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Foundation Calculus and Mathematical Techniques CELEN037

Practice Problems SET-4 Sample Solution

Type 1: Parametric Differentiation

4. Find the derivative $\dfrac{dy}{dx}$ for the function $x = \sin 2t,\ y = -\cos t$ at the point $t = \dfrac{\pi}{6}$.

$\dfrac{dx}{dt} = 2 \cdot \cos(2t)$

$\dfrac{dy}{dt} = \sin(t)$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sin(t)}{2\cos(2t)}$

Let $t = \dfrac{\pi}{6}$

$\dfrac{dy}{dx} = \dfrac{\sin(\frac{\pi}{6})}{2\cos(2 \times \frac{\pi}{6})} = \dfrac{1}{2}$

Type 2: Maclaurin's Series

6. Obtain the Maclaurin's Series expansions of the following functions: (v) $\dfrac{e^x + e^{-x}}{2}$.

Solution:

$$f(x) = \frac{e^x + e^{-x}}{2}, \quad f(0) = 1;$$
$$f'(x) = \frac{e^x - e^{-x}}{2}, \quad f'(0) = 0;$$
$$f''(x) = \frac{e^x + e^{-x}}{2}, \quad f''(0) = 1;$$
$$f^{(3)}(x) = \frac{e^x - e^{-x}}{2},\ f^{(3)}(x) = 0;$$
$$f^{(4)}(x) = \frac{e^x + e^{-x}}{2},\ f^{(3)}(x) = 1;$$
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$$
$$f(x) = 1 + x \cdot 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot 1 + \cdots$$
$$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Type 3: Equation of Tangent and Normal lines

12. The equation of a curve is given by $y(x)$ given by $x^2 - 4y^2 = 12$. Obtain the equations of (a) the tangent line and (b) the normal line to the curve at point $P(4, 1)$.

Solution:

$$\frac{d}{dx}(x^2 - 4y^2) = \frac{d}{dx}(12)$$
$$2x - 8y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{x}{4y}$$
$$\therefore\ m = \frac{x}{4y} = \frac{4}{4 \times 1} = 1$$

$\therefore$ (a.) tangent line equation is: $y - 1 = 1 \cdot (x - 4) \implies y = x - 3$

$\therefore$ (b.) normal line equation is: $y - 1 = -1 \cdot (x - 4) \implies y = -x + 5$

Type 4: Higher Order Derivatives

13. Given $y = C_1 e^{mx} + C_2 e^{-mx}$, where C_1, C_2 and m are arbitrary constants, show that $\frac{d^2y}{dx^2} - m^2 y = 0$.

Solution:

$$y = C_1 e^{mx} + C_2 e^{-mx}$$
$$\frac{dy}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}$$
$$\frac{d^2y}{dx^2} = m^2 C_1 e^{mx} + m^2 C_2 e^{-mx}$$
$$\frac{d^2y}{dx^2} = m^2(C_1 e^{mx} + C_2 e^{-mx}) = m^2(y)$$
$$\therefore\ \frac{d^2y}{dx^2} - m^2 y = 0$$

Type 5: Related Rates

23. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of $0.5m^2/s$ at what rate is the radius decreasing when the area of the sheet is $12m^2$? Solution:

$$A = \pi r^2$$

$$\frac{dA}{dr} = \frac{\frac{dA}{dt}}{\frac{dr}{dt}} = 2\pi r$$

$$\frac{-0.5}{\frac{dr}{dt}} = 2\pi r$$

$$\frac{dr}{dt} = -\frac{1}{4\pi r}$$

when $A = 12 = \pi r^2,\ r = \sqrt{\frac{12}{\pi}}$

$$\therefore\ \left.\frac{dr}{dt}\right|_{r=\sqrt{\frac{12}{\pi}}} = -\frac{1}{4\pi\sqrt{\frac{12}{\pi}}} = -0.040717$$