Lecture 2

Structure of lecture

1. The Chain Rule
2. Logarithmic Differentiation
3. Implicit Differentiation
4. Derivatives of Inverse Trigonometric Functions

Derivative of composite functions

We have seen how to differentiate the sum $f + g$, the difference $f - g$, the product fg and the quotient f/g of two functions. Now, we will learn how to find the derivative of a *composite function* $f \circ g$.

e.g. Let $f(x) = 3x^3 + 4x$, and $g(x) = 2\sqrt{x}$.

Question:

What is the derivative of the composite function

$y = g\big(f(x)\big) = 2\sqrt{3x^3 + 4x}$?

Derivative of composite functions

The derivative of composition $(f \circ g)$ of two functions f and g can be calculated by:

$$\frac{d}{dx}\left[(f \circ g)(x)\right] = \frac{d}{dx}\left[f\left(g(x)\right)\right] = f'[g(x)] \cdot g'(x)$$

Let $g(x) = u \Rightarrow \dfrac{du}{dx} = g'(x)$ Derivative of the inner function

$u = g(x)$: Inner function

so that $y = f(g(x)) = f(u) \Rightarrow \dfrac{dy}{du} = f'(u) = f'(g(x))$

$y = f(u)$: outer function

Derivative of the outer function

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Or, equivalently,

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'[g(x)] \cdot g'(x)$$

Chain Rule

Example 1: Given $y = \sin x^2$, find $\dfrac{dy}{dx}$.

Let $x^2 = u$ so that
$$\left[\begin{array}{l} \dfrac{du}{dx} = 2x \quad \text{and} \\ y = \sin u \Rightarrow \dfrac{dy}{du} = \cos u \end{array} \right.$$

Now, $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}}$ $\Rightarrow \dfrac{dy}{dx} = \cos u \cdot 2x$

Chain Rule

$$\therefore \frac{dy}{dx} = 2x \cdot \cos x^2$$

Chain Rule

Example 2: Given $y = \sin^2 x$, find $\dfrac{dy}{dx}$.

Note:

$$\sin^2 x = (\sin x)^2$$

Let $\sin x = u$ so that

$$\left[\begin{array}{l} \dfrac{du}{dx} = \cos x \text{ and} \\ y = (\sin x)^2 = u^2 \Rightarrow \dfrac{dy}{du} = 2u \end{array}\right.$$

Now, $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}} \Rightarrow \dfrac{dy}{dx} = 2u \cdot \cos x$

Chain Rule

$$= 2 \sin x \cdot \cos x = \sin 2x$$

Chain Rule

A composition of 3 functions

Example 3: Given $y = \ln\left(\sin x^3\right)$, find $\dfrac{dy}{dx}$.

Let $\left[\begin{array}{l} \sin x^3 = u \quad \text{and} \\ x^3 = v \end{array}\right.$ so that $\left[\begin{array}{l} \dfrac{dv}{dx} = 3x^2 \\ \dfrac{du}{dv} = \cos v \quad \text{and} \\ y = \ln\left(\sin x^3\right) = \ln u \end{array}\right.$

$$\Downarrow$$

$$u = \sin v$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{u}$$

Chain Rule

Now, $$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}}$$ **Chain Rule**

$$\frac{dv}{dx} = 3\,x^2$$

$$\frac{du}{dv} = \cos v$$

$$\therefore \frac{dy}{dx} = \frac{1}{u} \cdot \cos v \cdot 3\,x^2$$

$$= \frac{1}{\sin\left(x^3\right)} \cdot \cos\left(x^3\right) \cdot 3\,x^2$$

$$= 3\,x^2 \cdot \cot\left(x^3\right)$$

Chain Rule: Fast-track method

1) $y = \sin x^2 \Rightarrow \dfrac{dy}{dx} = \cos x^2 \cdot \boxed{\dfrac{d}{dx}\left(x^2\right)}$

$= 2x \cdot \cos x^2$

No need to specify the inner function u

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2) $y = \sin^2 x = (\sin x)^2$

