The University of Nottingham
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Science A Physics

Lecture 11:

Gauss's Law & Electric Potential

Aims of today's lecture

1. Electric Flux
2. Gauss's Law
3. Application of Gauss's Law
4. Electric Potential Energy and Potential Difference

Gauss's Law

Karl Friedrich Gauss (1777-1855)

- We can, in principle, using Coulomb's law, determine the electric field due to any given distribution of electric charge.

- The total electric field at any point will be the vector sum, or integral, of contributions from all charges present.

Gauss’s Law

Karl Friedrich Gauss (1777-1855)

- However, except for some simple cases, the sum or integral can be quite complicated to evaluate; we only looked at a few simple cases in our last lecture.

- Gauss developed a relation between electric charge and electric field, which is a more general and elegant form of Coulomb’s law, and makes problem solving easier.

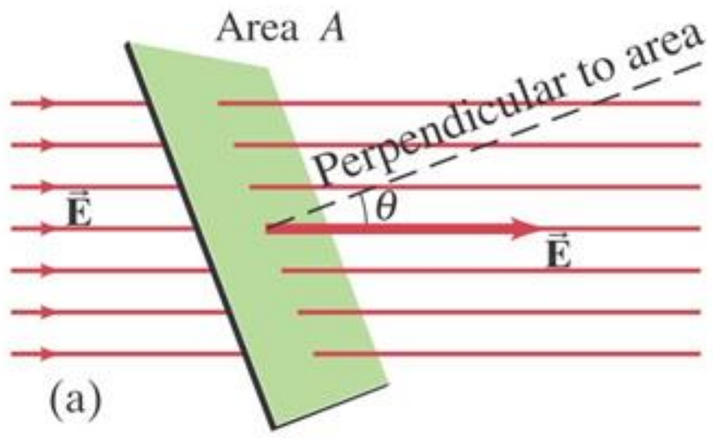
Gauss’s Law

Karl Friedrich Gauss (1777-1855)

- Before discussing Gauss’s law itself, though, we first must discuss the concept of **electric flux**.

1. Electric Flux

Electric Flux

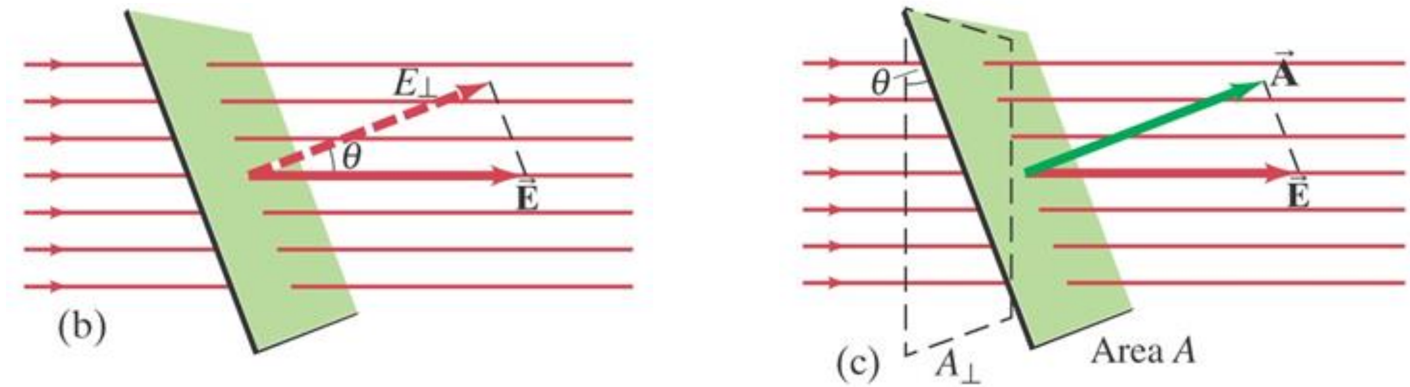


- Electric flux refers to the electric field passing through a given area. For a uniform electric field $\vec{E}$ passing through an area A, as shown above, the electric flux Φ_E is defined as

$$\Phi_E = EAcos\theta$$

where θ is the angle between the electric field direction and a line drawn perpendicular to the area.

Electric Flux

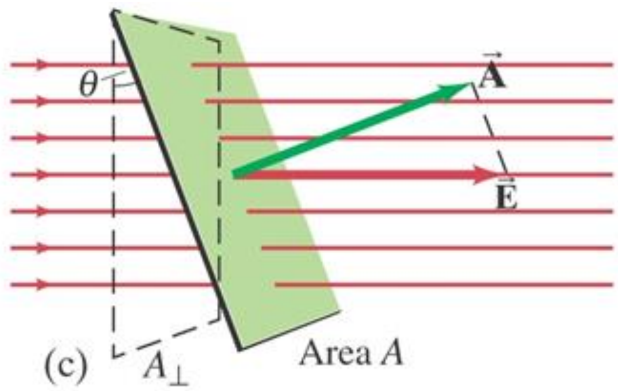


- The flux can also be written equivalently as

$$\Phi_E = E_\perp A = EA_\perp = EAcos\theta$$

where $E_\perp = Ecos\theta$ is the component of $\vec{E}$ along the perpendicular to the area (b), and similarly $A_\perp = Acos\theta$ is the projection of the area A perpendicular to the field $\vec{E}$ (c).

Electric Flux

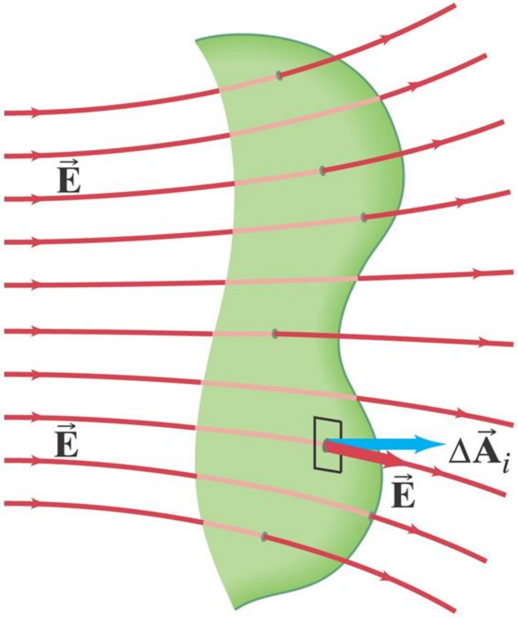


- Electric flux has an intuitive interpretation in terms of field lines. Field lines can always be drawn so that the number (N) passing through a unit area perpendicular to the field ($A_\perp$) is proportional to the magnitude of the field (E): $E \propto N/A_\perp$. Hence,

$$N \propto E_\perp A = \Phi_E$$

so the flux, Φ_E, through the area is proportional to the number of lines passing through that area.

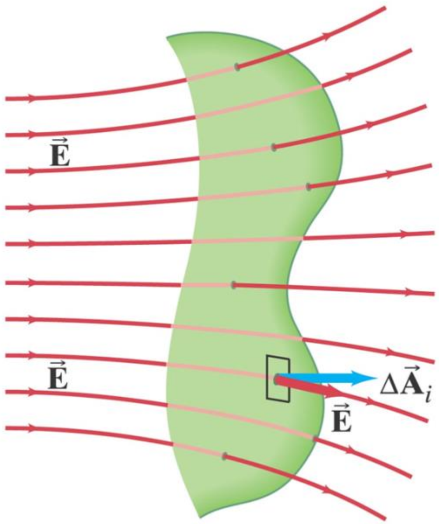
Electric Flux



- In the more general case, when the electric field $\vec{E}$ is not uniform and the surface is not flat, as shown above, we divide up the chosen surface into n small elements of surface whose areas are

$$\Delta A_1, \Delta A_2, \ldots, \Delta A_n$$

Electric Flux



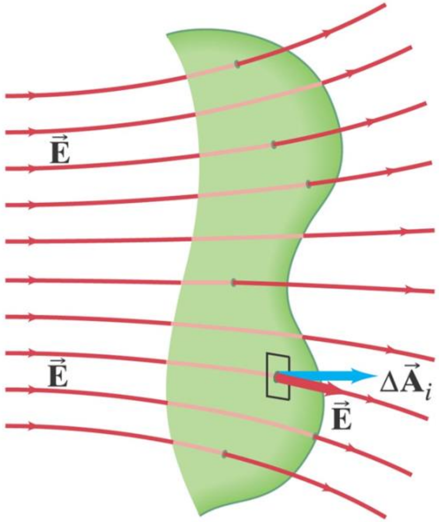
$$\Delta A_1, \Delta A_2, \ldots, \Delta A_n$$

- We can choose the division so that each ΔA_i is small enough that (1) it can be considered flat, and (2) the electric field varies so little over this small area that it can be considered uniform. Then the electric flux through the entire surface is approximately

$$\Phi_E \approx \sum_{i=1}^{n} \vec{E}_i \cdot \Delta\vec{A}_i,$$

where $\vec{E}_i$ is the field passing through $\Delta\vec{A}_i$.

