Lecture 04

Structure of lecture

1. Higher-order derivatives
2. Polynomial approximations
3. Maclaurin series
4. Parametric differentiation
5. Related rates problem

Early Module Feedback (EMF) (link)

Early Module Feedback

On Moodle page

Available from 09:00 Monday 10 March - 17:00 Sunday 16 March 2025

1. The module content was of sufficient quality to assist my learning on this module
2. Module materials were clear about what was expected of me.
3. I was given sufficient opportunity to contact my teachers/faculty on this module
4. The overall experience of studying this module has contributed to my learning
5. In your opinion, what is working well on the module so far? If there are any suggestions for the remaining weeks on the module, please also leave your comments here.

You are encouraged to complete this survey, as your views are important to us.

Higher-order derivatives

For a continuously differentiable function $y = f(x)$, the first and successive derivatives are denoted by:

$$\frac{dy}{dx}, \ \frac{d^2y}{dx^2}, \ \frac{d^3y}{dx^3}, \ \ldots\ldots, \ \frac{d^ny}{dx^n}$$

$$\text{or} \qquad f'(x), f''(x), f'''(x), f^{(4)}(x), \cdots, f^{(n)}(x)$$

The value of the n^{th} derivative when $x = a$ is denoted by

$$\left.\frac{d^ny}{dx^n}\right|_{x=a} \qquad \text{or} \qquad f^{(n)}(a)$$

Higher-order derivatives

Example: Given $y = \sec x$. Show that $\left.\dfrac{d^3y}{dx^3}\right|_{x=\pi/4} = 11\sqrt{2}$

$$y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \cdot \tan x$$

$$\therefore \quad \frac{d^2y}{dx^2} = \sec x \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(\sec x)$$

$$= \sec x \cdot (\sec^2 x) + \tan x \cdot (\sec x \, \tan x)$$

Higher-order derivatives

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec^3 x + \sec x \cdot \tan^2 x$$

$$= \sec^3 x + \sec x \cdot \left(\sec^2 x - 1\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2\sec^3 x - \sec x$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 6\sec^2 x \cdot \frac{d}{dx}(\sec x) - \sec x \tan x$$

$$= 6\sec^2 x \cdot (\sec x \, \tan x) - \sec x \tan x$$

Higher-order derivatives

$$\therefore \quad \left.\frac{d^3y}{dx^3}\right|_{x=\pi/4} = 6\sec^2\left(\frac{\pi}{4}\right)\cdot\left[\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)\right] - \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$$

$$= 6\,(2)\cdot\left(\sqrt{2}\right) - \sqrt{2}$$

$$= 11\sqrt{2}$$

Higher-order derivatives

Exercises:

1. Given $y = x \cdot \ln x$, find $\dfrac{d^3y}{dx^3}$.

   Answer: $\dfrac{d^3y}{dx^3} = -\dfrac{1}{x^2}$

2. Given $y = x^6 - 2x^4 + x^2 - 2$, find $\left.\dfrac{d^2y}{dx^2}\right|_{x=2}$.

   Answer: $\left.\dfrac{d^2y}{dx^2}\right|_{x=2} = 386$

Linear approximations

Much of the modern engineering/science analysis relies on numerical calculations, nearly always only approximate. Therefore, we often find the simpler functions to approximate the complicated ones.

The **tangent line approximation** or **linear approximation** of f at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

For values of x near a we can approximate values of $f(x)$ by

$$f(x) \approx f(a) + f'(a)(x - a)$$

Tangent line at $x = a$

Linear approximations

Example:

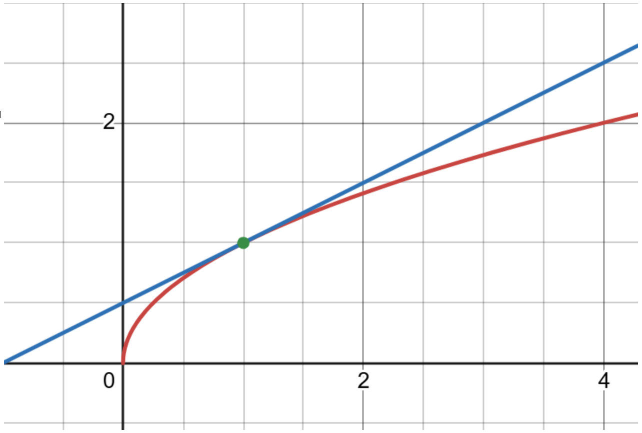
(a) Find the linear approximation of $f(x) = \sqrt{x}$ at $a = 1$.

