Lecture 7

Topics covered in this lecture session

1. Numerical Methods – An Introduction.
2. Bisection Method
3. Use of Calculator for Numerical Methods.
4. Iteration Method.

Numerical Methods – An Introduction

Some equations cannot be solved easily and some equations (e.g. $2\sin\theta = \theta$) are not possible to solve using the analytic methods (methods to find the exact solutions of equations (Quadratic equations, Trigonometric equations, Polynomial equations)).

Numerical methods are alternatives to such problems. These methods will allow us to find approximate roots of equations to any degree of accuracy.

Numerical Methods – An Introduction

Note:

Note that the roots of the equation $f(x) = 0$ are the values of x where the curve $y = f(x)$ cuts the X-axis.

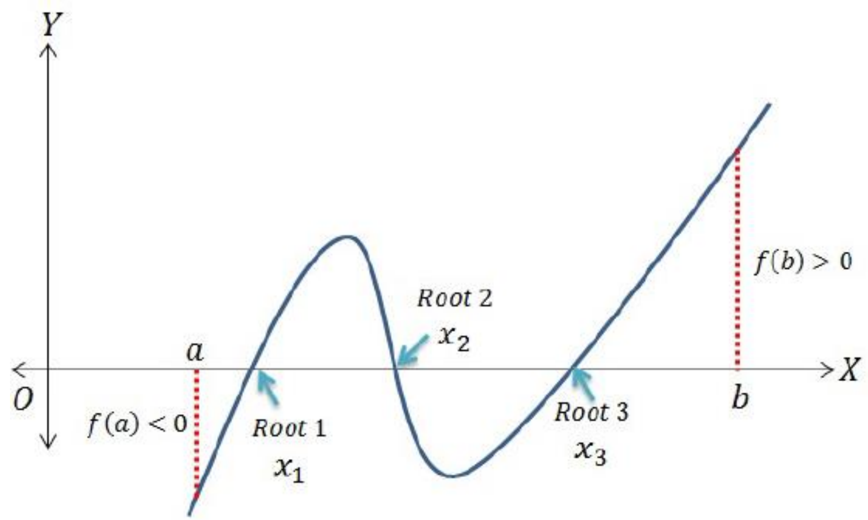
A fundamental result in the study of numerical methods is the Intermediate Value Theorem, which gives location of roots in the form of an interval.

Intermediate Value Theorem

If two values a and b can be found such that

(i) $a < b$ and

(ii) $f(a)$ and $f(b)$ have **different** signs.



Then, f(x) = 0 has at least one root in (a; b), provided that f(x) is continuous in the interval (a; b).

Example Show that $f(x) = x^3 - 3x^2 + 2 = 0$ has a root between $x = 2$ and $x = 3$.

Intermediate Value Theorem

Example Show that $f(x) = x^3 - 3x^2 + 2 = 0$ has a root between $x = 2$ and $x = 3$.

Solution:

$$f(2) = 2^3 - 3 \cdot 2^2 + 2 = -2$$

$$f(3) = 3^3 - 3 \cdot 2^3 + 2 = 2$$

$$\therefore f(2) \cdot f(3) = -4 < 0$$

There is at least a root in $(2, 3)$

Bisection Method

The Bisection method is useful to find an approximate solution/root of an equation of the form

$$f(x) = 0$$

where f is the given function.

It is based on

- Intermediate Value Theorem, and
- Bisection of an interval.

Bisection Method

1. We assume that the function f defined on $[a, b]$ is continuous with $f(a)$ and $f(b)$ having opposite signs.

2. In this method, we repeatedly halve/bisect the interval at each step and locate the half containing the required root x^*.

Bisection Method

Step 1

Choose a and b such that $f(a) \cdot f(b) < 0$.

i.e. signs of $f(a)$ and $f(b)$ are different.

so that the required root lies in the interval $[a, b]$.

Step 2

Take approximate solution of $f(x) = 0$ as $c = \dfrac{a+b}{2}$.

If $f(c) = 0$, then the required root is $x^* = c$.

If $f(c) \neq 0$, go to Step 3.

Bisection Method

Step 3

If $f(a) \cdot f(c) < 0$, the root x^* lies between a and c.

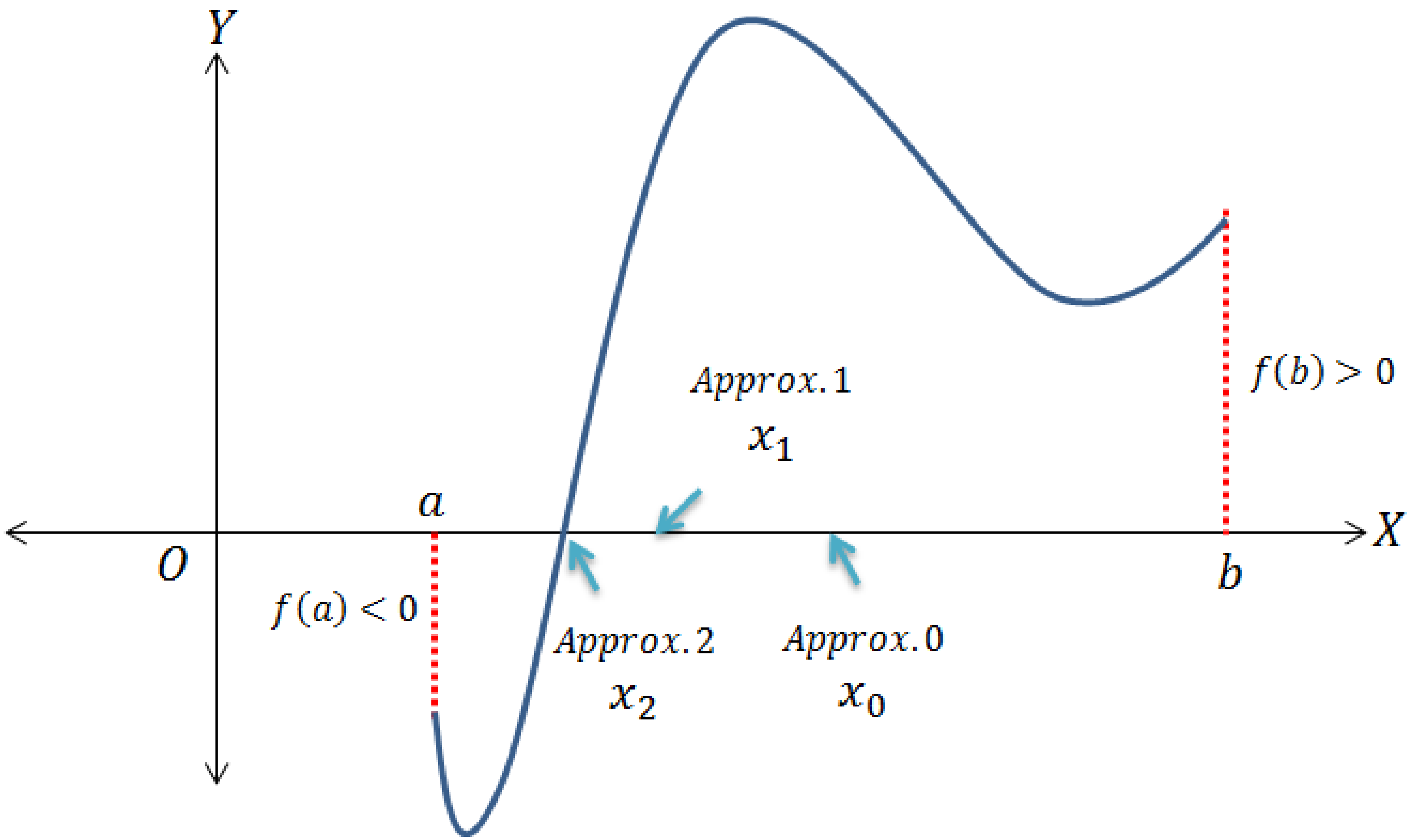
Replace b by c, that is, the required root $x^* \in [\,a, c\,]$.

If $f(b) \cdot f(c) < 0$, the root x^* lies between c and b.

Replace a by c, that is, the required root $x^* \in [\,c, b\,]$.

Repeat Steps 2 and 3 until the root with desired accuracy is obtained.

Bisection Method



Example

1. Use the Bisection method to calculate the root, correct to 4 decimal places, of the equation

$$x^3 - 3x^2 + 2 = 0$$

which lies between 2 and 3.

Solution: Here, $f(2) = -2$ and $f(3) = 2$

$$\Rightarrow f(2)\,f(3) < 0$$

$\therefore$ Root lies between 2 and 3.

Bisection Method

$$f(x) = x^3 - 3x^2 + 2 = 0$$

n	a	b	$x_n = c = \dfrac{a+b}{2}$	$f(a)$	$f(b)$	$f(c)$	$Replace$
0	2	3	$\frac{2+3}{2} = 2.5$	-2	2	-1.125	a by c
1	2.5	3	$\frac{2.5+3}{2} = 2.75$	-1.125	2	0.10938	b by c
2	2.5	2.75	$\frac{2.5+2.75}{2} = 2.625$	-1.125	0.10938	-0.58398	a by c
…	…….	…….	……………………………………	……..	……..	……..	……..
…	…….	…….	……………………………………	……..	……..	……..	……..
14	2.73199	2.73206	$\frac{2.73199+2.73206}{2} = 2.73203$	-0.00034	0.00003	-0.00015	a by c
15	2.73203	2.73206	$\frac{2.73199+2.73206}{2} = 2.73203$	Further calculation not required			

Bisection Method

What is meant by root correct to n decimal places (d.p) ?

It means the absolute difference between the calculated root and actual root is ≤ 0.0001

i.e. $|\text{calculated root} - \text{actual root}| \leq 10^{-4}$

But, in most cases, we do not know the actual root.

