Science A Physics

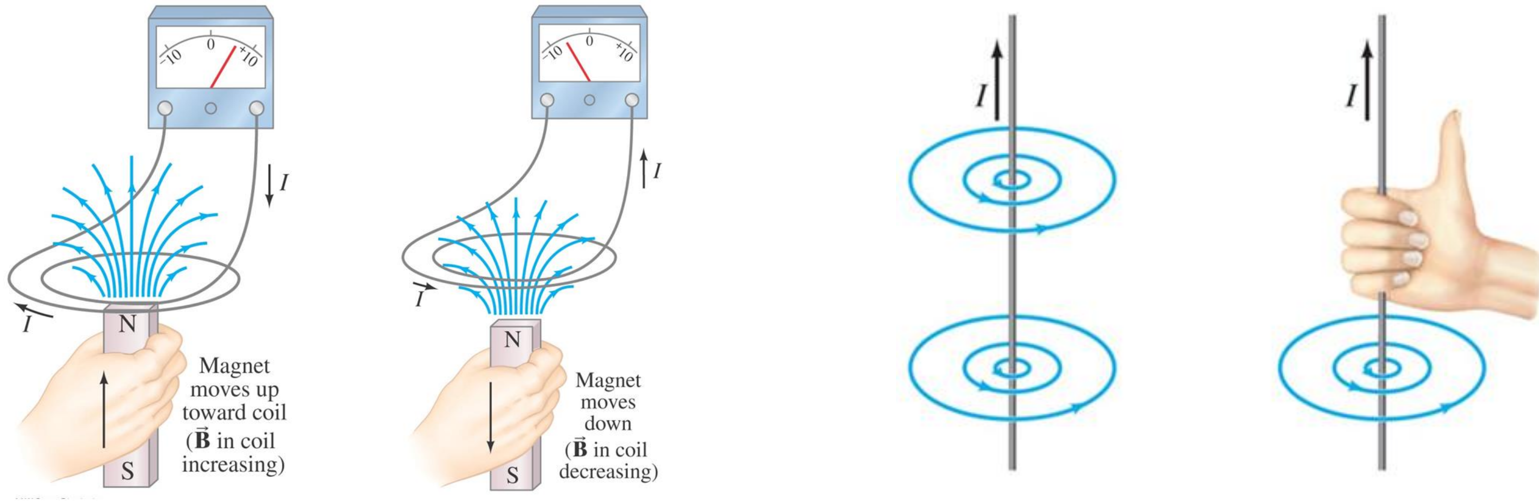
Lecture 19:

Inductance, and LR Circuits

Aims of today's lecture

1. Mutual Inductance
2. Self-Inductance
3. Energy Stored in a Magnetic Field
4. LR Circuits
5. LC Circuits and Electromagnetic Oscillations
6. LC Oscillations with Resistance (an *LRC* Circuit)

Key Ideas So Far

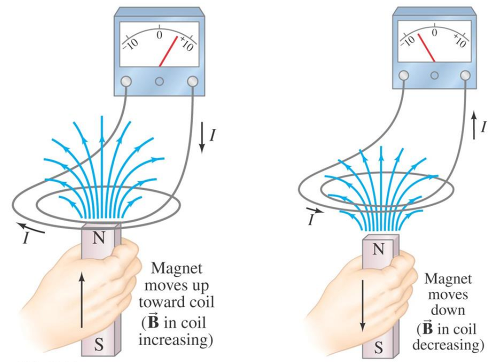


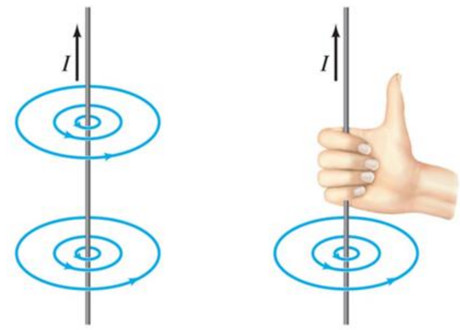
N.B. A changing magnetic flux through a circuit induces an emf in that circuit, which can produce an electric current.

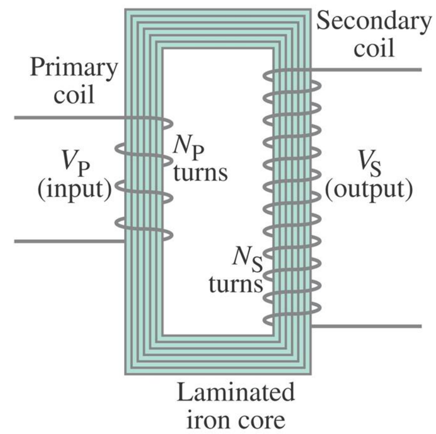
N.B. An electric current produces a magnetic field.

N.B. Combining these two ideas, we would predict that a changing current in one circuit (circuit 1) ought to induce an emf and a current in a second nearby circuit (circuit 2), which then induces an emf in circuit 1.

Key Ideas So Far





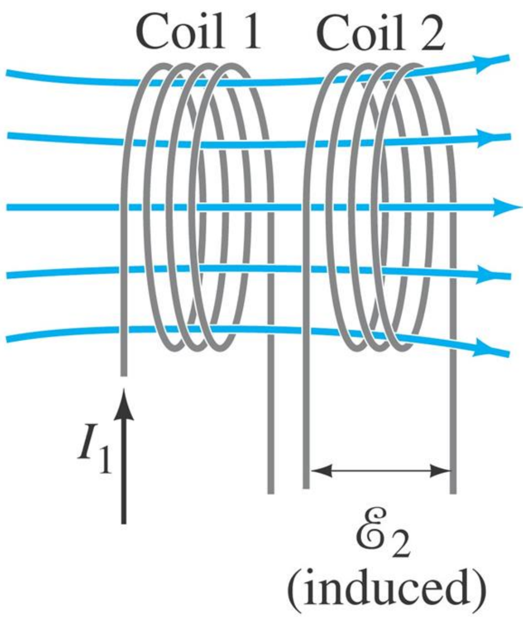


N.B. One example is a transformer. We call this effect **mutual inductance** (when two coils are involved), and, as we'll see later, **self-inductance** when only one coil is involved.

N.B. The concept of inductance also gives us a grounding to understand energy storage in a magnetic field as well as circuits that contain inductance, resistance and/or capacitance.

