Lecture 8

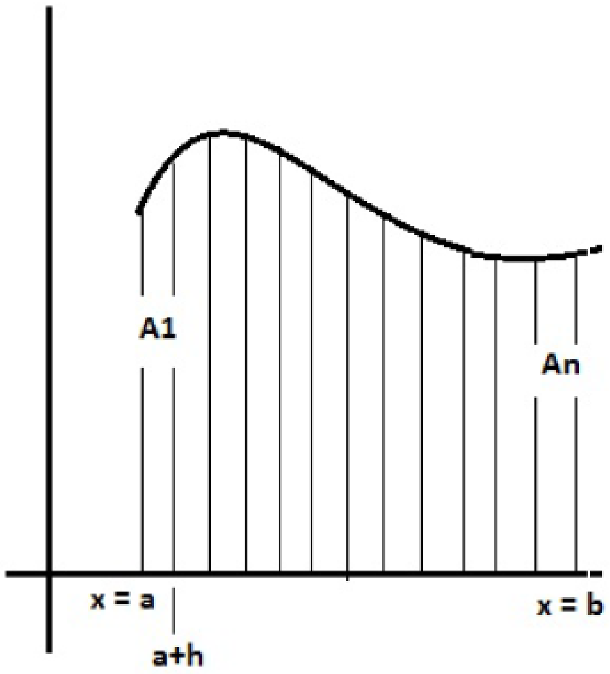
Lecture Content

- Definite integration as a limit of sums
- Properties of Definite Integration
- The method of substitution for <u>Definite Integration</u>
- Integration by parts for <u>Definite Integration</u>

(Definite) Integration as a limit of sum

Suppose we want to find the area of region R bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$.

Let the region R be subdivided into n thin strips of equal width h (say).



$$\therefore \; h = \frac{b-a}{n}$$

$$\equiv \quad \text{width of each strip}$$

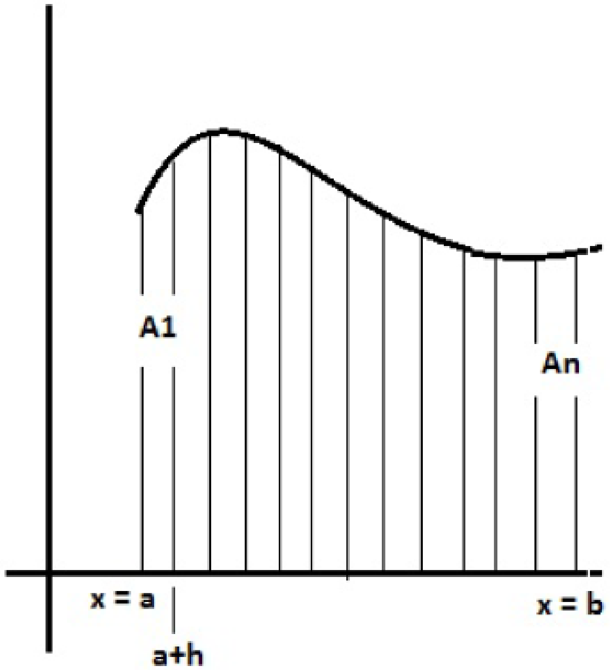
(Definite) Integration as a limit of sum

Now,

Total area $\boldsymbol{A}$ = sum of areas $\boldsymbol{A_1, A_2, A_3, \ldots \ldots, A_n}$

$$= \sum_{i=1}^{n} A_i$$

$$\approx \sum_{i=1}^{n} h \cdot f(a+ih)$$



(Definite) Integration as a limit of sum

If the number of strips are infinitely many, then

$$A = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a + ih)$$

which is same as $\displaystyle\int_{a}^{b} f(x)\, dx$

upper limit (b)

lower limit (a)

and is called **definite integral** from a to b.

(Definite) Integration as a limit of sum

Thus, integration as a limit of sum is defined by

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a+ih)}$$

$$\text{where} \quad h = \frac{b-a}{n}$$

(Definite) Integration as a limit of sum

Example

Evaluate $\int_0^1 x \, dx$ as a limit of sum

$$\int_a^b f(x) \; dx$$

$$\boxed{\begin{aligned} a &= 0 \\ b &= 1 \end{aligned} \quad \Rightarrow \quad h = \frac{b-a}{n} = \frac{1}{n}}$$

$$f(x) = x \quad \Rightarrow \quad f(a+ih) = f(0+ih) = ih$$

$$\boxed{\int_a^b f(x) \; dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a+ih)}$$

$$\therefore \int_0^1 x \; dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot (ih)$$

(Definite) Integration as a limit of sum

$$= \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot (ih)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \frac{n(n+1)}{2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \frac{n^2 \left(1 + \frac{1}{n}\right)}{2} \right]$$

$$= \frac{(1+0)}{2} = \frac{1}{2}$$

Thus, $\displaystyle\int_0^1 x\,dx = \frac{1}{2}$

(Definite) Integration as a limit of sum

Example

Evaluate $\int_0^4 x^3\,dx$ as a limit of sum

In solving definite integration as limit of sum, take note of the following formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Evaluating Definite integrals

Fundamental Theorem of Calculus

If f is continuous on any interval $[a,b]$ and F is any antiderivative of f in $[a,b]$, then

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)}$$

Example

$$\int_0^1 x\ dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Evaluating Definite integrals

