University of Nottingham
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Foundation Calculus and Mathematical Techniques CELEN037

Practice Problems SET-6 Sample Solution

Type 1: Integrals in the form $\displaystyle\int \sin Ax \cdot \cos Bx \, dx$

1. Evaluate the following integrals: (i) $\displaystyle\int \cos(5x) \cdot \cos(2x) \; dx$

   Solution:

$$\int \cos(5x) \cdot \cos(2x) \; dx$$

$$= \frac{1}{2} \int \left(\cos(7x) + \cos(3x)\right) \; dx$$

$$= \frac{1}{14} \sin(7x) + \frac{1}{6} \sin(3x) + C$$

Type 2: Integrals of the form $\displaystyle\int \sin^m x \cos^n x \; dx$

2. Evaluate the following integrals: (iv) $\displaystyle\int \sin^4 x \cdot \cos^5 x \; dx$

   Solution:

Let $\sin x = t$

$$\frac{dt}{dx} = \cos x \implies \cos x dx = dt$$

$$I = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \; dx$$

$$= \int \sin^4 x \cdot (1 - \sin^2 x)^2 \cdot \cos x \; dx$$

$$= \int t^4 (1 - t^2)^2 \, dt$$

$$= \int t^4 - 2t^6 + t^8 \, dt$$

$$= \frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$$

2. Evaluate the following integrals: (v) $\int \sin^4 x \, dx$

Solution:

$$I = \int \frac{1}{4}(1-\cos(2x))^2 \, dx$$

$$= \frac{1}{4}\int (1 - 2\cos(2x) + \cos^2(2x)) \, dx$$

$$= \frac{1}{4}\int (1 - 2\cos(2x) + \frac{1}{2}(1+\cos(4x))) \, dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right) \, dx$$

$$= \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$$

Type 3: Useful results (for fast integration)

3. Evaluate the following integrals: $(i) \int \frac{2+2\cos 3x}{3x+\sin 3x} \, dx$

Solution:

Let $f(x) = 3x + \sin 3x,$ therefore $f'(x) = 3 + 3\cos 3x$

$$I = \int \frac{2+2\cos 3x}{3x+\sin 3x} \, dx = \frac{2}{3}\int \frac{3+3\cos 3x}{3x+\sin 3x} \, dx = \frac{2}{3}\int \frac{f'(x)}{f(x)} \, dx$$

$$= \frac{2}{3}\ln|3x+\sin 3x| + C$$

3. Evaluate the following integrals: $(xiii) \int e^x \left[\frac{3-x}{(2-x)^2}\right] \, dx$

Solution:

$$\int\int e^x \left[\frac{3-x}{(2-x)^2}\right] \, dx = \int e^x \left[\frac{2-x+1}{(2-x)^2}\right] \, dx = \int e^x \left[\frac{1}{(2-x)} + \frac{1}{(2-x)^2}\right] \, dx$$

Let $f(x) = \frac{1}{(2-x)},$ therefore $f'(x) = \frac{1}{(2-x)^2}$

$$I = \int e^x \left[\frac{1}{(2-x)} + \frac{1}{(2-x)^2}\right] \, dx = \int e^x \left[f(x) + f'(x)\right] \, dx$$

$$= e^x \left(\frac{1}{2-x}\right) + C$$

Type 4: Integration by Completing the Square in the Denominator

4. Evaluate the following integrals: (iv) $\displaystyle\int \frac{1}{\sqrt{25x^2 - 20x + 14}}\, dx$

Solution:

$$I = \int \frac{1}{\sqrt{25x^2 - 20x + 4 + 10}}\, dx = \int \frac{1}{\sqrt{(5x-2)^2 + \sqrt{10}^2}}\, dx$$

Use the formula: $\int f(ax+b)\, dx = \frac{1}{a}F(ax+b) + C$

$$I = \frac{1}{5}\ln\left|5x - 2 + \sqrt{25x^2 - 20x + 14}\right| + C$$