(b) Use it to approximate the value of $\sqrt{1.1}$.

Solution:

(a) $\sqrt{x} \approx \sqrt{1} + \frac{1}{2\sqrt{1}}(x-1) = 1 + \frac{1}{2}(x-1)$.

(b) $\sqrt{1.1} \approx 1 + \frac{1}{2}(1.1-1) = 1.05$.



Approximation is slightly larger than the actual value

Quadratic approximations

Approximations by linear functions can be insufficiently accurate. Similarly, we can find approximations by polynomials of a higher order.

Suppose $f(x)$ is twice differentiable and we want to approximate it near $x = a$ by a quadratic polynomial:

$$p(x) = A + B(x-a) + C(x-a)^2$$

such that $f(a) = p(a)$, $f'(a) = p'(a)$, and $f''(a) = p''(a)$.

We see that $p'(x) = B + 2C(x-a)$ and $p''(x) = 2C$.

So, $A = p(a) = f(a)$, $B = p'(a) = f'(a)$ and $C = \frac{1}{2}p''(a) = \frac{1}{2}f''(a)$.

Quadratic approximations

For values of x near a, we can approximate $f(x)$ by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

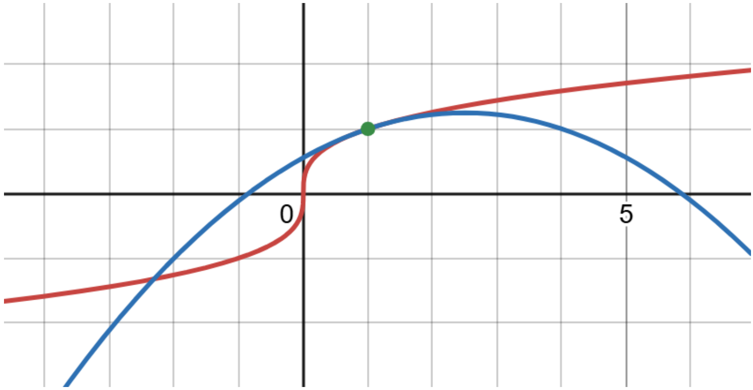
Example:

Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ at $a = 1$.

Solution:

$$\sqrt[3]{x} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$$



Higher-order approximations

We can improve the accuracy of an approximation by using a polynomial of a higher degree: $3, 4, 5, \ldots, n$.

If f can be differentiated n times at $a = 0$, then we define the $\boldsymbol{nth}$ **Maclaurin polynomial** for f to be

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

So, $f(x) \approx p(x)$, where p satisfies: $f(0) = p(0)$, $f'(0) = p'(0)$, $f''(0) = p''(0), \ldots, f^{(n)}(0) = p^{(n)}(0)$.

Maclaurin series

The Maclaurin's series of a continuously differentiable function f is an **infinite** series given by

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2\,!}\, f''(0) + \frac{x^3}{3\,!}\, f'''(0) \ldots\ldots + \frac{x^n}{n\,!}\, f^{(n)}(0) + \ldots\ldots$$

provided that $f(0),\ f'(0),\ f''(0),\ \ldots.,\ f^{(n)}(0)$ all have finite values.

Maclaurin series

Example: Find the Maclaurin's series for $f(x) = \sin x$.

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

Maclaurin series

Then, Maclaurin's series expansion of f is:

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2\,!}\, f''(0) + \frac{x^3}{3\,!}\, f'''(0)\ldots.. + \frac{x^n}{n\,!}\, f^{(n)}(0) + \ldots\ldots$$