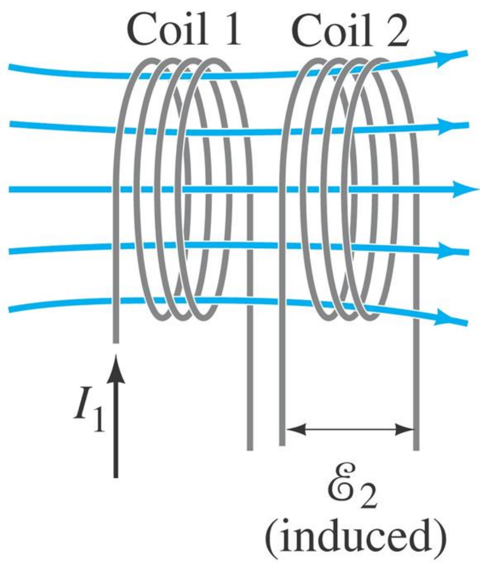
1. Mutual Inductance

Mutual Inductance



- If two coils of wire are placed near each other, as shown above, a changing current in one will induce an emf in the other.
- According to Faraday's law, the emf, $\mathcal{E}$ induced in coil 2 is proportional to the rate of change of magnetic flux passing through it.
- The flux is due to the current I_1 in coil 1, and it is often convenient to express the emf in coil 2 in terms of the current in coil 1.
- We let Φ_{21} be the magnetic flux in each loop of coil 2 created by the current in coil 1.

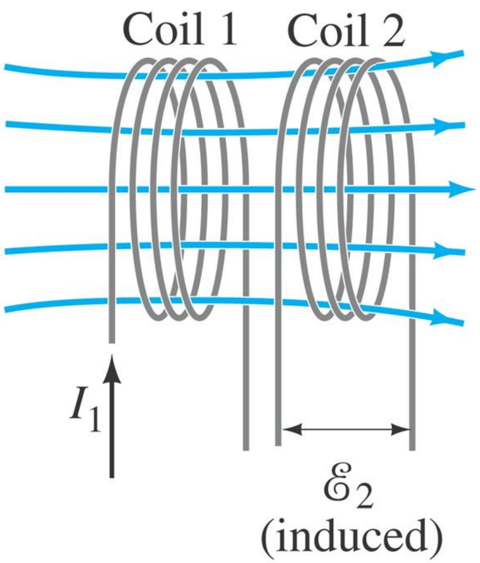
Mutual Inductance



- If coil 2 contains N_2 closely wrapped loops, then $N_2\Phi_{21}$is the total flux passing through coil 2.
- If the two coils are fixed in space, $N_2\Phi_{21}$ is proportional to the current I_1 in coil 1: the proportionality constant is called the mutual inductance, M_{21}, defined by

$$M_{21} = \frac{N_2\Phi_{21}}{I_1}$$

Mutual Inductance



$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}$$

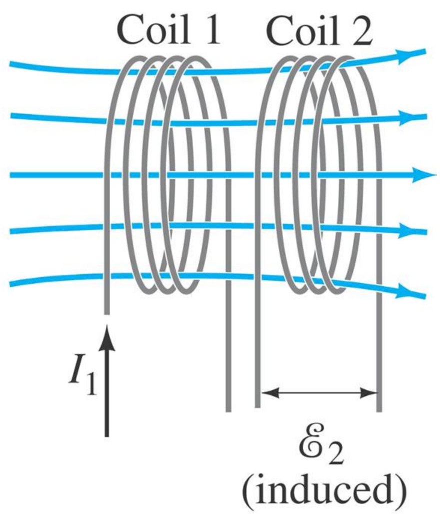
- The emf ε_2 induced in coil 2 due to a changing current in coil 1 is, by Faraday's law,

$$\varepsilon_2 = -N_2 \frac{d\Phi_{21}}{dt}$$

- We can combine the above two equations, writing $\Phi_{21} = M_{21} I_1 / N_2$, to obtain

$$\varepsilon_2 = -M_{21} \frac{dI_1}{dt}$$

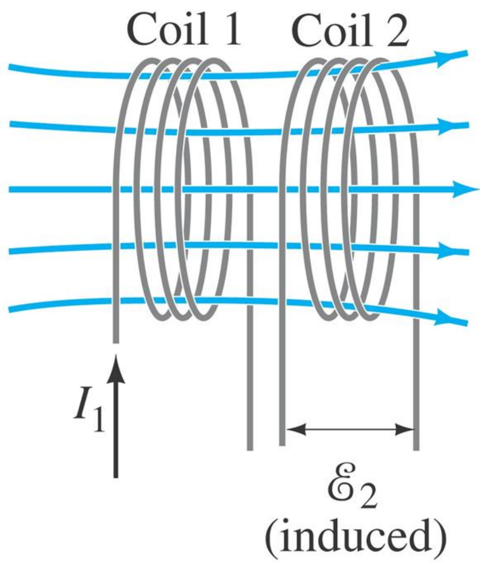
Mutual Inductance



$$\varepsilon_2 = -M_{21}\frac{dI_1}{dt}$$

- This relates the change in current in coil 1 to the emf it induces in coil 2.
- The mutual inductance of coil 2 with respect to coil 1, M_{21}, is a 'constant' in that it does not depend on I_1;
- M_{21} depends on 'geometric' factors such as the size, shape, number of turns, and relative positions of the two coils, and also on whether iron (or some other ferromagnetic material) is present.
- For some arrangements, the mutual inductance can be calculated; more often though, it is determined experimentally.

Mutual Inductance



- We can also consider the reverse situation: when a changing current in coil 2 induces an emf in coil 1. In this case,

$$\varepsilon_1 = -M_{12}\frac{dI_2}{dt}$$

where M_{12} is the mutual inductance of coil 1 with respect to coil 2.

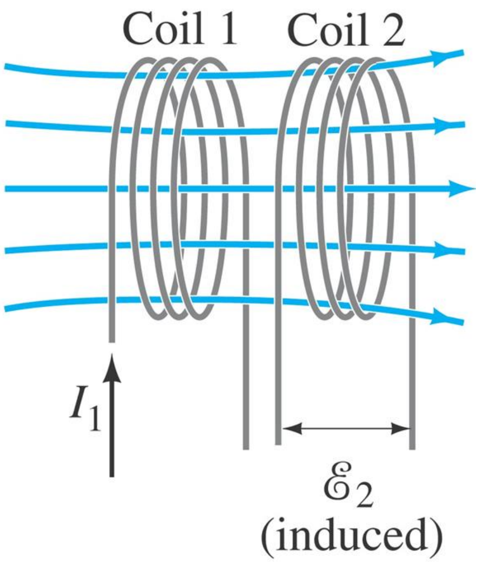
- Hence, for a given arrangement, we do not need the subscripts and we can let

$$M = M_{12} = M_{21}$$

So that $\varepsilon_1 = -M\frac{dI_2}{dt}$ and $\varepsilon_2 = -M\frac{dI_1}{dt}$

