Student Evaluation of Module (SEM)

Foundation Algebra for Physical Sciences & Engineering

- This is an evaluation of the module for **CELEN036**
- There are 5 questions and opportunity for some comments
- Scan the QR code below, login using your university account, and complete the survey.

Lecture 10

Topics covered in this lecture session

1. Partial sums and the sigma notation.
2. Progression and Series (Arithmetic series, Geometric series).
3. The sum of an infinite Geometric series.
4. Power Series.
5. Method of differences.

Partial Sums

Let $a_1,\ a_2,\ a_3, ...$ be a given sequence.

Define the sums:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ \vdots & \quad\;\; \vdots \quad \vdots \quad \vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots\ldots + a_n \end{aligned}$$

The sums defined as above are called partial sums.

Sigma notation

$$\sum_{k=1}^{n} a_k = \sum_{1}^{n} a_n = a_1 + a_2 + a_3 + ... + a_n$$

$$\sum_{1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

Using sigma notation, the partial sums for sequence $\{a_n\}$ is:

$$S_n = \sum_{k=1}^{n} a_k$$

Sigma notation

Some examples on the use of sigma notation:

$$\sum_{1}^{5} r^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$1 - x + x^2 - x^3 + ... = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$2 - 4 + 8 - 16 + ... + 128 = \sum_{n=1}^{7} (-1)^{n+1} 2^n$$

Sequences - Introduction

A sequence is an ordered list of numbers (objects).

Mathematically,

A sequence is a function defined on a set of natural numbers.

that is,

A sequence is a function $f : \mathbb{N} \to A,$ where A is any non-empty set of numbers (or objects).

Sequences - Introduction

Some examples of sequences are:

2, 4, 6, 8, 10, …. is a sequence of even numbers.

1, 2, 4, 8, 16, …. is a sequence of numbers of the form

$$2^{n-1} \quad ; \quad n = 1, 2, 3, 4, \ldots$$

4, 9, 16, 25, 36, …. is a sequence of numbers of the form

$$n^2 \quad ; \quad n = 2, 3, 4, \ldots$$

Sequences - Introduction

Each member of the set is called the term of the sequence, and is denoted by

$$a_1,\ a_2,\ a_3,\ \ldots. \quad \text{or} \quad T_1,\ T_2,\ T_3,\ \ldots. \quad \text{or} \quad f(1),\ f(2),\ f(3),\ \ldots..$$

Thus, a sequence may be denoted by: $\{a_n\}_{n=1}^{n=k}$

If k is a finite number, the sequence is called finite sequence; otherwise infinite.

Sequences - Introduction

Some sequences have a general formula, some not.

e.g.

- The sequence of numbers

  2, 5, 8, 11, ….. has a general formula

  $$f(n) = 3n - 1 \quad ; \quad n \in \mathbb{N}.$$

- The sequence of primes

  2, 3, 5, 7, 11, ….. has no general formula.

Arithmetic Sequence/Progression (A.P.)

An Arithmetic Progression (A.P.) is a sequence in which difference between any two consecutive terms is constant.

e.g. 1, 5, 9, 13, 17, 21, ….. is an A.P.

- The constant difference, called the common difference is denoted by d.
- The first term of the sequence is denoted by a.

Arithmetic Sequence/Progression (A.P.)

Some examples of A.P. are:

2, 4, 6, 8, 10, …. where $a = 2, \quad d = 2.$

5, 8, 11, 14, 17, …. where $a = 5, \quad d = 3.$

8, 5, 2, -1, - 4, …. where $a = 8, \quad d = -3.$

Thus, an A.P. takes the form:

$$a, \quad a + d, \quad a + 2d, \quad , \ldots.., \quad a + (n-1)\,d$$

$\therefore\ n^{th}$ term of an A.P. is: $\boxed{a_n = a + (n-1)\,d}$

Geometric Sequence/Progression (G.P.)

A Geometric Progression (G.P.) is a sequence in which ratio of any two consecutive terms is constant.

e.g. 4, 12, 36, 108, 324, ….. is a G.P.

- The constant ratio, called common ratio is denoted by r.
- The first term of the sequence is denoted by a.

Geometric Sequence/Progression (G.P.)

