frac{2\pi}{3}$

$\Rightarrow \frac{\pi}{3} + \frac{2\pi}{3} = \pi$

[2]

19. Without using a calculator, find the value of $\cot\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$

Let $\cos^{-1}\left(\frac{3}{5}\right) = \alpha$

$\therefore \cot\left(2\cos^{-1}\left(\frac{3}{5}\right)\right) = \cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1-\tan^2\alpha}{2\tan\alpha}$

$\because \cos^{-1}\left(\frac{3}{5}\right) = \alpha \quad \therefore \cos\alpha = \frac{3}{5}$

as $\alpha \in$ Restricted domain of cosine function

$\therefore \alpha \in [0, \pi] \quad \therefore \sin\alpha > 0$

$\therefore \sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{4}{5}$

$\therefore \tan\alpha = \frac{4}{3}$

$\therefore \cot\left(2\cos^{-1}\left(\frac{3}{5}\right)\right) = \frac{1-\left(\frac{4}{3}\right)^2}{2\cdot\frac{4}{3}} = -\frac{7}{24}$

[2]

20. Given $f(x) = \dfrac{3x^3 - 2x^2 + 9x + 2}{(x^2+3)(x-1)^2} = \dfrac{Ax+B}{x^2+3} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$,

find the values of the constants A, B and C.

$3x^3 - 2x^2 + 9x + 2 = (Ax+B)(x-1)^2 + C(x^2+3)(x-1) + D(x^2+3)$

Let $x=1 \Rightarrow 12 = D\cdot 4 \Rightarrow D = 3$

$$\left.\begin{array}{l}
\text{Let } x=0 \Rightarrow 2 = B - 3C + 9 \Rightarrow B - 3C = -7 \\
\text{Let } x=-1 \Rightarrow -12 = 4B - 4A - 8C + 12 \Rightarrow 4B - 4A - 8C = -24 \\
\text{Let } x=2 \Rightarrow 36 = 2A + B + 7C + 21 \Rightarrow B + 2A + 7C = 15
\end{array}\right\}$$

$$\therefore \begin{cases} A = 1 \\ B = -1 \\ C = 2 \\ D = 3 \end{cases} \qquad \therefore f(x) = \frac{x-1}{x^2+3} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

[4]