$$\therefore\ f(x) = \sin x$$

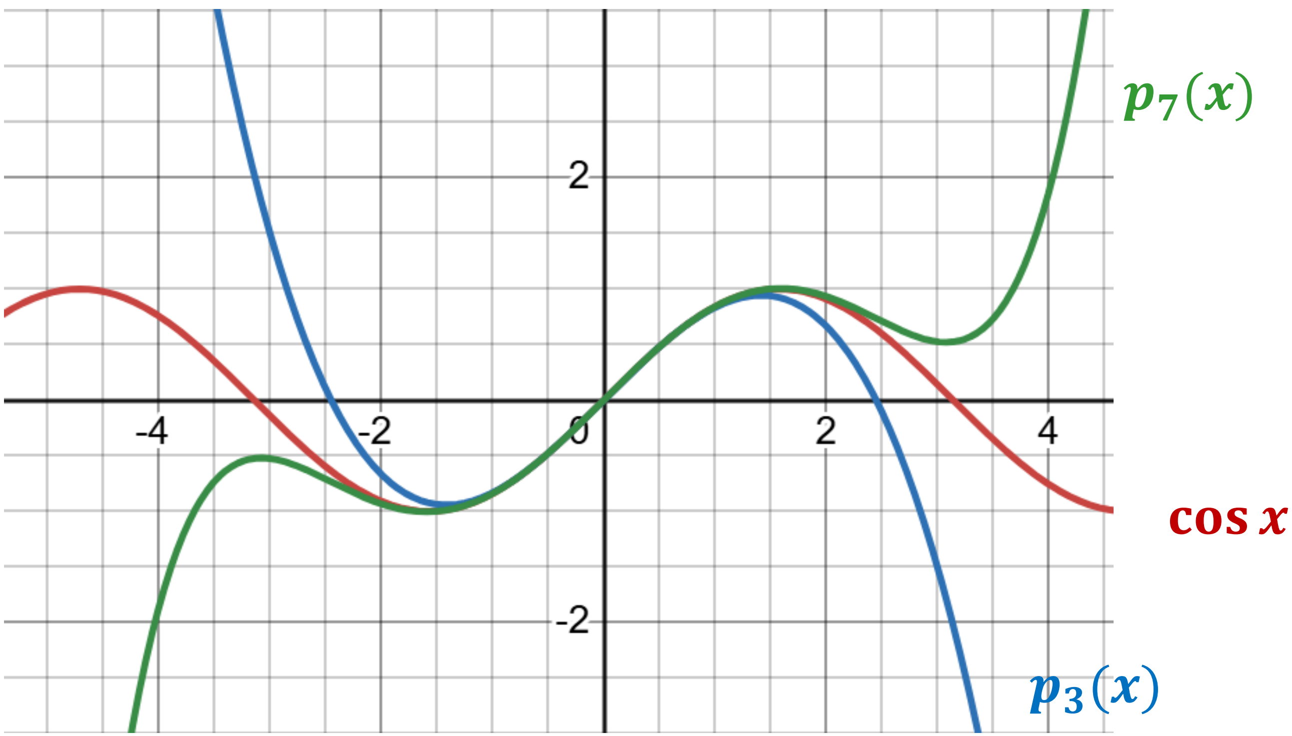
$$= 0 + x\,(1) + \frac{x^2}{2\,!}\,(0) + \frac{x^3}{3\,!}\,(-1) + \frac{x^4}{4\,!}\,(0) + \frac{x^5}{5\,!}\,(1) + \ldots.$$

$$\Rightarrow \sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \ldots..$$

$$\sin\left(\frac{\pi}{4}\right) \approx p_7\left(\frac{\pi}{4}\right) = 0.7071064696$$

Maclaurin series

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots..$$



Maclaurin series

Example: Show that when x is near 0,

$$e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2.$$

Let

$$f(x) = e^x \qquad f(0) = 1$$
$$f'(x) = e^x \qquad f'(0) = 1$$
$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1$$
$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$$

Maclaurin series

$$\boxed{f(x) = f(0) + x\, f'(0) + \frac{x^2}{2\,!}\, f''(0) + \frac{x^3}{3\,!}\, f'''(0)\ldots\ldots + \frac{x^n}{n\,!}\, f^{(n)}(0) + \ldots\ldots}$$

$$\therefore\ f(x) = e^x$$

$$= 1 + x\,(1) + \frac{x^2}{2\,!}\,(1) + \frac{x^3}{3\,!}\,(1) + \frac{x^4}{4\,!}\,(1) + \frac{x^5}{5\,!}\,(1) + \ldots.$$

$$\therefore\ e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \ldots.$$

Maclaurin series

$$\therefore\ e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \ldots.$$

$$\therefore\ e^{2x} = 1 + 2x + \frac{(2x)^2}{2\,!} + \ldots. = 1 + 2x + 2x^2 + \ldots.$$

and

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2\,!} + \ldots. = 1 - x + \frac{x^2}{2} + \ldots.$$

$$\therefore\ e^{2x} - e^{-x} = 3x + \frac{3}{2}x^2 + \ldots..$$

Thus, $e^{2x} - e^{-x} \approx 3x + \dfrac{3}{2}x^2.$

Maclaurin series

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2\,!}\, f''(0) + \frac{x^3}{3\,!}\, f'''(0) \ldots\ldots + \frac{x^n}{n\,!}\, f^{(n)}(0) + \ldots\ldots$$

Exercise:

Find the Maclaurin series for $f(x) = (1+x)^n$.

