Lecture 9

Lecture Content

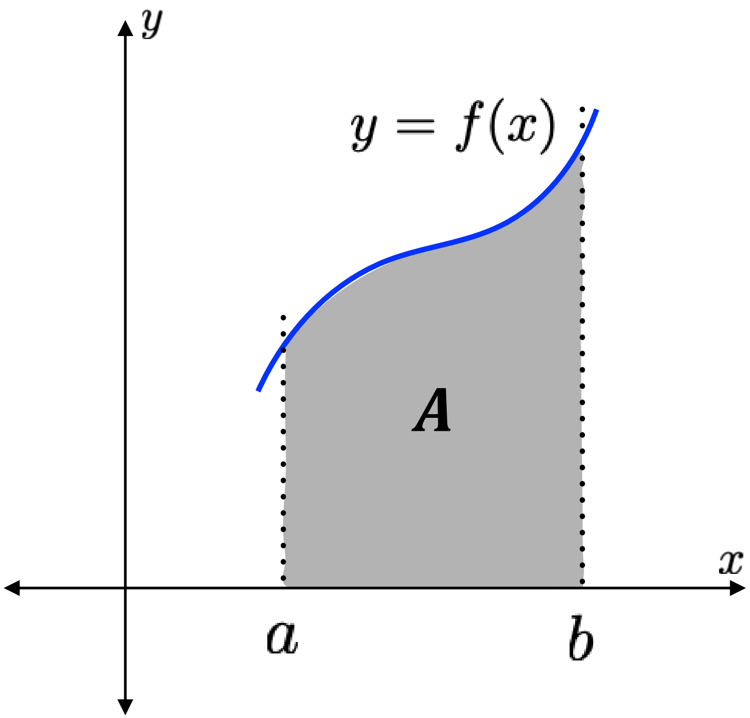
- Area of region bounded by two curves
- Solid of revolution
- Calculating volume of solid of revolution using Definite Integration
- Numerical Integration
  - Trapezoidal (or Trapezium) rule
  - Simpson's rule

Application of Integration (Area calculation)

Result 1

The area of region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$ and the X-axis is:

$$A = \int_a^b |y| \, dx = \int_a^b |f(x)| \, dx$$

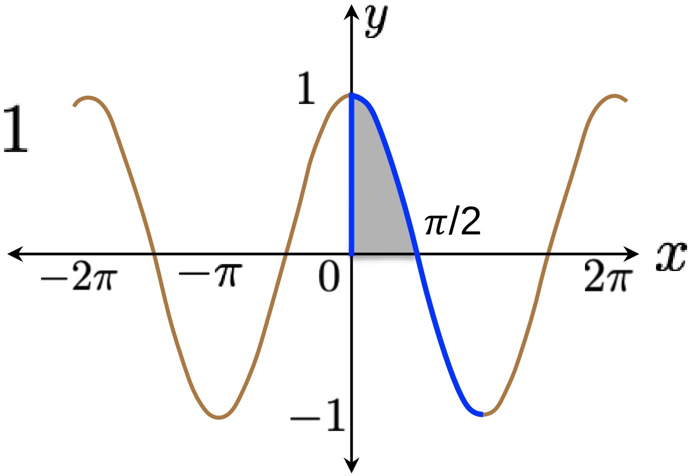


Application of Integration (Area calculation)

Example

Calculate the area of region bounded by the curve $y = \cos x$, lines $x = 0$, $x = \pi/2$ and the X-axis.

$$\text{Area, } A = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$$



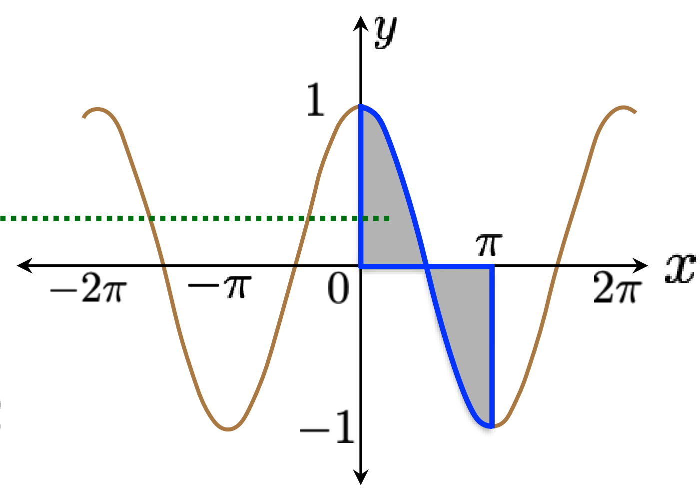
Application of Integration (Area calculation)

Example

Calculate the area of region bounded by the curve $y = \cos x$ and the X-axis in $[0, \pi]$.

$$\cancel{\text{Area, } A = \int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = 0}$$

$$\text{Area, } A = 2 \int_0^{\pi/2} \cos x \, dx = 2 \, [\sin x]_0^{\pi/2} = 2$$

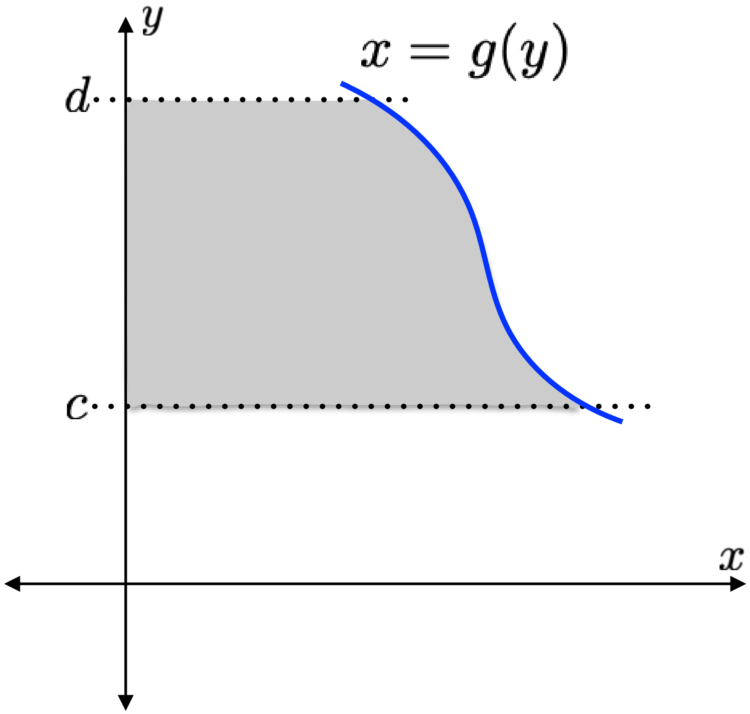


Application of Integration (Area calculation)

Result 2

The area of region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$ and the Y-axis is:

$$\boxed{A = \int_c^d |x| \; dy = \int_c^d |g(y)| \; dy}$$

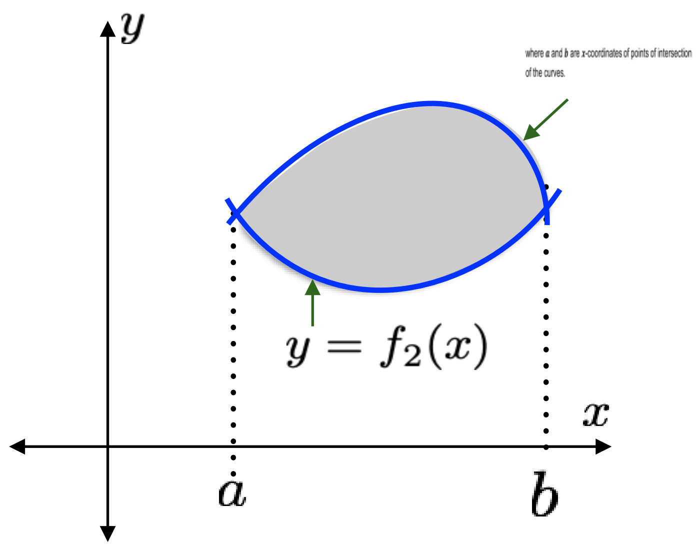


Application of Integration (Area calculation)

Result 3

The area of region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the X-axis is:

$$\boxed{A = \int_a^b \left| \, [f_1(x) - f_2(x)] \right| \, dx}$$



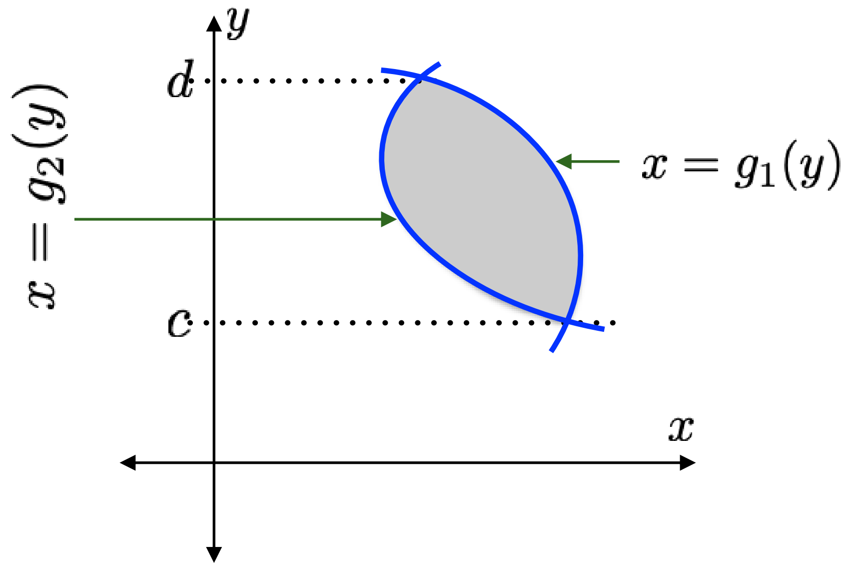
where $\boldsymbol{a}$ and $\boldsymbol{b}$ are $\boldsymbol{x}$-coordinates of points of intersection of the curves.

Application of Integration (Area calculation)

Result 4

The area of region bounded by the curves $x = g_1(y)$, $x = g_2(y)$ and the Y-axis is:

$$A = \int_c^d \Big| [g_1(y) - g_2(y)] \Big| \, dy$$



where c and d are x-coordinates of points of intersection of the curves.

Application of Integration (Area calculation)

Example

Calculate the area of region enclosed by $x = y^2$ and $y = x - 2$.