$\Rightarrow \dfrac{dy}{dx} = 2\,\sin x \cdot \boxed{\dfrac{d}{dx}(\sin x)} = 2\,\sin x \cdot \cos x$

$= \sin 2x$

Chain Rule: Fast-track method

3) $y = \ln\left(\sin x^3\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sin\left(x^3\right)} \cdot \boxed{\frac{d}{dx}\left(\sin\left(x^3\right)\right)}$$

$$= \frac{1}{\sin\left(x^3\right)} \cdot \cos\left(x^3\right) \cdot \boxed{\frac{d}{dx}\left(x^3\right)}$$

$$= 3\,x^2 \cdot \cot\left(x^3\right)$$

Chain Rule

$$\text{Derivative of } \ln y \text{ w.r.t. } x = \frac{1}{y} \cdot \boxed{\frac{dy}{dx}}$$

$$\text{Derivative of } y^3 \text{ w.r.t. } x = 3\,y^2 \cdot \boxed{\frac{dy}{dx}}$$

$$\text{Derivative of } e^y \text{ w.r.t. } x = e^y \cdot \boxed{\frac{dy}{dx}}$$

Implicit Functions

An equation of the form $y = f(x)$ is said to be an explicit function in the sense that

- the variable y appears alone on one side of the equation.
- y does not appear at all on the other side.

Functions that are not explicit are called implicit functions.

Implicit Functions

Examples

1. In $xy + y + 1 = x$, the variable y is not alone on one side.

   i.e. equation is not of the form $y = f(x)$.

   We say that such equation defines y implicitly as a function of x.

Implicit Functions

2. An equation in x and y can define more than one functions of x.

   e.g. if we solve the equation of the unit circle $x^2 + y^2 = 1$ we get two functions, namely

$$f_1(x) = \sqrt{1 - x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1 - x^2}$$

   We say that such equation defines y implicitly as a function of x.

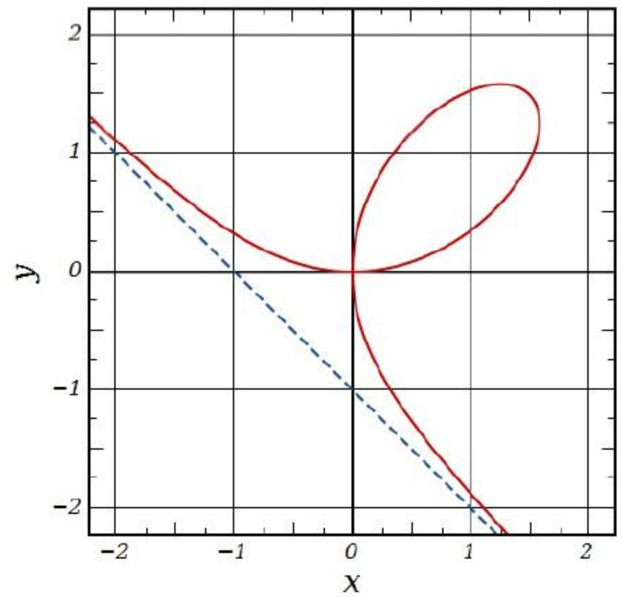
Implicit Functions

3. It is sometimes too complicated or impossible to solve y in terms of x.

   e.g. $x^3 + y^3 = 3xy$

   Folium of Descartes

   or $\sin(xy) = y$



We say that such equation defines y implicitly as a function of x.

Implicit Differentiation

To find derivatives of implicit functions, differentiate both sides with respect to x (independent variable).

Given $xy = 1$, find $\frac{dy}{dx}$.

