CELEN037
Foundation Calculus and Mathematical Techniques

Lecture 5

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Main content

- Antiderivatives and indefinite integrals
- Integrals of standard functions
- Integration by substitution
- Integration of a function of a linear polynomial
- Trigonometric substitution

Indefinite integration: Motivation

Suppose that an object is moving and we know $f(t)$ such that $x'(t) = f(t)$ where $x(t)$ is the position of the object at time t. Can we determine $x(t)$?

Suppose that water is leaking from a tank and we know $g(t)$ such that $V'(t) = g(t)$ where $V(t)$ is the volume of water in the tank at time t. Can we determine $V(t)$?

Indefinite integration (also called antidifferentiation) can be used to obtain solutions to such problems.

Definite integration: Motivation

Definite integration (taught in future lectures) can be used to compute areas and volumes.

The definite integral

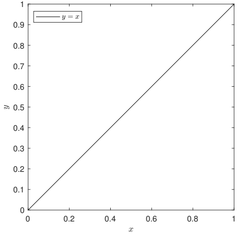
$$\int_0^1 x\,dx = \frac{1}{2},$$

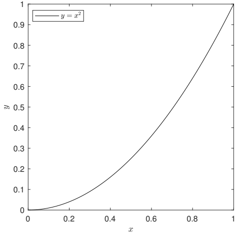
which is the area bounded by the line $y = x$, the line $x = 1$ and the x-axis.

The definite integral

$$\int_0^1 x^2\,dx = \frac{1}{3},$$

which is the area bounded by the line $y = x^2$, the line $x = 1$ and the x-axis.





Antiderivatives: Definition

An antiderivative of a function f is a function F which is such that

$$F'(x) = f(x)$$

for all x in the domain of f.

Result

If F is an antiderivative of f and C is a constant then G given by

$$G(x) = F(x) + C$$

is also an antiderivative of f.

Indefinite integrals

An integral of the form

$$\int f(x)\,dx$$

is an indefinite integral.

Indefinite integration

Result

If F is an antiderivative of f then

$$\int f(x)\,dx = F(x) + C.$$

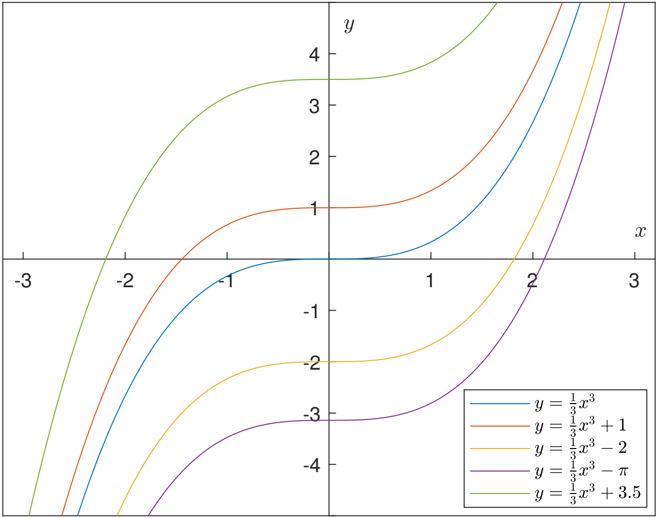
In the above line:

- $\int$ is the integral symbol.
- $f(x)$ is the integrand.
- x in the dx is the variable of integration.
- F is an antiderivative of f.
- C is the constant of integration.

The result of indefinite integration defines a family of functions. The form in which we have written this family is such that it consists of all of the antiderivatives of f if the domain of f is connected.

Indefinite integration: Example

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$



Property

If F is an antiderivative of f and G is an antiderivative of g then

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx = F(x) + G(x) + C$$

and

$$\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx = F(x) - G(x) + C.$$

Property

If F is an antiderivative of f and a is a constant then

$$\int af(x)\,dx = a\int f(x)\,dx = aF(x) + C.$$

Integrals of standard functions I

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \left(\text{since } \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \right).$$

Note that

$$\int 1 \, dx = x + C \quad \left(\text{since } \frac{d}{dx}(x) = 1 \right).$$

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Problem 1

Evaluate $\int (x^2 + x)\, dx$.

$$\int (x^2 + x)\, dx = \int x^2\, dx + \int x\, dx = \frac{x^3}{3} + \frac{x^2}{2} + C$$

Problem 2

Evaluate $\int (3x^6 - 2x^2 + 7x + 1)\, dx$.

$$\begin{aligned}\int (3x^6 - 2x^2 + 7x + 1)\, dx &= 3\int x^6\, dx - 2\int x^2\, dx + 7\int x\, dx + \int 1\, dx \\ &= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C\end{aligned}$$

Problem 3

Evaluate $\displaystyle\int \frac{2x^4 - x^2}{x^4}\,dx.$

$$\int \frac{2x^4 - x^2}{x^4}\,dx = \int \left(2 - \frac{1}{x^2}\right) dx = 2\int 1\,dx - \int x^{-2}\,dx = 2x + \frac{1}{x} + C$$

Problem 4

Evaluate $\displaystyle\int \frac{t^2 - 2t^4}{t^4}\,dt.$

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right) dt = \int t^{-2}\,dt - 2\int 1dt = -\frac{1}{t} - 2t + C$$

Integrals of standard functions II

$$\int \cos(x)\,dx = \sin(x) + C \qquad \left(\text{since } \frac{d}{dx}(\sin(x)) = \cos(x)\right),$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \left(\text{since } \frac{d}{dx}(-\cos(x)) = \sin(x)\right),$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \left(\text{since } \frac{d}{dx}(\tan(x)) = \sec^2(x)\right),$$

$$\int \operatorname{cosec}^2(x)\,dx = -\cot(x) + C \qquad \left(\text{since } \frac{d}{dx}(-\cot(x)) = \operatorname{cosec}^2(x)\right),$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C \qquad \left(\text{since } \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)\right),$$

$$\int \operatorname{cosec}(x)\cot(x)\,dx = -\operatorname{cosec}(x) + C \qquad \left(\text{since } \frac{d}{dx}(-\operatorname{cosec}(x)) = \operatorname{cosec}(x)\cot(x)\right).$$

Problem 5

Evaluate $\displaystyle\int 4\cos(x)\,dx$.

$$\int 4\cos(x)\,dx = 4\int \cos(x)\,dx = 4\sin(x) + C$$

Problem 6

Evaluate $\displaystyle\int \frac{\cos(x)}{\sin^2(x)}\,dx$.

$$\int \frac{\cos(x)}{\sin^2(x)}\,dx = \int \operatorname{cosec}(x)\cot(x)\,dx = -\operatorname{cosec}(x) + C$$

Integrals of standard functions III

For $a > 0$ but $a \neq 1$,

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C \quad \left(\text{since } \frac{d}{dx}\left(\frac{a^x}{\ln(a)}\right) = a^x\right).$$

Note that

$$\int e^x \, dx = e^x + C \quad \left(\text{since } \frac{d}{dx}(e^x) = e^x\right).$$

Problem 7

Evaluate $\int \left(\frac{2^x}{3} + x^2 + e^x + x^e \right) dx$.

$$\begin{aligned}
& \int \left(\frac{2^x}{3} + x^2 + e^x + x^e \right) dx \\
= \; & \frac{1}{3} \int 2^x \, dx + \int x^2 \, dx + \int e^x \, dx + \int x^e \, dx \\
= \; & \frac{2^x}{3 \ln(2)} + \frac{x^3}{3} + e^x + \frac{x^{e+1}}{e+1} + C
\end{aligned}$$

Integrals of standard functions IV

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C \quad \left(\text{since } \frac{d}{dx}\big(\sin^{-1}(x)\big) = \frac{1}{\sqrt{1-x^2}}\right),$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C \quad \left(\text{since } \frac{d}{dx}\big(\tan^{-1}(x)\big) = \frac{1}{1+x^2}\right).$$

Problem 8

Evaluate $\displaystyle\int \frac{x^2}{x^2+1}\,dx$.

$$\begin{aligned}
\int \frac{x^2}{x^2+1}\,dx &= \int \frac{x^2+1-1}{x^2+1}\,dx \\
&= \int \left(1-\frac{1}{x^2+1}\right)dx \\
&= \int 1\,dx - \int \frac{1}{x^2+1}\,dx \\
&= x - \tan^{-1}(x) + C
\end{aligned}$$

Problem 9

Evaluate $\displaystyle\int \frac{1-x^2-x^4}{1+x^2}\,dx$.

$$
\begin{aligned}
\int \frac{1-x^2-x^4}{1+x^2}\,dx &= \int \frac{1-x^2(1+x^2)}{1+x^2}\,dx \\
&= \int \left(\frac{1}{1+x^2} - x^2\right) dx \\
&= \int \frac{1}{1+x^2}\,dx - \int x^2\,dx \\
&= \tan^{-1}(x) - \frac{x^3}{3} + C
\end{aligned}
$$

Integrals of standard functions V

If $x > 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

If $x < 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{\frac{d}{dx}(-x)}{-x} = \frac{-1}{-x} = \frac{1}{x}.$$

Hence,

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

However, while the above line is sufficient for this module, we note that

$$F(x) = \begin{cases} \ln|x| + C_1 \text{ if } x < 0, \\ \ln|x| + C_2 \text{ if } x > 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants, is such that

$$F'(x) = \frac{1}{x}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Problem 10

Evaluate $\displaystyle\int \frac{(x+1)^2}{x^2}\,dx$.

$$\begin{aligned}
\int \frac{(x+1)^2}{x^2}\,dx &= \int \frac{x^2+2x+1}{x^2}\,dx \\
&= \int \left(1+\frac{2}{x}+\frac{1}{x^2}\right) dx \\
&= \int 1\,dx + 2\int \frac{1}{x}\,dx + \int x^{-2}\,dx \\
&= x + 2\ln|x| - \frac{1}{x} + C
\end{aligned}$$

Integration by substitution: Justification

Suppose that F is an antiderivative of f. Then, for x for which $g(x)$ is in the domain of f and $g'(x)$ exists,

$$\frac{d}{dx}(F(g(x))) = f(g(x))g'(x)$$

by the chain rule, and hence,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Also, since F is an antiderivative of f,

$$\int f(t)\,dt = F(t) + C.$$

Therefore, with $t = g(x)$,

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt.$$

Integration by substitution: Procedure

Suppose that we want to evaluate an integral of the form

$$\int f(g(x))g'(x)\,dx.$$

A procedure for doing this using substitution is:

1. Identify $g(x)$.
2. Let $t = g(x)$.
3. Obtain a relationship between dx and dt, for example $dt = g'(x)\,dx$.
4. Perform substitutions to convert the integral to one where the integrand is in terms of t (x should not appear) and the variable of integration is t (there should be a dt and not a dx).
5. Perform the integration with respect to t.
6. Convert the result of the integration to be in terms of the original variable, x.

Integration by substitution: Table of example substitutions

Integrand	Substitution to try
$f(x^n)x^{n-1}$	$t = x^n$
$f(\sin(x))\cos(x)$	$t = \sin(x)$
$f(\tan(x))\sec^2(x)$	$t = \tan(x)$
$\dfrac{f(\ln(x))}{x}$	$t = \ln(x)$
$\dfrac{f(\sqrt{x})}{\sqrt{x}}$	$t = \sqrt{x}$
$\dfrac{f(\tan^{-1}(x))}{1+x^2}$	$t = \tan^{-1}(x)$

Problem 11

Evaluate $\int \cos(e^x)e^x \, dx$.

Let $t = e^x$.
Then $\frac{dt}{dx} = e^x$ and $dt = e^x \, dx$.

$$\therefore \int \cos(e^x)e^x \, dx = \int \cos(t) \, dt = \sin(t) + C = \sin(e^x) + C$$

Problem 12

Evaluate $\int e^{\sin(x)} \cos(x)\, dx$.

Let $t = \sin(x)$.
Then $\dfrac{dt}{dx} = \cos(x)$ and $dt = \cos(x)\, dx$.

$$\therefore \int e^{\sin(x)} \cos(x)\, dx = \int e^t\, dt = e^t + C = e^{\sin(x)} + C$$

Problem 13

Evaluate $\displaystyle\int \frac{\sec^2(x)}{\sqrt{\tan(x)}}\,dx$.

Let $t = \tan(x)$.

Then $\dfrac{dt}{dx} = \sec^2(x)$ and $dt = \sec^2(x)\,dx$.

$$\therefore \int \frac{\sec^2(x)}{\sqrt{\tan(x)}}\,dx = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2}\,dt = 2t^{1/2} + C = 2\sqrt{\tan(x)} + C$$

Problem 14

Evaluate $\int \frac{1}{x \ln(x)} \, dx$.

Let $t = \ln(x)$.
Then $\frac{dt}{dx} = \frac{1}{x}$ and $dt = \frac{1}{x} \, dx$.

$$\therefore \int \frac{1}{x \ln(x)} \, dx = \int \frac{1}{t} \, dt = \ln |t| + C = \ln |\ln(x)| + C$$

Problem 15

Evaluate $\displaystyle\int \frac{\left(\tan^{-1}(x)\right)^2}{1+x^2}\,dx$.

Let $u = \tan^{-1}(x)$.

Then $\dfrac{du}{dx} = \dfrac{1}{1+x^2}$ and $du = \dfrac{1}{1+x^2}\,dx$.

$$\therefore \int \frac{\left(\tan^{-1}(x)\right)^2}{1+x^2}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\left(\tan^{-1}(x)\right)^3}{3} + C$$

Problem 16

Evaluate $\int x \sec^2\left(x^2\right) dx$.

Let $t = x^2$.
Then $\dfrac{dt}{dx} = 2x$ and $dt = 2x\,dx$.

$$\begin{aligned}
\therefore \int x \sec^2\left(x^2\right) dx &= \frac{1}{2}\int 2x \sec^2\left(x^2\right) dx \\
&= \frac{1}{2}\int \sec^2(t)\,dt \\
&= \frac{1}{2}\tan(t) + C \\
&= \frac{1}{2}\tan\left(x^2\right) + C
\end{aligned}$$

Problem 17

Evaluate $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

Let $t = \sqrt{x}$.
Then $\dfrac{dt}{dx} = \dfrac{1}{2\sqrt{x}}$ and $2dt = \dfrac{1}{\sqrt{x}}\, dx$.

$$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \int 2e^t\, dt = 2\int e^t\, dt = 2e^t + C = 2e^{\sqrt{x}} + C$$

Problem 18

Use the substitution $t = (x-1)^{3/2}$ to evaluate $\displaystyle\int x^2\sqrt{x-1}\,dx$.

Let $t = (x-1)^{3/2}$. Then $\dfrac{dt}{dx} = \dfrac{3}{2}\sqrt{x-1}$ and $\dfrac{2}{3}dt = \sqrt{x-1}\,dx$.
Moreover, $x^2 = \left(t^{2/3}+1\right)^2 = t^{4/3} + 2t^{2/3} + 1$.

$$
\begin{aligned}
\therefore \int x^2\sqrt{x-1}\,dx &= \int \frac{2}{3}\left(t^{4/3} + 2t^{2/3} + 1\right) dt \\
&= \frac{2}{3}\int t^{4/3}\,dt + \frac{4}{3}\int t^{2/3}\,dt + \frac{2}{3}\int 1\,dt \\
&= \frac{2t^{7/3}}{7} + \frac{4t^{5/3}}{5} + \frac{2}{3}t + C \\
&= \frac{2(x-1)^{7/2}}{7} + \frac{4(x-1)^{5/2}}{5} + \frac{2(x-1)^{3/2}}{3} + C
\end{aligned}
$$

Problem 19

Use the substitution $t = \sqrt{x-1}$ to evaluate $\displaystyle\int x^2\sqrt{x-1}\,dx$.

