Topic 1: Evaluating Definite Integrals

Fundamental Theorem of Calculus:

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$.

Illustration: Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2}\,dx$

$$\boxed{I = \int \frac{f'(x)}{f(x)}\,dx}$$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2}\,dx \\
&= \Big[\ln|\sin x + 2|\Big]_0^{\frac{\pi}{2}} \\
&= \Big[\ln(\sin x + 2)\Big]_0^{\frac{\pi}{2}} \\
&= \ln(\sin\frac{\pi}{2} + 2) - \ln(\sin 0 + 2) \\
&= \ln 3 - \ln 2
\end{aligned}$$

1. $\displaystyle\int_0^1 \frac{1}{4-x^2}\,dx$ **Answer:**	2. $\displaystyle\int_0^4 \frac{1}{\sqrt{x^2+9}}\,dx$ **Answer:**

1. $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$

Answer:

2. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\cos x + 2}\, dx$

Answer:

3. $\displaystyle\int_0^{\pi/4} \tan^2 x \; dx$

Answer:

4. $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+9}}\, dx$

Answer:

5. $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\, dx$

Answer:

6. $\displaystyle\int_1^e \frac{1}{x}\, dx$

Answer:

Topic 2: Method of Substitution for Definite Integrals

Remember to change the limits of integration for the transformed integral.

Illustration: Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{e^x(x+1)}{\cos^2(xe^x)} dx$.

Let $xe^x = t$. Then $(e^x + xe^x)\,dx = e^x(x+1)dx = dt$

Integration limits:

x	0	$\frac{1}{2}$
t	0	$\frac{\sqrt{e}}{2}$

$$\int_0^{\frac{1}{2}} \frac{e^x(x+1)}{\cos^2(xe^x)}\,dx = \int_0^{\frac{\sqrt{e}}{2}} \frac{1}{\cos^2 t}\,dt = \int_0^{\frac{\sqrt{e}}{2}} \sec^2 t\;dt$$

$$= \Big[\tan t\Big]_0^{\frac{\sqrt{e}}{2}}$$

$$= \tan\left(\frac{\sqrt{e}}{2}\right)$$

1. $\displaystyle\int_{1/3}^{1} \frac{1}{\sqrt{x}(x+1)}\,dx$

Answer:

2. $\displaystyle\int_{-1}^{1} \frac{x^2}{\sqrt{x^3+9}}\,dx$

Answer:

1. $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

Answer:

2. $\displaystyle\int_{0}^{\pi/2} \frac{\sin x}{\sqrt{5+\cos x}}\, dx$

Answer:

Topic 3: Integration by Parts for Definite Integrals

$$\int_a^b u \cdot \frac{dv}{dx}\, dx = \left[u \cdot v\right]_a^b - \int_a^b v \cdot \frac{du}{dx}\, dx$$

Illustration 1: Evaluate $\int_1^e x^2 \ln x \; dx$.

$$\text{Let} \quad u = \ln x \quad \text{and} \quad \frac{dv}{dx} = x^2$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad v = \frac{x^3}{3}$$

$$\Rightarrow \quad \int_1^e x^2 \ln x \; dx = \left[\ln x \cdot \frac{x^3}{3}\right]_1^e - \int_1^e \frac{x^3}{3} \cdot \frac{1}{x}\; dx$$

$$= \left[\ln x \cdot \frac{x^3}{3}\right]_1^e - \left[\frac{x^3}{9}\right]_1^e$$

$$= \frac{2e^3 + 1}{9}$$

1. $\int_0^1 x \cdot e^{2x} \; dx$

Answer:

2. $\int_{-1}^1 \ln(x+2) \; dx$

Answer:

Topic 3: Integration by Parts for Definite Integrals

$$\int_a^b u \cdot \frac{dv}{dx}\, dx = \left[u \cdot v\right]_a^b - \int_a^b v \cdot \frac{du}{dx}\, dx$$

Illustration 2: Evaluate $\displaystyle\int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx$.

$$\sec^{-1}\left(\sqrt{x}\right) = \cos^{-1}\left(\frac{1}{\sqrt{x}}\right) \Rightarrow \int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx = \int_1^4 \cos^{-1}\left(\frac{1}{\sqrt{x}}\right)\, dx$$

$$\text{Let} \quad u = \cos^{-1}\left(\frac{1}{\sqrt{x}}\right) \quad \text{and} \quad \frac{dv}{dx} = 1$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2x\sqrt{x-1}} \quad \text{and} \quad v = x$$

$$\Rightarrow \quad \int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx = \left[\cos^{-1}\left(\frac{1}{\sqrt{x}}\right) \cdot x\right]_1^4 - \int_1^4 \frac{1}{2x\sqrt{x-1}} \cdot x\, dx$$

$$= \left[\cos^{-1}\left(\frac{1}{\sqrt{x}}\right) \cdot x\right]_1^4 - \left[\sqrt{x-1}\right]_1^4 = \frac{4\pi}{3} - \sqrt{3}$$

1. $\displaystyle\int_0^1 \frac{\ln(x+1)}{(x+1)^2}\, dx$

Answer:

2. $\displaystyle\int_0^{\pi/4} \frac{x \cdot \sin x}{\cos^3 x}\, dx$

Answer:

Topic 4: Properties of Definite Integrals

1. If $a \in D_f$, then
$$\int_a^a f(x)\ dx = 0$$
2. If $f(x)$ is integrable on $[a, b]$, then
$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$
3. If $f(x)$ is integrable on an interval I, and $a, b, c \in I$, then
$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Illustration 1: Evaluate $\displaystyle\int_0^3 f(x)\ dx$ where $f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$.

$$\int_0^3 f(x)\ dx = \int_0^2 f(x)\ dx + \int_2^3 f(x)\ dx = \int_0^2 x^2\ dx + \int_2^3 (3x-2)\ dx$$
$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left[\frac{8}{3} - 0\right] + \left[\left(\frac{27}{2} - 6\right) - (6-4)\right] = \frac{49}{6}$$

1. $\displaystyle\int_0^2 f(x)\ dx$ where $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x^2, & 1 < x \leq 2 \end{cases}$ **Answer:**	2. $\displaystyle\int_{-1}^2 \lvert x \rvert\ dx$ **Answer:**

Topic 4: Properties of Definite Integrals

4. If $f(x)$ is integrable and EVEN on $[-a, a]$, then
$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

5. If $f(x)$ is integrable and ODD on $[-a, a]$, then
$$\int_{-a}^{a} f(x)\, dx = 0$$

6. If $f(x)$ is integrable on $[a, b]$, then
$$\int_{a}^{b} f(x)\, dx = \int_{a}^{b} f(a+b-x)\, dx$$

Illustration 2: Evaluate $\displaystyle\int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad \cdots\cdots(1)$$

$$I = \int_{0}^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\, dx$$

$$\Rightarrow I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \qquad \cdots\cdots(2)$$

$$(1)+(2) \Rightarrow 2I = \int_{0}^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \quad = \int_{0}^{\pi/2} 1\ dx$$

$$= [x]_{0}^{\pi/2} = \frac{\pi}{2} \qquad \Rightarrow I = \frac{\pi}{4}$$

1. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx$

Answer:

2. $\displaystyle\int_0^{4} \frac{\sqrt{4-x}}{\sqrt{x} + \sqrt{4-x}}\, dx$

Answer:

3. $\displaystyle\int_1^{8} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}}\, dx$

Answer:

4. $\displaystyle\int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}}\, dx$

Answer: