University of Nottingham Ningbo China

CENTRE FOR ENGLISH LANGUAGE EDUCATION

PRELIMINARY YEAR, SEMESTER TWO, 2024-25

FOUNDATION CALCULUS AND MATHEMATICAL TECHNIQUES

MOCK MID-SEMESTER EXAM

Time allowed: ONE HOUR

Candidates must write their ID number on this booklet and fill-in their attendance card but must NOT write anything else until the start of the exam is announced.

This paper contains TWENTY questions. The total number of points is 100.
Answer all questions.

Only general bilingual dictionaries are allowed. Subject-specific dictionaries are not permitted.

No electronic devices except for approved calculators (CASIO fx-82) can be used in this exam.

Do NOT open the examination paper until told to do so.

All answers must be written in this booklet.

ADDITIONAL MATERIAL: Formula Sheet

INFORMATION FOR INVIGILATORS:
1. A 15-minute warning should be given before the end of the exam.
2. Please collect this Booklet and Formula Sheet after the exam.
3. Please return this Booklets in ID order.

Student ID: ____________________

Seminar Group (*e.g.* A35): __________ **Marks (out of 100):** ______

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Section A: Multiple Choice Questions. Choose the CORRECT option.

1. Find the limit $\lim_{n \to 1} \frac{n^2 - 1}{n - 1}$. [4]

   (A) 2
   (B) -2
   (C) 1
   (D) -1

   Answer: A

2. Find the limit $\lim_{x \to \infty} \frac{3x^2 + 2x - 1}{5x^2 - 3x + 2}$. [4]

   (A) ∞
   (B) 0
   (C) $-\frac{3}{5}$
   (D) $\frac{3}{5}$

   Answer: D

3. Given that $y = e^x(x^{2022} - 2022x + 2022)$, find $\frac{dy}{dx}$. [4]

   (A) $e^x(x^{2022} + 2022x^{2021} + 2022x)$
   (B) $e^x(x^{2022} + 2022x^{2021} - 2022x)$
   (C) $x^{2022}(e^x + 2022x^{2021} - 2022x)$
   (D) $x^{2022}(e^x + 2022x^{2021} + 2022x)$

   Answer: B

4. Given that $y = \frac{1 - x^4}{1 + x^4}$, find $\frac{dy}{dx}$. [4]

   (A) $-\frac{4x^3}{(1 + x^4)^2}$
   (B) $-\frac{2x^3}{(1 + x^4)^2}$
   (C) $-\frac{8x^3}{(1 + x^4)^2}$
   (D) $\frac{4x^3}{(1 + x^4)^2}$

   Answer: C

5. Given $y = \tan\left(e^{x^2+3}\right)$, use the chain rule to find $\frac{dy}{dx}$. [4]

(A) $2x \cdot e^{x^2+3} \cdot \sec^2\left(e^{x^2+3}\right)$

(B) $(x^2+3) \cdot e^{x^2+3} \cdot \sec^2\left(e^{x^2+3}\right)$

(C) $e^{x^2+3} \cdot \sec^2\left(e^{x^2+3}\right)$

(D) $x^2 \cdot e^{x^2+3} \cdot \sec^2\left(e^{x^2+3}\right)$

Answer: A

6. Find the third order derivative of $y = e^{-5z} + 8\ln(2z^4)$ [4]

(A) $25e^{-5z} - 32z^{-2}$

(B) $-125e^{-5z} + 64z^{-3}$

(C) $125e^{-5z} - 64z^{-2}$

(D) $-25e^{-5z} + 32z^{-3}$

Answer: B

7. The function $f(x) = x^3 + 3ax^2 + 3bx - c$ has stationary points at $x = 1$ and $x = 2$, then the increasing interval is: [4]

(A) $(-\infty, 1)$ and $(2, +\infty)$

(B) $(1, 2)$

(C) $(-\infty, -1)$ and $(-2, +\infty)$

(D) $(-1, -2)$

Answer: A

8. Let $f(x) = e^x \cdot (x^2 + ax - 2a - 3)$, and $x = 2$ is a local minimum of the function, find the value of a. [4]

(A) 3

(B) -3

(C) 5

(D) -5

Answer: D

9. Evaluate the indefinite integral $\int \left(3x^4 - \frac{5}{x} + 2\cos(-2x)\right)\, dx$. [4]

(A) $-\frac{2}{x^2} + 4\sin(2x) + \frac{3x^5}{5} + C$

(B) $-\frac{2}{x^2} + \sin(2x) + \frac{3x^5}{5} + C$

(C) $-5\ln|x| + \sin(2x) + \frac{3x^5}{5} + C$

(D) $5\ln|x| + 4\sin(2x) + \frac{3x^5}{5} + C$

Answer: C

10. Evaluate $\int \cot x\, dx$ by using the result $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$. [4]

(A) $\ln|\cos x| + C$

(B) $-\ln|\cos x| + C$

(C) $\ln|\sin x| + C$

(D) $-\ln|\sin x| + C$

Answer: C

Section B: Short Answer Questions. Answers must be written with necessary steps.