Mutual Inductance

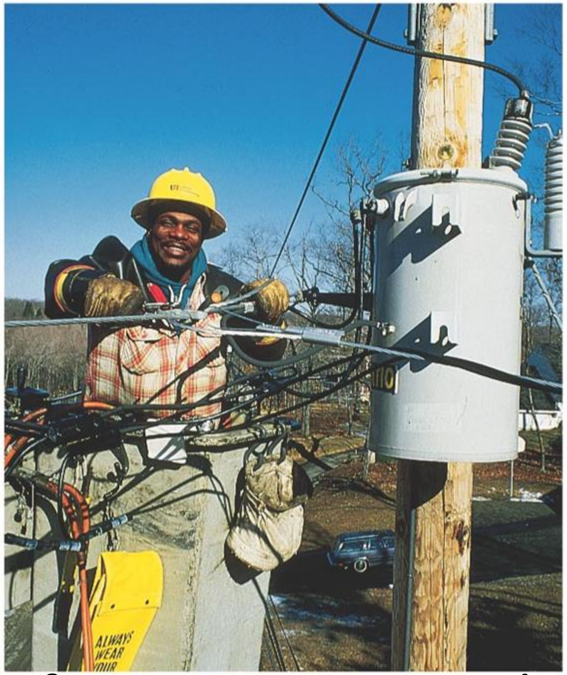
Henry (1797-1878)

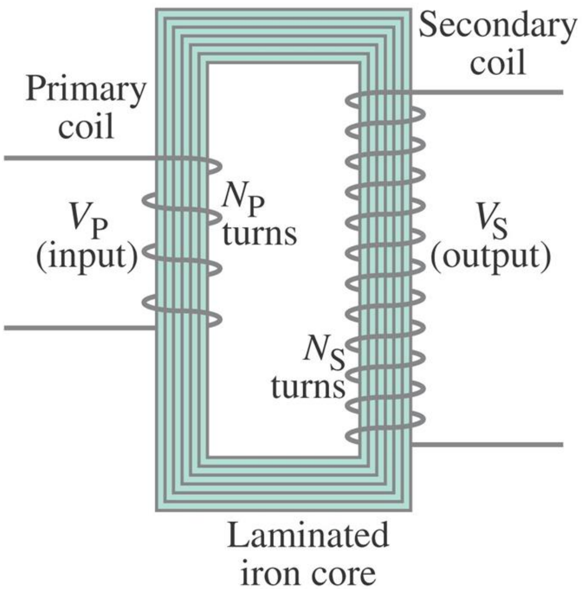


- The SI unit for mutual inductance is the henry (H), where

$$1\mathrm{H} = 1\mathrm{V} \cdot \frac{\mathrm{s}}{\mathrm{A}} = 1\Omega \cdot \mathrm{s}$$

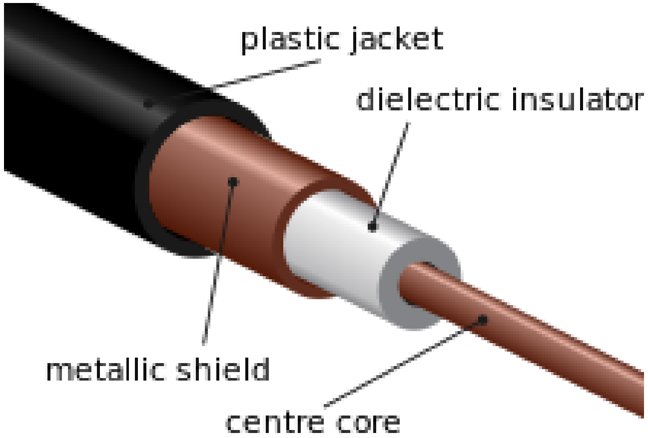
Mutual Inductance





- A transformer is an example of mutual inductance in which the coupling is maximised so that nearly all flux lines pass through both coils.
- Mutual inductance can sometimes be a problem, however.
- Any changing current in a circuit can induce an *emf* in another part of the same circuit or in a different circuit even though the conductors are not in the shape of a coil.
- The mutual inductance M is usually small though unless coils with many turns and/or iron cores are involved.

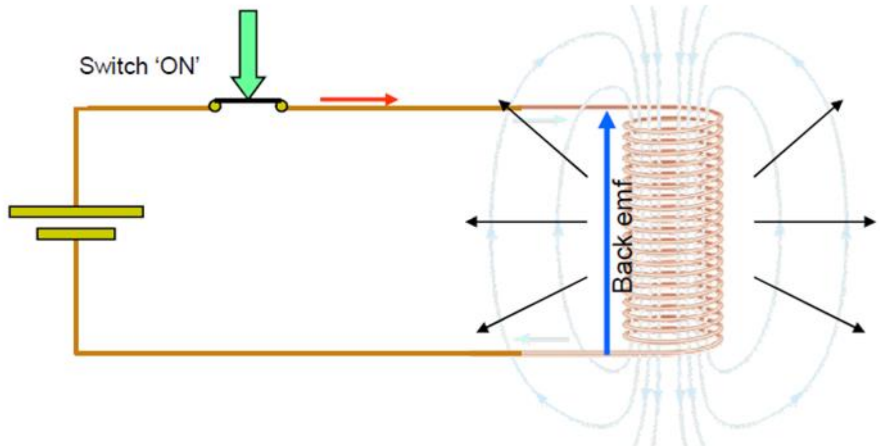
Mutual Inductance



- In situations where small voltages are being used, problems due to mutual inductance often arise.
- Shielded cable, in which an inner conductor is surrounded by a cylindrical grounded conductor, is often used to reduce this problem.