Direct Method:

$$y = \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

Implicit differentiation:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$\therefore \quad x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x^2}$$

Implicit Differentiation

Note:

Consider y is an (implicit) function of x, some results obtained using the chain rule and the product rule:

$$\frac{d}{dx}\left(y^2\right) = 2\,y\,\frac{dy}{dx}$$

$$\frac{d}{dx}\left(y^3\right) = 3\,y^2\,\frac{dy}{dx}$$

$$\frac{d}{dx}\left(e^y\right) = e^y\,\frac{dy}{dx}$$

$$\frac{d}{dx}\left(x^2 y\right) = x^2\,\frac{dy}{dx} + y \cdot (2x)$$

$$\frac{d}{dx}\left(xy^2\right) = x\,(2\,y)\,\frac{dy}{dx} + y^2$$

$$\frac{d}{dx}\left(xy\right) = x\,\frac{dy}{dx} + y$$

Implicit Differentiation

Example 1: Given $x^3 + y^3 = 3\,xy$. Find $\dfrac{dy}{dx}$.

Differentiating w.r.t. x

$$\Rightarrow \cancel{3}\,x^2 + \cancel{3}\,y^2\,\frac{dy}{dx} = \cancel{3}\left[x\,\frac{dy}{dx} + y\cdot(1)\right]$$

$$\Rightarrow x^2 + y^2\,\frac{dy}{dx} = x\,\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}\left[y^2 - x\right] = \left(y - x^2\right)$$

$$\therefore \frac{dy}{dx} = \left(\frac{y - x^2}{y^2 - x}\right)$$

Implicit Differentiation

Example 2: Given $\cos(xy) = \sqrt{x+y}$. Find $\dfrac{dy}{dx}$.

Differentiating w.r.t. x

$$\Rightarrow \; -\sin(xy) \cdot \frac{d}{dx}(xy) = \frac{1}{2\sqrt{x+y}} \cdot \frac{d}{dx}(x+y)$$

Chain Rule

$$\Rightarrow \; -\sin(xy) \cdot \left[x\frac{dy}{dx} + y \right] = \frac{1}{2\sqrt{x+y}} \cdot \left[1 + \frac{dy}{dx} \right]$$

Product and Sum Rule

$$\therefore \; \frac{dy}{dx} = -\left[\frac{1 + 2y\sqrt{x+y} \cdot \sin(xy)}{1 + 2x\sqrt{x+y} \cdot \sin(xy)} \right]$$

Implicit Differentiation

Example 3:

Prove that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, given $\frac{d}{dx}(e^x) = e^x$.

Let $y = \ln x$, then $x = e^y$.

Differentiate both sides w.r.t. x: $1 = e^y \cdot \frac{dy}{dx}$ Implicit differentiation

This leads to: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

This proof is much shorter than the one shown in Lecture 7

Implicit Differentiation

Example 4:

Find the gradient of $x^2 + 2xy - 2y^2 + x = 2$ at point $(-4, 1)$.

Differentiate w.r.t x:

$$\frac{d}{dx}(x^2+2xy - 2y^2 + x) = \frac{d}{dx}(2)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - 2\frac{d}{dx}(y^2) + \frac{d}{dx}(x) = \frac{d}{dx}(2)$$

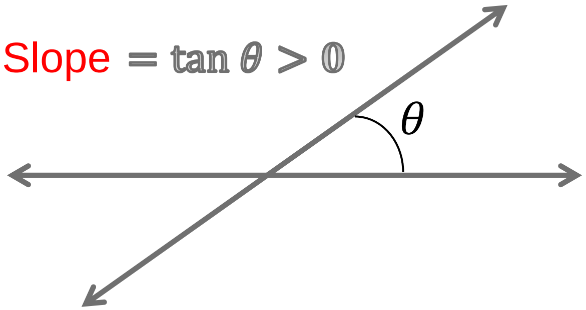
$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) - 2(2y)\cdot\frac{dy}{dx} + 1 = 0$$

Implicit Differentiation

Note: Slope $= \tan\theta$

$$\Rightarrow \quad 2x + 2x\frac{dy}{dx} + 2y - 4y\frac{dy}{dx} + 1 = 0$$

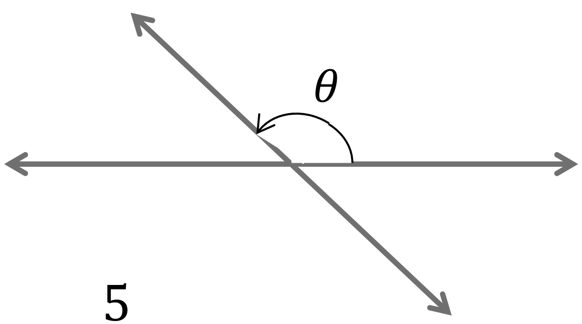
(Acute Angle) Slope $= \tan\theta > 0$



$$\Rightarrow \quad (2x - 4y)\cdot\frac{dy}{dx} = -1 - 2x - 2y$$

$$\therefore \frac{dy}{dx} = \frac{-(1 + 2x + 2y)}{2x - 4y}$$

(Obtuse Angle) Slope $= \tan\theta < 0$



$\therefore$ Gradient (slope) is: $\left.\dfrac{dy}{dx}\right|_{\substack{(-4,1)\\(x,y)}} = \dfrac{-(1 - 8 + 2)}{-8 - 4} = -\dfrac{5}{12}$

Logarithmic Differentiation

Logarithmic differentiation means finding the derivative of a function after taking logarithms.

The method is useful when either

- the function is raised to the power of variables or functions.

  e.g. $(\sin x)^{\tan x}$

OR

- the function is composed of a product of a number of parts.

  e.g. $$\left(\frac{\sqrt[3]{x^2-1}\cdot(1+e^x)^{2/3}}{(\sin x)^x}\right)$$

Logarithmic Differentiation

The method relies on the chain rule, the product rule, and the properties of logarithms.

The method consists of **3** main steps.

- Take logarithms on both the sides of the function.
- Apply rules of logarithms to simplify the expressions.
- Differentiate both the sides with respect to x, by using the chain rule and the product rule (where applicable).

Logarithmic Differentiation

Example 1: Differentiate $y = x^x$.

The Method:

- Take logarithms on both the sides.
- Apply rules of logarithms.
- Differentiate both the sides w.r.t. *x*.

Let $y = (x)^x$

$$\Rightarrow \ln y = \ln(x)^x$$

$$\Rightarrow \ln y = x \cdot \ln(x)$$

$$\boxed{\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}}$$

Differentiate with respect to x

$$\Rightarrow \boxed{\frac{1}{y}\frac{dy}{dx}} = x \cdot \frac{d}{dx}(\ln(x)) + \ln(x) \cdot \frac{d}{dx}(x)$$ Apply Product Rule

Logarithmic Differentiation

$$\Rightarrow \frac{dy}{dx} = y\left[\, x \cdot \frac{1}{x} + \ln(x) \cdot (1) \,\right]$$

$$= \; x^x\,(1 + \ln x)$$

Thus, $\dfrac{d}{dx}\left(x^x\right) = x^x \cdot (1 + \ln x)$

Logarithmic Differentiation

Example 2: Find $\frac{d}{dx}(\sin x)^{\tan x}$

Let $y = (\sin x)^{\tan x}$

$$\Rightarrow \ln y = \ln (\sin x)^{\tan x}$$

$$\Rightarrow \ln y = \tan x \cdot \ln(\sin x)$$

Differentiate with respect to x

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \ln(\sin x) \cdot \frac{d}{dx}(\tan x)$$

Apply Product Rule

The Method:

- Take logarithms on both the sides.
- Apply rules of logarithms.
- Differentiate both the sides w.r.t. *x*.

Logarithmic Differentiation

$$\therefore \ \frac{dy}{dx} = y \left[\tan x \cdot \frac{1}{\sin x} \ \cos x + \ln(\sin x) \ \sec^2 x \right]$$

$$= (\sin x)^{\tan x} \left[1 + \sec^2 x \ \ln(\sin x) \right]$$

Logarithmic Differentiation

Example 3: Find $\dfrac{d}{dx}\left(\dfrac{\sqrt[3]{x^2-1}\;(1+e^x)^{2/3}}{(\sin x)^x}\right)$

Let $y = \left(\dfrac{\sqrt[3]{x^2-1}\;(1+e^x)^{2/3}}{(\sin x)^x}\right)$

A lot of work if differentiating directly by product, quotient, power and chain rules!

