Introduction to Algorithms

Module Code: CELEN086

Lecture 10

(12/12/24)

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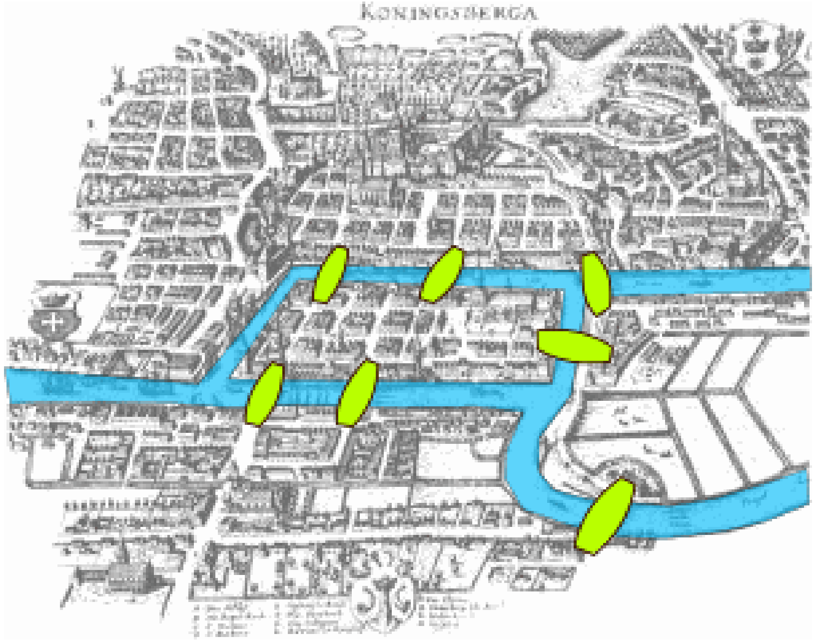
7 bridge problem

In the former city of Königsberg, East Prussia (now in Russia), the river Pregel had **7 bridges.**

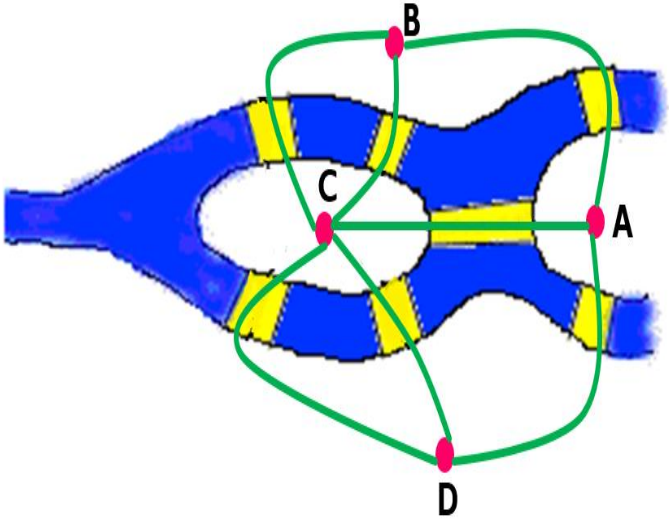
- Is there a path following which we can go cross all the 7 bridges exactly once?
- If such a path exists, is it possible that we end up at the same place as where we started?

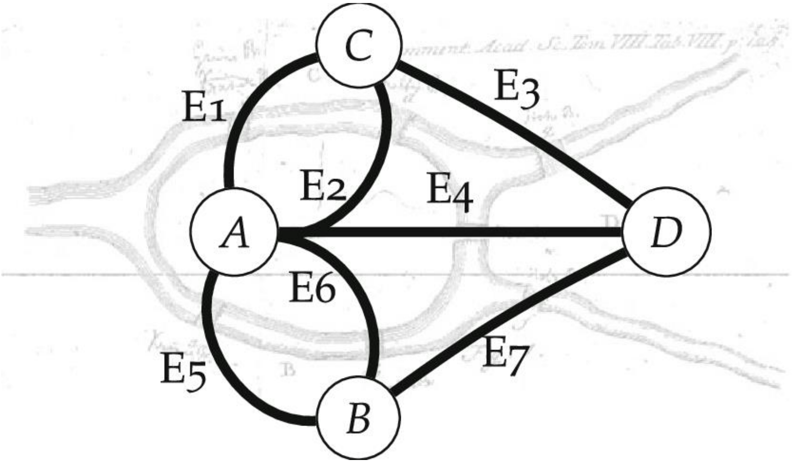


7 bridge problem

In 1736 Leonhard Euler explained why the problem of crossing the 7 bridges of Königsberg is impossible.

