Lecture 2

Topics covered in this lecture session

1. Quadratic Functions

2. Exponential function

3. Logarithmic function

Quadratic Equations

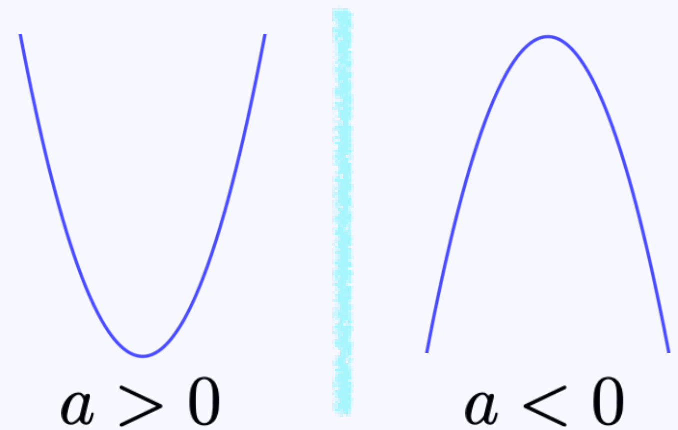
A general quadratic equation takes the form:

$$ax^2 + bx + c = 0 \quad ; \quad a \neq 0, \quad a, b, c \in \mathbb{R}$$

Roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Graphs of quadratic functions



Example: Solve $2x^2 + 3x - 1 = 0.$

Quadratic Equations (Nature of roots)

The nature of roots depends on

Discriminant $\Delta = b^2 - 4ac$

Discriminant $\Delta = b^2 - 4ac$	> 0	Roots are real and distinct
	$= 0$	Roots are real and equal (i.e. repeated roots)
	< 0	No real roots (i.e. roots are complex numbers)

Example: Find k if roots of the equation $2x^2 + 3x + k = 0$ are equal.

Quadratic functions (Completing the square)

Consider sketching graph of $f(x) = x^2 + bx + c$

Method:

$$x^2 + bx + c = \boxed{x^2 + bx + \left(\frac{b}{2}\right)^2} \underline{- \left(\frac{b}{2}\right)^2} + c$$

$$= \boxed{\left(x + \frac{b}{2}\right)^2} + c - \left(\frac{b}{2}\right)^2$$

For $g(x) = ax^2 + bx + c$,

first divide by a throughout and then apply the above method.

Quadratic functions (Completing the square)

Example: Complete the square to find the range of $f(x) = x^2 + 3x + 4$. Also sketch the graph f.

$$f(x) = x^2 + 3x + 4$$

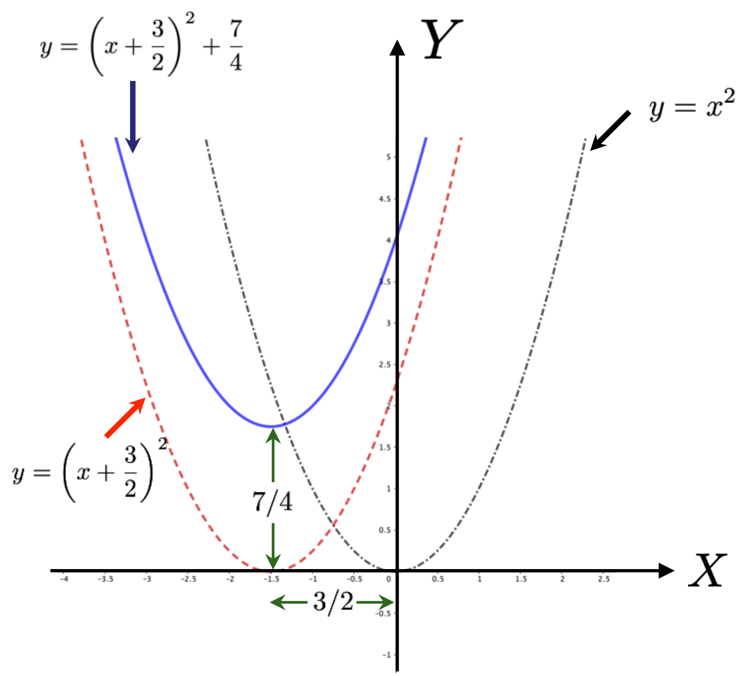
$$= \boxed{x^2 + 3x + \left(\frac{3}{2}\right)^2} \underline{- \left(\frac{3}{2}\right)^2} + 4$$

$$= \boxed{\left(x + \frac{3}{2}\right)^2} + 4 - \frac{9}{4}$$

$$= \left(x + \frac{3}{2}\right)^2 + \frac{7}{4}$$

Now, $\left(x + \frac{3}{2}\right)^2 \geq 0 \;\Rightarrow\; f(x) \geq \frac{7}{4}$