Here, $y^2 = x$ and $x = y + 2$

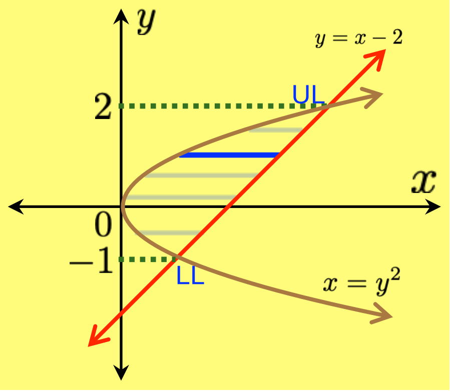
$\Rightarrow \; y^2 = y + 2$

i.e. $y^2 - y - 2 = 0$

$\Rightarrow \; (y+1)\cdot(y-2) = 0$

$\therefore \; y = -1 \text{ and } 2$

are y-coordinates of points of intersection.



$$\therefore \; \text{Area, } A = \int_{-1}^{2} \Big[\overset{\text{Right curve}}{(y+2)} - \overset{\text{Left curve}}{y^2} \Big] \; dx$$

$$= \left[\frac{y^2}{2} + 2\,y - \frac{y^3}{3} \right]_{-1}^{2}$$

$$= \frac{9}{2}$$

Exercise on area of region bounded by two curves

Find the area of the region bounded by the curve
$x = 1 - y^2$ and $x = y^2 - 1$.

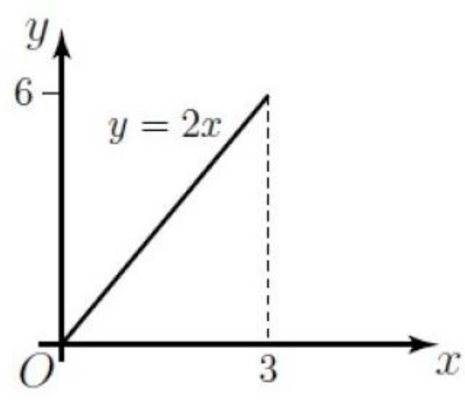
Solid of revolution

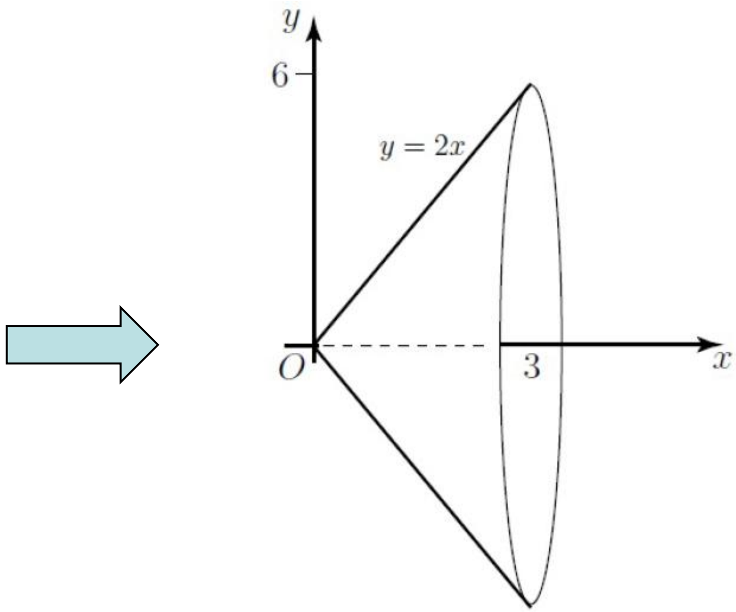
Imagine rotating the curve $y = f(x)$ between the points $x = a$ and $x = b$, by **one complete revolution** $(360^\circ$ or 2π radians) around the X-axis.

The three dimensional solid so formed is called a <u>solid of revolution</u>.

Solid of revolution

For example, when the graph of the function $y = 2x$ between the points $x = 0$ and $x = 3$ is rotated by one complete revolution about the X-axis, the solid of revolution formed is a cone as shown in the figure.



