University of Nottingham Ningbo China

CENTRE FOR ENGLISH LANGUAGE EDUCATION

PRELIMINARY YEAR, SEMESTER TWO, 2024-25

FOUNDATION CALCULUS AND MATHEMATICAL TECHNIQUES

SAMPLE MID-SEMESTER EXAM

Time allowed: ONE HOUR

Candidates must write their ID number on this booklet and fill-in their attendance card but must NOT write anything else until the start of the exam is announced.

This paper contains TWENTY questions. The total number of points is 100.

Answer all questions.

Only general bilingual dictionaries are allowed. Subject-specific dictionaries are not permitted.

No electronic devices except for approved calculators (CASIO fx-82) can be used in this exam.

Do NOT open the examination paper until told to do so.

All answers must be written in this booklet.

ADDITIONAL MATERIAL: Formula Sheet

INFORMATION FOR INVIGILATORS:

1. A 15-minute warning should be given before the end of the exam.
2. Please collect this Booklet and Formula Sheet after the exam.
3. Please return this Booklets in ID order.

Student ID: ______________________

Seminar Group (*e.g.* A35): __________ **Marks (out of 100):** ______

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Section A: Multiple Choice Questions. Choose the CORRECT option.

1. Find the limit $\lim_{\theta \to 0} \dfrac{\sin 2\theta}{\sin \theta}$. [4]

   (A) 1
   (B) 2
   (C) 0
   (D) ∞

   Answer: B

2. Find the limit $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^{3x}$ [4]

   (A) ∞
   (B) 0
   (C) $-e^6$
   (D) e^6

   Answer: D

3. Given that $y = (x+1)^2(2x-1)$, find $\dfrac{dy}{dx}$. [4]

   (A) $6x^2 + 6x$
   (B) $4x^2 + 3x$
   (C) $6x^2 + 3x$
   (D) $4x^2 + 6x$

   Answer: A

4. Given that $y = \dfrac{\sqrt{x}}{1 - \sin x}$, find $\dfrac{dy}{dx}$. [4]

   (A) $\sqrt{x}(\frac{1}{2x}(1 - \sin x) - \cos x)(1 - \sin x)^{-2}$
   (B) $(\frac{1}{2x})\sqrt{x}((1 - \sin x) + \cos x)(1 - \sin x)^{-2}$
   (C) $\sqrt{x}(\frac{1}{2x}(1 - \sin x) + \cos x)(1 - \sin x)^{-2}$
   (D) $(\frac{1}{2x})\sqrt{x}((1 - \sin x) - \cos x)(1 - \sin x)^{-2}$

   Answer: C

5. Given $y = \sin(\tan(e^x))$, use the chain rule to find $\dfrac{dy}{dx}$. [4]

(A) $\cos(\tan(e^x)) \cdot \sec^2(e^x) \cdot e^x$

(B) $\cos\left(\sec^2(e^x)\right) \cdot \tan(e^x) \cdot e^x$

(C) $\sin(\tan(e^x)) \cdot \sec^2(e^x) \cdot e^x$

(D) $\sin\left(\sec^2(e^x)\right) \cdot \tan(e^x) \cdot e^x$

Answer: A

6. Given $y = x^{99}$, find $\dfrac{d^{99}y}{dx^{99}}$. [4]

(A) $\dfrac{1}{99!}$

(B) $99!$

(C) 1

(D) 0

Answer: B

7. Given $f(1) = 2$, $f'(1) = 0$, $g(1) = 3$, and $g'(1) = 4$, find $h'(1)$ if $h(x) = \dfrac{f(x)}{g(x)}$. [4]

(A) $-\dfrac{8}{9}$

(B) $\dfrac{8}{9}$

(C) 1

(D) 2

Answer: A

8. Let $f(x) = (x-a)^2 \cdot \ln x$, and $x = e$ is a local maximum, find the value of a. [4]

(A) $-e$

(B) $-3e$

(C) e

(D) $3e$

Answer: D

9. Evaluate the indefinite integral $\int \left(3e^{3x} - \frac{6}{x} + \operatorname{cosec}^2(2x)\right)\, dx$. [4]

   (A) $-6\ln|x| - \frac{\cot 2x}{2} + \frac{e^{3x}}{3} + C$

   (B) $-6\ln|x| - \cot 2x + \frac{e^{3x}}{3} + C$

   (C) $-6\ln|x| - \frac{\cot 2x}{2} + e^{3x} + C$

   (D) $-6\ln|x| - \cot 2x + e^{3x} + C$

   Answer: C

10. Evaluate $\int e^x \left(\frac{\cos x \sin x - \cos^2 x - \sin^2 x}{\sin^2 x}\right) dx$ by using the result $\int e^x(f(x)+f'(x))\, dx = e^x(f(x)) + C$. [4]

   (A) $e^x(\sec x) + C$

   (B) $e^x(\operatorname{cosec} x) + C$

   (C) $e^x(\cot x) + C$

   (D) $e^x(\tan x) + C$

   Answer: C

Section B: Short Answer Questions. Answers must be written with necessary steps.

