Student Evaluation for Module (SEM):

[CELEN037 Foundation Calculus and Mathematical Techniques (Chenyang Xue)]

Topic 1: Solutions of Ordinary Differential Equations

A function $f(x)$ is called a solution of a differential equation if the differential equation is satisfied when $y = f(x)$ and its derivatives are substituted into the given differential equation.

Illustration 1: Show that $y = C_1 \sin 4x + C_2 \cos 4x$ is a solution of the ODE

$$\frac{d^2y}{dx^2} + 16y = 0, \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

$$\frac{dy}{dx} = 4C_1 \cos 4x - 4C_2 \sin 4x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -16C_1 \sin 4x - 16C_2 \cos 4x$$

$$= -16(C_1 \sin 4x + C_2 \cos 4x) = -16y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 16y = 0$$

$\therefore y = C_1 \sin 4x + C_2 \cos 4x$ is a solution of the ODE.

Illustration 2: Show that $y = e^{-x} + ax + b$ is a solution of the ODE

$$e^x \cdot \frac{d^2y}{dx^2} - 1 = 0, \text{ where } a \text{ and } b \text{ are arbitrary constants.}$$

$$y = e^{-x} + ax + b \qquad \Rightarrow \qquad \frac{dy}{dx} = -e^{-x} + a$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = e^{-x}$$

$$\Rightarrow \quad e^x \cdot \frac{d^2y}{dx^2} - 1 = e^x \cdot e^{-x} - 1 = 1 - 1 = 0$$

$\therefore y = e^{-x} + ax + b$ is a solution of the ODE.

1. Show that $y = C_1 e^{2x} + C_2 e^{3x}$ is a solution of the ODE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, where C_1 and C_2 are arbitrary constants.

Answer:

2. Show that $y = C_1 e^{-2x} + C_2 e^{x}$ is a solution of the ODE $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, where C_1 and C_2 are arbitrary constants.

Answer:

3. Show that $y = a\cos^{-1} x + b$ is a solution of the ODE $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$, where a and b are arbitrary constants.

Answer:

4. Show that $y = \frac{a}{x} + b$ is a solution of the ODE $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$, where a and b are arbitrary constants.

Answer:

Topic 2: Solving ODE of Variable-Separable Form

The variable-separable ordinary differential equation can be written in the form

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(x)}{g(y)} \\ \Rightarrow \quad g(y)\,dy &= f(x)\,dx \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \Rightarrow \quad \int g(y)\,dy &= \int f(x)\,dx \\ \Rightarrow \quad G(y) &= F(x) + C \end{aligned}$$

where $G(y)$ and $F(x)$ denote the antiderivatives of $g(y)$ and $f(x)$, respectively.

Illustration 1: Solve the variable-separable ODE: $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \int y\,dy = -\int x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$ general solution of the given ODE

$$\Rightarrow \quad y^2 = -x^2 + C$$

Solve the following Variable-Separable ODEs:

1. $\dfrac{dy}{dx} = \dfrac{y}{x}$ **Answer:**	2. $\dfrac{dy}{dx} = \dfrac{1+x^2}{1+y^2}$ **Answer:**
3. $\dfrac{dy}{dx} = x^2(1+y^2)$ **Answer:**	4. $y\dfrac{dy}{dx} = (1+y^2)\tan x$ **Answer:**

Topic 2: Solving ODE of Variable-Separable Form

Illustration 2: Solve the variable-separable ODE: $\ln(\sin x)\dfrac{dy}{dx} = \cot x$.

$$\ln(\sin x)\frac{dy}{dx} = \cot x$$

$$\Rightarrow \quad dy = \frac{\cot x}{\ln(\sin x)}\,dx$$

$$\Rightarrow \quad dy = \frac{1}{\ln(\sin x)}\,\frac{\cos x}{\sin x}\,dx$$

$$\Rightarrow \int dy = \int \frac{1}{\ln(\sin x)}\,\frac{\cos x}{\sin x}\,dx$$

$$\boxed{\text{Let } \ln(\sin x) = t \quad \Rightarrow \quad \frac{\cos x}{\sin x}\,dx = dt}$$

$$\Rightarrow \quad y = \int \frac{1}{t}\,dt = \ln|t| + C$$

$$\Rightarrow \quad y = \ln|\ln(\sin x)| + C \quad \boxed{\text{general solution of the given ODE}}$$

