Topic 1: Area under the curve using Definite Integrals

- The area of the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is:

$$A = \int_a^b |y| \; dx = \int_a^b |f(x)| \; dx$$

- The area of the region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is:

$$A = \int_c^d |x| \; dy = \int_c^d |g(y)| \; dy$$

Illustration 1: Find the area of the region bounded by the curve $y = x^3$, lines $x = -2, x = 2$, and the X-axis.

The function $y = x^3$ is an odd function.

$$A = \int_{-2}^{2} |x^3| \; dx = \left| \int_{-2}^{0} x^3 \; dx \right| + \int_0^2 x^3 \; dx = -\int_{-2}^{0} x^3 \; dx + \int_0^2 x^3 \; dx$$

$$= 2\int_0^2 x^3 \; dx \quad = 2\left[\frac{x^4}{4}\right]_0^2 = 8$$

Illustration 2: Find the area of the region bounded by the curve $y = e^{\sin x} \sin 2x$ and the X-axis, where $x \in \left[0, \frac{\pi}{2}\right]$.

$$A = \int_0^{\frac{\pi}{2}} |y| \; dx = \int_0^{\frac{\pi}{2}} e^{\sin x} \sin 2x \; dx \qquad \boxed{\sin 2x = 2\sin x \cdot \cos x}$$

$$= 2\int_0^{\frac{\pi}{2}} \sin x \cdot e^{\sin x} \cos x \; dx$$

$$= 2\int_0^1 te^t \; dt \qquad (\sin x = t)$$

x	0	$\frac{\pi}{2}$
t	0	1

$$= 2\left(\left[te^t\right]_0^1 - \int_0^1 e^t \; dt\right) \qquad \left(u = t, \frac{dv}{dt} = e^t; \quad \frac{du}{dt} = 1, v = e^t\right)$$

$$= 2e - 2\left[e^t\right]_0^1 = 2$$

1. The curve $y = e^{\sqrt{x}}$ and the X–axis, where $0 \leq x \leq 1$. **Answer:**	2. The curve $y = 2x \cdot e^{x}$ and the X–axis, where $x \in [0, 1]$. **Answer:**
3. The curve $x = y \cdot \sin y$; $y \in \left[0, \frac{\pi}{2}\right]$; and the Y–axis. **Answer:**	4. The curve $x = \frac{y}{(y+1)^2}$ and the lines $y = 0$, $y = 2$, and Y–axis. **Answer:**

Topic 2: Area of the region between two curves using Definite Integrals

- The area of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ the limites $x = a$, $x = b$ is:

$$A = \int_a^b \Big| [f_1(x) - f_2(x)] \Big| \, dx$$

- The area of the region bounded by the curve $x = g_1(y)$, $x = g_2(y)$, the limits $y = c$, $y = d$ is:

$$A = \int_c^d \Big| [g_1(y) - g_2(y)] \Big| \, dy$$

- If the intervals are not specified $\Rightarrow$ Find out the points of intersection

Illustration 1: Find the area of the region bounded by the curves $y = \sec^2 x$, $y = 2$ and lines $x = -\frac{\pi}{4}, x = \frac{\pi}{4}$.

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |(\sec^2 x - 2)| \, dx = \left| \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 2) \, dx \right| = \left| \Big[\tan x - 2x \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right|$$

$$= \left| \left(\tan \frac{\pi}{4} - 2 \cdot \frac{\pi}{4} \right) - \left[\tan(-\frac{\pi}{4}) - 2 \cdot (-\frac{\pi}{4}) \right] \right|$$

$$= \left| \left(1 - \frac{\pi}{2} \right) - \left(-1 + \frac{\pi}{2} \right) \right| = |2 - \pi| = \pi - 2$$

Illustration 2: Find the area of the region bounded by the curve

$y = x^2$ and $y = 2x - x^2$.