11. Given $e^{x^2 y} + \sin(xy^2) + \ln(x+y) = 5$, use implicit differentiation to find $\dfrac{dy}{dx}$. [5]

$$\frac{d}{dx}\left(e^{x^2y} + \sin(xy^2) + \ln(x+y)\right) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}(e^{x^2y}) + \frac{d}{dx}(\sin(xy^2)) + \frac{d}{dx}(\ln(x+y)) = 0$$

$$e^{x^2y}\cdot\frac{d}{dx}(x^2y) + \cos(xy^2)\cdot\frac{d}{dx}(xy^2) + \frac{1}{x+y}\cdot\frac{d}{dx}(x+y) = 0$$

$$e^{x^2y}\cdot\left(x^2\frac{dy}{dx} + 2xy\right) + \cos(xy^2)\left(2xy\frac{dy}{dx} + y^2\right) + \frac{1+\frac{dy}{dx}}{x+y} = 0$$

$$e^{x^2y}\cdot x^2\cdot\frac{dy}{dx} + 2xye^{x^2y} + 2xy\cos(xy^2)\frac{dy}{dx} + \cos(xy^2)y^2 + \frac{1}{x+y}\frac{dy}{dx} + \frac{1}{x+y} = 0$$

$$\left(e^{x^2y}x^2 + 2xy\cos(xy^2) + \frac{1}{x+y}\right)\frac{dy}{dx} = -\left(2xye^{x^2y} + \cos(xy^2)y^2 + \frac{1}{x+y}\right)$$

$$\frac{dy}{dx} = -\frac{2xye^{x^2y} + \cos(xy^2)y^2 + \frac{1}{x+y}}{e^{x^2y}x^2 + 2xy\cos(xy^2) + \frac{1}{x+y}}$$

12. Given $y = \dfrac{\sqrt[3]{(x-1)^2}\cdot(x-3)^5}{e^{3x}}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$. [5]

$$\ln y = \ln\left(\frac{\sqrt[3]{(x-1)^2}\cdot(x-3)^5}{e^{3x}}\right)$$

$$\ln y = \frac{2}{3}\ln(x-1) + 5\ln(x-3) - 3x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{2}{3}\ln(x-1) + 5\ln(x-3) - 3x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3}\cdot\frac{1}{x-1} + 5\cdot\frac{1}{x-3} - 3$$

$$\frac{dy}{dx} = y\cdot\left(\frac{2}{3x-3} + \frac{5}{x-3} - 3\right)$$

or $$\frac{dy}{dx} = \frac{\sqrt[3]{(x-1)^2}\cdot(x-3)^5}{e^{3x}}\left(\frac{2}{3x-3} + \frac{5}{x-3} - 3\right)$$

13. Given the curve described by parametric equations $x = e^{-t} \cdot \cos(2t)$, $y = e^{-2t} \cdot \sin(2t)$; [8]
(a) find $\dfrac{dy}{dx}$.

$$\frac{dy}{dt} = 2e^{-2t}\cos(2t) - 2\sin(2t) \cdot e^{-2t}$$

$$\frac{dx}{dt} = -2e^{-t}\sin(2t) - \cos(2t)\, e^{-t}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^{-2t}\cos(2t) - 2\sin(2t) \cdot e^{-2t}}{-2e^{-t}\sin(2t) - \cos(2t)\, e^{-t}}$$

$$= \frac{2e^{-t}\sin(2t) - 2e^{-t}\cos(2t)}{2\sin(2t) + \cos(2t)}$$

(b) Hence, find $\left.\dfrac{dy}{dx}\right|_{t=0}$.

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{2e^{0}\sin 0 - 2e^{0}\cos 0}{2\sin 0 + \cos 0} = \frac{2\cdot 1\cdot 0 - 2\cdot 1\cdot 1}{2\cdot 0 + 1}$$

$$= -2$$

(c) Also, find the equation of the tangent line to the curve when $t = 0$.

$$y|_{t=0} = e^{0} \cdot \sin 0 = 0$$

$$x|_{t=0} = e^{0} \cdot \cos 0 = 1$$

$$m = -2$$

$$\therefore \quad y - 0 = -2(x - 1)$$

$$y = -2x + 2$$

14. A conical water tank with vertex down has a radius of $5\mathrm{m}$ at the top and a height of $10\mathrm{m}$. Water is being pumped into the tank at a rate of $3\mathrm{m}^3/\mathrm{min}$. How fast is the water level rising when the water depth $h = 6\mathrm{m}$? ($V = \frac{1}{3}\pi r^2 h$) [5]

