Foundation Physics, Online quiz 4 Question 4

A toy rocket with a mass of 300 g takes off in a vertical direction under the influence of gravity. It burns fuel at the rate of 25 g/s. The exhaust speed of the gases is 80 m/s. What is the speed of the rocket at the end of 10 seconds?

Answer: We will use Newton's second law to solve this problem. However, Newton's second law is used here in the form of "relation between impulse and change of momentum" (please check Physics Lecture 5 slides, p.62 & p.65). That is,

$$\sum \vec{F}\,dt = d\vec{p} = d(m\vec{v}) = m\,d\vec{v} \qquad \text{Eqn. 1}$$

We will take { rocket + the part of fuel/gas the rocket is going to spew } together as a whole, to be our system.

Notation: The original mass of the rocket, M_0. The mass of the rocket, M. The rate at which the fuel is burnt, m. The exhaust (i.e., *relative*) speed of the gases, v_r. The velocity of the rocket, v. We choose upwards as positive direction.

At any specific moment t, let's consider the two sides of the above Eqn. 1

- On one hand, the only force acting on rocket is their weight which equals $Mg = (M_0 - mt)g$.
  - Hence **the impulse** during that dt-period will be $(-Mg \cdot dt)$.
- On the other, we need to calculate **change of momentum** of the system, which is two-fold.
  - Change of velocity of the rocket. The rocket's experiencing a change of velocity of dv, it has a mass of M.
  - Change of velocity of the part of fuel/gas. The gas experiencing a change of velocity of v_r, it has a mass of $m\,dt$. By definition, "exhaust speed" of gas is the relative speed of gas to the rocket, after spewed out.

Therefore by adding these up, we obtain from Newton's second law (Eqn. 1)

$$\boxed{-Mg \cdot dt = M \cdot dv + (m\,dt) \cdot (v_r)} \qquad \boxed{\text{Eqn. 2}}$$

Divide M on both sides and then rewrite the above equation, we obtain

$$\left(\frac{-Mg - mv_r}{M}\right) dt = dv \qquad \text{Eqn. 3}$$

We have just successfully turned a physics problem, into a (pure) math problem.

The rest of solution, amounts to solving this math problem represented by the above Eqn. 3. A bit knowledge of integral calculus is necessary here. However, if you are not very familiar with integral calculus, no worries at all. Just skip this part, and go to Eqn. 5.

Recall that $M = (M_0 - mt)$, so $t = {}^{(M_0 - M)}/_{m}$ and $dt = {}^{-dM}/_{m}$. Substitute t with M in Eqn. 3,

$$\left(\frac{g}{m} + \frac{v_r}{M}\right) dM = dv \qquad \text{Eqn. 4}$$

Integrate Eqn. 4 on both sides (i.e., $M_0 \to M$ on the left and $v_0 \to v$ on the right), we obtain

$$v - v_0 = \int_{M_0}^{M} \left(\frac{g}{m} + \frac{v_r}{M}\right) dM = \left[\frac{gM}{m} + v_r \ln M\right]_{M_0}^{M} = \frac{g(M - M_0)}{m} + v_r\,(\ln M - \ln M_0) = -gt + v_r \ln\left(\frac{M}{M_0}\right)$$

Set $v_0 = 0$, since the rocket stays at rest when $t = 0$.

$$v = -gt + v_r \ln\left(\frac{M}{M_0}\right) = -gt - v_r \ln\left(\frac{M_0}{M_0 - mt}\right) \qquad \text{Eqn. 5}$$

Substitute variables with their values. $M_0 = 300\text{g}$, $m = 25\,\text{g/s}$, $v_r = -80\,\text{m/s}$ (downward hence -ve), $t = 10\text{s}$ and $g = 9.8\,\text{m/s}^2$

$$v = -gt + v_r \ln\left(\frac{M}{M_0}\right) = -9.8\,\text{m/s}^2 \cdot 10\text{s} - 80\,\text{m/s} \cdot \ln\left(\frac{300\text{g} - 25\text{g/s} \cdot 10\text{s}}{300\text{g}}\right) = -98\,\text{m/s} + 80\,\text{m/s} \cdot \ln(6.0) = \boxed{45.3\,\text{m/s}}.$$

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