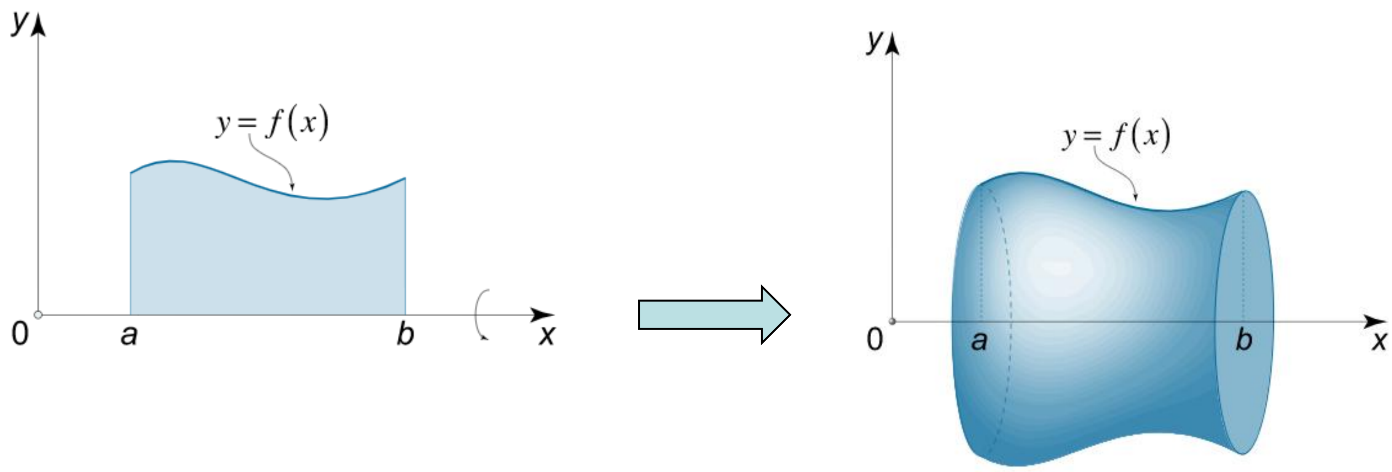


Calculating volume of solid of revolution using Definite Integration

Result 1

If the region R bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is revolved about X-axis, then the volume of the solid of revolution that is generated is:

$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [\, f(x)\,]^2 \, dx$$

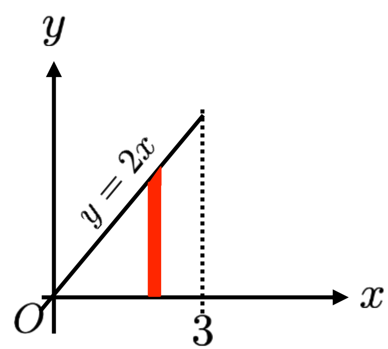


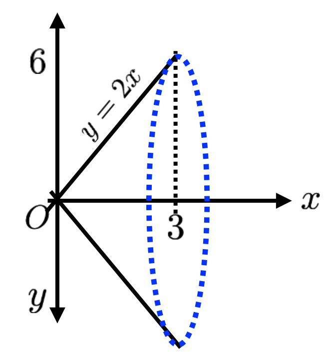
Example: Find the volume of the solid that is generated when the region bounded by $y = 2x$, $x = 3$ and the X-axis is revolved about the X-axis.

$$V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 4x^2 \, dx$$

$$= 4\pi \left[\frac{x^3}{3} \right]_0^3$$

$$= 36\pi$$





Example: Find the volume of a sphere of radius r.

A sphere is obtained when a semi-circle is revolved about the X-axis.

Also, the equation of a semi-circle is: $x^2 + y^2 = r^2 \Rightarrow \; y^2 = r^2 - x^2$

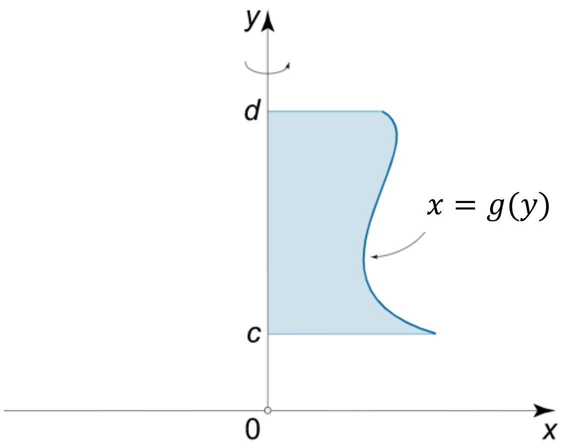
$$\therefore \; V = \pi \int_{-r}^{r} y^2 \, dx = \pi \int_{-r}^{r} \left(r^2 - x^2 \right) \, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4 \pi r^3}{3}$$

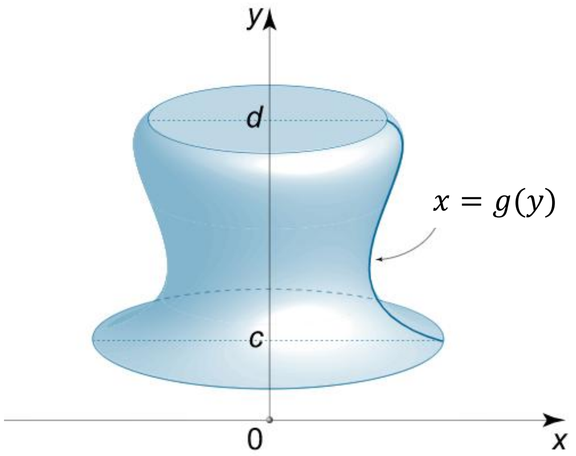
Calculating volume of solid of revolution using Definite Integration