In such instances, if you are asked to obtain the root correct to n d.p., fix your calculator to $(n+1)$ digits.

Bisection Method

Note:

If the equation involves Trigonometric functions, set the calculator to Radian mode. $\boxed{\text{shift}}$ $\boxed{\text{mode}}$ $\boxed{4}$

However, if the angle is given in degree measure, change the calculator to Degree mode. $\boxed{\text{shift}}$ $\boxed{\text{mode}}$ $\boxed{3}$

Advantages:

(i) The Bisection method is always convergent.

(ii) The error bound decreases by half with each iteration.

Drawbacks:

(i) The method converges very slowly.

(ii) The bisection method cannot give multiple roots.

Iteration Method (Fixed-point Iteration)

The Iteration method start by transforming algebraically the given equation $f(x) = 0$ in the form

$$x = g(x)$$

By assuming a guess value x_0, the method is aimed at calculating better and better approximations

$$x_1, \; x_2, \dots \dots .$$

to the solution of $f(x) = 0$.

Iteration Method (Fixed-point Iteration)

Step 1

Transform equation (1) algebraically in the form $x = g(x)$.

Step 2

Choose x_0 Hint: Take $x_0 = \dfrac{a+b}{2}$ if the interval $[a, b]$ is given

or use IVT to find a and b

Step 3

Calculate $x_1 = g\left(x_0\right)$, $x_2 = g\left(x_1\right)$, and in general

$$x_{n+1} = g\left(x_n\right)$$

Repeat the process until the root with desired accuracy is obtained.

Iteration Method (Fixed-point Iteration)

Example

1. Use the Iteration method to calculate the root, correct to 4 decimal places, of the equation

$$x^3 - 3x^2 + 2 = 0$$

which lies between 2 and 3.

Solution: Rearranging the given equation in the form

$$x = g(x) \;\Rightarrow\; x = g(x) = 3 - \frac{2}{x^2}$$

Iteration Method (Fixed-point Iteration)

Thus, the iterative formula obtained is:

$$x_{n+1} = g(x_n) = 3 - \frac{2}{x_n^2}$$

n	x_n	x_{n+1}
0	2.5	2.68
1	2.68	2.72154
2	2.72154	2.72998
3	2.72998	2.73164
4	2.73164	2.73197
5	2.73197	2.73204
6	2.73204	2.73205

Take $x_0 = \frac{2+3}{2} = 2.5$

because the root lies

between 2 and 3

Iteration Method (Fixed-point Iteration)

From the above example, it is clear that the Iteration method converges much faster in comparison to the Bisection method (which took 15 approximations).

Convergence

The iterative formula obtained in previous example is NOT the only available formulation.

We can obtain a number of different forms of the iterative formula.

Iteration Method (Fixed-point Iteration)

$$x_{n+1} = g_1(x_n) = 3 - \frac{2}{x_n^2}$$

$$x_{n+1} = g_2(x_n) = \sqrt[3]{3x_n^2 - 2}$$

$$x_{n+1} = g_3(x_n) = \sqrt{\frac{2}{3 - x_n}}$$

$$x_{n+1} = g_4(x_n) = \sqrt{\frac{x_n^3 + 2}{3}}$$

The behaviour of corresponding iterative sequences

$$x_0 , x_1 , \quad \dots \dots$$

with regard to different iterative formulae, may differ, in particular, with respect to their speed of convergence.

Iteration Method (Fixed-point Iteration)

Some iterative formula may converge to the other root, and some may NOT converge at all.

e.g. If we use the iterative formula

$$x_{n+1} = g_1(x_n) = \frac{1}{1+x_n^2}$$

to solve the equation $x^3 + x - 1 = 0$

then the successive approximations are:

$$x_0 = 1, \quad x_1 = 0.5, \quad x_2 = 0.8, \quad x_3 = 0.610,$$
$$x_4 = 0.729, \quad x_5 = 0.653, \quad x_6 = 0.701, \ldots..$$

which approaches the solution $x = 0.682328$ of the equation $x^3 + x = 1$

Iteration Method (Fixed-point Iteration)

However, if we use the iterative formula

$$x_{n+1} = g_2(x_n) = 1 - x_n^3$$

then the successive approximations are:

$$x_0 = 1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 0,$$
$$x_4 = 1, \quad x_5 = 0, \quad x_6 = 1, \ldots..$$

which oscillates between 0 and 1; and so the sequence of approximations does not converge.

Iteration Method (Fixed-point Iteration)

Example:

Consider solving numerically the equation

$$2x^3 - x - 4 = 0. \qquad (1)$$

Show that equation (1) can be rearranged to give the iterative formula

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{1}{2}}. \qquad (2)$$

Use (2) with $x_0 = 1.35$ to find the root of (1), correct to 3 decimal places.

Iteration Method (Fixed-point Iteration)

Example:

Consider solving numerically the equation

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Use (2) with $x_0 = 1.35$ to find the root of (1), correct to 3 decimal places.

THANKS FOR YOUR ATTENTION