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Foundation Algebra for Physical Sciences & Engineering CELEN036

Practice Problems SET-6 Sample Solution

Type 1: Real and imaginary parts of complex numbers

2. Find the real and imaginary parts of w defined by $w = \dfrac{1+z}{1-z}$, where $z = x + iy$ for some $x, y \in \mathbb{R}$.

Solution:

$$w = \frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy}$$

Let w to be multiply and divided by the conjugate of the denominator

$$\begin{aligned} w &= \frac{1+x+iy}{1-x-iy} \cdot \frac{1-x+iy}{1-x+iy} \\ &= \frac{(1+x)(1-x)+(1-x)iy+(1+x)iy-y^2}{(1-x)^2+y^2} \\ &= \frac{1-x^2-y^2+iy}{1-2x+x^2+y^2} \end{aligned}$$

$$\therefore Re(w) = \frac{1-x^2-y^2}{1-2x+x^2+y^2}, \quad Im(w) = \frac{y}{1-2x+x^2+y^2}$$

Type 2: Expressing complex numbers in the form $a + i\,b$

6. Simplify $(1+i)^6 - (1-i)^3$.

Solution:

$$\begin{aligned} (1+i)^6 - (1-i)^3 &= ((1+i)^2)^3 - (1-i)^2(1-i) \\ &= (2i)^3 - (-2i \cdot (1-i)) \\ &= -8i - (-2i-2) \\ &= 2 - 6i \end{aligned}$$

Type 3: Solving equations

9. Solve the following polynomial equations for $z \in \mathbb{C}$: $(i)\ z^2 + 6z + 10 = 0$

Solution:

The root of quadratic equation is:

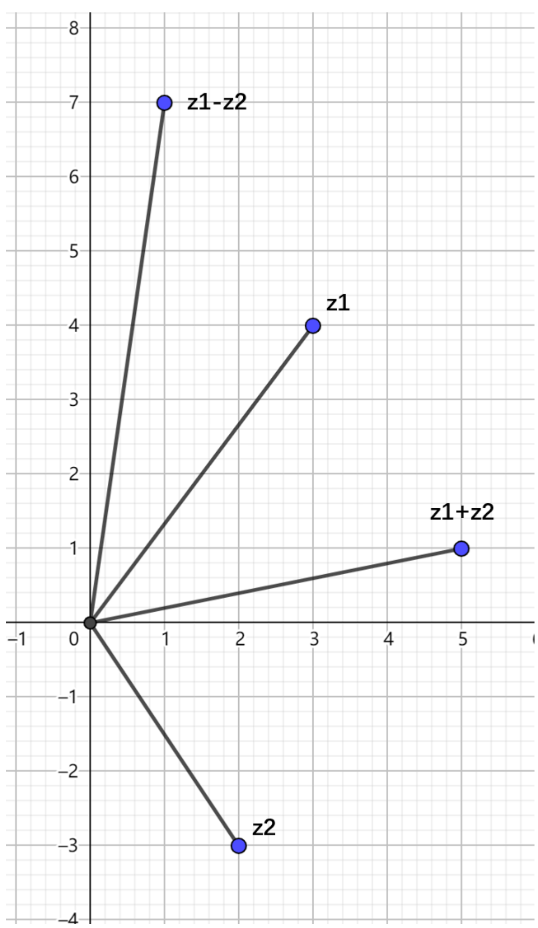
$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \therefore z &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= \frac{-6 \pm 2i}{2} = -3 \pm i \end{aligned}$$

Type 4: Argand diagram

11. For the given complex numbers z_1 and z_2, plot $z_1 + z_2$ and $z_1 - z_2$ on the Argand diagram:

$(i) z_1 = 3 + 4i, \qquad z_2 = 2 - 3i$

Solution:



Type 5: Modulus and argument

12. Given $z_1 = 3 - 2i$, $z_2 = 1 + 4i$, and $z_3 = 4 + 5i$, find the following values: $(i)\ \left|\dfrac{z_1\, z_3}{z_2}\right|$

Solution:

$$|z_1| = \sqrt{3^2 + (-2)^2} = \sqrt{13}, \quad |z_2| = \sqrt{1^2 + 4^2} = \sqrt{17}, \quad |z_3| = \sqrt{4^2 + 5^2} = \sqrt{41}$$

Use the properties of complex number modulus:

$$\left|\frac{z_1\, z_3}{z_2}\right| = \frac{|z_1| \cdot |z_3|}{|z_2|} = \frac{\sqrt{13} \cdot \sqrt{41}}{\sqrt{17}} = \sqrt{\frac{533}{17}}$$

Type 6: Polar form of complex numbers

15. Find the polar form of the following complex numbers:

$$z_1 = 2 + 2i \qquad z_2 = 2 - 2i$$

Hence find the modulus r and principal argument $(\theta \in (-\pi, \pi])$ of the complex numbers $z_1 \cdot z_2$

Solution:

Modulus: $r_1 = \sqrt{2^2 + 2^2} = 2\sqrt{2},\ r_2 = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

Argument of $z_1 = x_1 + iy_1 = 2 + 2i$:

$x_1 = 2 > 0$ and $y_1 = 2 > 0,$ belong to Quadrant I

therefore use the formula: $\theta_1 = \tan^{-1}\left|\dfrac{y_1}{x_1}\right| = \tan^{-1}\left|\dfrac{2}{2}\right| = \dfrac{\pi}{4}$

Argument of $z_2 = x_2 + iy_2 = 2 - 2i$:

$x_2 = 2 > 0$ and $y_2 = -2 > 0,$ belong to Quadrant IV

therefore use the formula: $\theta_2 = -\tan^{-1}\left|\dfrac{y_2}{x_2}\right| = -\tan^{-1}\left|\dfrac{2}{2}\right| = -\dfrac{\pi}{4}$

$$\therefore z_1 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right), \quad z_2 = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right) = 8\left(\cos(0) + i\sin(0)\right) = 8$$