Euler laid the foundations of graph theory and prefigured the idea of topology (a mathematical branch).

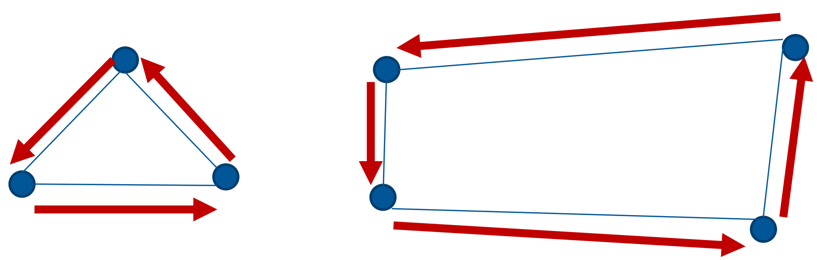




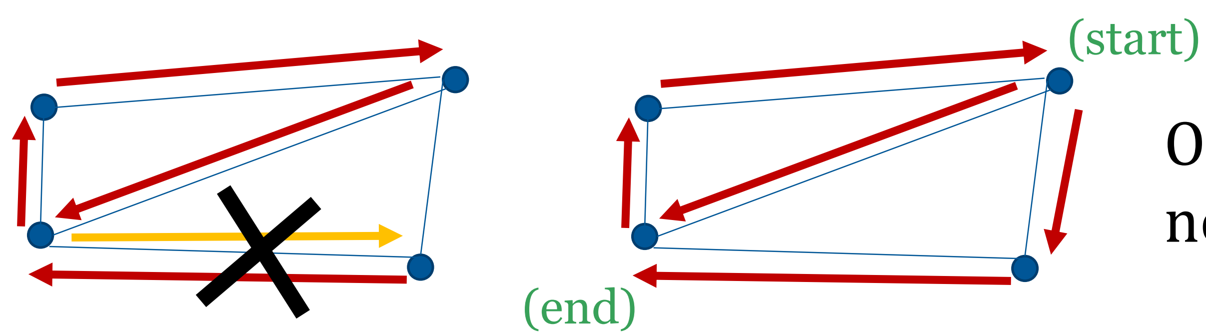
Euler path and Euler circuit

Euler path (Eulerian path, Euler trail) is a path that passes through every edge of the graph exactly once.

If the Euler path returns to the origin, forming a closed path, then the path is called Euler circuit (Euler cycle, Euler tour).



Both have Euler circuits.

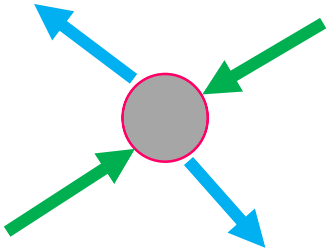


Only has Euler path, no Euler circuit.

Theorem

- If a graph has Euler circuit, then the degree of every vertex must be an even number.