Electric Flux

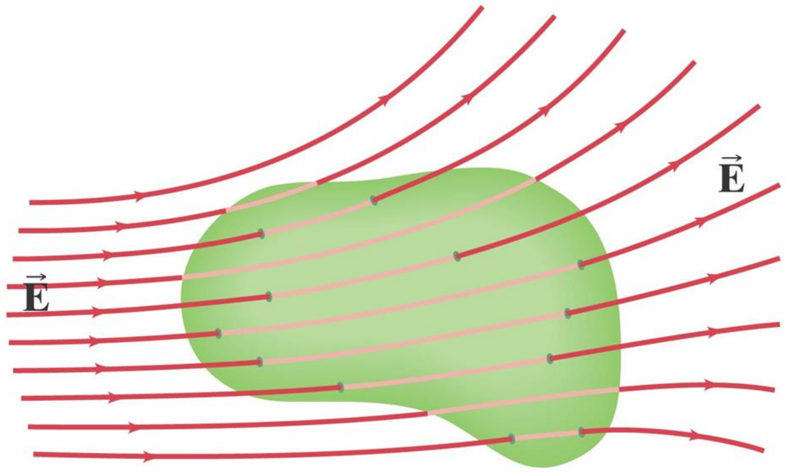


$$\Phi_E \approx \sum_{i=1}^{n} \vec{E}_i \cdot \Delta\vec{A}_i,$$

- In the limit as we let $\Delta\vec{A}_i \to 0$, the sum becomes an integral over the entire surface and the relation becomes

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

Electric Flux

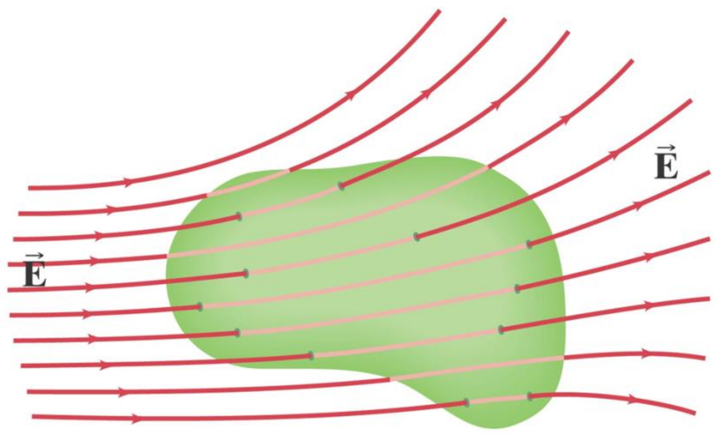


- Gauss's law involves the total flux through a closed surface– a surface of any shape that completely encloses a volume of space, such as that shown above. In this case, the net flux through the enclosing surface is given by

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

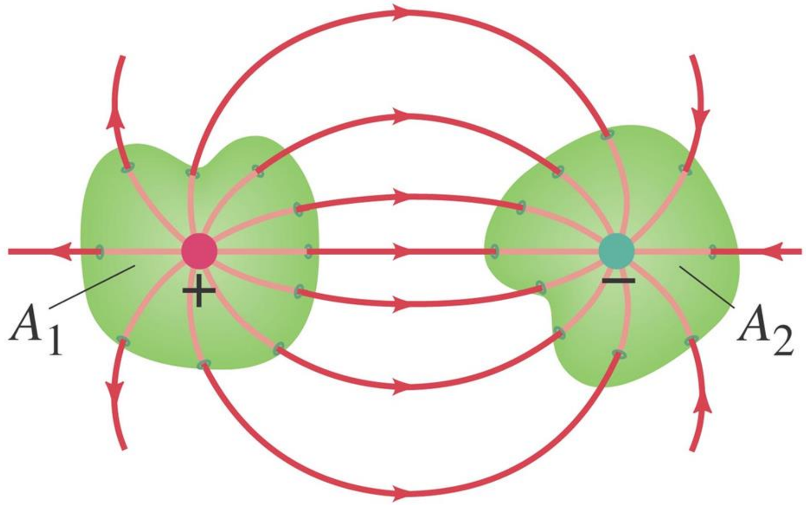
where the integral sign is written as shown to indicate that the integral is over the value of $\vec{E}$ at every point on an enclosing surface.

Electric Flux



- The flux will be nonzero only if one or more lines start or end within the surface. Since electric field lines start and stop only on electric charges, the flux will be nonzero only if the surface encloses a net charge.

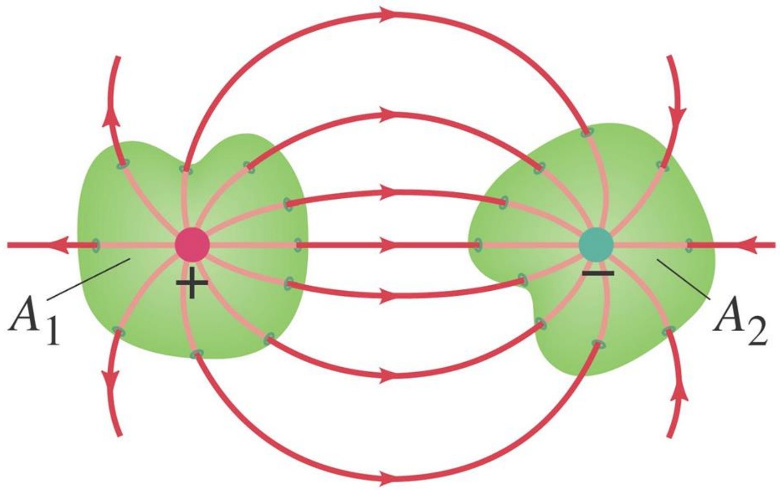
Electric Flux



- For example, the surface labelled A_1 in the above figure encloses a positive charge, and thus, there is a net outward flux through this surface ($\Phi_E > 0$).
- The surface A_2 encloses an equal magnitude negative charge and there is a net inward flux ($\Phi_E < 0$).
- The value of Φ_E depends on the charge enclosed by the surface, and this is what Gauss's law is all about.