Result 2

If the region R bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is revolved about Y-axis, then the volume of the solid of revolution that is generated is:

$$V = \pi \int_c^d x^2 \; dy = \pi \int_c^d [\, g(y) \,]^2 \; dy$$

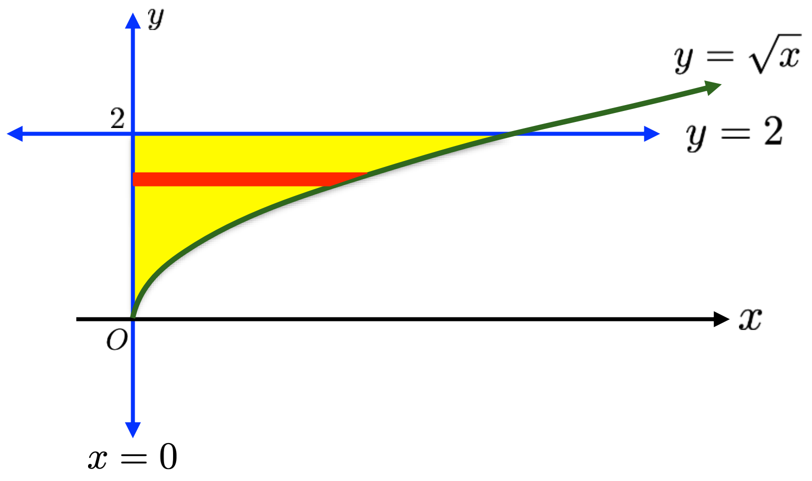




Example: Find the volume of the solid generated when the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is rotated about the Y-axis.

$$V = \pi \int_0^2 x^2 \, dy = \pi \int_0^2 y^4 \, dy$$

$$= \pi \left[\frac{y^5}{5} \right]_0^2 = \frac{32\pi}{5}$$

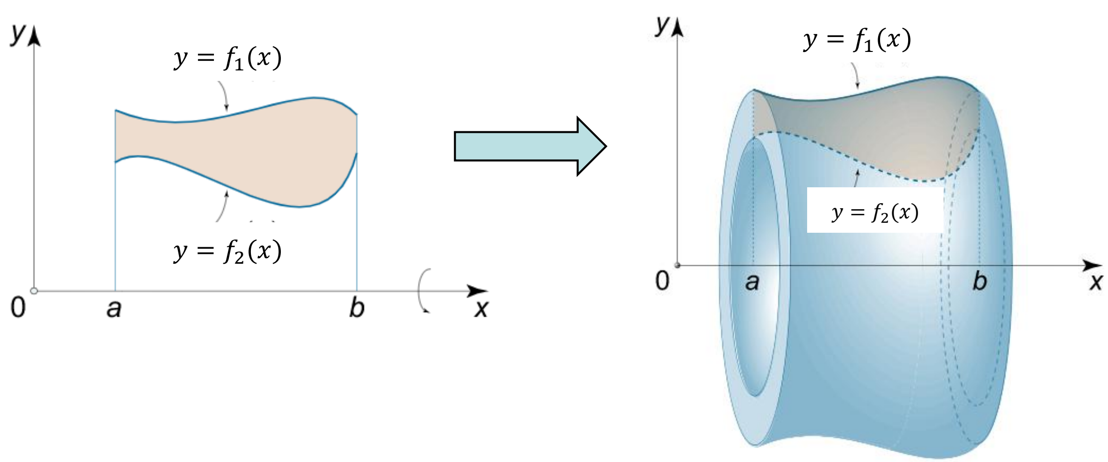


Calculating volume of solid of revolution using Definite Integration

Result 3

If the region bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ between the points (of intersection) $x = a$, $x = b$ is revolved about the X-axis, then the volume of the solid of revolution generated is:

$$V = \pi \int_a^b \left| [f_1(x)]^2 - [f_2(x)]^2 \right| \, dx$$

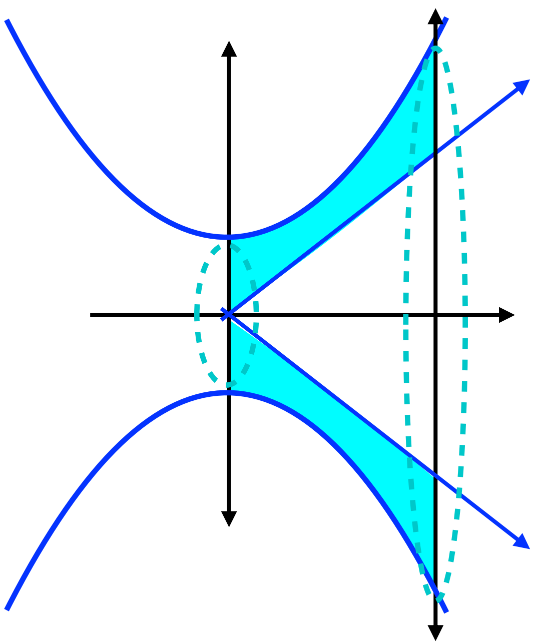


Example: Find the volume of the solid generated when the region bounded by $y = \frac{1}{2} + x^2$ and $y = x$ over $[0, 2]$ is rotated about the X-axis.