Answer: $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

This is also called a binomial series.

Maclaurin series

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{2\,!}\ f''(0) + \frac{x^3}{3\,!}\ f'''(0)\ldots.. + \frac{x^n}{n\,!}\ f^{(n)}(0) + \ldots\ldots$$

Exercise:

Expand $f(x) = \cos x$ using Maclaurin's series.

Answer: $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

Taylor series (extended topic)

Maclaurin series is an expansion around $x = 0$. If we want to approximate values of $f(x)$ in the neighborhood of $x = a$, we can use the **Taylor series**:

If f has derivatives of all orders at $x = a$, then the Taylor series for f about $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor series (extended topic)

$$f(x) = f(a) + (x-a)\ f'(a) + \frac{(x-a)^2}{2\,!}\ f''(a) + \frac{(x-a)^3}{3\,!}\ f'''(a)\ldots.. + \frac{(x-a)^n}{n\,!}\ f^{(n)}(a) + \ldots\ldots$$

Example: Obtain the Taylor series expansion of

$f(x) = \ln x$ around $x = 1$.

$$f(x) = \ln x \;\Rightarrow\; f(1) = 0 \qquad f''(x) = \frac{-1}{x^2} \;\Rightarrow\; f''(1) = -1$$

$$f'(x) = \frac{1}{x} \;\Rightarrow\; f'(1) = 1 \qquad f'''(x) = \frac{2}{x^3} \;\Rightarrow\; f'''(1) = 2$$

Taylor series (extended topic)

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2\,!}\, f''(a) + \frac{(x-a)^3}{3\,!}\, f'''(a)\ldots\ldots + \frac{(x-a)^n}{n\,!}\, f^{(n)}(a) + \ldots\ldots$$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$

$$= 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \cdots$$

$$\therefore\ \ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots .$$

Taylor series (extended topic)

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2\,!}\, f''(a) + \frac{(x-a)^3}{3\,!}\, f'''(a)\ldots\ldots + \frac{(x-a)^n}{n\,!}\, f^{(n)}(a) + \ldots\ldots$$

Exercise:

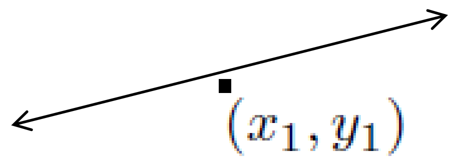
Expand $f(x) = \sqrt{x}$ using Taylor's series around $x = 1$.

Answer: $\sqrt{x} = 1 + \dfrac{x-1}{2} - \dfrac{(x-1)^2}{8} + \dfrac{(x-1)^3}{16} - \cdots$

Revisiting Topics from Coordinate Geometry

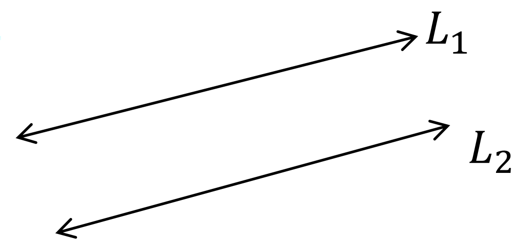
1. The equation of a straight line passing through the point (x_1, y_1) and having slope m is:

$$y - y_1 = m(x - x_1)$$



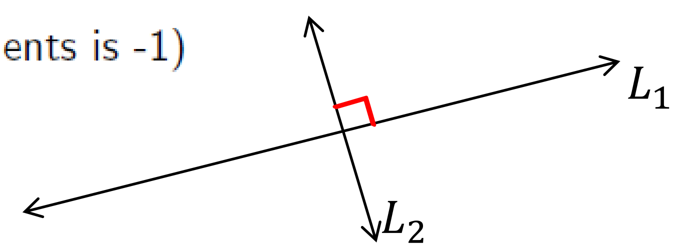
2. Two lines $L_1 : y = m_1 x + c_1$ and $L_2 : y = m_2 x + c_2$ are **parallel** if

$$m_1 = m_2 \qquad \text{(i.e. slopes/gradients are equal)}$$



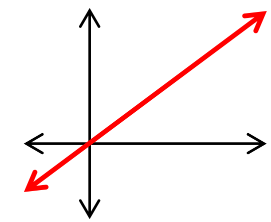
3. Two lines $L_1 : y = m_1 x + c_1$ and $L_2 : y = m_2 x + c_2$ are **perpendicular** (orthogonal) if

$$m_1 \cdot m_2 = -1 \qquad \text{(i.e. product of slopes/gradients is -1)}$$

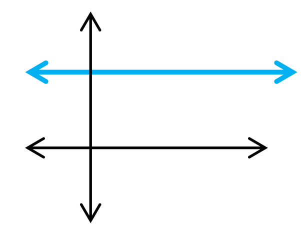


Revisiting Topics from Coordinate Geometry

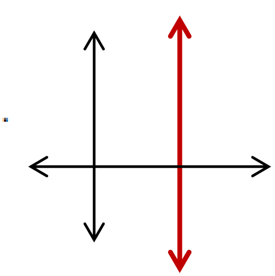
4. If the line $y = mx + c$ **passes through the origin** then, $c = 0$.