11. Given $x^3y + xy^3 = \sin(x^3y)$, use implicit differentiation to find $\dfrac{dy}{dx}$. [5]

$$\frac{d}{dx}(x^3y + xy^3) = \frac{d}{dx}(\sin(x^3y))$$

$$3x^2y + x^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = \cos(x^3y)\cdot\left(3x^2y + x^3\frac{dy}{dx}\right)$$

$$(x^3 + 3xy^2)\frac{dy}{dx} + (3x^2y + y^3) = 3x^2y\cos(x^3y) + x^3\cos(x^3y)\frac{dy}{dx}$$

$$\left(x^3 + 3xy^2 - x^3\cos(x^3y)\right)\frac{dy}{dx} = 3x^2y\cos(x^3y) - 3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{3x^2y\cos(x^3y) - 3x^2y - y^3}{x^3 + 3xy^2 - x^3\cos(x^3y)}$$

12. Given $y = (\tan x)^{e^x}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$. [5]

$$\ln y = \ln\left[(\tan x)^{e^x}\right]$$

$$\ln y = e^x\cdot\ln(\tan(x))$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(e^x\cdot\ln(\tan x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = e^x\ln(\tan x) + e^x\cdot\frac{1}{\tan x}\cdot\sec^2 x$$

$$\frac{dy}{dx} = y\cdot e^x\left(\ln(\tan x) + \frac{\sec^2 x}{\tan x}\right)$$

or $$\frac{dy}{dx} = (\tan x)^{e^x}\cdot e^x\left(\ln(\tan x) + \frac{\sec^2 x}{\tan x}\right)$$

13. Given the curve described by parametric equations $x = \tan\theta - \sec\theta$, $y = \tan\theta + \sec\theta$; $\theta \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$ [8]

(a) find $\dfrac{dy}{dx}$.

$$\frac{dy}{d\theta} = \sec^2\theta + \sec\theta\tan\theta ,$$

$$\frac{dx}{d\theta} = \sec^2\theta - \sec\theta\tan\theta .$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec^2\theta - \sec\theta\tan\theta} = \frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta}$$

(b) Hence, find $\left.\dfrac{dy}{dx}\right|_{\theta=\pi/4}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)}{\sec\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

(c) Also, find the equation of the tangent line to the curve when $\theta = \pi/4$.

$$x\Big|_{\theta=\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \sec\left(\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

$$y\Big|_{\theta=\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}$$

$$m = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\therefore\ y - (1+\sqrt{2}) = \frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\left(x - (1-\sqrt{2})\right) \Rightarrow y = \frac{\sqrt{2}+1}{\sqrt{2}-1}x + \frac{1}{\sqrt{2}-1} + 1 + \sqrt{2}$$

14. A circular disc of radius 2 cm is being heated. Due to expansion, its radius increases at a rate of 0.025 cm/sec. Find the rate at which its area is increasing when its radius is 2.1 cm. (Area: $A = \pi r^2$) [5]

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot \frac{dr}{dt}$$

$$= 2 \cdot \pi \cdot 2.1 \cdot 0.025 = 0.105\pi \text{ cm}^2/\text{sec}$$

15. Evaluate the integral $\int \frac{(\ln x)^n}{x}\, dx$ by using the substitution $\ln x = t$. [3]

$\ln x = t \quad \therefore \frac{dt}{dx} = \frac{1}{x} \quad \therefore dt = \frac{1}{x}\, dx$

$\therefore I = \int t^n\, dt \qquad$ If $n \neq -1 \quad I = \frac{t^{n+1}}{n+1} + c = \frac{(\ln x)^{n+1}}{n+1} + c$

If $n = -1 \quad I = \ln|t| + c = \ln|\ln x| + c$

16. Use an appropriate substitution to evaluate the integral $\int \sin 2x \sqrt{\cos x}\, dx$. [4]

$I = 2\int \sin x \cos x \sqrt{\cos x}\, dx$

Let $\cos x = t \quad \therefore \frac{dt}{dx} = -\sin x \quad \therefore \sin x\, dx = -dt$