$$V = \pi \int_0^2 \left[\left(\frac{1}{2} + x^2 \right)^2 - (x)^2 \right] dx$$

$$= \pi \int_0^2 \left[\frac{1}{4} + x^2 + x^4 - x^2 \right] dx$$

$$= \pi \left[\frac{x}{4} + \frac{x^5}{5} \right]_0^2 = \frac{69\,\pi}{10}$$

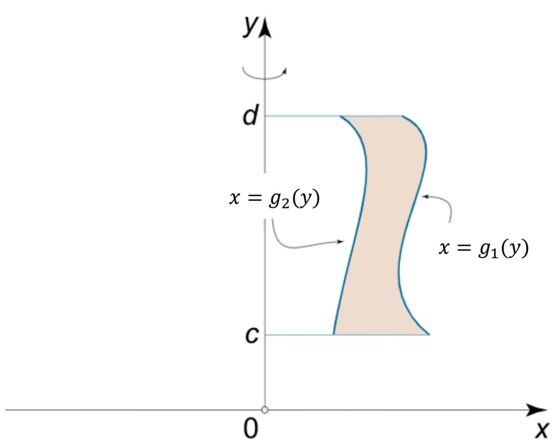


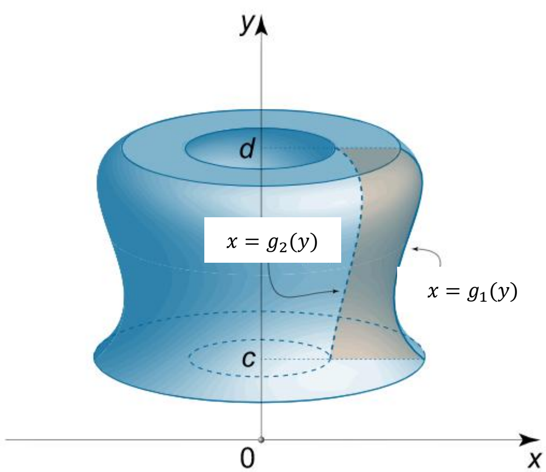
Calculating volume of solid of revolution using Definite Integration

Result 4

If the region bounded by two curves $x = g_1(y)$ and $x = g_2(y)$ between the points (of intersection) $y = c$, $y = d$ is revolved about the Y-axis, then the volume of the solid of revolution generated is:

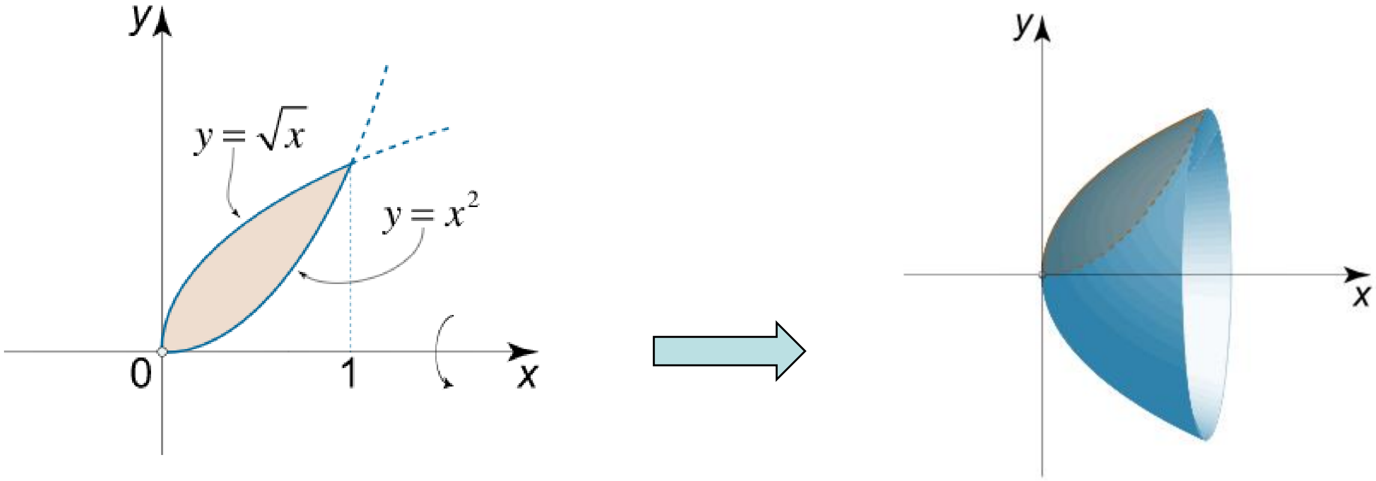
$$V = \pi \int_c^d \left| [g_1(y)]^2 - [g_2(y)]^2 \right| \, dy$$





Example

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$ and $y = \sqrt{x}$ around the x−axis.



Find the volume of solid of revolution when the region bounded by curves $y = \sin x$, $y = \cos x$ and lines $x = 0$, $x = \dfrac{\pi}{4}$ is revolved about the X-axis.