$\therefore$ Range of f is $\left[\frac{7}{4}, \infty\right)$



Exponential function

- The exponential function is the function

$$y = f(x) = a^x \quad \text{where} \quad a > 0$$

- A particularly important exponential function is

$$y = f(x) = e^x, \quad \text{where} \quad e \approx 2.718281828.$$

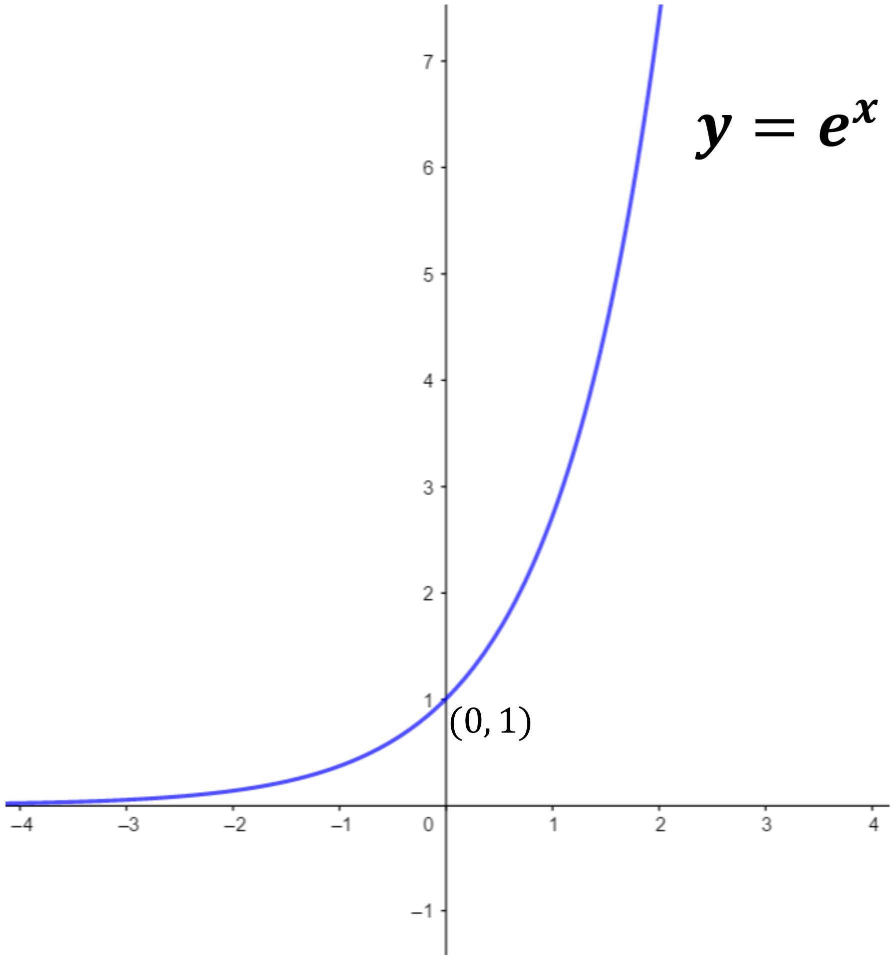
This is often called the exponential function.

- The exponential function is widely used in physics, chemistry, engineering, mathematical biology, economics and mathematics.