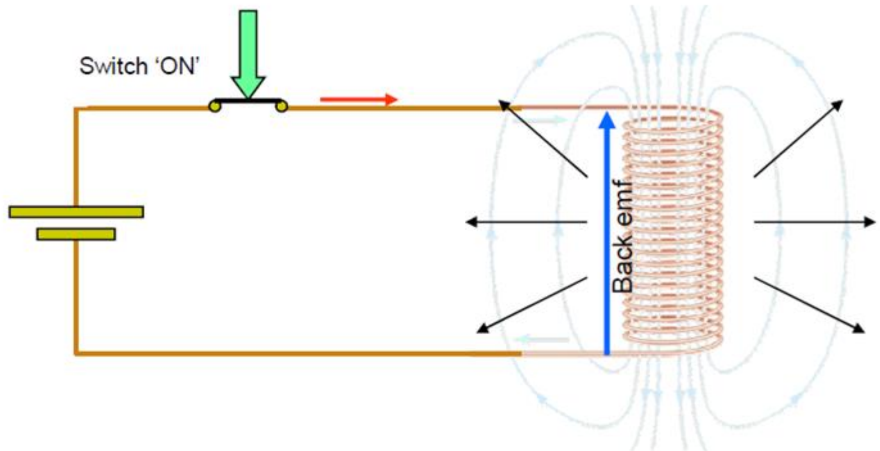
2. Self-Inductance

Self-Inductance



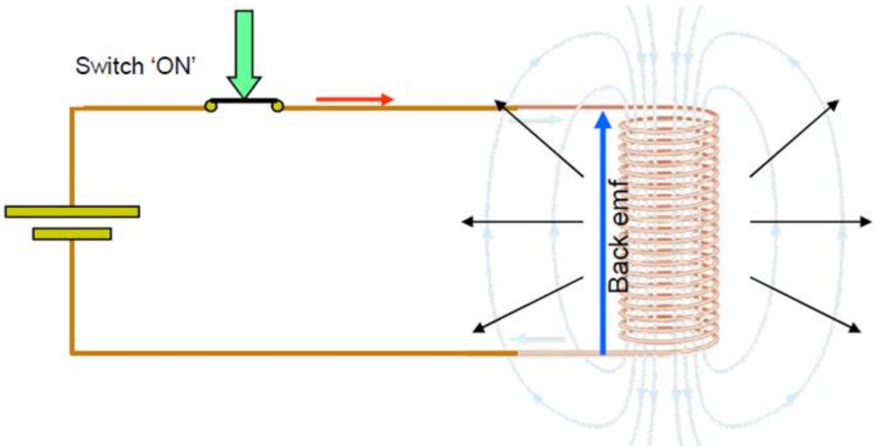
- The idea of inductance also applies to a single isolated coil of N turns. Because only one coil is involved, we call it **self-inductance**.
- When a changing current passes through a coil (or solenoid), a changing magnetic flux is produced inside the coil, and this in turn induces what we call a **back emf** in that same coil.
- This induced **back emf** opposes the change in flux (Lenz's law).

Self-Inductance



- For example, if the current through the coil is increasing, the increasing magnetic flux induces an emf that opposes the original current and tends to retard its increase.
- If the current is decreasing in the coil, the decreasing flux induces an emf in the same direction as the current, thus tending to maintain the original current.

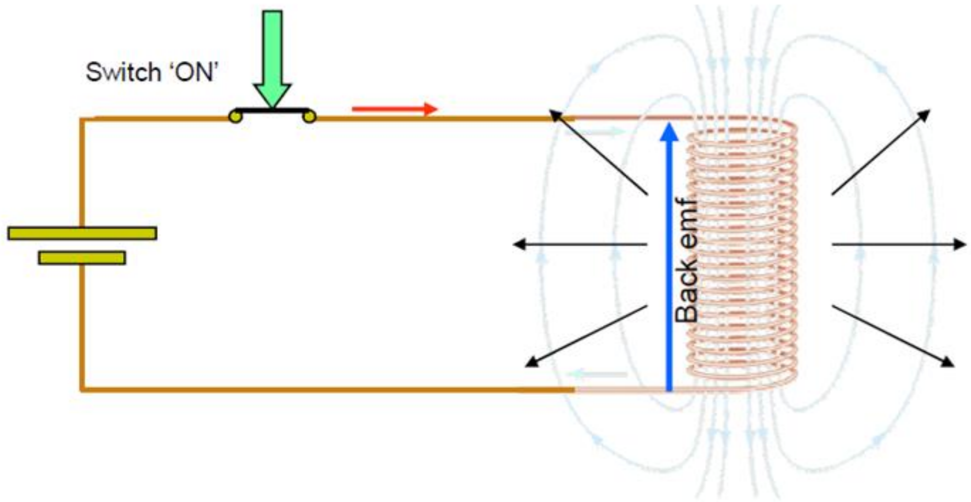
Self-Inductance



- The magnetic flux Φ_B passing through the N turns of a coil is proportional to the current I in the coil, so we define the **self-inductance** L (in analogy to mutual inductance) as

$$L = \frac{N\Phi_B}{I}$$

Self-Inductance

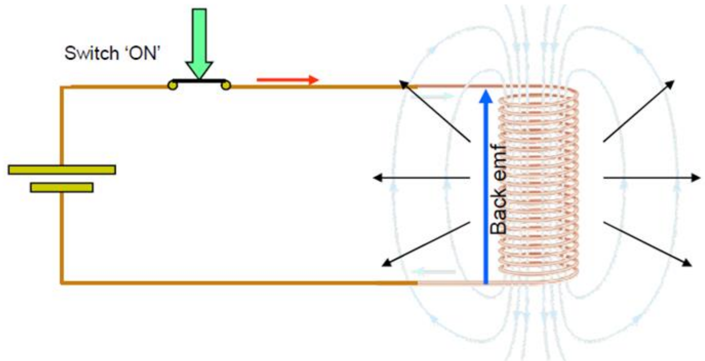


- Then the emf ε induced in a coil of self-inductance L is, from Faraday's law,

$$\varepsilon = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

- Like mutual inductance, self-inductance is measured in Henrys.
- The magnitude of L depends on the geometry and on the presence of a ferromagnetic material.
- Self-inductance (inductance, for short) can be defined, as above, for any circuit or part of a circuit.