$\frac{dv}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{\pi}{3} h \cdot 2r \frac{dr}{dt}$

as $\frac{h}{r} = \frac{10}{5} \Rightarrow h = 2r \Rightarrow \frac{dh}{dt} = 2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$

$\therefore \frac{dv}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{\pi}{3} h \cdot r \cdot \frac{dh}{dt}$.

when $h = 6$, $r = 3$, $\frac{dv}{dt} = 3$

$\therefore 3 = \frac{\pi}{3} \cdot 3^2 \cdot \frac{dh}{dt} + \frac{\pi}{3} \cdot 6 \cdot 3 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{3\pi}$ m/min.

15. Evaluate the integral $\int \tan^5 x \cdot \sec^2 x \, dx$ by using the substitution $\tan x = t$. [3]

$\tan x = t \quad \therefore \frac{dt}{dx} = \sec^2 x \Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int t^5 \cdot dt$

$= \frac{t^6}{6} + C = \frac{\tan^6 x}{6} + C$

16. Evaluate the integral $\int \sin 6x \cdot \sin 3x \, dx$. [4]

$I = \frac{1}{2}\int (\cos(6x - 3x) - \cos(6x + 3x))\, dx$

$= \frac{1}{2}\int (\cos 3x - \cos 9x)\, dx$

$= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$

17. Evaluate the integral $\int \sin^5 x\, dx$. [5]

Let $\cos x = t$

$\therefore \frac{dt}{dx} = -\sin x \Rightarrow \sin x \; x = -dt$

$I = \int \sin^4 x \sin x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \, dx$

$= -\int (1-t^2)^2 \, dt = -\int (1 - 2t^2 + t^4) \, dt$

$= -t + \frac{2}{3}t^3 - \frac{t^5}{5} + C$

$= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$

18. Use an appropriate substitution to evaluate the integral $\int \frac{\ln(\ln x)}{x \ln x} dx$. [5]

Let $\ln x = t \Rightarrow \frac{dt}{dx} = \frac{1}{x} \therefore \frac{1}{x} dx = dt$

$\therefore I = \int \frac{\ln t}{t} dt$

Let $\ln t = u \Rightarrow \frac{du}{dt} = \frac{1}{t} \therefore \frac{1}{t} dt = du$

$\therefore I = \int u \, du$

$= \frac{u^2}{2} + C$

$= \frac{(\ln t)^2}{2} + C$

$= \frac{(\ln(\ln x))^2}{2} + C$

19. The Newton-Raphson iteration formula is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, [10]

(a) Consider solving $f(x) = x^3 + 3x + 1 = 0$, show that $x_{n+1} = \dfrac{2x_n^3 - 1}{3x_n^2 + 3}$.

$f'(x) = 3x^2 + 3$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

$$= \frac{3x_n^3 + 3x_n - x_n^3 - 3x_n - 1}{3x_n^2 + 3}$$

$$= \frac{2x_n^3 - 1}{3x_n^2 + 3}$$

(b) Starting with $x_0 = 1$, determine the root of $f(x) = 0$ that lies in the interval $(-1, 3)$, correct to 5 decimal places. List all x_n values until the approximation is achieved.

n	x_n
0	1
1	0.16667
2	-0.32132
3	-0.32219
4	-0.32219

$\therefore x^* = -0.32219$

20. Given $f(x) = \sqrt[4]{1-x}$, $-1 < x < 1$. [10]

(a) Obtain the Maclaurin's expansion of $f(x)$ up to the terms with x^2

$f(x) = \sqrt[4]{1-x}$ $\quad f(0) = 1$

$f'(x) = -\frac{1}{4}(1-x)^{-\frac{3}{4}}$ $\quad f'(0) = -\frac{1}{4}$

$f''(x) = -\frac{3}{16}(1-x)^{-\frac{7}{4}}$ $\quad f''(0) = -\frac{3}{16}$

$\therefore f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \cdots$

$\approx 1 - \frac{1}{4}x - \frac{3}{32}x^2$

(b) Use the substitution $x = \dfrac{1}{16}$ in the expansion above to approximate the value of $\sqrt[4]{15}$. Give your answer correct to 4 decimal places.

$f(\frac{1}{16}) = \sqrt[4]{1-\frac{1}{16}} = \frac{\sqrt[4]{15}}{2}$

$f(\frac{1}{16}) \approx 1 - \frac{1}{4}\cdot\frac{1}{16} - \frac{3}{32}\cdot(\frac{1}{16})^2$

$= 0.9840$

$\therefore \frac{\sqrt[4]{15}}{2} = 0.9840$

$\sqrt[4]{15} = 1.9680$

You may use this space for rough work.

All answers must be written in the Answer Booklet.