First find the points of intersection

$y = x^2$ and $y = 2x - x^2 \quad \Rightarrow \quad x_1 = 0,\ y_1 = 0$ or $x_2 = 1,\ y_2 = 1$

$$A = \int_0^1 \left| (x^2 - 2x + x^2) \right| \, dx = \left| 2 \int_0^1 (x^2 - x) \, dx \right|$$

$$= \left| 2 \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \right| = \left| 2 \left(-\frac{1}{6} \right) \right| = \frac{1}{3}$$

1. The curves $y = x^2,\ y = \sqrt{x}$ and the lines $x = \frac{1}{4},\ x = 1$. **Answer:**	2. The curves $y = x^2$ and $y = \lvert x \rvert$. **Answer:**
3. The curves $x^2 = y$, and $x = y - 2$. **Answer:**	4. The curves $x = 1 - y^2$ and $x = y^2 - 1$. **Answer:**

Topic 3: Volume Calculation using Definite Integrals

1. If the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$

Illustration 1: Find the volume of solid of revolution formed when the region bounded by the curve $y = x^2$, lines $x = 0,\ x = 4$, and the X-axis, is revolved about the X-axis.

$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$

Here $a = 0,\ b = 4,$

and $y = f(x) = x^2.$

$$\therefore\ V = \pi \int_0^4 \left(x^2\right)^2 \, dx$$

$$= \pi \int_0^4 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^4$$

$$= \pi \left(\frac{4^5}{5} - \frac{0^5}{5}\right)$$

$$= \frac{1024\pi}{5}$$

1. Find the volume of solid of revolution when the region bounded by the curve $y = \sin x$, lines $x = 0, x = \pi$ and the X-axis is revolved about the X-axis.

Answer:

2. Find the volume of solid of revolution when the region bounded by the curve $y = (x + 2)^2$, lines $x = 0, x = 1$ and the X-axis is revolved about the X-axis.

Answer:

Topic 3: Volume Calculation using Definite Integrals

2. If the region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is revolved about the Y-axis, then the volume of the solid of revolution is:

$$V = \pi \int_c^d x^2 \, dy = \pi \int_c^d [g(y)]^2 \, dy$$

Illustration 2: Find the volume of solid of revolution formed when the region bounded by the curve $x = y^2$, lines $y = 0,\ y = 2$, and the Y-axis, is revolved about the Y-axis.

$$V = \pi \int_c^d x^2 \, dy = \pi \int_c^d [g(y)]^2 \, dy$$

Here $c = 0,\ d = 2,$

and $x = g(y) = y^2.$

$$\therefore\ V = \pi \int_0^2 \left(y^2\right)^2 \, dy$$

$$= \pi \int_0^2 y^4 \, dy$$

$$= \pi \left[\frac{y^5}{5}\right]_0^2$$

$$= \pi \left(\frac{2^5}{5} - \frac{0^5}{5}\right)$$

$$= \frac{32\pi}{5}$$

1. Find the volume of solid of revolution when the region bounded by the curve $x = \sqrt{y}$, lines $y = 0, y = 2$ and the Y-axis is revolved about the Y-axis.

Answer:

2. Find the volume of solid of revolution when the region bounded by the curve $x = e^y$, lines $y = 0, y = 1$ and the Y-axis is revolved about the Y-axis

Answer:

Topic 3: Volume Calculation using Definite Integrals

3. If the region bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ between the lines $x = a$, $x = b$ is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \int_a^b \left| [f_1(x)]^2 - [f_2(x)]^2 \right| \, dx$$

Illustration 3: Find the volume of solid of revolution formed when the region bounded by the curve $y = x$ and $y = x^2$ is revolved about the X-axis.

First, find the points of intersection of the two curves.

Let, $\quad y = f_1(x) = x \quad$ and $\quad y = f_2(x) = x^2 \quad \Rightarrow \quad x^2 = x \quad \Rightarrow \quad x^2 - x = x(x-1) = 0$

$\therefore \quad x = a = 0, \quad$ or $\quad x = b = 1, \quad$ Now, $\quad V = \pi \int_a^b \left| [f_1(x)]^2 - [f_2(x)]^2 \right| \, dx$

$$\therefore \quad V = \pi \int_0^1 \left| [(x)^2 - (x^2)^2] \right| \, dx = \pi \left| \int_0^1 (x^2 - x^4) \, dx \right|$$

$$= \pi \left| \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \right| = \pi \left| \left(\frac{1^3}{3} - \frac{1^5}{5} \right) - \left(\frac{0^3}{3} - \frac{0^5}{5} \right) \right| = \frac{2\pi}{15}$$

1. Find the volume of solid of revolution formed when the region bounded by curves $y = \sin x$, $y = \cos x$ and lines $x = 0, x = \pi/4$ is revolved about the X-axis.