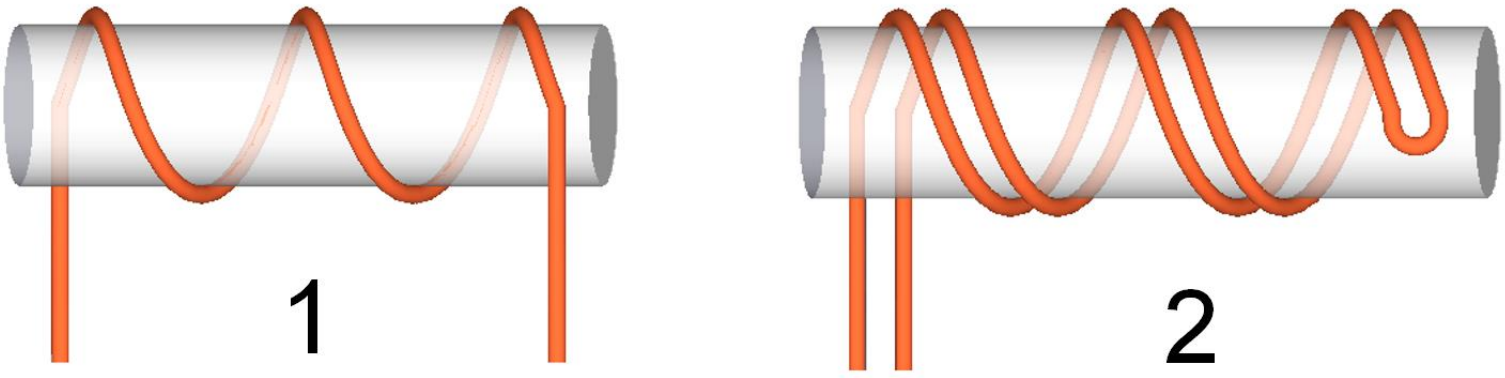
Self-Inductance



$$\varepsilon = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

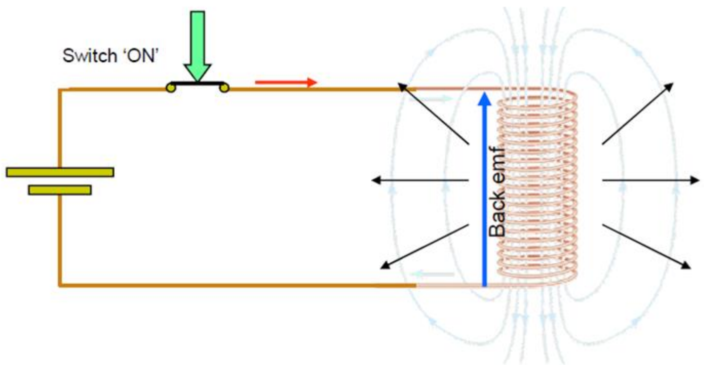
- Circuits always contain some inductance, but often it is quite small unless the circuit contains a coil of many turns.
- A coil that has significant self-inductance L is called an inductor.
- Inductance is shown on circuit diagrams by the symbol
- Any resistance that an inductor has should also be shown separately.

Self-Inductance



- Inductance can both be useful and non-useful in circuits.
- Precision resistors are normally wire wound and thus would have inductance as well as resistance.
- The inductance can be minimised by winding the insulated wire back on itself in the opposite sense so that the current going in opposite directions produces little net magnetic flux; this is called **non-inductive winding**.

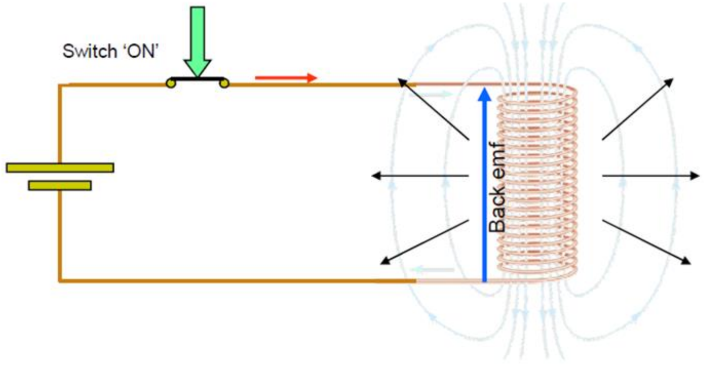
Self-Inductance



$$\varepsilon = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

- If an inductor has negligible resistance, it is the inductance (or induced emf) that controls a changing current.
- If a source of changing or alternating voltage is applied to the coil, this applied voltage will just be balanced by the induced emf of the coil.

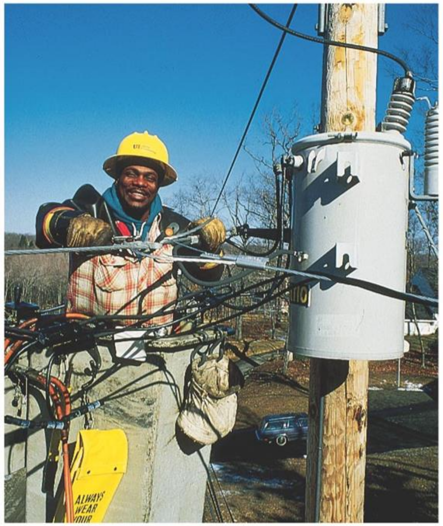
Self-Inductance

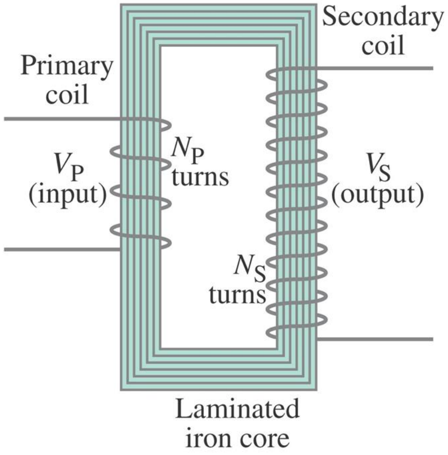


$$\varepsilon = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

- We can see from the above equation that for a given emf, if the inductance L is large, the change in the current will be small, and therefore the current itself, if it is ac, will be small.
- The greater the inductance, the less the ac current.
- An inductance thus acts something like a resistance to impede the flow of alternating current. We use the term **reactance** or **impedance** for this quality of an inductor.

Self-Inductance





- The resistance of the primary in a transformer is usually quite small, perhaps less than 1Ω.
- If resistance alone limited the current in a transformer, tremendous currents would flow when a high voltage was applied. Indeed a dc voltage applied to a transformer can burn it out.
- It is the induced emf (or reactance) of the coil that limits the current to a reasonable value.

3. Energy Stored in a Magnetic Field

Energy Stored in a Magnetic Field

- When an inductor of inductance L is carrying a current I which is changing at rate dI/dt, energy is being supplied to the inductor at a rate

$$P = I\varepsilon = LI\frac{dI}{dt}$$

where P stands for power.

Energy Stored in a Magnetic Field

- Let us calculate the work needed to increase the current in an inductor from zero to some value I.
- The work dW done in a time dt is

$$dW = Pdt = LIdI,$$

- The total work done to increase the current from zero to I is

$$W = \int dW = \int_{0}^{I} LIdI = \frac{1}{2}LI^2$$

Energy Stored in a Magnetic Field

- The work done is equal to the energy U stored in the inductor when it is carrying a current I (and we take $U = 0$ when $I = 0$):