$\therefore I = -2\int t\sqrt{t}\, dt = -2\int t^{\frac{3}{2}}\, dt = -2\, \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + c$

$$= -\frac{4}{5} t^{\frac{5}{2}} + c$$

$$= -\frac{4}{5} (\cos x)^{\frac{5}{2}} + c$$

17. Evaluate the integral $\int \sin^7 x \cdot \cos^4 x \, dx$. [6]

Let $\cos x = t$ $\therefore \frac{dt}{dx} = -\sin x$ $\therefore \sin x \, dx = -dt$

$\therefore I = \int \sin^6 x \cdot \cos^4 x \cdot \sin x \, dx = \int (1-\cos^2 x)^3 \cos^4 x \sin x \, dx$

$= -\int (1-t^2)^3 t^4 \, dt$

$= -\int (t^4 - 3t^6 + 3t^8 - t^{10}) \, dt$

$= -\frac{t^5}{5} + \frac{3t^7}{7} - \frac{3t^9}{9} + \frac{t^{11}}{11} + C$

$= -\frac{\cos^5 x}{5} + \frac{3\cos^7 x}{7} - \frac{\cos^9 x}{3} + \frac{\cos^{11} x}{11} + C$

18. Evaluate the integral $\int \cos 5x \cdot \cos 2x \, dx$. [4]

$I = \frac{1}{2} \int (\cos(5x+2x) + \cos(5x-2x)) \, dx$

$= \frac{1}{2} \int (\cos 7x + \cos 3x) \, dx$

$= \frac{1}{14} \sin 7x + \frac{1}{6} \sin 3x + C$

19. The Newton-Raphson iteration formula is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, [10]

(a) Consider solving $f(x) = 4x^3 + x^2 - 3x - 10 = 0$, show that $x_{n+1} = \dfrac{8x_n^3 + x_n^2 + 10}{12x_n^2 + 2x_n - 3}$.

$f'(x) = 12x^2 + 2x - 3$

$\therefore x_{n+1} = x_n - \dfrac{4x_n^3 + x_n^2 - 3x_n - 10}{12x_n^2 + 2x_n - 3}$

$= \dfrac{12x_n^3 + 2x_n^2 - 3x_n - 4x_n^3 - x_n^2 + 3x_n + 10}{12x_n^2 + 2x_n - 3}$

$= \dfrac{8x_n^3 + x_n^2 + 10}{12x_n^2 + 2x_n - 3}$

(b) Starting with $x_0 = 1.5$, determine the root of $f(x) = 0$ that lies in the interval $(1, 2)$, correct to 6 decimal places. List all x_n values until the approximation is achieved.

n	x_n
0	1.5
1	1.453704
2	1.452108
3	1.452106
4	1.452106

$x^* = 1.452106$

20. Given $f(x) = \sqrt[3]{1-x}$, $-1 < x < 1$. [10]

(a) Obtain the Maclaurin's expansion of $f(x)$ up to the terms with x^2

$f(x) = \sqrt[3]{1-x}$ $\quad$ $f(0) = 1$

$f'(x) = -\dfrac{1}{3(1-x)^{\frac{2}{3}}}$ $\quad$ $f'(0) = -\dfrac{1}{3}$

$f''(x) = -\dfrac{2}{9(1-x)^{\frac{5}{3}}}$ $\quad$ $f''(0) = -\dfrac{2}{9}$

$f(x) = f(0) + x f'(0) + x^2 \dfrac{f''(0)}{2!} + \cdots$

$= 1 - \dfrac{1}{3}x - \dfrac{1}{9}x^2 + \cdots$

(b) Use the substitution $x = \dfrac{5}{40}$ in the expansion above to approximate the value of $\sqrt[3]{7}$.

Give your answer correct to 6 decimal places.

$f(\frac{5}{40})$ $\sqrt[3]{1-\frac{5}{40}} = \sqrt[3]{\frac{35}{40}} = \sqrt[3]{\frac{7}{8}} = \dfrac{\sqrt[3]{7}}{2}$

$f(\frac{5}{40}) = 1 - \frac{1}{3}\cdot\frac{5}{40} - \frac{1}{9}\cdot(\frac{5}{40})^2 = \dfrac{\sqrt[3]{7}}{2}$

$\therefore \sqrt[3]{7} = 2 \times (1 - \frac{1}{3}\cdot\frac{5}{40} - \frac{1}{9}(\frac{5}{40})^2)$

$= 1.913194$

You may use this space for rough work.

All answers must be written in the Answer Booklet.