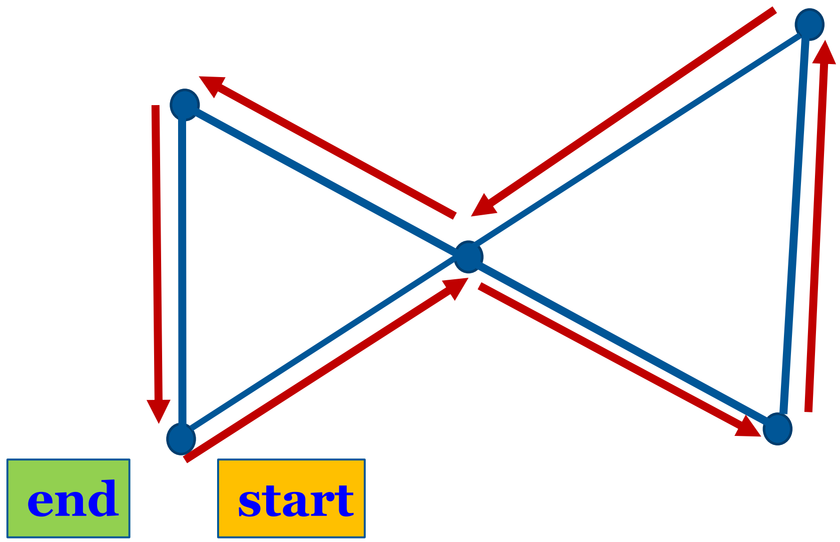
For each vertex that we enter during the tour via an edge, we must also be able to exit via another edge.



The number of edges entering each vertex, must be equal to the number of edges exiting that vertex.

- If a graph has Euler path, then it can have at most two vertices (starting/ending vertices) with odd degrees.

Example



Degrees of all vertices are even.

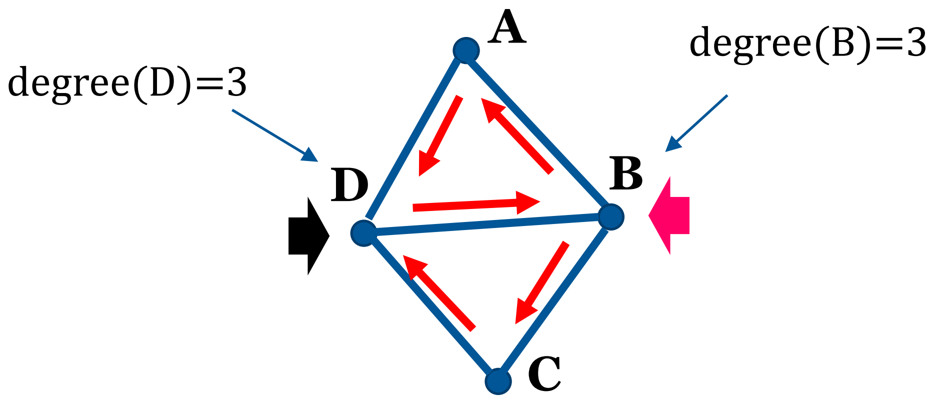
Euler circuit Yes

Euler path Yes

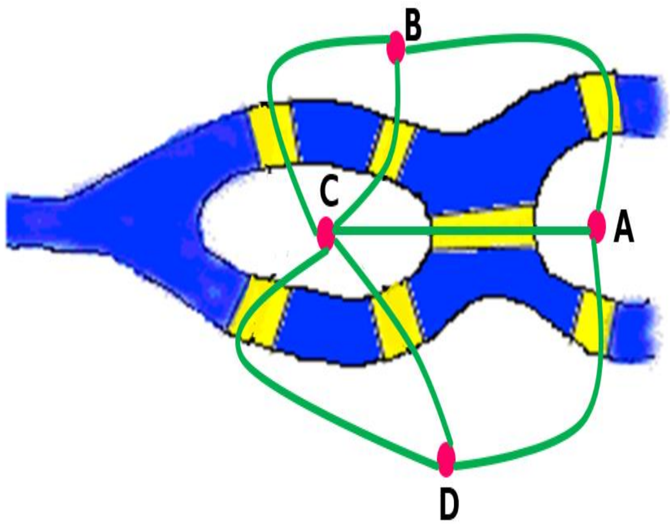
Two vertices are of odd degrees.