Examples

$$(i) \quad \int_{1}^{4} 2 \; dx \; = \left[\, 2x \,\right]_1^4 \; = 8 - 2 = 6$$

$$(ii) \quad \int_{-1}^{2} x^2 \; dx \; = \left[\frac{x^3}{3} \right]_{-1}^{2} \; = \frac{2^3}{3} - \frac{(-1)^3}{3} = 3$$

$$(iii) \quad \int_{0}^{\pi/2} \cos x \; dx \; = \left[\, \sin x \,\right]_0^{\pi/2} \; = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1$$

Properties of Definite Integration

1. If $a \in D_f$, then $\displaystyle\int_a^a f(x)\ dx = 0$

2. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

Properties of Definite Integration

3. If f is integrable on a closed interval containing three points a, b, and c, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Properties of Definite Integration

Example

Evaluate $$\int_0^9 f(x)\ dx$$

where $$f(x) = \begin{cases} \sin x & ; \quad 0 \le x \le \pi/2 \\ 1 & ; \quad \pi/2 \le x \le 5 \\ e^x - 5 & ; \quad 5 \le x \le 9 \end{cases}$$

$$I = \int_0^{\pi/2} \sin x\, dx + \int_{\pi/2}^{5} 1\, dx + \int_5^9 e^x - 5\, dx = e^5\left(e^4 - 1\right) - \frac{\pi}{2} - 14$$

Properties of Definite Integration

4. If f is integrable on $[0, a]$, then

$$\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$$

Example

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\ dx$ **(1)**

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\ dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\ dx \qquad \textbf{(2)}$$

Properties of Definite Integration

(1) + (2) gives

$$I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$\therefore\ 2I = \int_0^{\pi/2} 1\ dx$$

$$\therefore\ I = \frac{1}{2}\,[x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{4}$$

Properties of Definite Integration

Example

Evaluate $\displaystyle\int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\; dx$ **(1)**

$$I = \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{3-(3-x)}}\; dx$$

using the property

$$\int_0^a f(x)\;\, dx = \int_0^a f(a-x)\;\, dx$$

$$= \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}}\; dx$$ **(2)**

(1) + (2) gives $\displaystyle 2I = \int_0^3 1\;\, dx \qquad \therefore\; I = \frac{1}{2}\left[\, x \,\right]_0^3 = \frac{3}{2}$

Properties of Definite Integration

5. If f is integrable on $[\,a, b\,]$, then

$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$$

6. If f is EVEN integrable on $[\,-a, a\,]$, then

$$\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$$

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Properties of Definite Integration

7. If f is ODD integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\ dx = 0$$

Properties of Definite Integration

Example

Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin x}{x^4 + x^2 + \cos x}\, dx$

Let $f(x) = \dfrac{\sin x}{x^4 + x^2 + \cos x}$

$$\Rightarrow f(-x) = \frac{\sin(-x)}{(-x)^4 + (-x)^2 + \cos(-x)} = \frac{-\sin x}{x^4 + x^2 + \cos x} = -f(x)$$

$$\therefore\ f \text{ is odd function } \Rightarrow \int_{-1}^{1} f(x)\, dx = 0$$

The method of Substitution for Definite Integrals

When using substitution, remember to change the limits of integration for the newly formed integral.

Example

Evaluate $\displaystyle\int_{1}^{2} x \cdot e^{x^2} \; dx$

Let $x^2 = t \;\Rightarrow\; x\,dx = \dfrac{1}{2}\,dt$

x	1	2
t	1	4

$$\therefore\; I = \int_{1}^{4} e^t \,\frac{dt}{2} = \frac{e}{2}\,\left(e^3 - 1\right)$$

Practice Question

Write the definite integral obtained when $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\sin x} \cos x dx$

is evaluated by using the substitution $\sin x = t$

A $\int_{0}^{\pi} e^t dt$

B $\boxed{\int_{-1}^{1} e^t dt}$

C $\int_{0}^{1} e^t dt$

Integration by parts for Definite Integrals

$$\boxed{\int_a^b u \cdot \frac{dv}{dx} \; dx = [u\,v]_a^b - \int_a^b v \cdot \frac{du}{dx} \; dx}$$

Example

Evaluate $\displaystyle\int_0^1 x \cdot e^x \; dx$

Let $u = x \;\Rightarrow\; \dfrac{du}{dx} = 1$

and $\dfrac{dv}{dx} = e^x \Rightarrow v = \displaystyle\int e^x \; dx = e^x$

$$\therefore \; I = \left[\, x \cdot e^x \,\right]_0^1 - \int_0^1 e^x \cdot (1)\; dx$$

$$= (e - 0) - \left(e^1 - e^0\right) = 1$$

Evaluating Definite Integrals

(using substitution and integration by parts)

$$(i) \quad \int_0^1 e^{\sqrt{x}}\, dx = 2$$

$$(ii) \quad \int_0^{\pi/2} e^{\sin x} \sin 2x \, dx = 2$$

$$(iii) \quad \int_0^1 x^3 \, e^{x^2} \, dx = \frac{1}{2}$$

Practice Problems

1. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{1/n}}{(\sin x)^{1/n} + (\cos x)^{1/n}} dx$

2. Evaluate $\displaystyle\int_0^1 e^{\sqrt{x}} dx$

Thank You