$$U = \frac{1}{2}LI^2$$

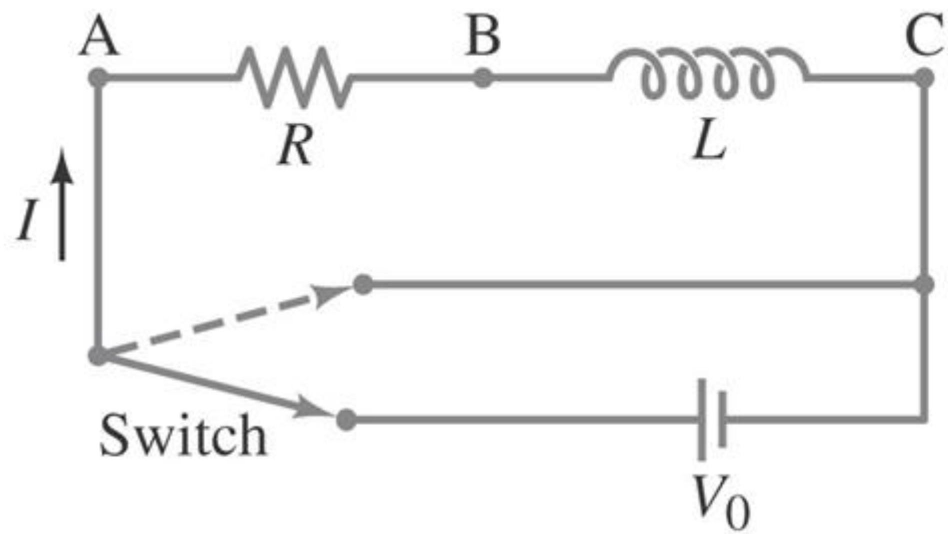
- The above can be compared to the energy stored in a capacitor, C, when the potential difference across it is V:

$$U = \frac{1}{2}CV^2$$

- Just as the energy stored in a capacitor can be considered to reside in the electric field between its plates, so the energy in an inductor can be considered to be stored in its magnetic field.

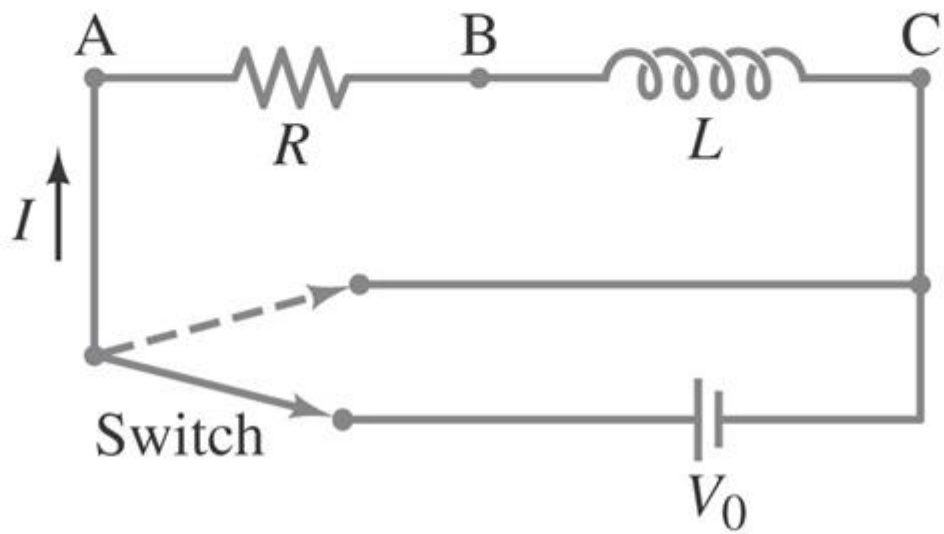
4. LR Circuits

LR Circuits



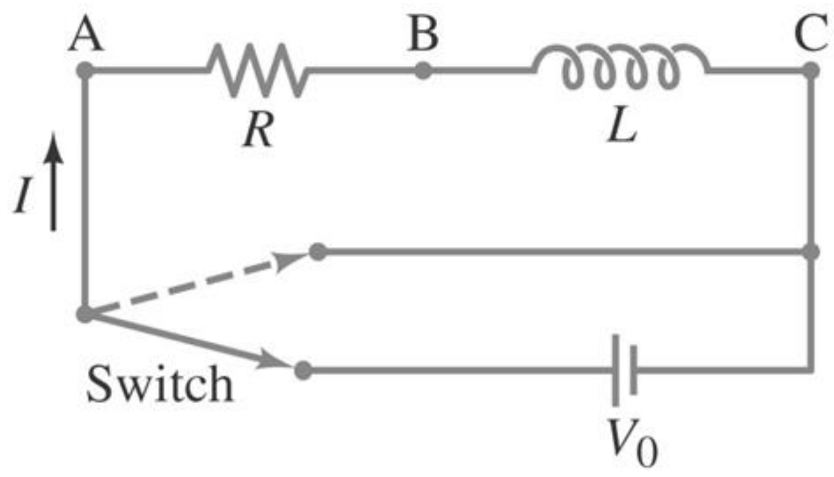
- Any inductor will have some resistance; we can represent this situation by drawing its inductance L and its resistance R separately, as shown above.
- The resistance R could also include any other resistance present in the circuit.
- Now we can ask, what happens when a battery or other source of dc voltage V_0 is connected in series to such an LR circuit?

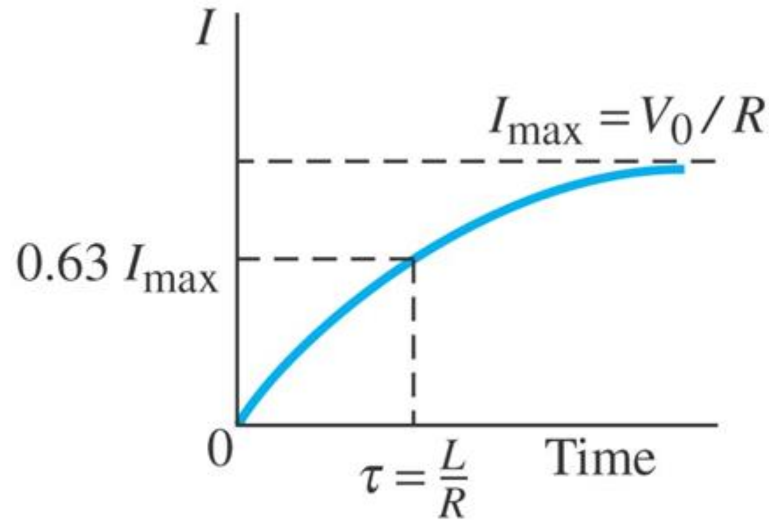
LR Circuits



- At the instant the switch connecting the battery is closed, the current starts to flow.
- It is opposed by the induced emf in the inductor which means point B in the above is positive relative to point C.
- However, as soon as current starts to flow, there is also a voltage drop of magnitude IR across the resistance.