Graph of the exponential function

Observations:

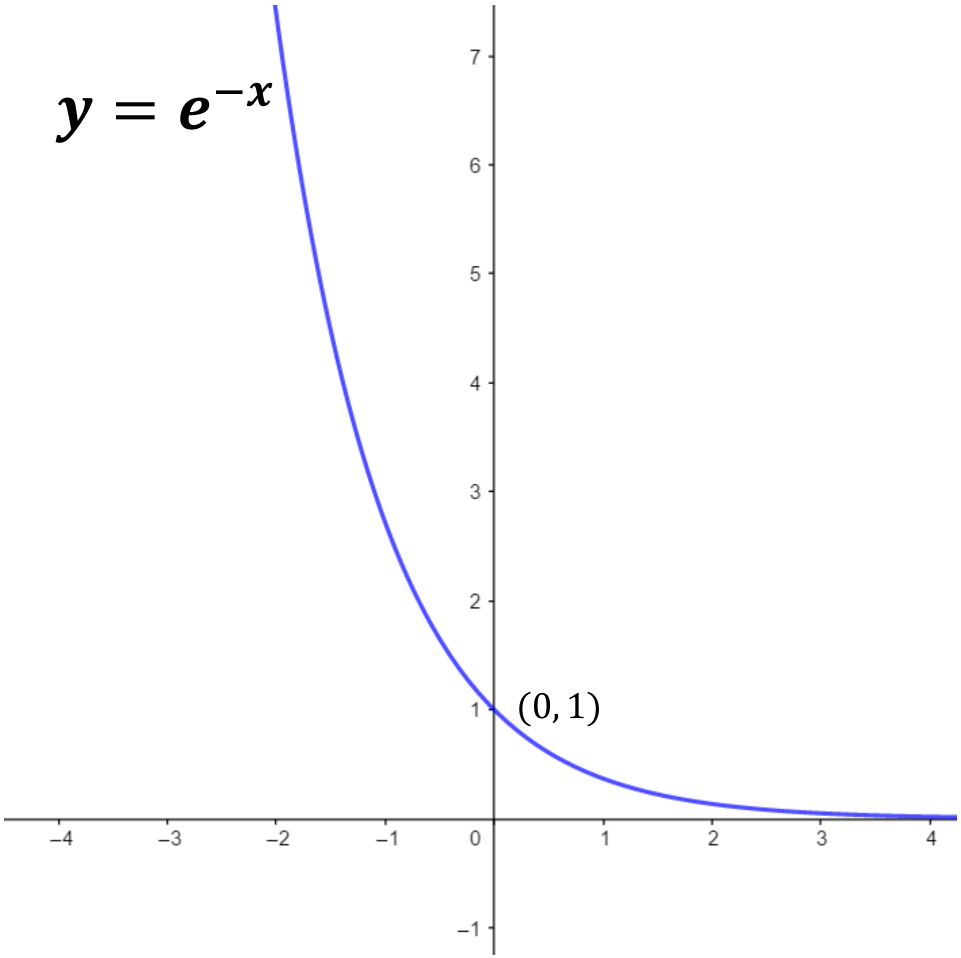
- The graph of $y = e^x$ is upward- sloping.
- It increases rapidly as x increases.
- It always lies above the X –axis, and gets arbitrarily close to it for negative x.
- Thus, the X – axis is a horizontal asymptote



Graph of the exponential function

Observations:

- The graph of $y = e^{-x}$ is upward- sloping (for negative x).
- It reduces rapidly as x increases.
- It always lies above the X $-$axis, and gets arbitrarily close to it for positive x.
- Thus, the X $-$ axis is a horizontal asymptote.



Laws of indices

$$a^0 = 1 \qquad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(ab)^n = a^n \cdot b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^m)^n = a^{mn} = (a^n)^m$$

$$a^{1/n} = \sqrt[n]{a}$$

In particular, $a^{\frac{1}{2}} = \sqrt{a}$

and $a^{\frac{1}{3}} = \sqrt[3]{a}$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Worked Examples

1. Simplify:

$(i)\quad \dfrac{x^8 \cdot x^{-3}}{x^{-5} \cdot x^2}$

$(ii)\quad \left(\dfrac{x^2}{y^3}\right)^{\frac{1}{3}} \cdot \left(\dfrac{y^2}{x^3}\right)^{\frac{1}{2}}$

$(iii)\quad \left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} \cdot \left(\sqrt[5]{x^6}\right)^{\frac{5}{12}}$