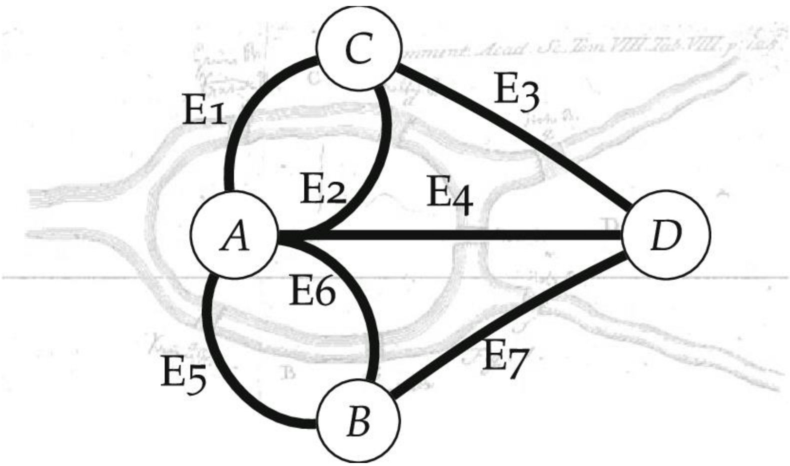
Euler circuit No

Euler path Yes



Example





Vertex	Degree
A	3
B	3
C	5
D	3

More than two vertices have odd degrees.

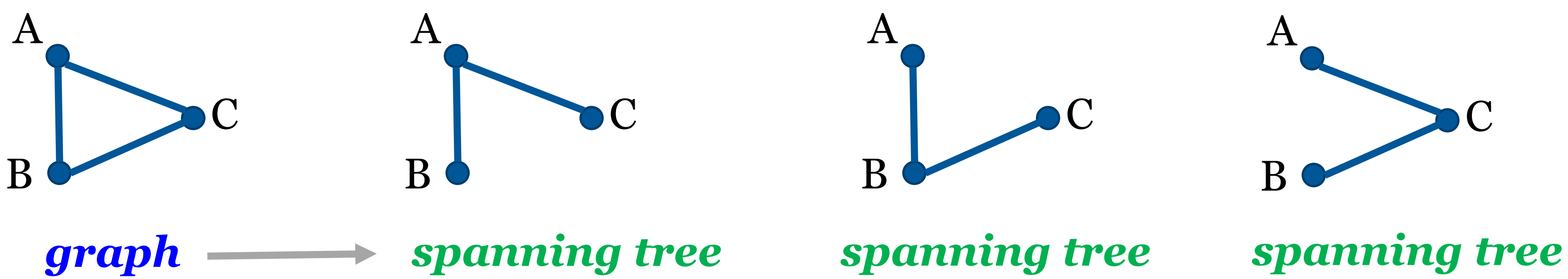
Euler circuit No

Euler path No

Spanning tree

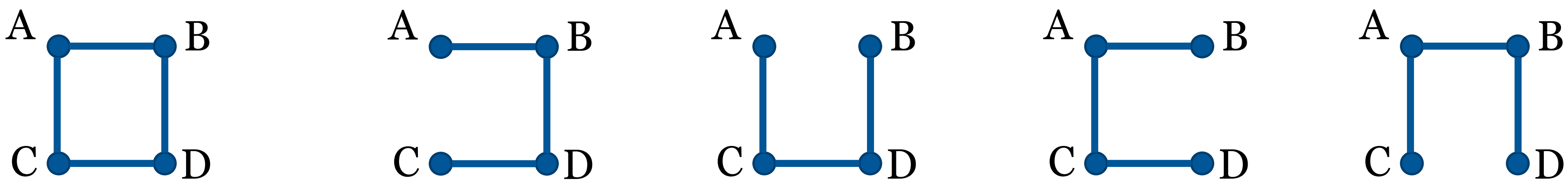
Tree is a special connected and undirected graph with no cycles.