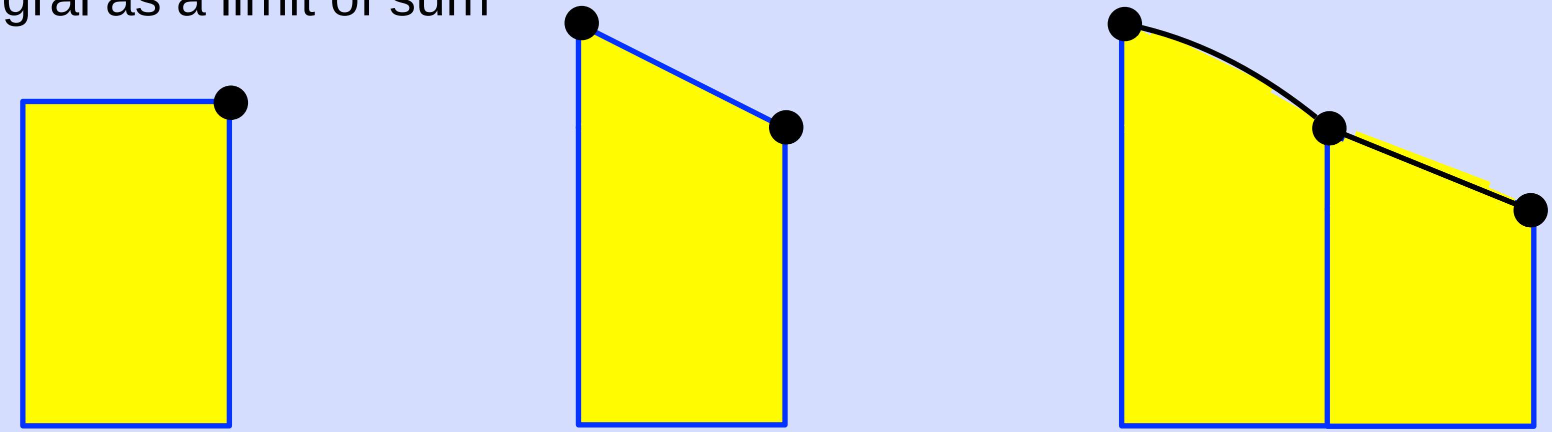
A $\pi/2$
B π
C $\pi/6$

Numerical Integration

Can we evaluate $\int_0^1 e^{x^2}\,dx$ **analytically?**

We cannot evaluate the integral by **the** known analytical methods. **We need Numerical Methods to evaluate such integrals.**

Integral as a limit of sum



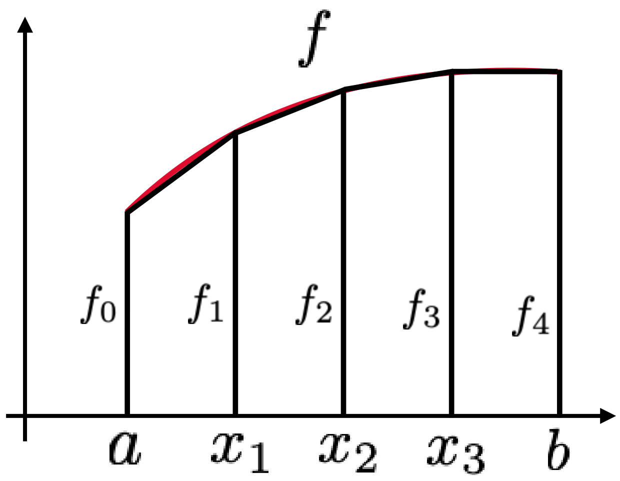
Riemann Sums | Trapezoidal rule | Simpson's rule

Numerical Integration (Trapezoidal/Trapezium rule)

The general idea is to use trapezoids instead of rectangles to approximate the area under the curve.

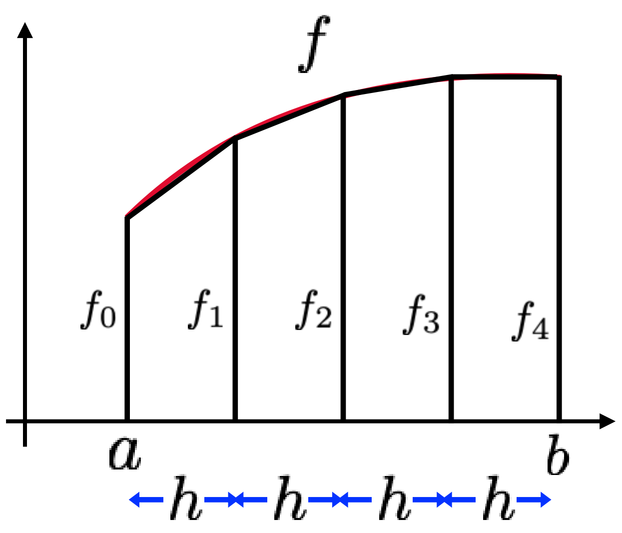
We subdivide the interval $[a, b]$ into n subintervals of equal width $h = \frac{b-a}{n}$ so that

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots\cdots\cdots \leq x_n = b$$



Numerical Integration (Trapezoidal/Trapezium rule)

Area of trapezoid = $\frac{1}{2}$(sum of parallel sides) × (width of the subinterval)



$$\therefore\ A_1 = \frac{h}{2}(f_0 + f_1)$$

$$A_2 = \frac{h}{2}(f_1 + f_2)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A_n = \frac{h}{2}(f_{n-1} + f_n)$$

Numerical Integration (Trapezoidal/Trapezium rule)

Now, total area under the curve $=$ sum of areas of trapezoids

$$\therefore \int_a^b f(x)\,dx \approx \frac{h}{2}\left[\, f_0 + f_1 + f_1 + f_2 + f_2 + f_3 + f_3 + \cdots\cdots + f_{n-1} + f_{n-1} + f_n \,\right]$$

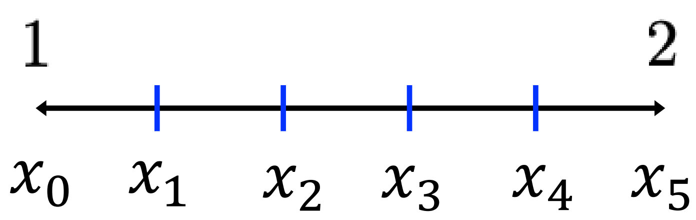
$$\therefore \int_a^b f(x)\,dx \approx \frac{h}{2}\left[\, f_0 + 2(f_1 + f_2 + f_3 + \cdots\cdots + f_{n-1}) + f_n \,\right]$$