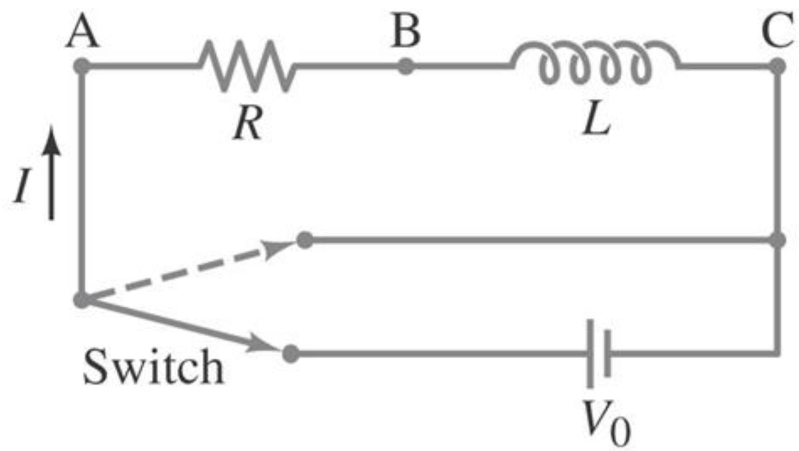
LR Circuits





- Hence, the voltage applied across the inductance is reduced and the current increases less rapidly.
- The current thus rises gradually as shown in the above, and approaches the steady value $I_{max} = V_0/R_0$ for which all the voltage drop is across the resistance.

LR Circuits

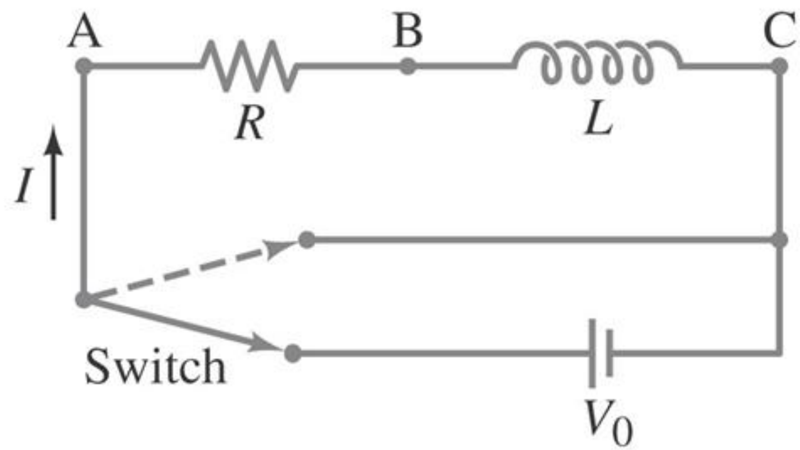


- We can also apply Kirchoff's loop rule to the above circuit.
- The $emfs$ in the circuit are the battery voltage V_0 and the emf $\varepsilon = -L\left(\frac{dI}{dt}\right)$ in the inductor opposing the increasing current.
- Hence, the sum of the potential changes around the loop is

$$V_0 - IR - L\frac{dI}{dt} = 0$$

where I is the current in the circuit at any instant.

LR Circuits



$$V_0 - IR - L\frac{dI}{dt} = 0$$

We can arrange the above to obtain

$$L\frac{dI}{dt} + RI = V_0$$

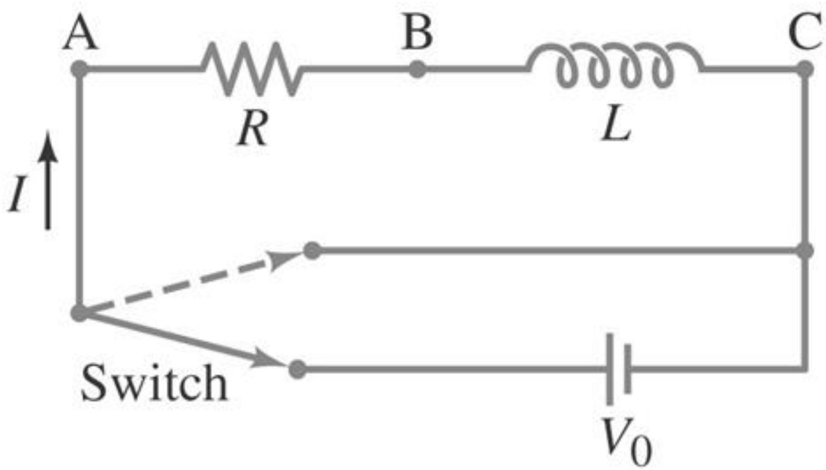
We can re-arrange the above further, and integrate it,

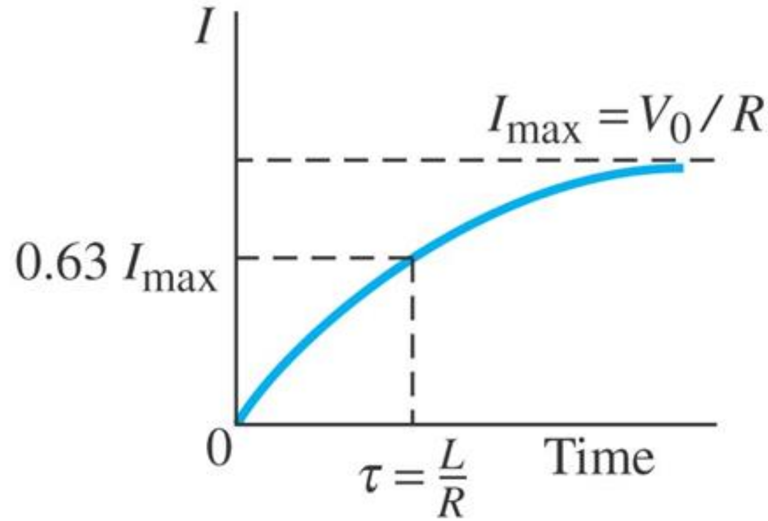
$$\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{L}$$

Then

$$-\frac{1}{R} ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$$

LR Circuits





$$\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{L}$$

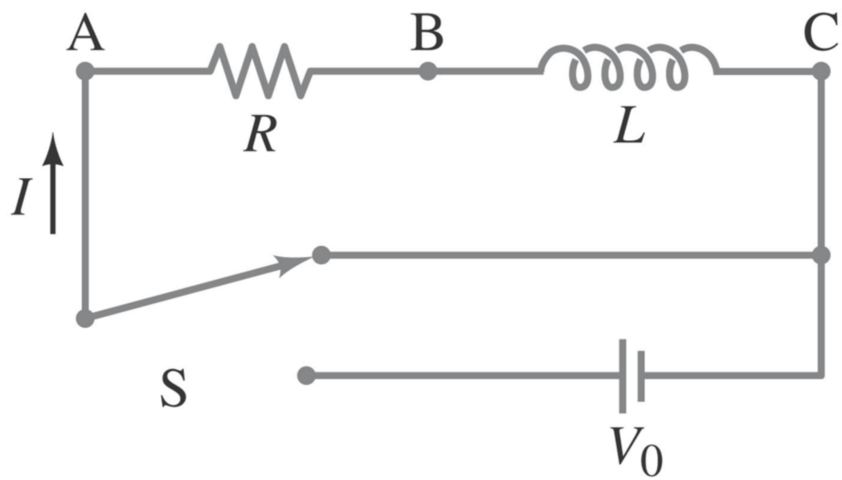
or

$$I = \frac{V_0}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

where $\tau = \frac{L}{R}$ is the time constant of the *LR* circuit.