Some examples of G.P. are:

$$2, \quad 4, \quad 8, \quad 16, \quad 32, \; \ldots\ldots \qquad \text{where} \quad a = 2, \quad r = 2$$

$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \; \ldots\ldots \qquad \text{where} \quad a = 1, \quad r = \frac{1}{2}$$

Thus, a G.P. takes the form:

$$a, \quad a\,r, \quad a\,r^2, \quad a\,r^3, \quad ,\ldots\ldots, \quad a\,r^{n-1}$$

$\therefore n^{th}$ term of a G.P. is: $\boxed{a_n = a\,r^{n-1}}$

Worked Examples

1. The eighth term of an A.P. is 11 and its fifteenth term is 21. Find the common difference, the first term of the sequence, and the n^{th} term.
2. Find the G.P. of positive terms such that its first term is 4 and the fifth term is 324.
3. The fourth term of a G.P. is 24 and its ninth term is 768. Find its eleventh term.

Harmonic Sequence

A general harmonic progression (or harmonic sequence) is a progression formed by taking the reciprocals of an arithmetic progression.

i.e. it is a sequence of the form: $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots..$

A particular case of general harmonic sequence is given by:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.., \frac{1}{n}, \ldots...$$

$\therefore n^{th}$ term of a Harmonic sequence is: $\boxed{f(n) = \frac{1}{n}}$

Fibonacci Sequence

- Fibonacci sequence is named after Leonardo Fibonacci.
- It consists of (Fibonacci) numbers in the following integer sequence: $0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots\ldots$

Mathematically, Fibonacci numbers is defined by the recurrence relation:

$$\boxed{f(n) = f(n-1) + f(n-2)} \quad ; \quad n \in \mathbb{N},\ n > 1$$

with $f(0) = 0$ and $f(1) = 1.$

Fibonacci Sequence and the Golden ratio

- The ratio of neighbouring Fibonacci numbers tends to the Golden ratio.

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$= \boxed{1.6180339887}$$

Golden Ratio

2 / 1 = 2.0

3 / 2 = 1.5

5 / 3 = 1.67

8 / 5 = 1.6

13 / 8 = 1.625

21 / 13 = 1.615

34 / 21 = 1.619

55 / 34 = 1.618

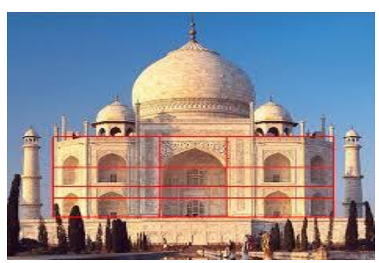
89 / 55 = 1.618

Fibonacci Sequence and the Golden ratio

Many artists and architects have been fascinated by the presumption that the golden rectangle (with length of sides as neighboring Fibonacci numbers) is considered aesthetically pleasing.

1.618

1



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Series

A series $\{S_n\}$ is a sequence whose terms are partial sums of terms of a given sequence $\{a_n\}$.

e.g. if a given sequence is 2, 4, 6, 8, 10, ….

then the corresponding (associated) series is:

$$2, \quad 2+4, \quad 2+4+6, \quad 2+4+6+8, \quad .....$$

i.e. $2, \quad 6, \quad 12, \quad 20, \quad .....$

Series

n^{th} term of series $\{S_n\}$ can be obtained from sequence $\{a_n\}$, using

$$S_n = \sum_{k=1}^{n} a_k$$

On the other hand,

$$\begin{aligned} S_n - S_{n-1} &= (a_1 + a_2 + ... + a_{n-1} + a_n) \\ &\qquad - (a_1 + a_2 + ... + a_{n-1}) \\ &= a_n \end{aligned}$$

$$\therefore \boxed{a_n = S_n - S_{n-1}}$$

Arithmetic Series

Consider the sum S_n of the first n terms of an A.P.

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots\ldots + [a+(n-2)d] + [a+(n-1)d]$$

Writing $l = a + (n-1)\,d$

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ \ldots\ldots + (l-d) + l$$

Reversing the sum

$$S_n = l + (l-d) + (l-2d) + (l-3d) + \ \ldots\ldots + (a+d) + a$$

Adding

$$2S_n = (a+l) + (a+l) + (a+l) + (a+l) + \ \ldots\ldots + (a+l) + (a+l)$$

n times

Arithmetic Series

$$\therefore\ 2S_n = n\,(a+l) \Rightarrow S_n = \frac{n}{2}\,(a+l) = \frac{n}{2}\,[a+a+(n-1)\,d]$$

Thus, the sum of the first n terms of an A.P. is:

$$\boxed{S_n = \frac{n}{2}\,[2a+(n-1)\,d]}$$

Example:

The eighth term of an A.P. is 23 and its 24th term is 103.

Find the sum of its first 30 terms.

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Find the sum of its first 30 terms.

$$8^{th}\text{ term} = 23 \rightarrow a + 7d = 23$$

$$24^{th}\text{ term} = 103 \rightarrow a + 23d = 103$$

$$\therefore 16d = 80$$

$$\boxed{d = 5}$$

$$a = 23 - 7d = 23 - 35$$

$$\boxed{a = -12}$$

Example:

The eighth term of an A.P. is 23 and its 24th term is 103.

Find the sum of its first 30 terms.

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]}$$

Sum of first 30 terms i.e. $S_{30} = \frac{30}{2}[2(-12) + (30-1)5]$

$$\boxed{S_{30} = 1815}$$

Geometric Series

The sum of the first n terms of a G.P. is:

$$\boxed{S_n = \begin{cases} na & ; \quad r = 1 \\ a\left(\dfrac{1-r^n}{1-r}\right) & ; \quad r \neq 1 \end{cases}}$$

Example:

If $r = \dfrac{1}{3}$, $S_4 = 150$, find the first term a.

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If $r = \frac{1}{3}$, $S_4 = 150$, find the first term a.

For a G.P. : $S_n = a\left(\frac{1-r^n}{1-r}\right) \quad r \neq 1$

$$r = \frac{1}{3}; S_4 = 150; \ a = ?$$

$$150 = a\left(\frac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right)$$

$$101.25 + 33.75 + 11.25 + 3.75 + \cdots \qquad \boxed{a = 101.25}$$

Sum of Infinite Geometric Series

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \quad ; \quad r \neq 1$$

If $|\, r \,| < 1$ then, $\lim\limits_{n\to\infty} r^n = 0$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[a\left(\frac{1-r^n}{1-r}\right)\right] = \frac{a}{1-r} \Rightarrow \boxed{S = \frac{a}{1-r}}$$

Example: Find the sum of the infinite geometric series:

$$5 + 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\ldots$$

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Find the sum of the infinite geometric series:

$$5 + 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots\dots$$

From the given series: $a = 5$ and $\mathrm{r} = \frac{1}{5}$

$$S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{5}} \quad \therefore \boxed{S = \frac{25}{4}}$$

Harmonic Series

The harmonic series is the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots.$$

Power Series

If $k \in \mathbb{N}$, the series:

$$1^k + 2^k + 3^k + \ldots\ldots + n^k = \sum_{a=1}^{n} a^k$$

is called the Power Series.

Power Series

- When $k = 1,$

$$1 + 2 + 3 + \ldots\ldots + n = \sum_{a=1}^{n} a = \frac{n(n+1)}{2}$$

- When $k = 2,$

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{a=1}^{n} a^2 = \frac{n(n+1)(2n+1)}{6}$$

- When $k = 3,$

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum_{a=1}^{n} a^3 = \frac{n^2(n+1)^2}{4}$$

Power Series (Worked Example)

Find the sum: $1+(1+2)+(1+2+3)+\ldots.\quad(\text{up to } n \text{ terms})$

Solution:

$$\boxed{\text{Sum} = \sum n^{th} \text{ term}}$$

$$\therefore \text{Sum} = \sum(1+2+3+\ldots.+n)$$

$$=\sum\left(\sum n\right) = \sum \frac{n(n+1)}{2}$$

Power Series (Worked Example)

$$\therefore \text{Sum} = \frac{1}{2}\sum (n^2+n)$$

$$= \frac{1}{2}\left(\sum n^2 + \sum n\right)$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$ (upon simplification)

Method of differences

Find the sum: $$\sum_{1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

Solution:

$$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\ldots}+\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

Method of differences

Find the sum: $\displaystyle\sum_{1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

Solution:

$$=\lim_{n\to\infty}\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\ldots}\cancel{+}\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=1$$

THANKS FOR YOUR ATTENTION