2. Gauss's Law

Gauss's Law



Karl Friedrich Gauss (1777-1855)

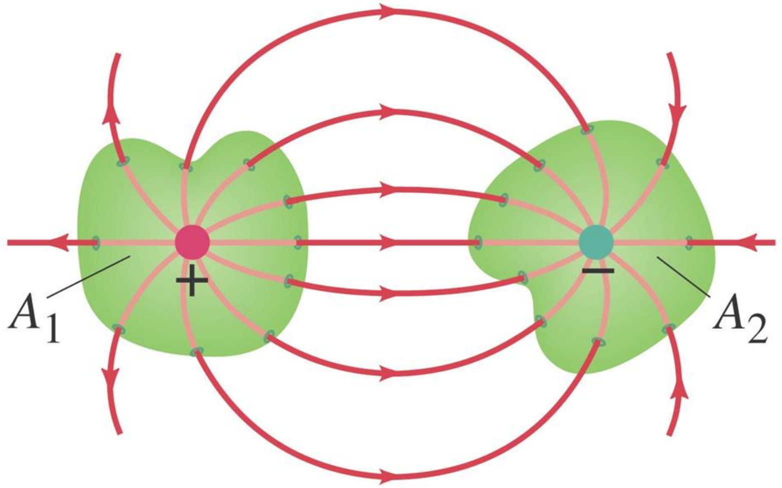
- The precise relation between the electric flux through a closed surface and the net charge Q_{encl} enclosed within that surface is given by Gauss's law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$F = \frac{kQ_1Q_2}{r^2} \qquad k = \frac{1}{4\pi\epsilon_0}$$

Coulomb's law

Gauss's Law

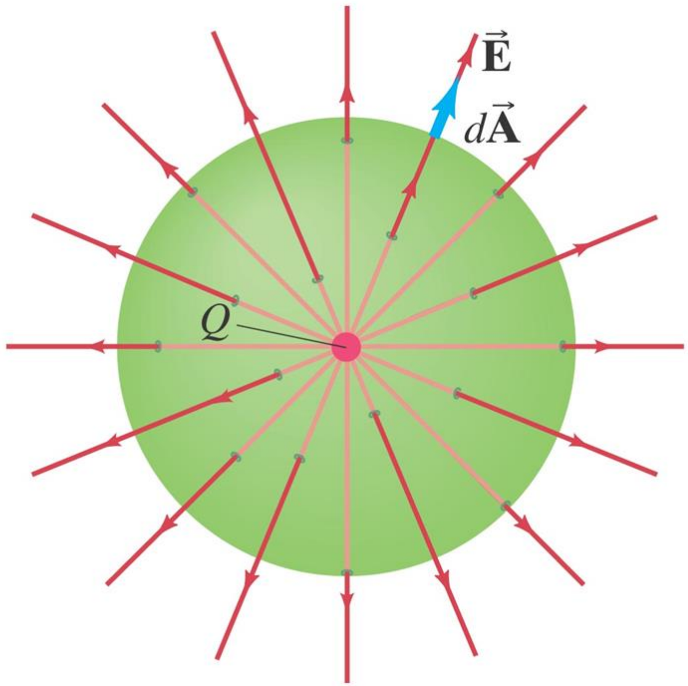


Karl Friedrich Gauss (1777-1855)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

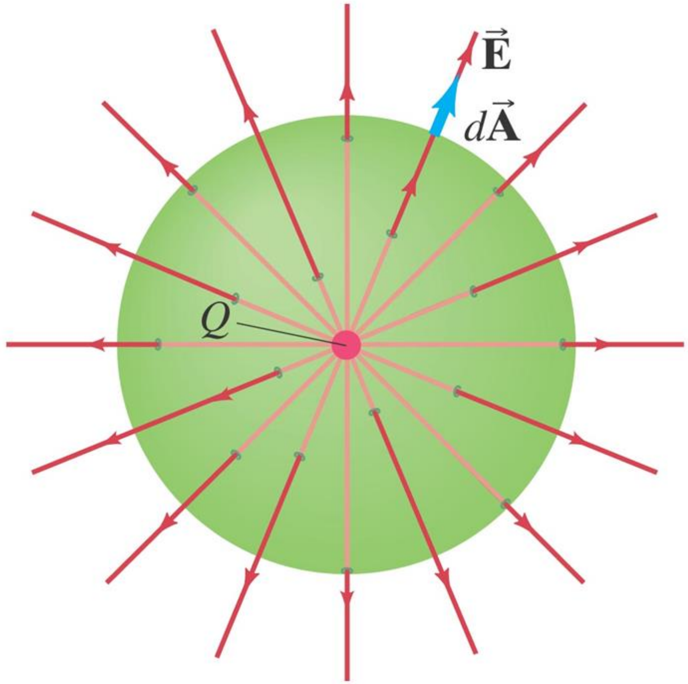
- For example, Q_{encl} for the gaussian surface A_1 in the above figure would be the positive charge enclosed by A_1; the negative charge does contribute to the electric field at A_1, but it is not enclosed by surface A_1 and so is not included in Q_{encl}.

How Gauss's Law is Related to Coulomb's Law



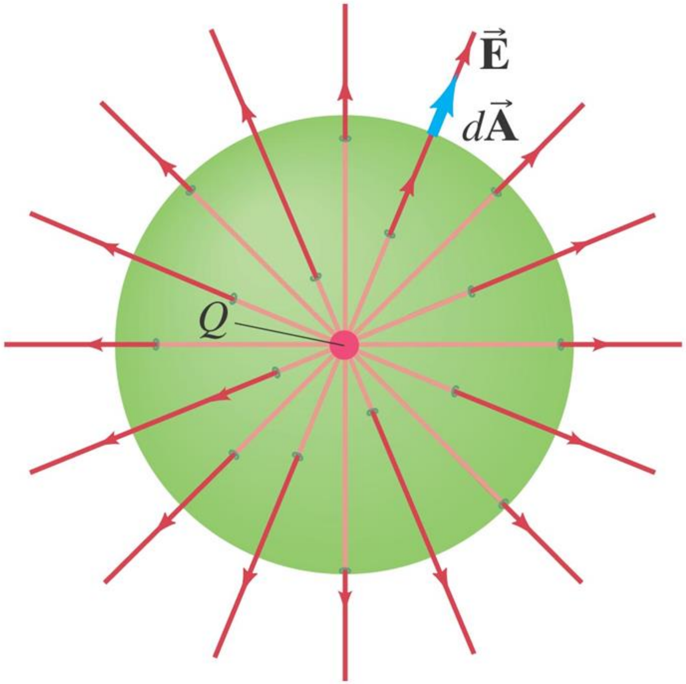
- In the above figure, we have a single isolated charge Q, and for our gaussian surface, we choose an imaginary sphere of radius r centred on the charge.

How Gauss's Law is Related to Coulomb's Law



- Because of the symmetry of this (imaginary) sphere about the charge at its centre, we know that $\vec{E}$ must have the same magnitude at any point on the surface, and that $\vec{E}$ points radially outward (inward for a negative charge) perpendicular to $d\vec{A}$, an element of the surface area.

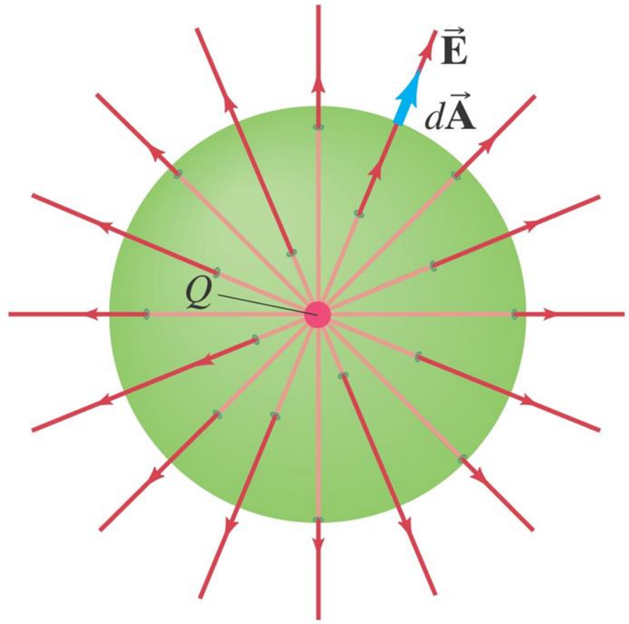
How Gauss's Law is Related to Coulomb's Law



- We can write the integral in Gauss's law as

$$\oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = E(4\pi r^2)$$

How Gauss's Law is Related to Coulomb's Law

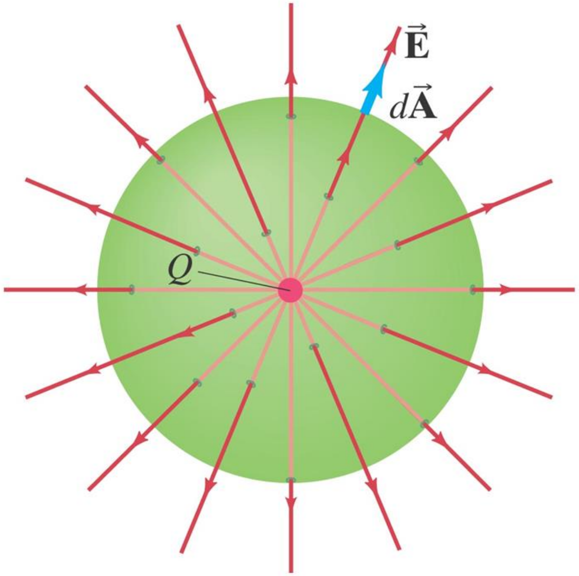


- Thus, Gauss's law becomes, with $Q_{encl} = Q$,

$$\frac{Q}{\epsilon_0} = \oint \vec{E} \cdot d\vec{A} = E(4\pi r^2)$$

Because $\vec{E}$ and $d\vec{A}$ are both perpendicular to the surface at each point, and $cos\theta = 1$.