5. The slope of a **horizontal** line (i.e. line parallel to the x-axis) is 0.



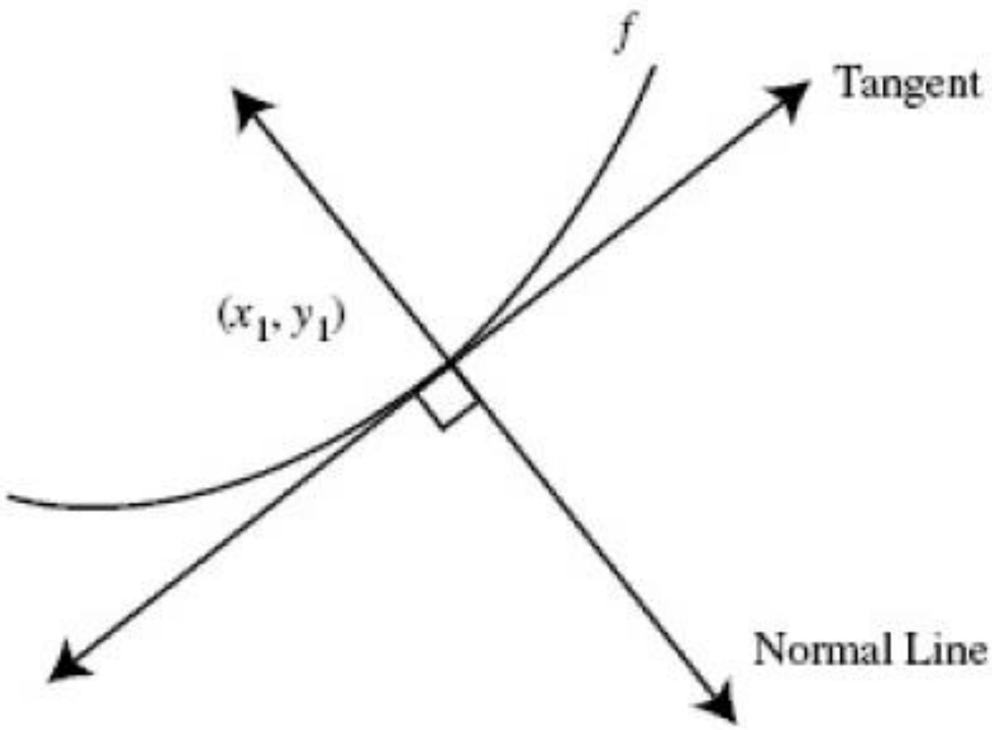
6. The slope of a **vertical** line (i.e. line parallel to the y-axis) is not defined.



7. If the line/curve $y = f(x)$ **cuts**/intersects the x-axis, put $y = 0$.

8. If the line/curve $y = f(x)$ **cuts**/intersects the y-axis, put $x = 0$.

Tangent and Normal Lines



The point (x_1, y_1) must be ***on the curve****.*

For tangents from points outside the curve, the method is different.

A **normal** line is a line that is perpendicular (orthogonal) to the tangent line at the point of contact.

Equation of a Tangent Line

The equation of a tangent line at the point (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = m\,(x - x_1)$$

where $m = \left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}$

$=$ slope of the tangent line at point (x_1, y_1).

