Foundation Calculus and Mathematical Techniques CELEN037

Practice Problems SET-2 Sample Solution

Type 1: Chain Rules

5. Find the derivative of $K(x) = \dfrac{1+e^{-2x}}{x+\tan(12x)}$ using the Chain Rule.

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^{-2x}+1}{\tan(12x)+x}\right) &= \frac{\frac{d}{dx}\left(e^{-2x}+1\right)(\tan(12x)+x)-\left(e^{-2x}+1\right)\frac{d}{dx}(\tan(12x)+x)}{(\tan(12x)+x)^2} \\
&= \frac{\left(e^{-2x}\frac{d}{dx}(-2x)+0\right)(\tan(12x)+x)-\left(e^{-2x}+1\right)\left((\sec(12x))^2\frac{d}{dx}(12x)+1\right)}{(\tan(12x)+x)^2} \\
&= \frac{-2\cdot 1e^{-2x}(\tan(12x)+x)-\left(e^{-2x}+1\right)\left(12(\sec(12x))^2 1+1\right)}{(\tan(12x)+x)^2} \\
&= \frac{-2e^{-2x}(\tan(12x)+x)-\left(e^{-2x}+1\right)\left(12(\sec(12x))^2+1\right)}{(\tan(12x)+x)^2} \\
&= -\frac{2e^{-2x}}{\tan(12x)+x}-\frac{\left(e^{-2x}+1\right)\left(12(\sec(12x))^2+1\right)}{(\tan(12x)+x)^2}
\end{aligned}$$

Type 2: Logarithmic Differentiation

10. Use logarithmic differentiation to find the derivative $\dfrac{dy}{dx}$ of $y = \sec(x^{\ln x})$.

Solution:

$$\frac{d}{dx}\left(\sec(x^{\ln(x)})\right) = \sec(x^{\ln(x)})\tan(x^{\ln(x)})\frac{d}{dx}\left(x^{\ln(x)}\right)$$

Use logarithmic differentiation to find: $\dfrac{d}{dx}\left(x^{\ln(x)}\right)$

Let $u = x^{\ln(x)}$

$$\ln(u) = \ln(x)\cdot\ln(x)$$

Differentiate both sides w.r.t x

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\left(\ln(x)\ln(x)\right)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\left((\ln(x))^2\right)$$

$$\frac{1}{u}\frac{du}{dx} = 2\ln(x)\frac{d}{dx}\left(\ln(x)\right)$$

$$\frac{1}{u}\frac{du}{dx} = 2\ln(x)\frac{1}{x}$$

$$\therefore \quad \frac{du}{dx} = u\cdot 2\ln(x)\frac{1}{x}$$

$$\frac{d}{dx}\left(x^{\ln(x)}\right) = x^{\ln(x)}\cdot 2\ln(x)\frac{1}{x}$$

$$\frac{d}{dx}\left(x^{\ln(x)}\right) = \frac{x^{\ln(x)}\cdot 2\ln(x)}{x}$$

$$\therefore \quad \frac{d}{dx}\left(\sec(x^{\ln(x)})\right) = \frac{2\sec(x^{\ln x})\cdot\tan(x^{\ln x})\cdot x^{\ln x}\cdot\ln x}{x}$$

Type 3: Implicit Differentiation

15. Find the gradient of $y^2e^{2x} = 3y + x^2$ at $(0, 3)$.

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(y^2e^{2x}\right) &= \frac{d}{dx}\left(3y + x^2\right) \\
e^{2x}\frac{d}{dx}\left(y^2\right) + y^2\frac{d}{dx}\left(e^{2x}\right) &= \frac{d}{dx}\left(x^2 + 3y\right) \\
2y \cdot \frac{dy}{dx}e^{2x} + \frac{d}{dx}\left(e^{2x}\right)y^2 &= \frac{d}{dx}\left(x^2 + 3y\right) \\
2y \cdot \frac{dy}{dx}e^{2x} + 2\left(e^{2x}\right)y^2 &= \frac{d}{dx}\left(x^2\right) + 3\frac{d}{dx}y \\
2y \cdot \frac{dy}{dx}e^{2x} + 2\left(e^{2x}\right)y^2 &= 2x + 3\frac{dy}{dx}
\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2x - 2e^{2x}y^2}{-3 + 2e^{2x}y}$$

Let $x = 0,\ y = 3$

$\therefore$ The gradient $\dfrac{dy}{dx} = -6$

Type 4: Derivatives of Inverse Functions

19. Use the definition of the derivative of an inverse function to find $\dfrac{dy}{dx}$ for $x = \cos^{-1}(\sqrt{1-y^2})$.

Hint: $\dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)}$.

Solution:

$$\begin{aligned}
\frac{dx}{dy} &= -\frac{1}{\sqrt{1-\left(\sqrt{1-y^2}\right)^2}}\frac{d}{dy}\left(\sqrt{1-y^2}\right) \\
&= -\frac{\frac{1}{2}\left(1-y^2\right)^{\frac{1}{2}-1}\frac{d}{dy}\left(1-y^2\right)}{|y|} \\
&= -\frac{\frac{d}{dy}1 - \frac{d}{dy}\left(y^2\right)}{2\sqrt{1-y^2}\,|y|} \\
&= \frac{y}{\sqrt{1-y^2}\,|y|}
\end{aligned}$$

As $0 < y < 1$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\therefore \frac{dy}{dx} = \sqrt{1-y^2}$$