Let $t = \sqrt{x-1}$. Then $t^2 = x - 1$ and $x = t^2 + 1$. Hence, $x^2\sqrt{x-1} = \left(t^2+1\right)^2 t = t^5 + 2t^3 + t$. Also, $\dfrac{dx}{dt} = 2t$ and $dx = 2t\,dt$.

$$
\begin{aligned}
\therefore \int x^2\sqrt{x-1}\,dx &= \int 2\left(t^5 + 2t^3 + t\right)t\,dt \\
&= 2\int \left(t^6 + 2t^4 + t^2\right)dt \\
&= \frac{2t^7}{7} + \frac{4t^5}{5} + \frac{2t^3}{3} + C \\
&= \frac{2(x-1)^{7/2}}{7} + \frac{4(x-1)^{5/2}}{5} + \frac{2(x-1)^{3/2}}{3} + C
\end{aligned}
$$

Class activity

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\big(x^2+1\big) + C \tag{1}$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\big(2x^2+2\big) + C \tag{2}$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\left(\frac{x^2+1}{2}\right) + C \tag{3}$$

Which, if any, of the above are correct?

Integration of a function of a linear polynomial: Justification

Suppose that a is a nonzero real number and that b is a real number. Also, suppose that

$$\int f(x)\,dx = F(x) + C.$$

Let $t = ax + b$.
Then $\dfrac{dt}{dx} = a$ and $dx = \dfrac{1}{a}\,dt$.

$$\therefore \int f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt = \frac{1}{a}F(t) + C = \frac{1}{a}F(ax+b) + C$$

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Integration of a function of a linear polynomial: Result

Result

If a is a nonzero real number, b is a real number and

$$\int f(x)\,dx = F(x) + C$$

then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

Problem 20

Evaluate $\int \cos(2x+3)\,dx$.

$$\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$$

Problem 21

Evaluate $\int \frac{1}{5x-7}\,dx$.

$$\int \frac{1}{5x-7}\,dx = \frac{1}{5}\ln|5x-7| + C$$

Problem 22

Evaluate $\int e^{4x-9}\,dx$.

$$\int e^{4x-9}\,dx = \frac{1}{4}e^{4x-9} + C$$

Problem 23

Evaluate $\int \sec^2(3x+5)\,dx$.

$$\int \sec^2(3x+5)\,dx = \frac{1}{3}\tan(3x+5) + C$$

Extended topic: Trigonometric substitution

Integrand contains	Substitution to try
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$x = a\sin(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$
$\dfrac{1}{x^2 + a^2}$	$x = a\tan(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$x = \begin{cases} a\sec(t) \text{ with } 0 < t < \dfrac{\pi}{2} \text{ if } \dfrac{x}{a} > 1 \\ a\sec(t) \text{ with } \dfrac{\pi}{2} < t < \pi \text{ if } \dfrac{x}{a} < -1 \end{cases}$

Problem 24

Show that, for all positive real numbers a,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Let $x = a\sin(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Then $t = \sin^{-1}\left(\dfrac{x}{a}\right)$ since t is in the range of $\sin^{-1}$.

Moreover, $\dfrac{dx}{dt} = a\cos(t)$ and $dx = a\cos(t)\,dt$.