Equation of a Tangent Line

The equation of a normal line at the point (x_1, y_1) to the curve $y = f(x)$ is given by

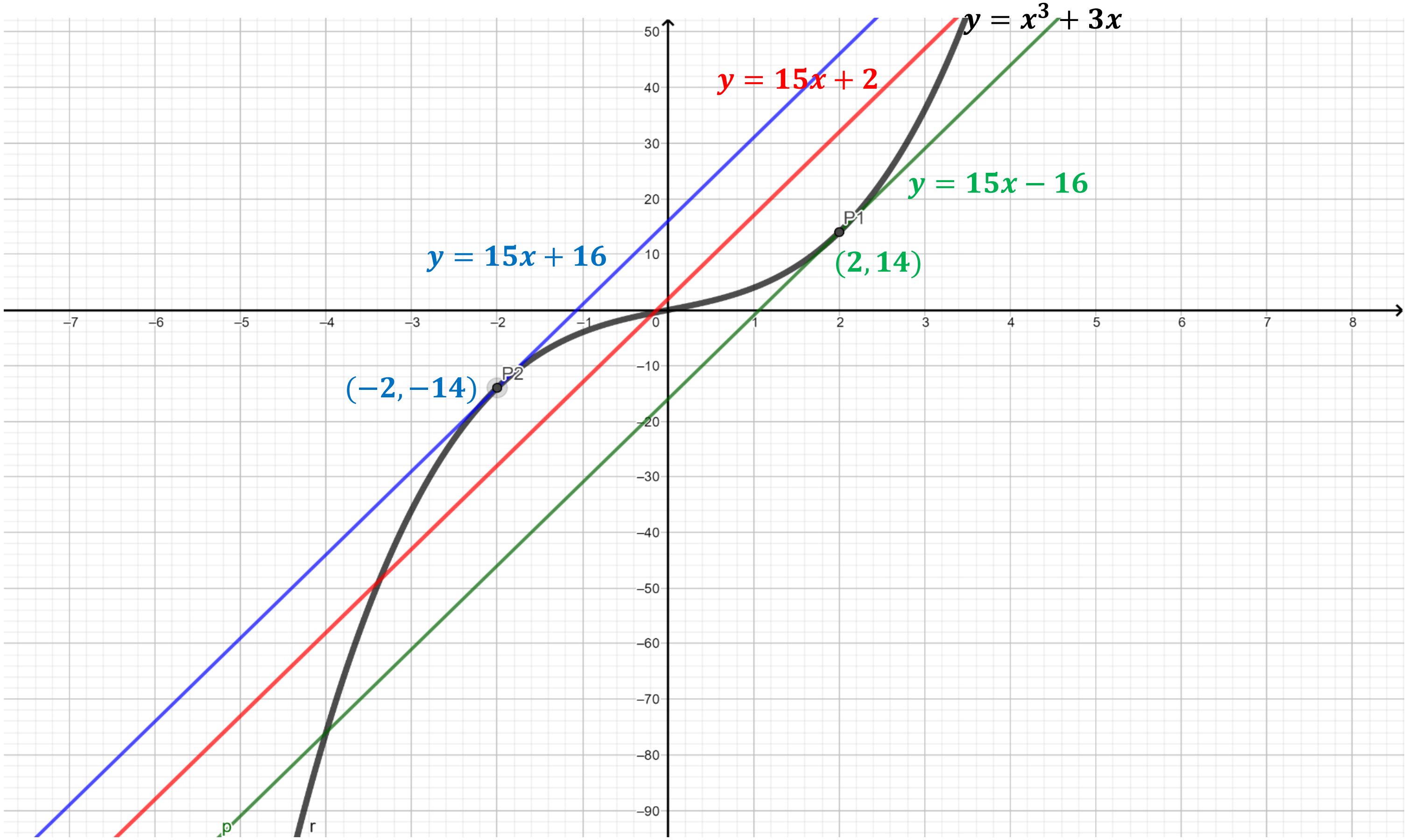
$$y - y_1 = n\,(x - x_1)$$

where $n = \dfrac{-1}{\left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}} = -\left.\dfrac{dx}{dy}\right|_{(x_1, y_1)}$

$=$ slope of the normal line at point (x_1, y_1).

Practice Problem

Find the equations of the tangents to $y = x^3 + 3x$ which are parallel to the line $y = 15x + 2$.



Parametric Differentiation

When we define $\dfrac{dy}{dx}$, the variable x is independent and the variable y is dependent on x.

i.e. the value of y changes with the change in value of x.

and the rate of change of y with change in x is given by $\dfrac{dy}{dx}$.

We now consider the case when both x and y depend on an independent third variable, generally the time variable t.