Numerical Integration (Trapezoidal/Trapezium rule)

Example

Evaluate the definite integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using trapezoidal rule, by dividing $[1, 2]$ into 5 sub-intervals of equal width, give your answer correct to 4 d.p.

$$h = \frac{2-1}{5} = \frac{1}{5} = 0.2 \qquad \text{and} \qquad f(x) = \frac{1}{x}$$



Numerical Integration (Trapezoidal/Trapezium rule)

$$\therefore \int_1^2 \frac{1}{x}\,dx \approx \frac{h}{2}\left[\, f_0 + 2(f_1 + f_2 + f_3 + f_4) + f_5 \,\right]$$

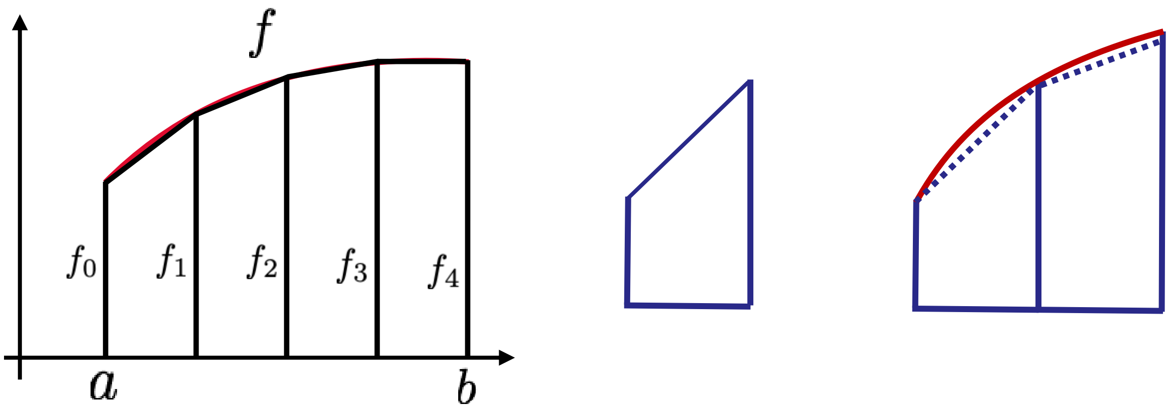
x	1	1.2	1.4	1.6	1.8	2
$f(x) = \frac{1}{x}$	1	0.8333	0.7143	0.6250	0.5556	0.5
f_n	f_0	f_1	f_2	f_3	f_4	f_5

$$\therefore I \approx \frac{0.2}{2}\left[1 + 2(0.8333 + 0.7143 + 0.625 + 0.5556) + 0.5\right] \approx 0.6956$$

Numerical Integration (Simpson's Rule)

Simpson's rule: is another technique that can be used to approximate the value of a definite integral.

Simpson's method replaces the Trapezoid with parabolas.



Numerical Integration (Simpson's Rule)

We state the formula without proof.

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[\, f_0 + 2(f_2 + f_4 + f_6 + \cdots\cdots + f_{n-2}) \right.$$
$$\left. + 4\,(f_1 + f_3 + f_5 + \cdots\cdots + f_{n-1}) + f_n \,\right]$$

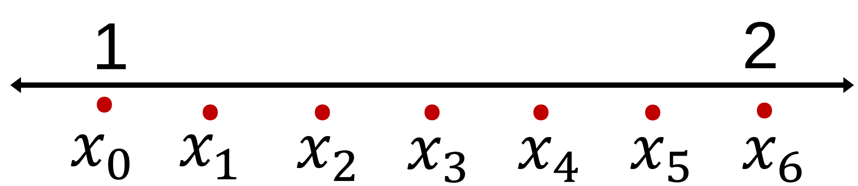
Note: The method works only for **even number of sub-intervals**.

Numerical Integration (Simpson's Rule)

Example

Evaluate the definite integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using Simpson's rule, by dividing $[1, 2]$ into 6 sub-intervals of equal width, give your answer correct to 4 d.p.

$$h = \frac{2-1}{6} = \frac{1}{6} \quad \text{and} \quad f(x) = \frac{1}{x}$$



Numerical Integration (Simpson's Rule)

$$\therefore \int_1^2 \frac{1}{x}\,dx \approx \frac{h}{3}\left[\, f_0 + 2(f_2 + f_4) + 4(f_1 + f_3 + f_5) + f_6 \,\right]$$

x	1	7/6	8/6	9/6	10/6	11/6	2
$f(x) = \frac{1}{x}$	1	6/7	6/8	6/9	6/10	6/11	1/2
f_n	f_0	f_1	f_2	f_3	f_4	f_5	f_6

$$\therefore \int_1^2 \frac{1}{x}\,dx \approx \frac{(1/6)}{3}\left[1 + 2\left(\frac{6}{8} + \frac{6}{10}\right) + 4\left(\frac{6}{7} + \frac{6}{9} + \frac{6}{11}\right) + \frac{1}{2}\right] \approx 0.6932$$

Thank You!