Answer:

Topic 3: Volume Calculation using Definite Integrals

4. If the region bounded by two curves $x = g_1(y)$ and $x = g_2(y)$ between the lines $y = c$, $y = d$ is revolved about the Y-axis, then the volume of the solid of revolution is:

$$V = \pi \int_c^d \left|[g_1(y)]^2 - [g_2(y)]^2\right| \, dy$$

Illustration 4: Find the volume of solid of revolution formed when the region bounded by the curve $x = y^2$ and $x = y + 2$ is revolved about the Y-axis.

Let, $x = g_1(y) = y^2$ and $x = g_2(y) = y + 2$

$\Rightarrow \quad y^2 = y + 2 \quad \Rightarrow \quad y^2 - y - 2 = (y-2)(y+1) = 0 \quad \therefore \quad y = c = -1 \quad$ or $\quad y = d = 2$

Now, $V = \pi \int_c^d \left|[g_1(y)]^2 - [g_2(y)]^2\right| \, dy \quad \Rightarrow \quad V = \pi \int_{-1}^2 \left|\left[(y+2)^2 - (y^2)^2\right]\right| \, dy$

$$= \pi \left| \int_{-1}^2 \left[(y+2)^2 - y^4\right] \, dy \right| = \pi \left| \int_{-1}^2 \left(-y^4 + y^2 + 4y + 4\right) \, dy \right|$$

$$= \pi \left| \left[-\frac{y^5}{5} + \frac{y^3}{3} + 2y^2 + 4y \right]_{-1}^2 \right| = \frac{72\pi}{5}$$

1. Find the volume of solid of revolution formed when the region bounded by the curves $y = x^2$ and $x = y^2$ is revolved about the Y-axis.

Answer:

Topic 4: Numerical Integration - Trapezoidal Rule

We divide the interval $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$. Then

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f_0 + 2\left(f_1 + f_2 + \cdots + f_{n-1}\right) + f_n\Big]$$

$$\big(f_i = f(a + ih),\ i = 0, 1, 2, \ldots, n\big)$$

Illustration 1: Using the Trapezoidal rule, evaluate $\int_0^2 e^{x^2}\,dx$ by dividing $[0, 2]$ into 5 sub-intervals of equal width. Give approximation to 3 decimal places (d.p.).

x	f_n	$f(x) = e^{x^2}$
0	f_0	1
2/5	f_1	1.174
4/5	f_2	1.896
6/5	f_3	4.221
8/5	f_4	12.936
2	f_5	54.598

Here $f(x) = e^{x^2}$ and $h = \dfrac{b-a}{n} = \dfrac{2-0}{5} = \dfrac{2}{5}$

Using the Trapezoidal rule:

$$I = \int_0^2 e^{x^2}\,dx \approx \frac{h}{2}\left[f_0 + 2\left(f_1 + f_2 + f_3 + f_4\right) + f_5\right]$$

$$\approx \frac{1}{5}\left[1 + 2(1.174 + 1.896 + 4.221 + 12.936) + 54.598\right]$$

$$\approx 19.210$$

Use TABLE Mode in Calculator:

Step 1: Fix calculator to 3 d. p.: **Shift Mode 6 3**

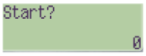
Step 2: Set calculator to TABLE mode: **Mode 3**

Step 3: Enter the integrand function $f(x)$

On calculator to use e: **ALPHA** $\times 10^x$

On calculator to use variable X: **ALPHA)**

Step 4: Enter lower limit value:



Enter **0** and press "="

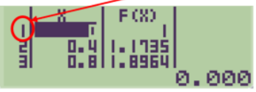
Step 5: Enter upper limit value:

Enter **2** and press "="

Step 6: Enter step width h:

Enter **0.4** and press "="

Step 7: Create table:



Note the difference of starting index

Please use the buttons:  to get detailed values in required d.p. from the table in the calculator

x	f_n	$f(x) = e^{x^2}$
0	f_0	1
$\frac{2}{5}$	f_1	1.174
$\frac{4}{5}$	f_2	1.896
$\frac{6}{5}$	f_3	4.221
$\frac{8}{5}$	f_4	12.936
2	f_5	54.598

dividing the given interval into 5 sub-intervals of equal width. Give approximation to 5 d.p.