Parametric Differentiation

To find $\dfrac{dy}{dx}$, we find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ separately,

and then use the formula for parametric differentiation:

$$\boxed{\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}}$$

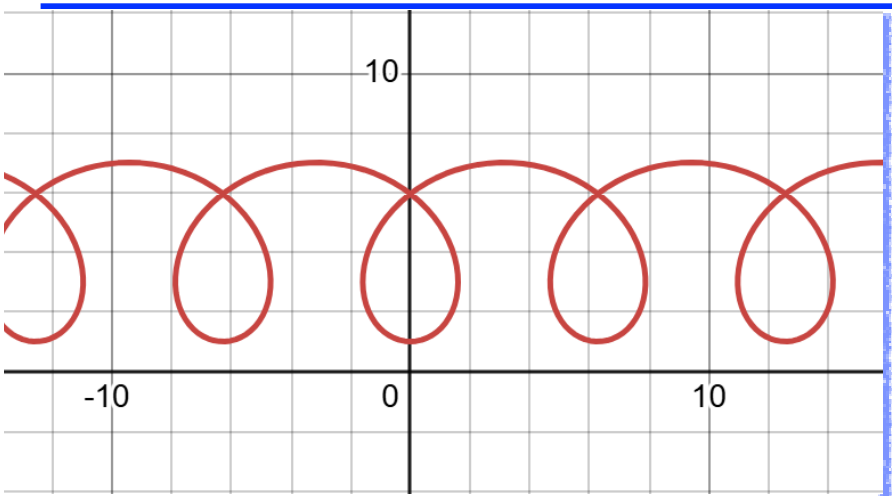
Note: Remember to put $\dfrac{dy}{dt}$ in the numerator.

This method for finding $\dfrac{dy}{dx}$ for parametric equations is called Parametric Differentiation.

Parametric Differentiation

Example: Given that the parametric equations of the curve are:

$x = t - 3\sin t,\ y = 4 - 3\cos t\ ;\ t \in \mathbb{R}$. Find $\dfrac{dy}{dx}$.



$$\begin{aligned} x &= t - 3\sin t \\ \Rightarrow \frac{dx}{dt} &= 1 - 3\cos t \end{aligned}$$

$$\begin{aligned} y &= 4 - 3\cos t \\ \Rightarrow \frac{dy}{dt} &= 3\sin t \end{aligned}$$

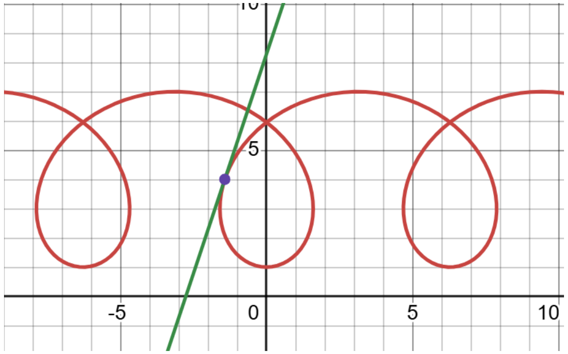
$$\therefore\ \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{3\sin t}{1 - 3\cos t}$$

Parametric Differentiation

Example: Find the line tangent to the curve when $t = \pi/2$.

$$x = t - 3\sin t,\ y = 4 - 3\cos t\ ;\ t \in \mathbb{R}.$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3\sin t}{1 - 3\cos t} \Rightarrow \text{Slope of the tangent: } \frac{3\sin(\pi/2)}{1-3\cos(\pi/2)} = 3$$



When $t = \pi/2$, $x = \frac{\pi}{2} - 3$ and $y = 4$

The tangent line is: $y - 4 = 3\left(x - \frac{\pi}{2} + 3\right)$ or $y = 3x + 13 - \frac{3\pi}{2}$