Illustration 3: Solve the variable-separable ODE: $\dfrac{dy}{dx} = \dfrac{xe^x}{y\sqrt{1+y^2}}$.

$$\frac{dy}{dx} = \frac{xe^x}{y\sqrt{1+y^2}}$$

$$\Rightarrow \quad y\sqrt{1+y^2}\,dy = xe^x\,dx$$

using substitution

integration by parts

$$\Rightarrow \int y\sqrt{1+y^2}\,dy = \int xe^x\,dx$$

$$\Rightarrow \quad \frac{1}{3}(1+y^2)^{\frac{3}{2}} = xe^x - \int e^x\,dx$$

$$\Rightarrow \quad \frac{1}{3}(1+y^2)^{\frac{3}{2}} = xe^x - e^x + C$$

$$\Rightarrow \quad (1+y^2)^{\frac{3}{2}} = 3xe^x - 3e^x + C \quad \boxed{\text{general solution of the given ODE}}$$

Solve the following Variable-Separable ODEs:

1. $\dfrac{dy}{dx} = \dfrac{\tan y}{x \sec^2 y}$

Answer:

2. $(e^x + e^{-x})\dfrac{dy}{dx} = e^x - e^{-x}$

Answer:

3. $\dfrac{dy}{dx} = \dfrac{y \cos x}{1 + \sin x}$

Answer:

4. $y \ln y dx = x dy$

Answer:

Topic 3: Solving Initial Value Problem (IVP) of Variable-Separable ODE

Illustration 1: Solve the IVP of the variable-separable ODE:

$$\frac{dy}{dx} = \frac{x^2}{y^2}; \quad y(0) = 2$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\Rightarrow \quad y^2\,dy = x^2\,dx$$

$$\Rightarrow \int y^2\,dy = \int x^2\,dx$$

general solution of the given ODE

$$\Rightarrow \quad \frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y(0) = 2 \;\Rightarrow\; x = 0, y = 2$$

$$\Rightarrow \quad \frac{2^3}{3} = \frac{0^3}{3} + C$$

intial value

$$\Rightarrow \quad C = \frac{8}{3}$$

particular solution of the given ODE

$$\Rightarrow \quad \frac{y^3}{3} = \frac{x^3}{3} + \frac{8}{3}$$

$$\Rightarrow \quad y^3 = x^3 + 8$$

Illustration 2: Solve the IVP of the variable separable ODE:

$$e^{\frac{dy}{dx}} = x + 1 \quad (x > -1); \quad y(0) = 3.$$

$$e^{\frac{dy}{dx}} = x + 1 \quad \Rightarrow \quad \frac{dy}{dx} = \ln(x+1) \quad \Rightarrow \quad dy = \ln(x+1)\,dx$$

$$\Rightarrow \int dy = \int \ln(x+1)\,dx$$

apply integration by parts

$$\Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{x+1}\,dx$$

$$\Rightarrow \quad y = x\ln(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \quad y = x\ln(x+1) - x + \ln(x+1) + C$$

general solution

$$y(0) = 3 \;\Rightarrow\; x = 0, y = 3 \quad \Rightarrow \quad 3 = C$$

intial value

$$\Rightarrow \quad y = (x+1)\ln(x+1) - x + 3$$

particular solution of the given ODE

Solve the following IVPs:

1. $\dfrac{dy}{dx} + 4xy^2 = 0;\ y(0) = 1$

Answer:

2. $\dfrac{dy}{dx} = \dfrac{y \sin x}{1 + y^2};\ y(0) = 1$

Answer:

Solve the following IVPs:

3. $\dfrac{dy}{dx} = y\tan x;\ y(0) = 1$

Answer:

4. $x + 2y\sqrt{x^2+1}\dfrac{dy}{dx} = 0;\ y(0) = 1$

Answer:

Topic 4: Applications of Ordinary Differential Equation (ODE)

Illustration: The rate of population increase of insects used in an experiment is proportional to the insect population (P).