Taking logarithms on both the sides

$$\therefore \ln y = \ln(x^2-1)^{1/3} + \ln(1+e^x)^{2/3} - \ln(\sin x)^x$$

Logarithmic Differentiation

$$\therefore \ln y = \frac{1}{3}\ \ln(x^2-1) + \frac{2}{3}\ \ln(1+e^x) - x\ \ln(\sin x)$$

Differentiate with respect to x

$$\Rightarrow \frac{1}{y}\ \frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{(x^2-1)} \cdot \frac{d}{dx}\left(x^2-1\right)$$

$$+\frac{2}{3} \cdot \frac{1}{(1+e^x)} \cdot \frac{d}{dx}\left(1+e^x\right)$$

$$-x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) - \ln(\sin x) \cdot \frac{dx}{dx}$$

Apply Product Rule

Logarithmic Differentiation

$$\frac{dy}{dx} = y\left[\frac{1}{3}\cdot\frac{1}{(x^2-1)}\cdot 2x + \frac{2}{3}\cdot\frac{1}{(1+e^x)}\cdot e^x\right.$$

$$\left. - x\cdot\frac{1}{\sin x}\cdot\cos x - \ln(\sin x)\cdot(1)\right]$$

$$\therefore\ \frac{dy}{dx} = \left(\frac{\sqrt[3]{x^2-1}\ (1+e^x)^{2/3}}{(\sin x)^x}\right)$$

$$\cdot\left[\frac{2x}{3}\,\frac{1}{(x^2-1)} + \frac{2}{3}\,\frac{e^x}{(1+e^x)} - x\cot x - \ln(\sin x)\right]$$

Derivatives of Inverse Functions

Let $y = f^{-1}(x)$, which is equivalent to writing $x = f(y)$.

Differentiating with respect to x, we obtain

$$\frac{dx}{dx} = \frac{d}{dy}\,[f(y)] \cdot \frac{dy}{dx} \qquad \text{(by Chain Rule)}$$

$$\Rightarrow \quad 1 = \frac{d}{dy}\,[f(y)] \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{d}{dy}\,[f(y)]} \qquad \Rightarrow \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{\frac{d}{dy}\,[f(y)]}$$

Derivatives of Inverse Functions

If f is a differentiable and one-to-one function, then

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right]} = \frac{1}{\boxed{\frac{d}{dy}\left[f(y)\right]}} \qquad \text{where} \quad y = f^{-1}(x)$$

$$\text{provided} \qquad \frac{d}{dy}\left[f(y)\right] \neq 0$$

Also, $x = f(y)$ gives the alternative and more useful form

$$\boxed{\boxed{\frac{dy}{dx}} = \frac{1}{\boxed{\left(\frac{dx}{dy}\right)}}}$$

Derivatives of Inverse Trig Functions

Example: Prove the following derivative formula using the chain rule and implicit differentiation.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Let $y = \sin^{-1} x$, then $x = \sin y$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Differentiate both sides w.r.t. x: $1 = \cos y \cdot \frac{dy}{dx}$

So, we obtain: $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad ; \quad |\, x \,| < 1$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad |\, x \,| < 1$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \quad ; \quad x \in \mathbb{R}$$

Derivatives of Inverse Trig Functions

Example: Given $y = \sin^{-1}\left(x^3\right)$. Find $\dfrac{dy}{dx}$ where $|\,x\,| < 1$

Note: $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1 - x^2}}$

$$\therefore \; \frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^3)^2}} \cdot \boxed{\frac{d}{dx}\left(x^3\right)} = \frac{3\,x^2}{\sqrt{1 - x^6}}$$

Chain Rule