Problems on related rates

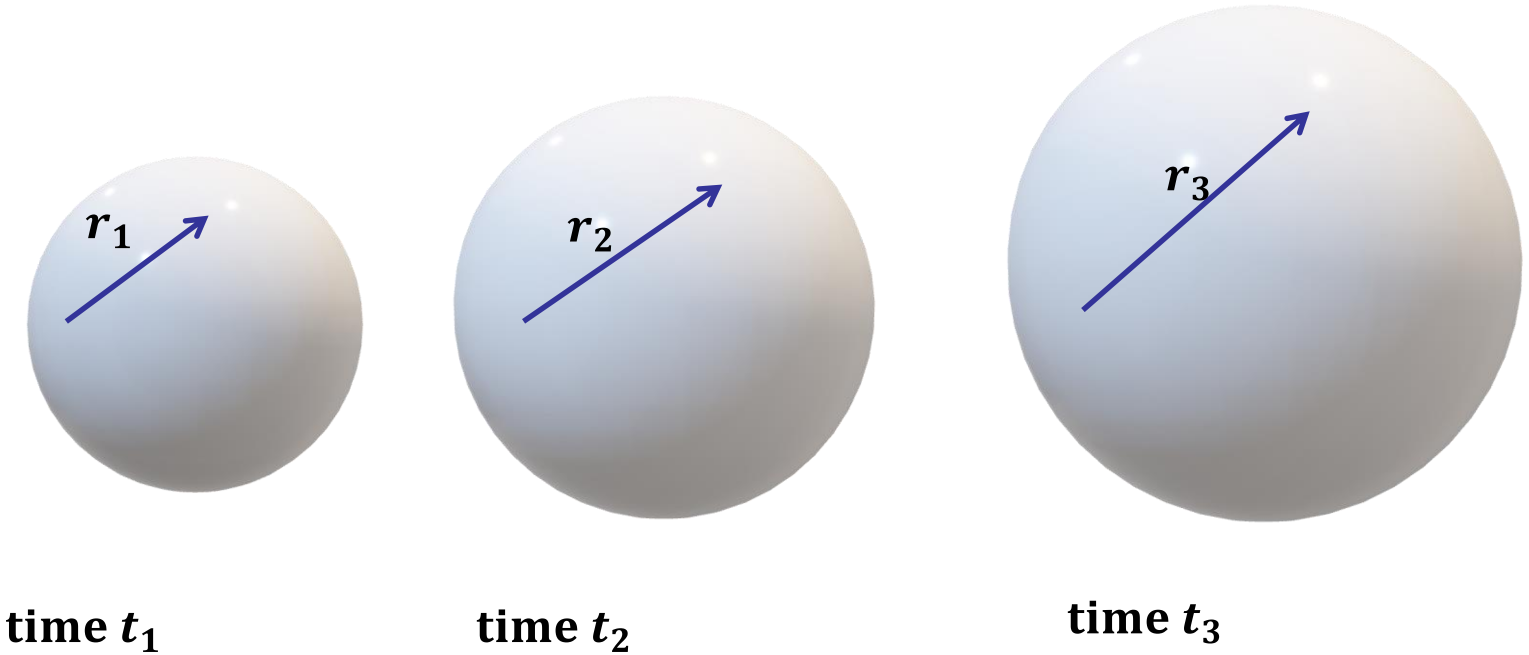
Given $y = f(x)$ where y and x changes w.r.t. time t, then:

Chain Rule: $$\frac{dy}{dt} = \frac{d}{dx}[f(x)] \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Differentiation: $$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Problems on related rates

Example: The radius of a sphere changes at the rate of 0.25 cm/s, find the rate at which the volume is changing at the instant when the radius is 5 cm .



Problems on related rates

Example: The radius of a sphere changes at the rate of 0.25 cm/s, find the rate at which the volume is changing at the instant when the radius is 5 cm .

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{d}{dr}(V) \cdot \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt}$$

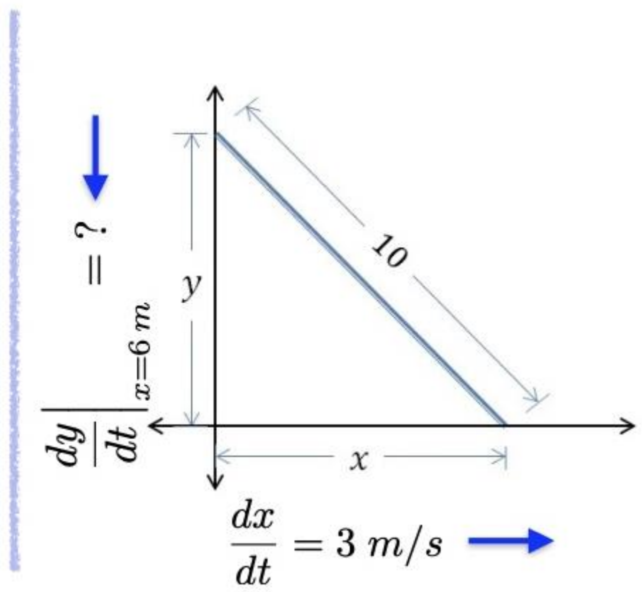
$$\Rightarrow \frac{dV}{dt} = (4\pi r^2) \cdot \frac{dr}{dt}$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=5} = (4\pi(5)^2) \cdot (0.25) = 25\pi \mathrm{cm/s}$$

Problems on related rates

Practice Problem:

A 10m ladder is leaning against the wall, and the base of the ladder is sliding away from the building at a rate of 3m/s. How fast is the top of the ladder sliding down the wall when the base ladder is 6m away from the wall?



Frequently used MacLaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

Thank You!