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Now,

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2 - x^2}}\, dx &= \int \frac{a\cos(t)}{\sqrt{a^2 - a^2\sin^2(t)}}\, dt \\
&= \int \frac{a\cos(t)}{\sqrt{a^2\left(1 - \sin^2(t)\right)}}\, dt \\
&= \int \frac{a\cos(t)}{\sqrt{a^2}\sqrt{\cos^2(t)}}\, dt \\
&= \int \frac{a\cos(t)}{a\cos(t)}\, dt
\end{aligned}$$

since $a > 0$ and $\cos(t) > 0$ as $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int 1\, dt = t + C = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Problem 25

Show that, for all nonzero real numbers a,

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Let $x = a\tan(t)$ with $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.

Then $t = \tan^{-1}\left(\dfrac{x}{a}\right)$ since t is in the range of $\tan^{-1}$.

Moreover, $\dfrac{dx}{dt} = a\sec^2(t)$ and $dx = a\sec^2(t)\,dt$.

Now,

$$\begin{aligned}\int \frac{1}{x^2+a^2}\,dx &= \int \frac{a\sec^2(t)}{a^2\tan^2(t)+a^2}\,dt\\ &= \int \frac{a\sec^2(t)}{a^2(\tan^2(t)+1)}\,dt\\ &= \int \frac{a\sec^2(t)}{a^2\sec^2(t)}\,dt\\ &= \int \frac{1}{a}\,dt.\end{aligned}$$

$$\therefore \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\int 1\,dt = \frac{1}{a}t + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Problem 26

Show that, for all positive real numbers a,

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \sec^{-1}\left(\frac{|x|}{a}\right) + C$$

if $x > a$.

Let $x = a\sec(t)$ with $0 < t < \frac{\pi}{2}$.

Then, since t is in the range of $\sec^{-1}$, $t = \sec^{-1}\left(\frac{x}{a}\right) = \sec^{-1}\left(\frac{|x|}{a}\right)$.

Moreover, $\frac{dx}{dt} = a\sec(t)\tan(t)$ and $dx = a\sec(t)\tan(t)\, dt$.

Now,

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-a^2}}\,dx &= \int \frac{a\sec(t)\tan(t)}{a\sec(t)\sqrt{a^2\sec^2(t)-a^2}}\,dt \\ &= \int \frac{a\sec(t)\tan(t)}{a\sec(t)\sqrt{a^2(\sec^2(t)-1)}}\,dt \\ &= \int \frac{\tan(t)}{\sqrt{a^2}\sqrt{\tan^2(t)}}\,dt \\ &= \int \frac{\tan(t)}{a\tan(t)}\,dt\end{aligned}$$

since $a > 0$ and $\tan(t) > 0$ as $0 < t < \frac{\pi}{2}$.

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\int 1\,dt = \frac{1}{a}t + C = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$$

Problem 27

Show that, for all positive real numbers a,

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \sec^{-1}\left(\frac{|x|}{a}\right) + C$$

if $x < -a$.

Let $x = a\sec(t)$ with $\frac{\pi}{2} < t < \pi$.

Then, since t is in the range of $\sec^{-1}$, $t = \sec^{-1}\left(\frac{x}{a}\right) = \pi - \sec^{-1}\left(\frac{|x|}{a}\right)$.

Moreover, $\frac{dx}{dt} = a\sec(t)\tan(t)$ and $dx = a\sec(t)\tan(t)\, dt$.

Now,

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2-a^2}}\,dx &= \int \frac{a\sec(t)\tan(t)}{a\sec(t)\sqrt{a^2\sec^2(t)-a^2}}\,dt \\
&= \int \frac{a\sec(t)\tan(t)}{a\sec(t)\sqrt{a^2(\sec^2(t)-1)}}\,dt \\
&= \int \frac{\tan(t)}{\sqrt{a^2}\sqrt{\tan^2(t)}}\,dt \\
&= \int \frac{\tan(t)}{-a\tan(t)}\,dt
\end{aligned}$$

since $a > 0$ and $\tan(t) < 0$ as $\frac{\pi}{2} < t < \pi$.

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = -\frac{1}{a}\int 1\,dt = -\frac{1}{a}t + c = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$$

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