How Gauss's Law is Related to Coulomb's Law



$$\frac{Q}{\epsilon_0} = \oint \vec{E} \cdot d\vec{A} = E(4\pi r^2)$$

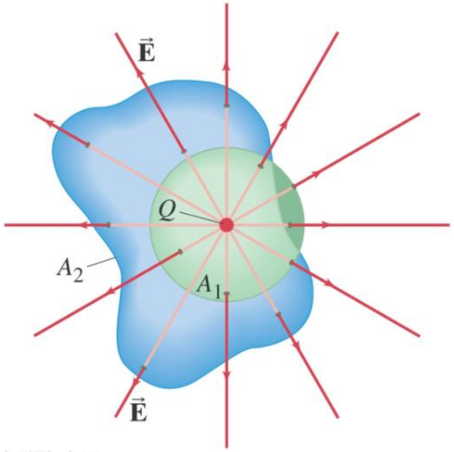
Thus,

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

which is the electric field form of Coulomb's law.

3. Applications of Gauss’s Law

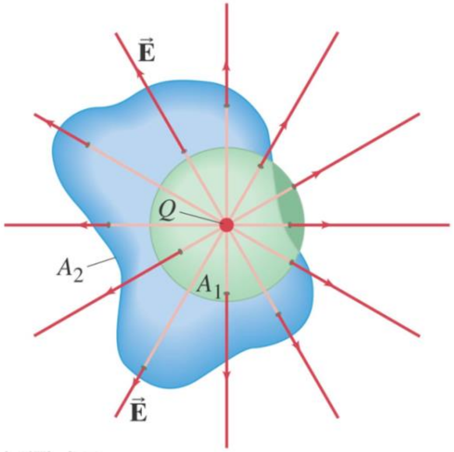
Applications of Gauss's Law



$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

- As we have seen, Gauss's law is a very compact and elegant way to write the relation between electric charge and electric field.
- It also offers a simple way to determine the electric field when the charge distribution is simple and/or possesses a high degree of symmetry.

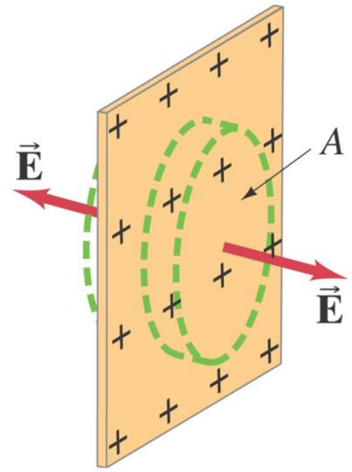
Applications of Gauss's Law



$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

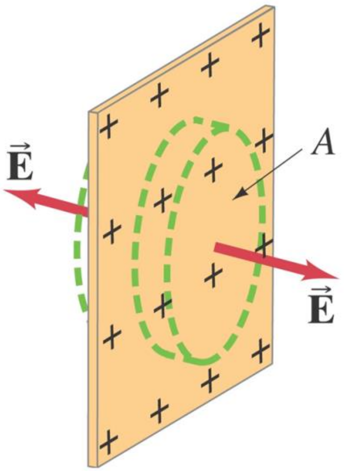
- In order to apply Gauss's law, however, we must choose the 'gaussian surface' very carefully (for the integral on the left side of Gauss's law) so we can determine $\vec{E}$.
- We normally try to think of a surface that has just the symmetry needed so that E will be constant on all or on parts of its surface.

An Infinite Plane of Charge



Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

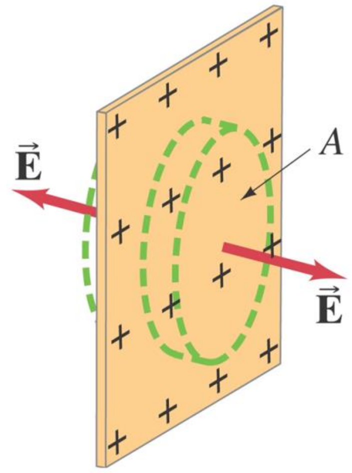
An Infinite Plane of Charge



Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

- We choose as our gaussian surface a small closed cylinder whose axis is perpendicular to the plane and which extends through the plane as shown above.

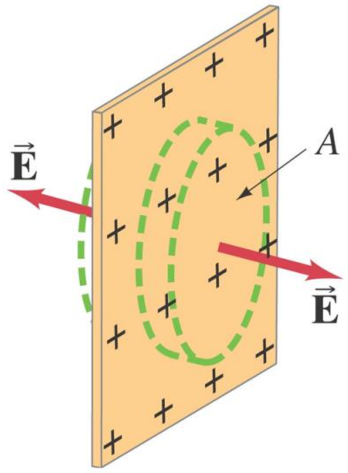
An Infinite Plane of Charge



Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

- Because of the symmetry, we expect $\vec{E}$ to be directed perpendicular to the plane on both sides as shown, and to be uniform over the end caps of the cylinder, each of whose area is A.

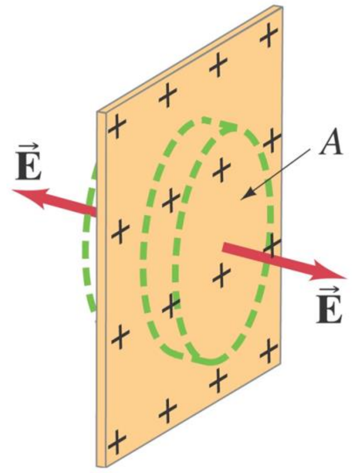
An Infinite Plane of Charge



Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

- Since no flux passes through the curved sides of our chosen cylindrical surface, all the flux is through the two end caps. So . . .

An Infinite Plane of Charge

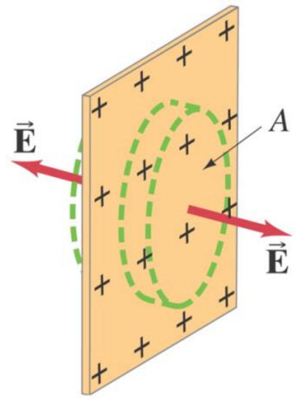


Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

. . . Gauss's law gives

$$\oint \vec{E} \cdot d\vec{A} = 2EA = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

An Infinite Plane of Charge

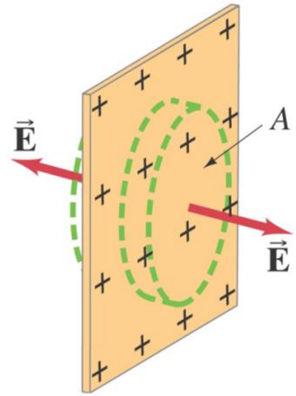


Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

$$\oint \vec{E} \cdot d\vec{A} = 2EA = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

- where $Q_{encl} = \sigma A$ is the charge enclosed by our gaussian cylinder.

An Infinite Plane of Charge



Q. Charge is distributed uniformly, with a surface charge density σ ($\sigma = charge\ per\ unit\ area = dQ/dA$), over a very large, but very thin non-conducting flat plane surface. Determine the electric field at points near the plane.

- The electric field is then

$$E = \frac{\sigma}{2\epsilon_0}$$

The field is uniform for points far from the ends of the plane, and close to its surface.