A spanning tree is a subset of a connected and undirected graph, which has all vertices covered with minimum possible number of edges.



$$|V| = 3, |E| = 3, |E_s| = 2$$

Number of spanning trees



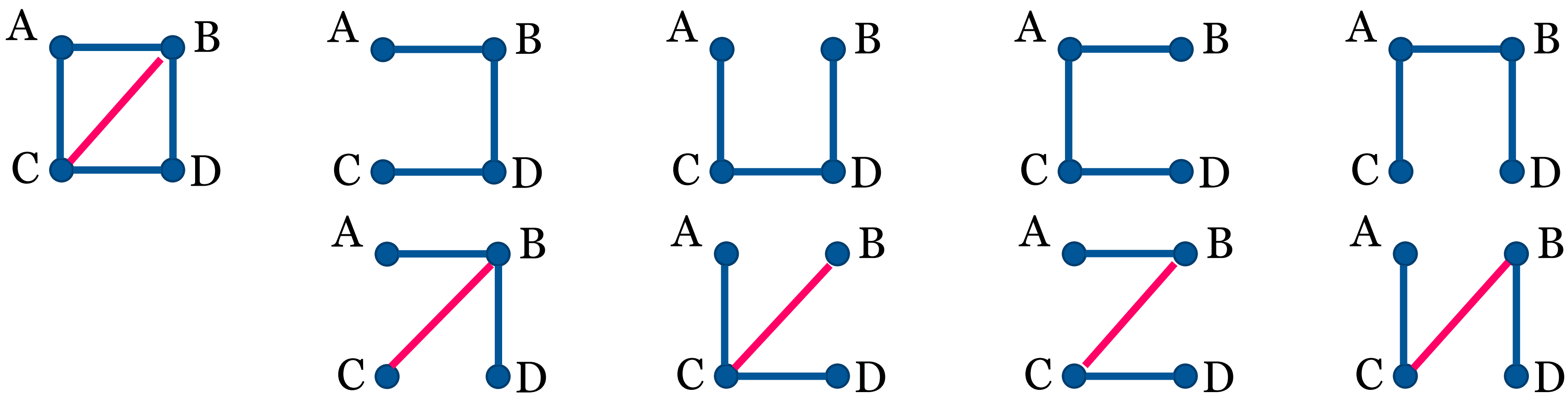
$$|V| = 4, |E| = 4, |E_s| = 3$$

$$\Rightarrow \binom{4}{3} = 4$$

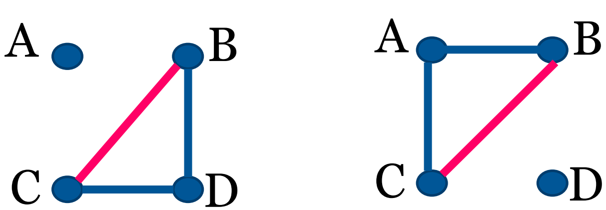
Property: $|E_s| = |V| - 1$

This can help us identifying total number of edges needed in the spanning tree of a connected graph.

Number of spanning trees



$$|V| = 4, |E| = 5, |E_s| = 3 \quad \Rightarrow \quad \binom{5}{3} = 10$$



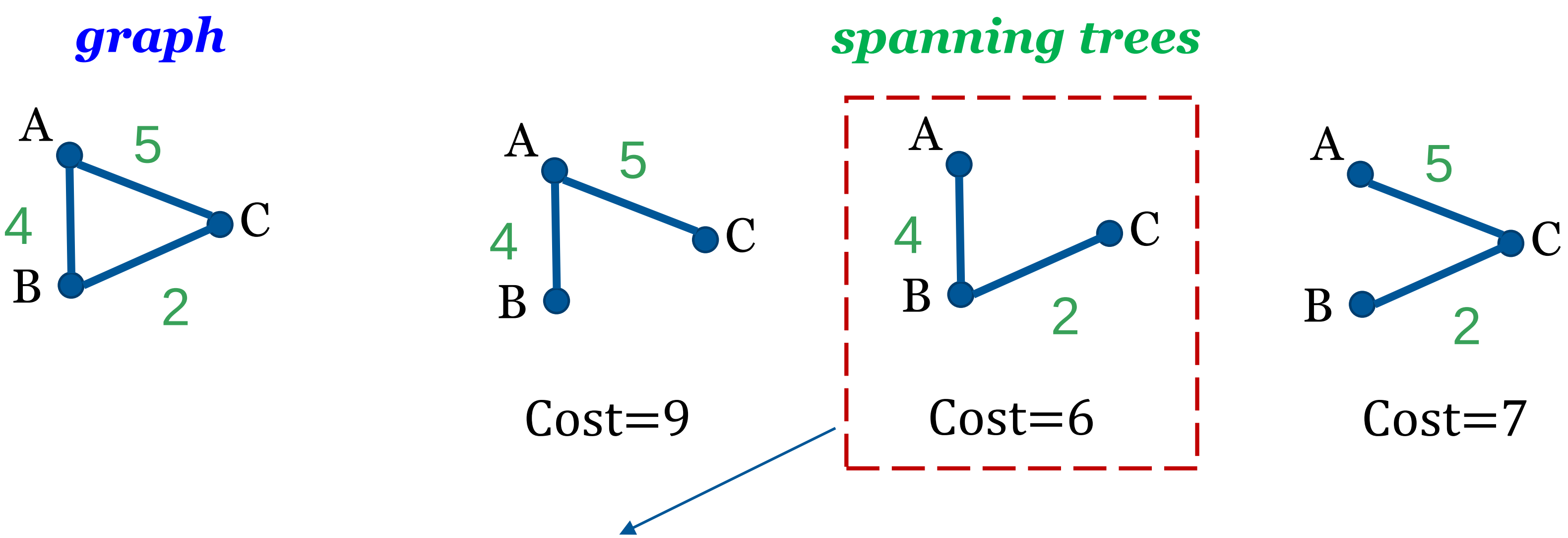
Number of spanning tree

$= 10 - 2 = 8$

Exclude 2 cases with cycles!

Minimum spanning tree

Minimum spanning tree is a sub graph of an undirected graph such that the subgraph spans(includes) all nodes, is connected, is acyclic, and has total minimum edge weight.



Minimum spanning tree (MST)

Minimum cost = 6

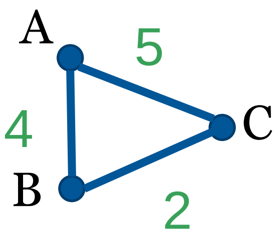
Minimum spanning tree

For a weighted graph, we are interested in finding its spanning tree with minimum sum of its weighted edges.

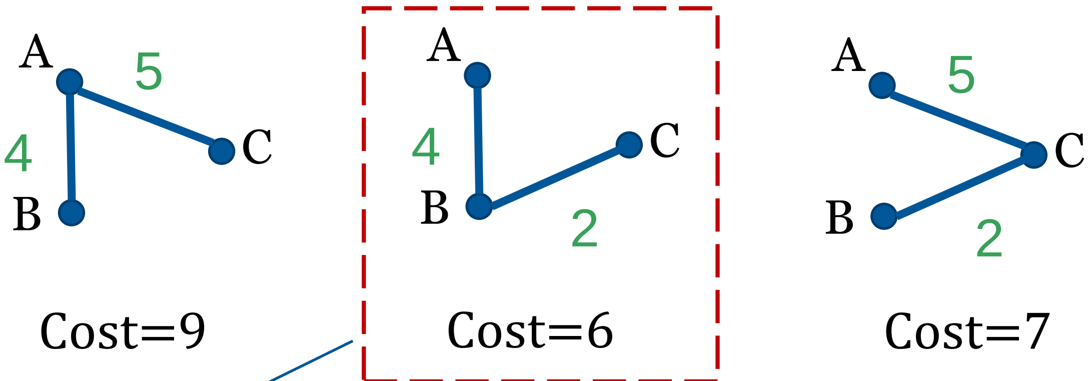
Such spanning tree is called minimum spanning tree of the graph.

This sum is called the minimum cost.

graph



spanning trees

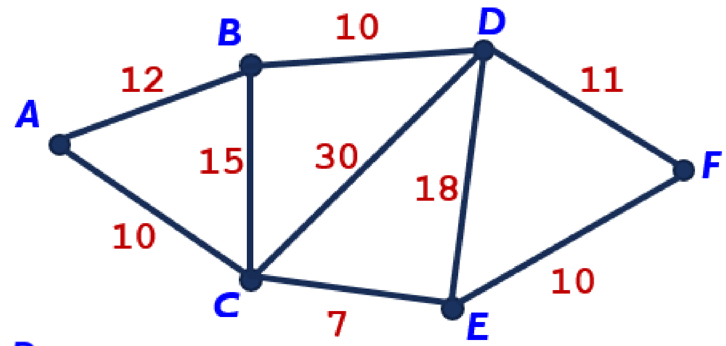


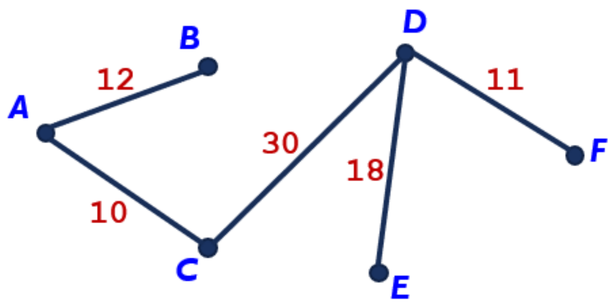
Minimum spanning tree (MST)

Minimum cost = 6

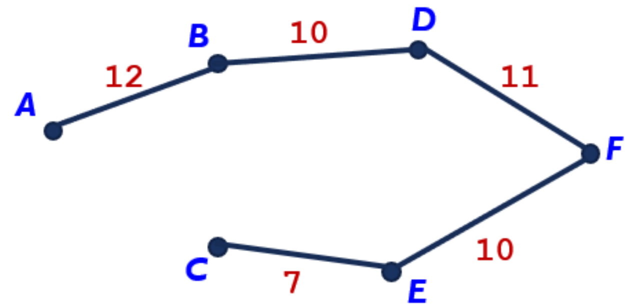
Example

If the weighted and connected graph is complex, it is not a simple task to identify the minimum spanning tree and its minimum cost.





Cost=12+10+30+18+11



Cost=12+10+11+10+7

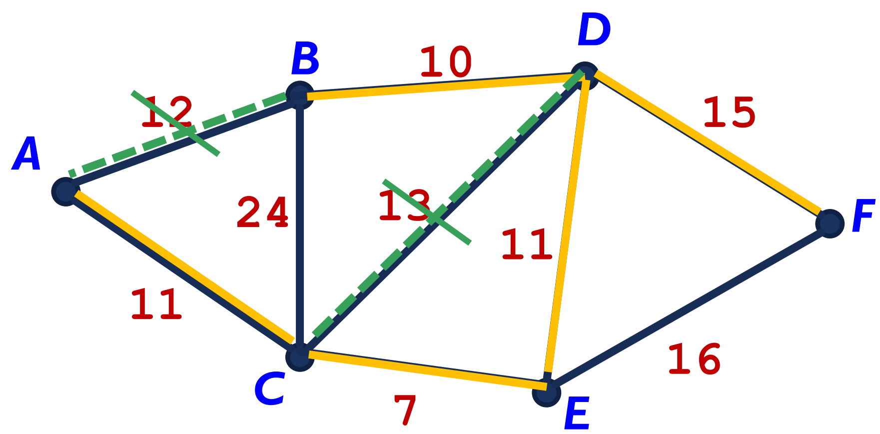
…

We can apply two classical algorithms for solving such a task:

Kruskal's algorithm Prim's algorithm

Kruskal's algorithm

$$|E_s| = |V| - 1 = 6 - 1 = 5$$



Edge selected

CE

BD

AC (same weight as DE, choose either first)

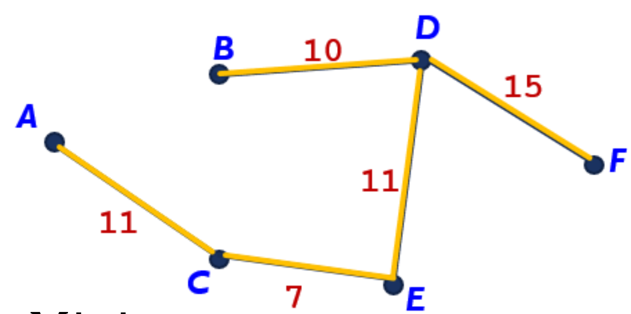
DE

DF (avoid making a cycle)

5 edges selected, done.

- At each step, select the edge with smallest weight.
- Repeat this process. Avoid forming cycles.
- Stop when all vertices are connected.

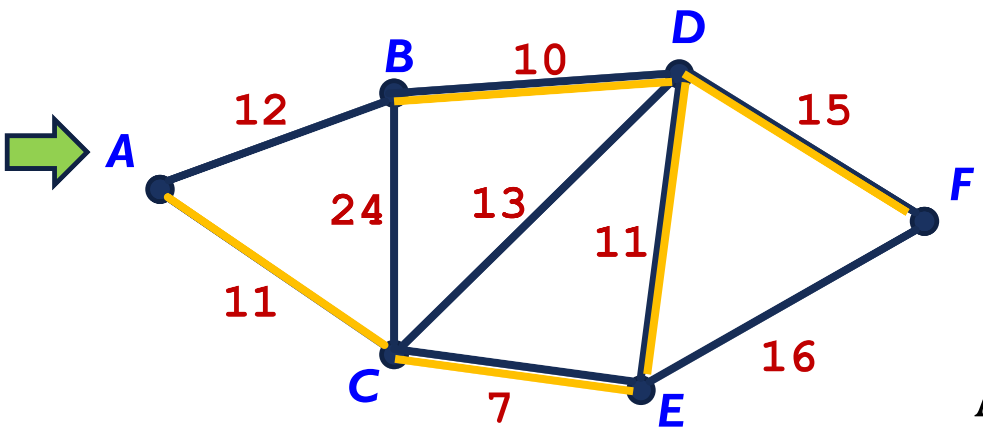
Minimum spanning tree:



Minimum cost

$$7 + 10 + 11 + 11 + 15 = 54$$

Prim's algorithm

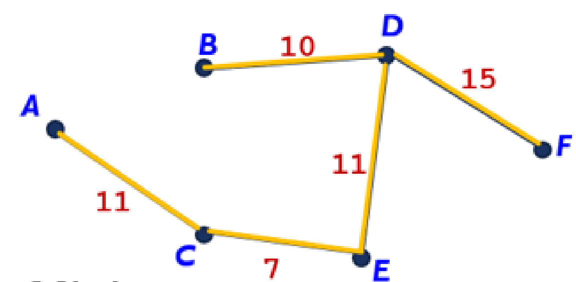


Visited	Unvisited	Edge selected
A	B,C,D,E,F	AC
A,C	B,D,E,F	CE
A,C,E	B,D,F	ED
A,C,E,D	B,F	DB
A,C,E,D,B	F	DF
A,C,E,D,B,F		

5 edges selected, done.

- Start with any vertex (mark as visited).
- Choose the edge with smallest weight connecting visited and unvisited vertices.
- Mark the connected vertex as visited, repeat the previous step.
- Stop when all vertices are connected.

Minimum spanning tree:



Minimum cost

$$7 + 10 + 11 + 11 + 15 = 54$$

Shortest path algorithms (summary)

Dijkstra's algorithm | Kruskal's/Prim's algorithms

They are all greedy algorithms on weighted graphs.

The optimized solution may not be unique.

Dijkstra's algorithm	Kruskal's/Prim's algorithms
To find shortest paths among vertices.	To find the minimum spanning tree.
Work on undirected graph and directed graph.	Work on undirected graph.

Review

Question1: Which of the following graphs has an Euler Tour?

(a) A graph with vertices of degrees 2, 2, 2, 2

(b) A graph with vertices of degrees 3, 3, 3, 3

(c) A graph with vertices of degrees 4, 4, 4, 4

(d) A graph with vertices of degrees 1, 1, 1, 1

Question 2: What is the primary purpose of a Minimum Spanning Tree (MST)?

(a) To find the longest path in a graph

(b) To connect all vertices with the minimum total edge weight

(c) To find the shortest path between two vertices

(d) To detect cycles in a graph