1. $\displaystyle\int_4^9 \frac{1}{\sqrt{x}}\,dx$ **Answer:**	2. $\displaystyle\int_0^3 \frac{1}{4x+1}\,dx$ **Answer:**
3. $\displaystyle\int_1^3 e^{-2x}\,dx$ **Answer:**	4. $\displaystyle\int_0^8 \sqrt{x+1}\,dx$ **Answer:**

Topic 4: Numerical Integration – Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + 2(f_2 + f_4 + \cdots + f_{n-2}) + 4(f_1 + f_3 + \cdots + f_{n-1}) + f_n\right]$$

$h = \dfrac{b-a}{n}$, and n is an **even** number.

Illustration 2: Using Simpson's rule, evaluate $\int_0^\pi \sin x\,dx$ by dividing $[0, \pi]$ into 6 sub-intervals of equal width. Give approximation to 3 decimal places.

Here $f(x) = \sin(x)$ and $h = \dfrac{b-a}{n} = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$

Using Simpson's rule:

$$I = \int_0^\pi \sin x\,dx \approx \frac{h}{3}\left[f_0 + 2\left(f_2 + f_4\right) + 4\left(f_1 + f_3 + f_5\right) + f_6\right]$$

x	f_n	$f(x) = \sin x$
0	f_0	0
$\pi/6$	f_1	0.500
$2\pi/6$	f_2	0.866
$3\pi/6$	f_3	1.000
$4\pi/6$	f_4	0.866
$5\pi/6$	f_5	0.500
π	f_6	0.000

$$I \approx \frac{\pi}{18}\left[0 + 2(0.866 + 0.866) + 4(0.500 + 1.000 + 0.5000) + 0\right]$$

$$\approx 2.001$$

dividing the given interval into 8 sub-intervals of equal width. Give approximation to 5 d.p.

1. $\displaystyle\int_0^4 \sin\left(\sqrt{x}\right)\ dx$ **Answer:**	2. $\displaystyle\int_{-1}^3 x\sqrt{2+x^3}\ dx$ **Answer:**
3. $\displaystyle\int_0^1 x\sin(x^2)\ dx$ **Answer:**	4. $\displaystyle\int_0^3 x\sin x\ dx$ **Answer:**

Topic 4: Numerical Integration – Exam Style Question

We divide the interval $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$. Then

Trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f_0 + 2\,(f_1 + f_2 + \cdots + f_{n-1}) + f_n\Big]$$

$$(f_i = f(a + ih),\ i = 0, 1, 2, \ldots, n)$$

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\,[f_0 + 2(f_2 + f_4 + \cdots + f_{n-2}) + 4(f_1 + f_3 + \cdots + f_{n-1}) + f_n]$$

$h = \dfrac{b-a}{n}$, and n is an **even** number.

Illustration 3: Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ by using

(a) Direct integration of the definite integral.

(b) Numerical method with

- (i) the Trapezoidal rule
- (ii) the Simpson's rule, by dividing the interval $[0, 1]$ into 4 subintervals of equal width. Give approximation to 4 d.p.

(c) Comment on the results

(a) $I = \displaystyle\int_0^1 \frac{1}{1+x}\,dx = [\ln|1+x|]_0^1 = \ln(2) - \ln(1) = \ln(2) \approx 0.6931$

(b) Numerical method Here $f(x) = \dfrac{1}{1+x}$ and $h = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4}$

(i) Using the Trapezoidal rule: $I = \displaystyle\int_0^1 \frac{1}{1+x}\,dx \approx \frac{h}{2}\,[f_0 + 2\,(f_1 + f_2 + f_3) + f_4]$

$$\approx \frac{1}{8}\Big[1 + 2(0.8000 + 0.6667 + 0.5714) + 0.5000\Big] \approx 0.6970$$

(ii) Using Simpson's rule: $I = \displaystyle\int_0^1 \frac{1}{1+x}\,dx \approx \frac{h}{3}\,[f_0 + 2\,(f_2) + 4\,(f_1 + f_3) + f_4]$

$$\approx \frac{1}{12}\Big[1 + 2(0.6667) + 4(0.8 + 0.5714) + 0.5000\Big] \approx 0.6933$$

(c)

Method	$\int_0^1 \frac{1}{1+x}\,dx$
Direct integration	0.6931
Trapezoidal rule	0.6970
Simpson's rule	0.6933

Compared to the Trapezoidal rule, the value of the integral from Simpson's rule is closer to the value obtained from direct integration. Hence, the value of the integral obtained from Simpson's rule is more accurate than that of the Trapezoidal rule.