(i) Formulate a differential equation to show that the population of the insect at time t is $P = P_0 \cdot e^{kt}$, where $k > 0$ is constant, and P_0 is the initial population.

(ii) If the population increased from 1000 to 1300 after 20 days, find the population after 35 days.

(iii) How long will it take for the population to reach 2000?

(i) $$\frac{dP}{dt} \propto P$$

The ODE is variable-separable

$$\Rightarrow \quad \frac{dP}{dt} = kP \quad (k > 0) \quad \Rightarrow \quad \frac{1}{P}dP = k\,dt$$

$$\Rightarrow \quad \int \frac{1}{P}dP = k \int dt$$

$$\Rightarrow \quad \ln P = kt + C$$

general solution of the ODE

$$\text{Now,} \quad t = 0, \ \Rightarrow \ P(0) = P_0$$

intial value

$$\Rightarrow \quad \ln P_0 = k \cdot 0 + C$$

$$\Rightarrow \quad C = \ln P_0$$

$$\Rightarrow \quad \ln P = kt + \ln P_0$$

particular solution of the ODE

$$\Rightarrow \quad \ln\left(\frac{P}{P_0}\right) = kt$$

$$\Rightarrow \quad \frac{P}{P_0} = e^{kt}$$

$$\therefore \quad P = P_0 e^{kt}$$

Topic 4: Applications of Ordinary Differential Equation (ODE)

(ii) If the population increased from 1000 to 1300 after 20 days, find the population after 35 days.

(ii) $$P_0 = 1000 \quad \Rightarrow \quad P = P(t) = 1000\, e^{kt}$$

When $t = 20$

$$P(20) = 1000\, e^{k\cdot 20} = 1300$$

t	0	20	35
$P(t)$	1000	1300	?

$$\Rightarrow \quad \frac{13}{10} = e^{20k}$$

$$\Rightarrow \quad k = \frac{1}{20}\ln\left(\frac{13}{10}\right)$$

$$\Rightarrow \quad P(t) = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t}$$

When $t = 35$

$$\Rightarrow \quad P(35) = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]\cdot 35} \quad \approx 1583 \text{ insects}$$

(iii) How long will it take for the population to reach 2000?

t	0	20	35	?
P	1000	1300	1583	2000

From (ii), $P(t) = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t}$

When $P(t) = 2000$

$$2000 = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t} \quad \Rightarrow \quad 2 = e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t}$$

$$\Rightarrow \quad \ln 2 = \left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t$$

$$\Rightarrow \quad t = \frac{20\cdot \ln 2}{\ln(13/10)} \quad \approx 52.84 \text{ days}$$

1. The population of a city P increases exponentially at the rate of $r = 2\%$ per year.
   How many years will it take for the population to double.

Answer:

2. The decay rate of caffeine level in your system is proportional to the amount (m) of the caffeine at that time.

(a) Formulate a differential equation to show that the amount of the caffeine at time t is $m = m_0 \cdot e^{kt}$, where $k < 0$ is a constant and m_0 is the initial amount.

(b) Assume that it takes 1 hour for the caffeine level to decrease from 20 mg/L to 5 mg/L. How much will remain in 3 hours?

Answer:

3. Scientists can determine the age of objects containing organic material by a method called carbon dating or radiocarbon dating. Cosmic rays hitting the atmosphere convert nitrogen into a radioactive isotope of carbon ^{14}C, with a half-life of about **5730** years.

Vegetation absorbs carbon dioxide from the atmosphere through photosynthesis, and animals acquire ^{14}C by eating plants. When a plant or an animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay.

Let $Q(t)$ denote the amount of ^{14}C in the plant or the animal t years after it dies. The number of radioactive decays per year is propotional to the amount of ^{14}C at time t.

(i) Formulate a differential equation to show that the amount of ^{14}C at time t is $Q = Q_0 \cdot e^{-kt}$, where $k > 0$ is a constant, and Q_0 is the initial amount.

(ii) A particular piece of parchment contains about 64% as plants do today. Estimate the age of the parchment.

Answer: