Seminar 2

In this seminar you will study:

- The Nature of Roots
- The Method of Completing the Square
- Graphing and Finding Range of Quadratic Functions
- Exponential Functions and Equations
- Logarithmic Functions and Equations

The Nature of Roots of Quadratic equations

For Quadratic equations $f(x) = ax^2 + bx + c = 0 \quad a \neq 0$

Discriminant $\Delta = b^2 - 4ac$	> 0	Roots are real and distinct
	$= 0$	Roots are real and equal (i.e. repeated roots)
	< 0	No real roots (i.e. roots are complex numbers)

Example: If roots of the equation $kx^2 - x + 3 = 0$ are equal, find k.

Solution:

Here, $a = k,\ b = -1,$ and $c = 3 \ \Rightarrow\ \triangle = 1 - 12k$

Now, Roots are equal $\Rightarrow\ \triangle = 0$

$$\Rightarrow\ 1 - 12k = 0$$

$$\Rightarrow\ k = \frac{1}{12}$$

The method of completing the square

The general form of quadratic functions

$$\boxed{f(x) = ax^2 + bx + c, \quad a \neq 0}$$

Case (i): $a = 1 \Rightarrow \qquad f(x) = x^2 + bx + c$

Add and subtract $\left(\frac{b}{2}\right)^2$ to express $f(x)$ in the form:

$$\boxed{f(x) = (x+p)^2 + q}$$

$$x^2 + bx + c = \underline{x^2 + bx + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 + c$$

$$= \underline{\left(x + \frac{b}{2}\right)^2} + c - \left(\frac{b}{2}\right)^2$$

The method of completing the square

Example: Find the range of the function $f(x) = x^2 - 5x + 9$ by completing the square.

Solution: $f(x) = x^2 - 5x + 9 = x^2 + bx + c$

$$\therefore \quad b = -5,\ c = 9.$$

$$x^2 - 5x + 9 = \underline{x^2 - 5x + \left(\frac{5}{2}\right)^2} - \left(\frac{5}{2}\right)^2 + 9$$

$$= \underline{\left(x - \frac{5}{2}\right)^2} - \frac{25}{4} + 9$$

$$= \left(x - \frac{5}{2}\right)^2 + \frac{11}{4}$$

Since $\left(x - \frac{5}{2}\right)^2 \geq 0$

$$\Rightarrow \quad \left(x - \frac{5}{2}\right)^2 + \frac{11}{4} \geq \frac{11}{4} \quad \Rightarrow f(x) \geq \frac{11}{4}, \quad \therefore \text{Range of } f(x) \text{ is } \left[\frac{11}{4}, +\infty\right)$$

The method of completing the square

The general form of quadratic functions

$$\boxed{f(x) = ax^2 + bx + c, \quad a \neq 0}$$

Case (ii): $a \neq 1 \Rightarrow \qquad f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

Take the coefficient a of x^2 as a common factor, then follow the steps as in **Case (i).**

Example: Find the range of the function $f(x) = 2x^2 - 7x + 4$ by completing the square.

Solution:

$$f(x) = 2\left(x^2 - \frac{7}{2}x + 2\right) = 2\left(x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + 2\right)$$

$$= 2\left[\left(x - \frac{7}{4}\right)^2 - \frac{17}{16}\right] = 2\left(x - \frac{7}{4}\right)^2 - \frac{17}{8}$$

$$\Rightarrow f(x) \geq -\frac{17}{8} \quad \therefore \quad R_f = \left[-\frac{17}{8}, +\infty\right)$$

Laws of exponents/indices

$$a^0 = 1 \qquad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(ab)^n = a^n \cdot b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^m)^n = a^{mn} = (a^n)^m$$

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

In particular,

$$a^{\frac{1}{2}} = \sqrt{a} \quad \text{and} \quad a^{\frac{1}{3}} = \sqrt[3]{a}$$

Exponential Functions and Equations

Example 1: Simplify $\dfrac{\sqrt[3]{x^4}}{x^3} \cdot \sqrt{\left(\dfrac{x^2}{\sqrt[3]{x}}\right)^3}$

Solution:

$$\begin{aligned}\frac{\sqrt[3]{x^4}}{x^3} \cdot \sqrt{\left(\frac{x^2}{\sqrt[3]{x}}\right)^3} &= x^{\frac{4}{3}} \cdot x^{-3} \cdot (x^2 \cdot x^{-\frac{1}{3}})^{\frac{3}{2}} \\ &= x^{-\frac{5}{3}} \cdot (x^{\frac{5}{3}})^{\frac{3}{2}} = x^{-\frac{5}{3}} \cdot x^{\frac{5}{2}} \\ &= x^{\frac{5}{6}}\end{aligned}$$

Example 2: Solve $e^{2x} - 2e^x - 3 = 0$.

Solution:

Let $e^x = t$

$\Rightarrow \quad t^2 - 2t - 3 = 0$

$\Rightarrow \quad t = -1 \text{ or } 3$

$\therefore \quad e^x = -1 \text{ or } e^x = 3$

But, $e^x > 0, \ \forall\, x \in \mathbb{R} \ \Rightarrow \ e^x \neq -1$

$\therefore \quad e^x = 3$

$\Rightarrow \quad x = \ln 3$

Rules of Logarithms

$$\log_a 1 = 0 \qquad (a > 0)$$

$$\log_a a = 1$$

$$\log_a (xy) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_y x = \frac{\log_a x}{\log_a y}$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_a x} = x$$

$$\boxed{a^x = y \quad \Leftrightarrow \quad x = \log_a y}$$

Logarithmic Functions and Equations

Example: Solve $\log_{10}(2x) + \log_{10}(x-5) = 2$

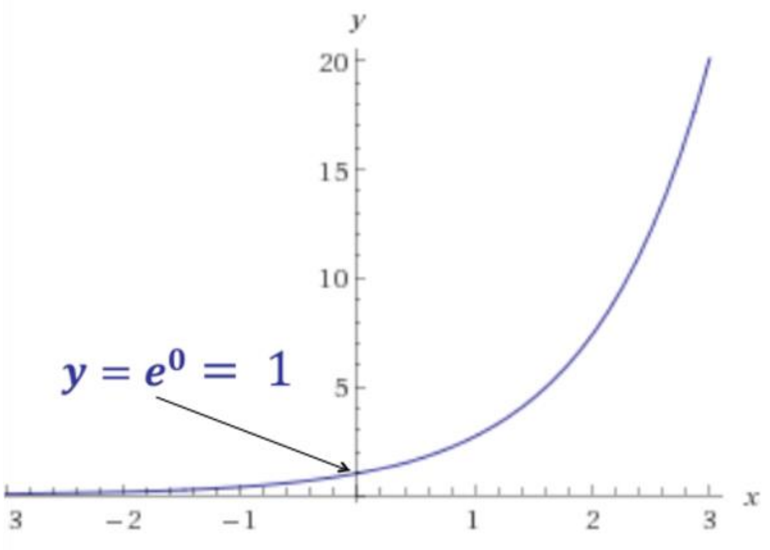
Solution:

$$
\begin{aligned}
& \log(2x) + \log(x-5) = 2 \\
\Rightarrow\quad & \log[2x(x-5)] = 2\log 10 \\
\Rightarrow\quad & \log(2x^2 - 10x) = \log 10^2 \\
\Rightarrow\quad & 2x^2 - 10x = 100 \\
\Rightarrow\quad & 2x^2 - 10x - 100 = 0 \\
\Rightarrow\quad & x^2 - 5x - 50 = 0 \\
\Rightarrow\quad & (x-10)(x+5) = 0 \\
\Rightarrow\quad & x = 10 \text{ or } x = -5
\end{aligned}
$$

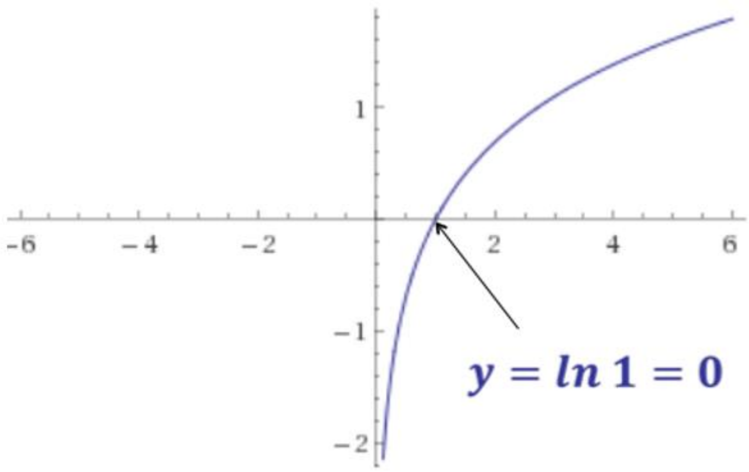
But, $x = -5$ is not an acceptable solution

$$\therefore \quad x = 10$$

Graphs of Exponential and Logarithmic Functions



(a) Graph of $y = e^x$



(b) Graph of $y = \ln x$