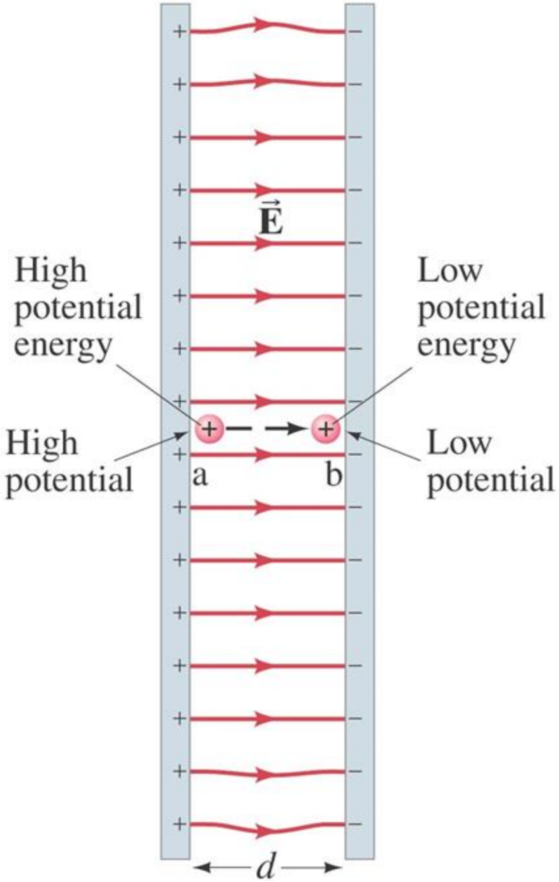
4. Electric Potential Energy and Potential Difference

Electric Potential Energy and Potential Difference

Karl Friedrich Gauss (1777-1855)

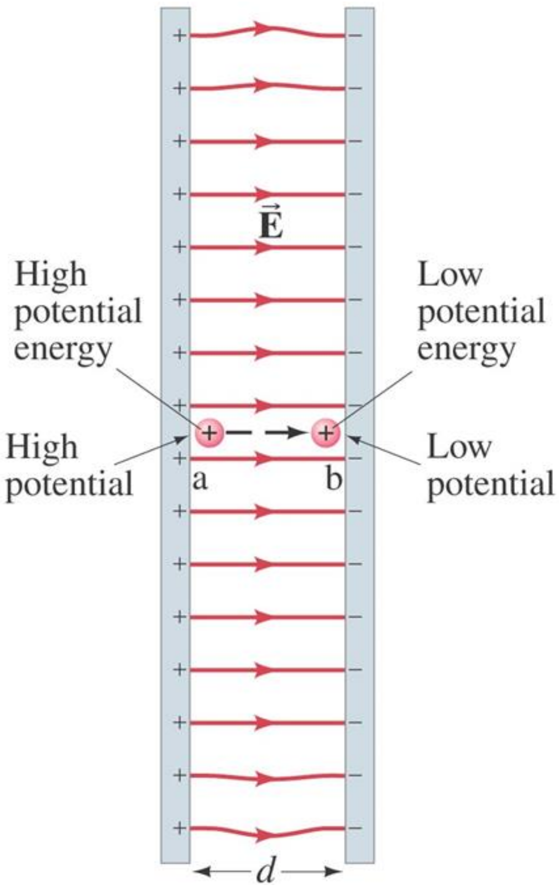
- As stated in previous lectures, in respect of mechanics, the energy point of view is useful for understanding electricity.
- It not only extends the law of conservation of energy, but it gives us another way to view electrical phenomena.
- It is also a powerful tool for solving problems more easily in many cases than by using forces and electric fields.

Electric Potential Energy



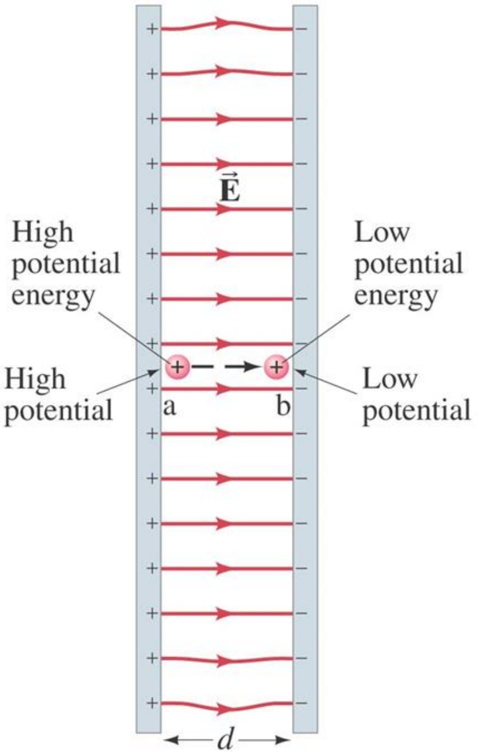
- To apply the conservation of energy principle, we need to define electric potential energy as we do for other types of potential energy.

Electric Potential Energy



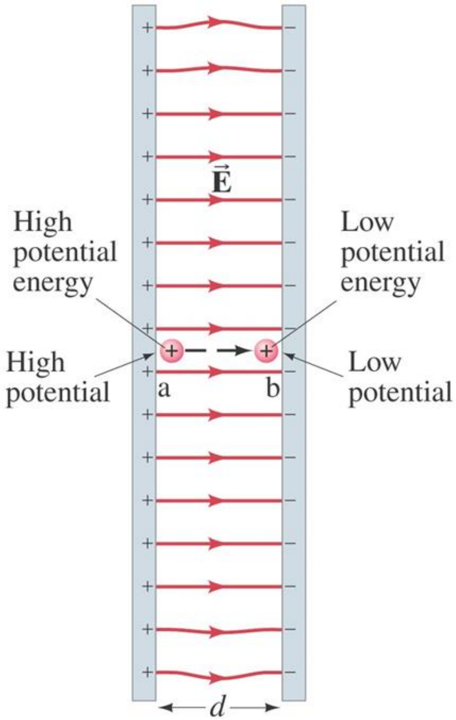
- The change in potential energy between two points, a and b, equals the negative of the work done by the conservative force (the electrical force) as an object moves from a to b: $\Delta U = -W$.

Electric Potential Energy



- In other words, the negative work represents the loss in potential energy that our charge (object) undergoes as it moves from a to b. The lost potential energy is gained by the object in terms of an increase in its kinetic energy.

Electric Potential Energy

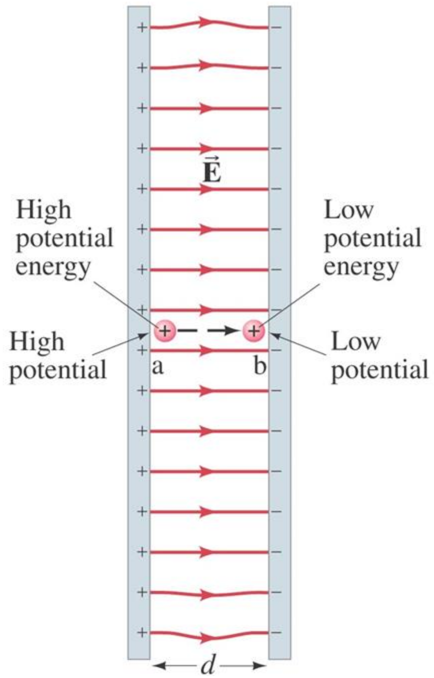


- The work W done by the electric field E to move the charge a distance d is

$$W = Fd = qEd$$

where $F = qE$.

Electric Potential Energy

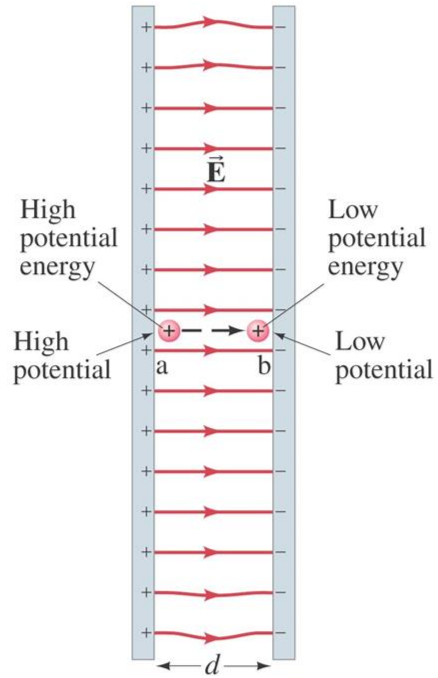


- The change in electric potential energy equals the negative of the work done by the electric force:

$$U_b - U_a = -W = -qEd$$

For this case of an assumed uniform electric field $\vec{E}$.