$(iv)\quad \sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}.$

2. Prove that $\left(\dfrac{x^a}{x^b}\right)^{a+b} \cdot \left(\dfrac{x^b}{x^c}\right)^{b+c} \cdot \left(\dfrac{x^c}{x^a}\right)^{c+a} = 1.$

Worked Examples

(iv) $\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}.$

$$\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}$$

$$\left(\sqrt[3]{4x^2 - 4x + 1}\right)^3 = \left(\sqrt[3]{x}\right)^3$$

$$4x^2 - 5x + 1 = 0$$

$$(4x - 1)(x - 1) = 0$$

$$x = \frac{1}{4} \quad \text{or} \quad x = 1$$

Worked Examples

2. Prove that $\left(\frac{x^a}{x^b}\right)^{a+b} \cdot \left(\frac{x^b}{x^c}\right)^{b+c} \cdot \left(\frac{x^c}{x^a}\right)^{c+a} = 1.$

$$\left(x^{a-b}\right)^{a+b} \cdot \left(x^{b-c}\right)^{b+c} \cdot \left(x^{c-a}\right)^{c+a}$$

$$x^{(a-b)(a+b)} \cdot x^{(b-c)(b+c)} \cdot x^{(c-a)(c+a)}$$

$$x^{a^2-b^2} \cdot x^{b^2-c^2} \cdot x^{c^2-a^2}$$

$\boldsymbol{Note}: u^2 - v^2 = (u-v)(u+v)$

$$x^{\cancel{a^2}-\cancel{b^2}+\cancel{b^2}-\cancel{c^2}+\cancel{c^2}-\cancel{a^2}}$$

$$\underline{\boldsymbol{x^0 = 1}}$$

Logarithmic function

- The logarithmic and exponential functions are inverses of each other

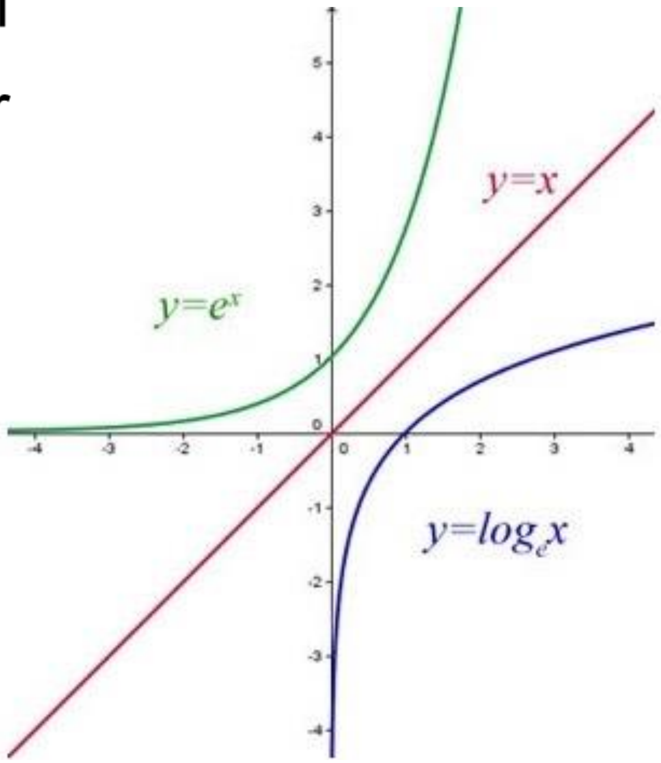
  (so reflection of graphs in the line $y = x$).

- The logarithm is denoted by

$$\log_a x$$

pronounced as $\log$ of x to base a.

$$a \in \mathbb{R}^+ - \{1\} \quad \text{and} \quad x > 0.$$



Bases of logarithms

Three choices for bases of logarithms are particularly common.

$$a = 10, \quad a = e \approx 2.718281828, \quad \text{and} \quad a = 2.$$

Logarithms with base e are called natural logarithms and written as $\ln x$.

In mathematical analysis, the logarithm to base e is widespread.

Logarithm with base 10 is called decimal/standard logarithm and written as $\log x$.