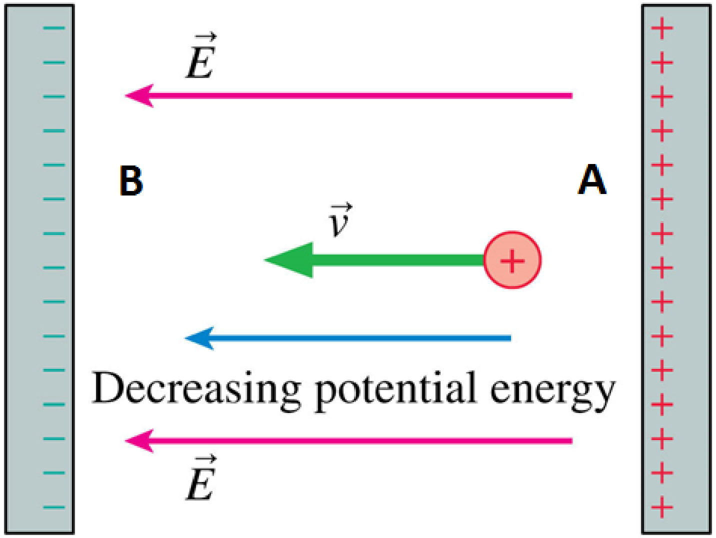
Electric Potential Energy



- Note that the positive charge q has its greatest potential energy at point a, near the positive plate.

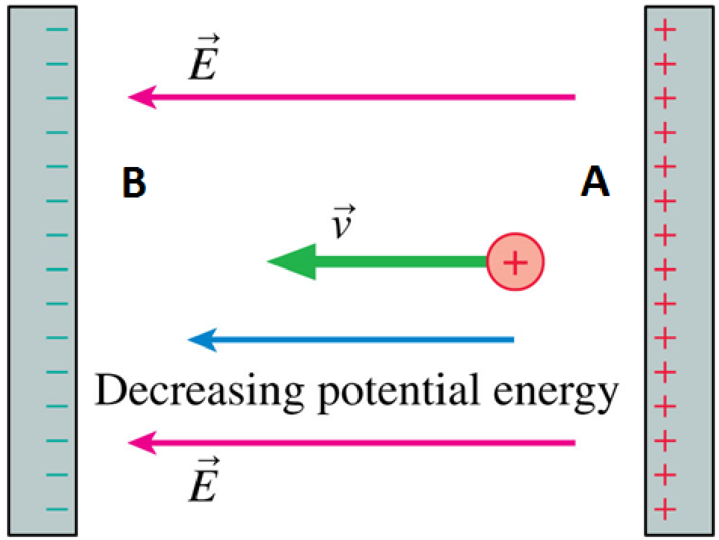
- The reverse is true for a negative charge: its potential energy is greatest near the negative plate.

Electric Potential and Potential Difference



- It is useful to define the **electric potential** (or simply the **potential** with the word 'electric' being implied) as the electric potential energy per unit charge.

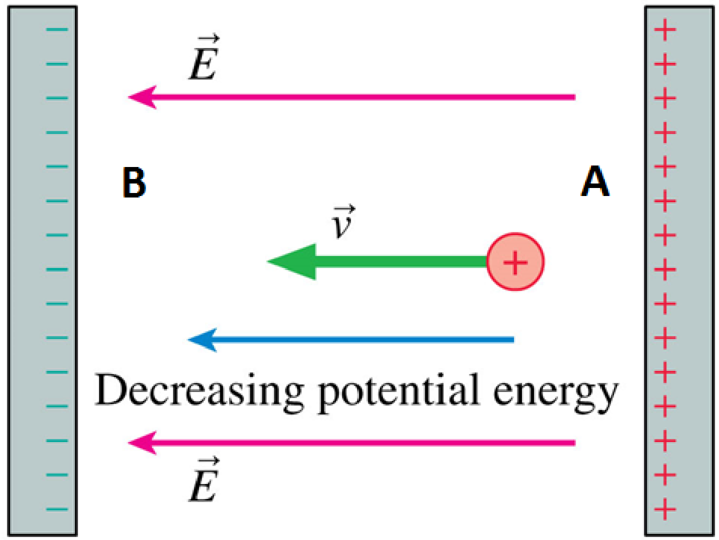
Electric Potential and Potential Difference



- Electric potential is given the symbol V. If a positive test charge q in an electric field has electric potential energy U_A at some point A (relative to some zero potential energy), the electric potential V_A at this point is

$$V_A = \frac{U_A}{q}$$

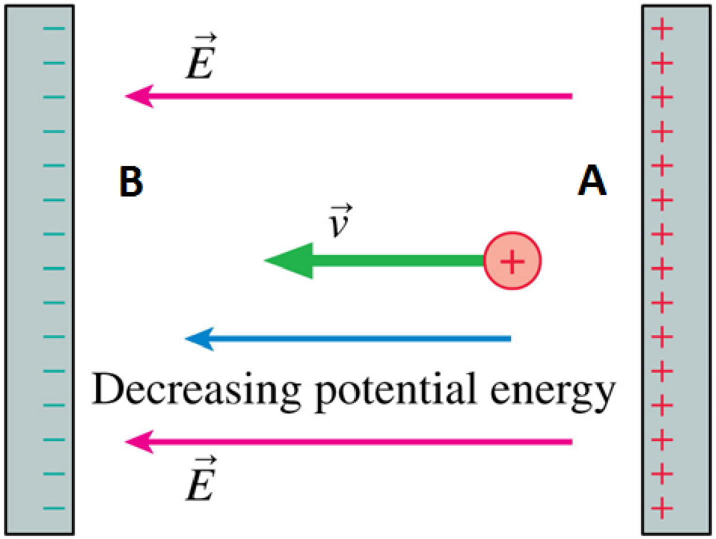
Electric Potential and Potential Difference



- Only the difference in potential, or the potential difference, between two points A and B (such as above) is measurable.
- As we've just seen, as the electric force does positive work on a charge, its kinetic energy increases and the potential energy decreases.

$$V_A = \frac{U_A}{q}$$

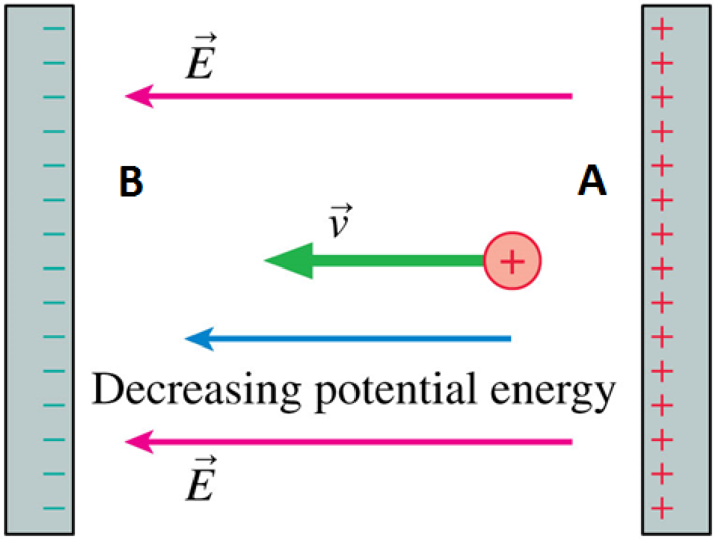
Electric Potential and Potential Difference



- The difference in potential energy, $U_B - U_A$, is equal to the negative of the work, W_{BA}, done by the electric field as the charge moves from A to B, so the potential difference V_{BA} is

$$V_{BA} = \Delta V = V_B - V_A = \frac{U_B - U_A}{q} = -\frac{W_{BA}}{q}$$

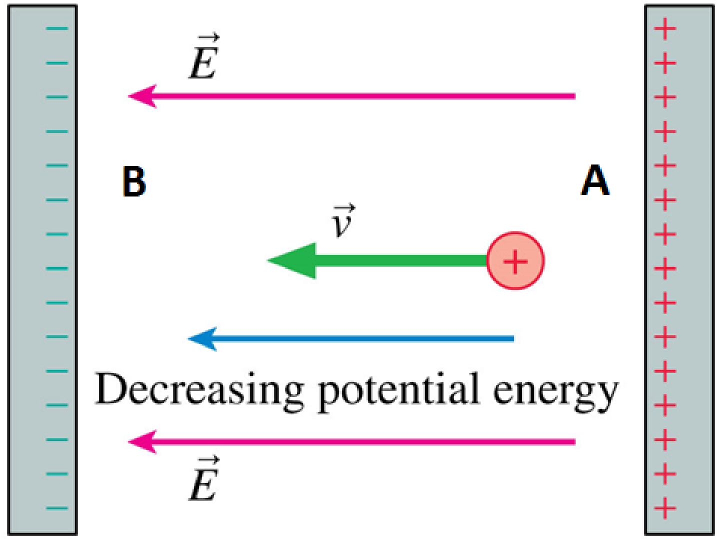
Electric Potential and Potential Difference



$$V_{BA} = \Delta V = V_B - V_A = \frac{U_B - U_A}{q} = -\frac{W_{BA}}{q}$$

- Note that electric potential, like electric field, does not depend on our test charge q.
- V depends on the other charges that create the field, not on q; q acquires potential energy by being in the potential V due to the other charges.

Electric Potential and Potential Difference



$$V_{BA} = \Delta V = V_B - V_A = \frac{U_B - U_A}{q} = -\frac{W_{BA}}{q}$$

- We can see from our definition that the positive plate in the above figure is at a higher potential than the negative plate. Thus, a positively charged object moves naturally from a high potential to a low potential. A negative charge does the reverse.

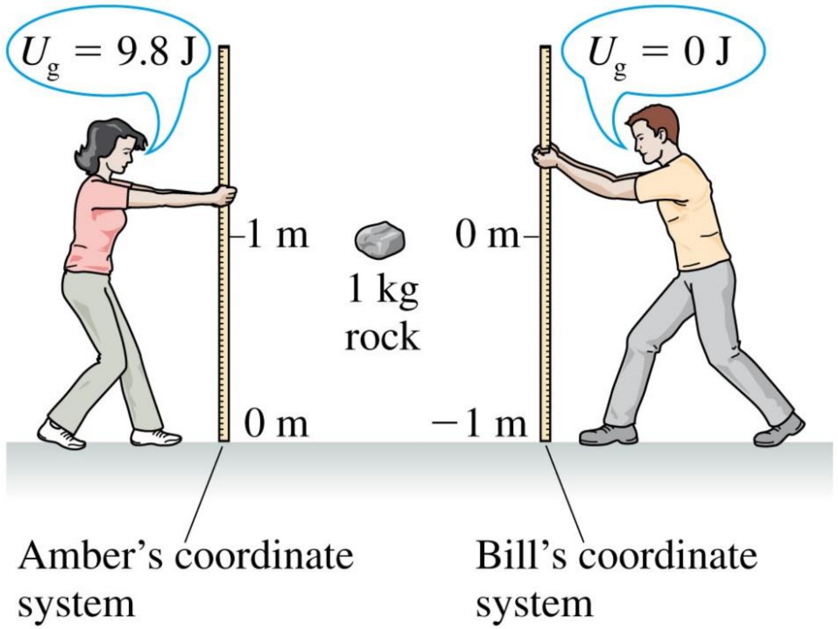
Electric Potential and Potential Difference

Alessandra Volta, 1745-1827

$$V_B = \Delta V = V_B - V_A = \frac{U_B - U_A}{q} = -\frac{W_{BA}}{q}$$

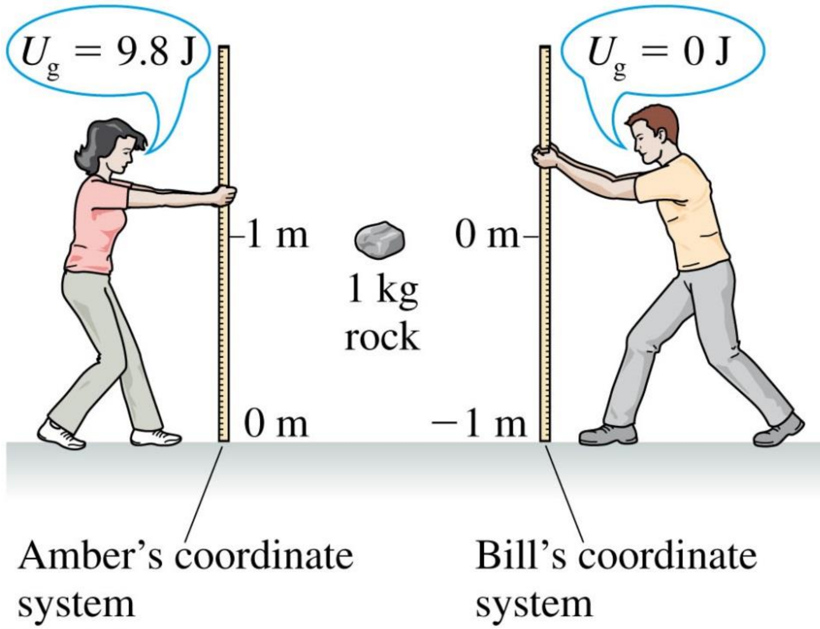
- The unit of electric potential, and of potential difference, is joules/coulomb and is given a special name, the **volt**, in honor of Alessandro Volta.
- The volt is abbreviated V, so $1\mathrm{V} = 1\ \mathrm{J/C}$.
- Potential difference, since it is measured in volts is often referred to as **voltage**.

Electric Potential and Potential Difference



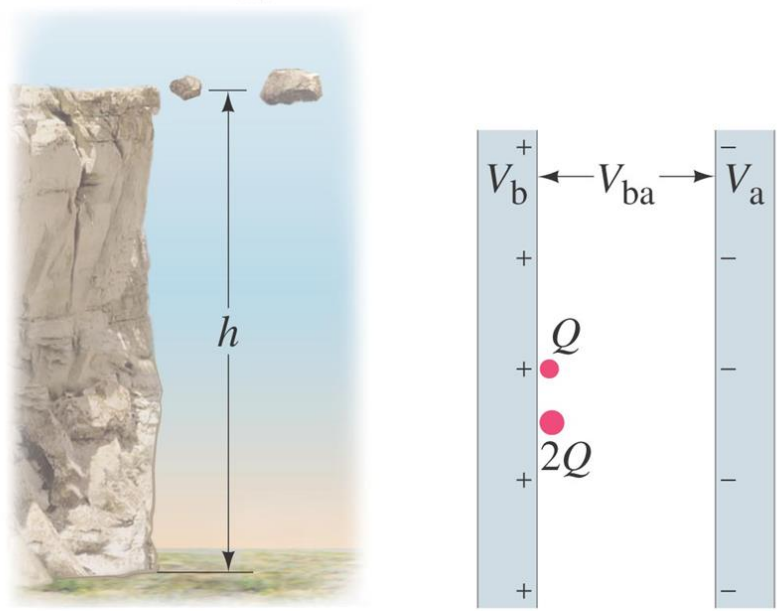
- If we wish to speak of the potential V_A at some point A, we must be aware that V_A depends on where the potential is chosen to be zero.
- The zero for electric potential energy can be chosen randomly, just as for potential energy, because only differences in potential energy can be measured.

Electric Potential and Potential Difference



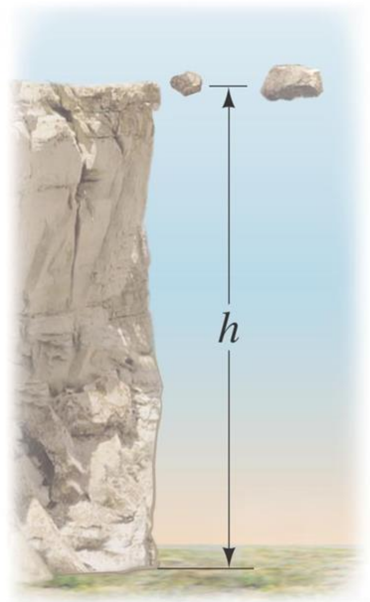
- Often the ground, or a conductor connected directly to the ground (the Earth), is taken as zero potential, and other potentials are given with respect to the ground.
- In other cases, as we will see, we may choose the potential to be zero at an infinite distance.

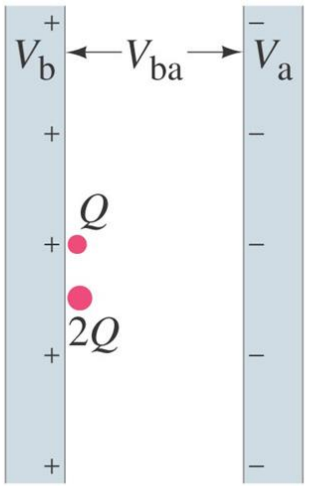
Electric Potential and Potential Difference



- To better understand electric potential, let's make a comparison to the gravitational case when a rock falls from the top of a cliff.
- The greater the height, h, of a cliff, the more potential energy $(= mgh)$ the rock has at the top of the cliff, relative to the bottom, and the more kinetic energy it will have when it reaches the bottom.

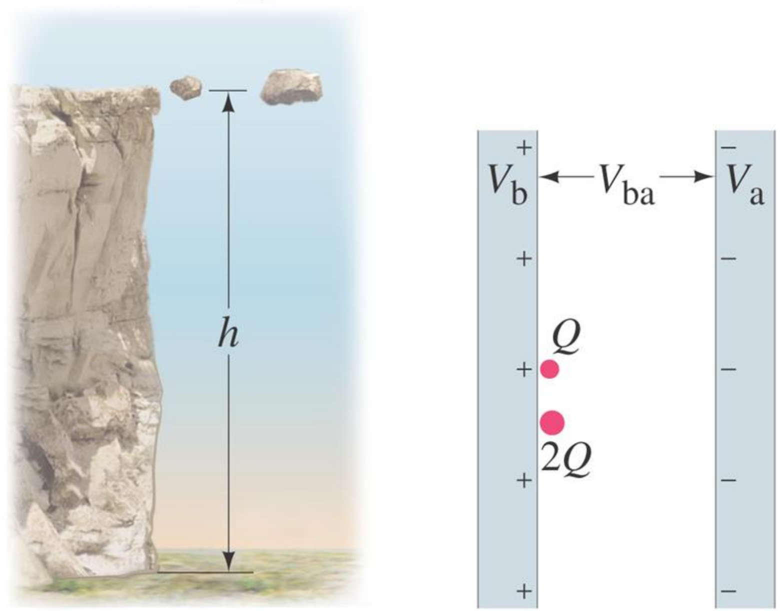
Electric Potential and Potential Difference





- The actual amount of kinetic energy it will acquire, and the amount of work it can do, depends both on the height of the cliff and the mass m of the rock.
- A large rock and a small rock can be at the same height h and thus have the same 'gravitational potential', but the larger rock has the greater potential energy (it has more mass).

Electric Potential and Potential Difference



- The electrical case is similar, as shown by the figure on the right.
- The potential energy change, or the work that can be done, depends both on the potential difference (corresponding to the height of the cliff) and on the charge (corresponding to mass).
- However, electric charge comes in two types, + and − , whereas gravitational mass is always +.

Summary of today's lecture

1. Electric Flux
2. Gauss's Law
3. Electric Potential Energy and Potential Difference

Home Work

Do not forget to attempt the **Additional Problems** for this lecture before logging in to **Mastering Physics** to complete your assignments.

Lecture 12: Optional Reading

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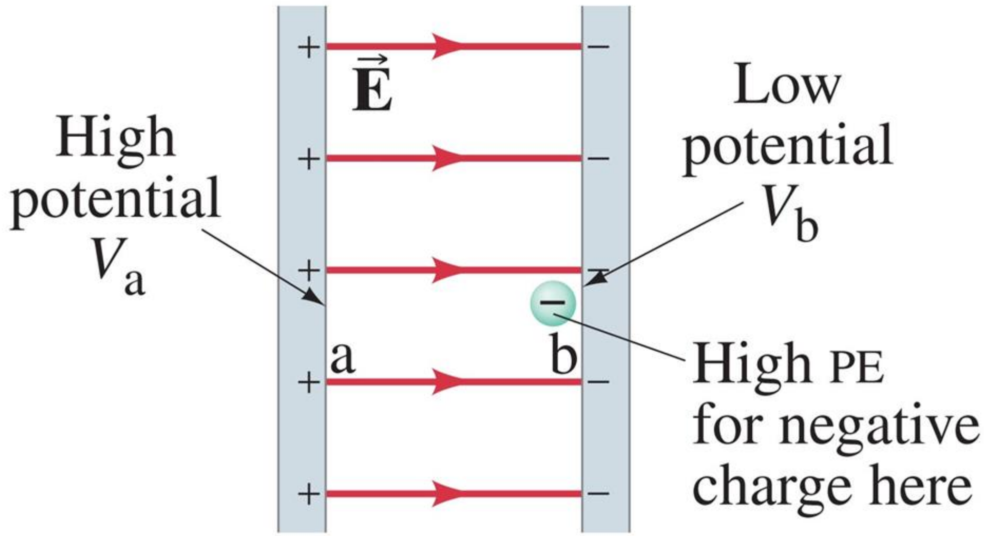
- **Ch. 22.1**, Electric Flux; p.684-685.
- **Ch. 22.2**, Gauss's Law; p.685
- **Ch. 22.3**, Applications of Gauss's Law; p.687-690
- **Ch. 23.2**, Relation between Electric Potential and Electric Field; p.706-708

Home Work

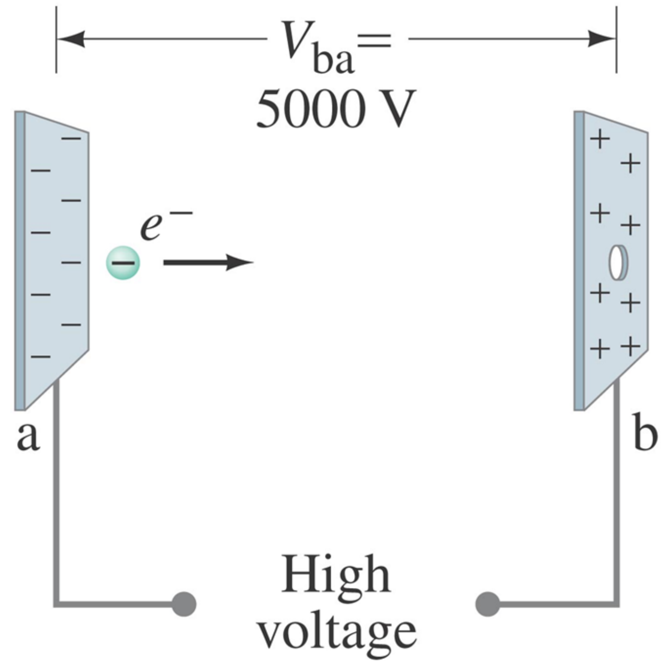
Do not forget to attempt the **Additional Problems** for this lecture before logging in to **Mastering Physics** to complete your assignments.

Possible Exam Question: Have a Read (p.705)



Q.1 Suppose a negative charge, such as an electron, is placed near the negative plate in the above figure. If the electron is free to move, will its electric potential energy increase or decrease? How will the electric potential change?

Possible Exam Question: Have a Read (p.706)



Q.2 Suppose an electron in a cathode ray tube is accelerated from rest through a potential difference $V_b - V_a = V_{ba} = +5000V$.

(a) What is the change in electric potential energy of the electron?

(b) What is the speed of the electron ($\mathrm{m} = 9.1 \times 10^{-31}\mathrm{kg}$) as a result of this acceleration?