Base 10 logarithms are easy to use for manual calculations in the decimal number system.

The logarithm to base 2 is used in Computer Science, and in Music Theory.

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The logarithm to base 2 is used in Computer Science, and in Music Theory.

Uses of logarithms

- to quantify the loss of voltage levels in transmitting electrical signals;
- to describe power levels of sounds in acoustics;
- to determine the strength of an earthquake by measuring (on the Richter scale) the common logarithm of the energy emitted at the quake;
- to determine the brightness of stars;
- to determine pH value;
- to scale both the axes logarithmically to draw log-log graphs.

Relation between Exponential & logarithmic functions

The logarithmic and exponential functions are connected by the relation:

$$a^x = y \quad \Leftrightarrow \quad x = \log_a y$$

For example,

$$2^5 = 32 \quad \Leftrightarrow \quad 5 = \log_2 32$$

$$\log_3 81 = 4 \quad \Leftrightarrow \quad 81 = 3^4$$

Laws of logarithms

$$\log_a 1 = 0 \qquad (a > 0)$$

$$\log_a a = 1$$

$$\log_a (xy) = \log_a x + \log_a y$$

(Product Rule)

$$\mathbf{log_a(x+y)} \ \neq \ \mathbf{log_a x} \ \times \ \mathbf{log_a y}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

(Quotient Rule)

$$\mathbf{log_a(x-y)} \ \neq \ \mathbf{log_a x} \ \div \ \mathbf{log_a y}$$

$$\log_y x = \frac{\log_a x}{\log_a y}$$

(Change of base rule)

$$\log_b a = \frac{1}{\log_a b}$$

$$\log_a x^n = n \log_a x$$

(Logarithm of a Power)

$$a^{\log_a x} = x$$

Worked Examples

Solve: $e^{4-x} = 10$

$$\Rightarrow 4 - x = \ln 10$$

$$\Rightarrow 4 - x \approx 2.3026$$

$$\Rightarrow x \approx 4 - 2.3026$$

$$\Rightarrow x \approx 1.6974$$

obtained using calculator

Solve: $e^{2x} + e^{x} - 6 = 0$

Let $e^{x} = t$

$$\Rightarrow t^2 + t - 6 = 0$$

$$\Rightarrow t = 2 \quad or \quad -3$$

$$\Rightarrow e^{x} = 2 \quad or \quad \cancel{e^{x} = -3}$$

But, $e^{x} > 0$ for $x \in R$

$$\Rightarrow e^{x} = 2 \Rightarrow x = \ln 2$$

Worked Examples

1. Simplify:

$$(i) \quad \ln 3x^2 + \ln 2x - \ln 6x^3 \qquad\qquad (ii) \quad 3\ln x - \ln x^2$$

$$(iii) \quad \frac{1}{3}\left(\ln 9x + \ln 3x^2\right)$$

2. Solve: $\ln(x+2) + \ln(x-2) = 1.3.$

Worked Examples

3. Solve: $\ln(x+2)+\ln(x-2)=1.3.$

$$\ln[(x+2)(x-2)] = 1.3$$

$$\ln[(x^2-4)] = 1.3$$

$$\mathrm{e}^{\ln[(x^2-4)]} = e^{1.3} \quad \therefore \quad (x^2-4) = e^{1.3}$$

$$x^2 = e^{1.3}+4 \quad \therefore \quad x \approx \pm\sqrt{7.6693}$$ ***with calculator***

$$\underline{x = 2.7693}$$

This is because at $x = -2.77$, $\ln(x+2)$ is undefined

Class Activity

Solve the equation: $2^{2x} - 10(2^x) + 16 = 0$

Hint let $2^x = t$

A. $\boxed{x = 3 \; or \; x = 1}$

B. $x = -3 \; or \; x = 1$

C. $x = -3 \; or \; x = -1$

Further Reading (click on links)

College Algebra by J. W. Coburn & J. P. Coffelt (3rd edition)

(Chapter 4 and 5)

Foundation Algebra by P. Gajjar.

(Chapter 3 and 4)

Openstax Resource Openstax (Exponential Functions)

HELM HELM Mathematics resource

(Section 3 and 6)

THANKS FOR YOUR ATTENTION