The τ symbol represents the time required for the current I to reach $\left(1 - \frac{1}{e}\right) = 0.63$ or 63% of its maximum value (V_0/R).

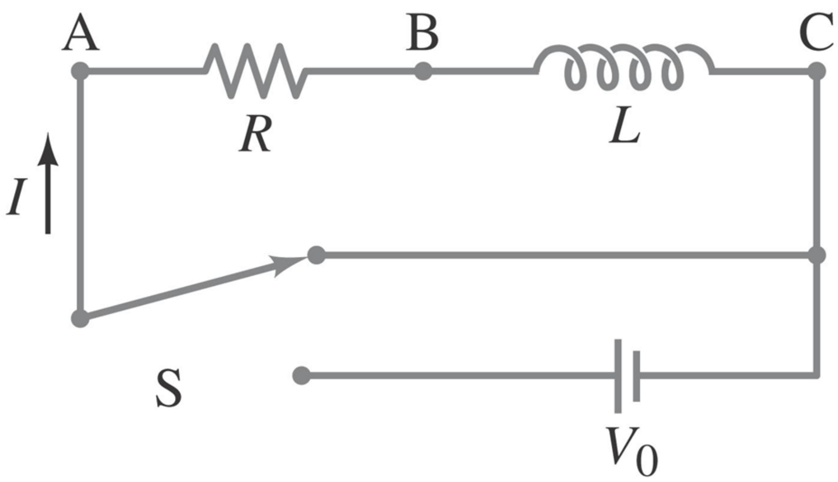
LR Circuits



- Now let's flip the switch in the above so that the battery is taken out of the circuit, and points A and C are connected together as shown.
- At this moment when the switching occurs, we can call it $t = 0$ and the current is I_0.
- Then the differential equation becomes (since $V_0 = 0$):

$$L\frac{dI}{dt} + RI = 0$$

LR Circuits



$$L\frac{dI}{dt} + RI = 0$$

- We can rearrange the above equation and integrate to obtain

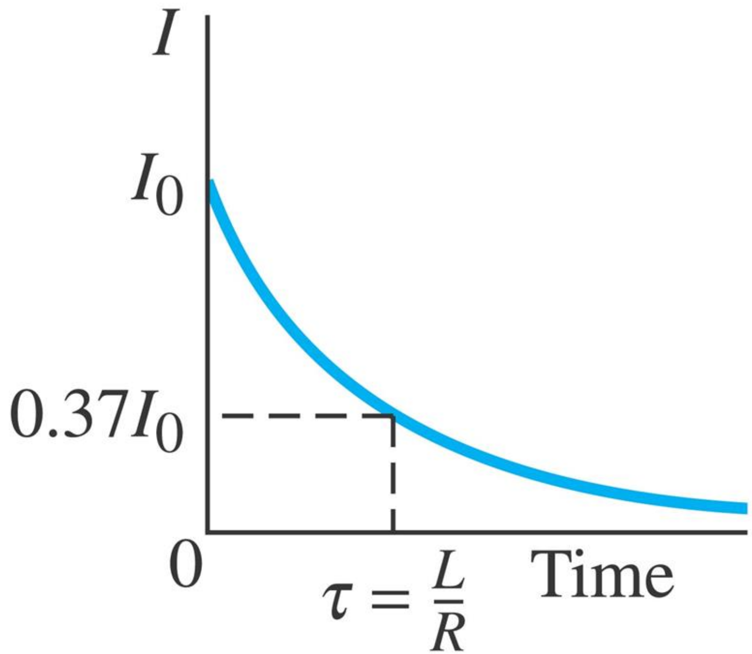
$$\int_{I_0}^{I} \frac{dI}{I} = -\int_0^t \frac{R}{L}\,dt$$

$$ln\frac{I}{I_0} = -\frac{R}{L}t$$

or

$$I = I_0 e^{-t/\tau}$$

LR Circuits

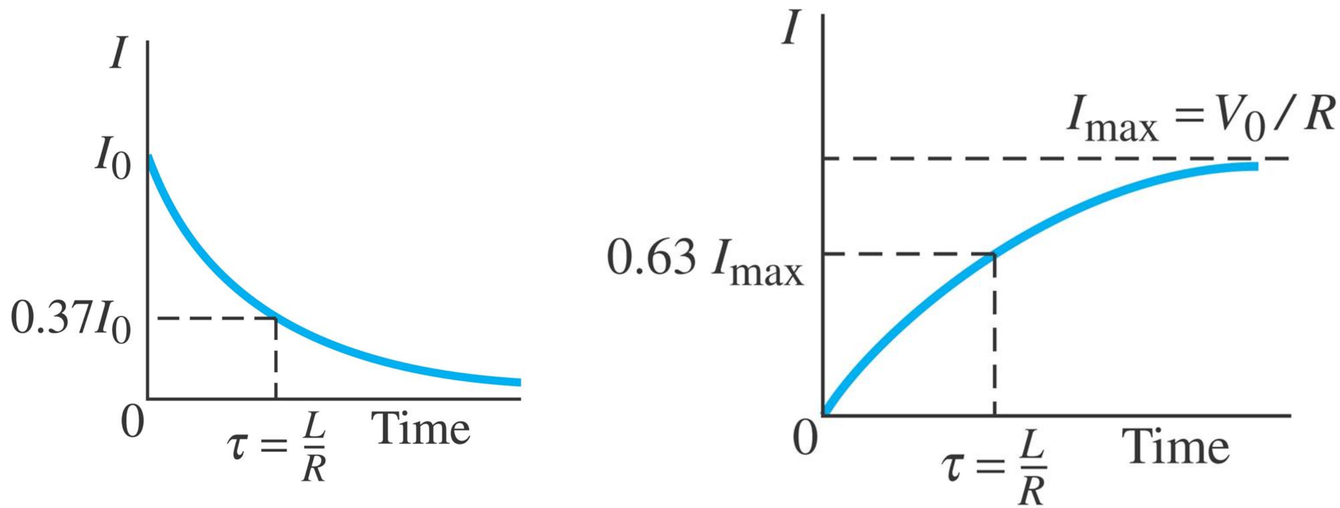


$$I = I_0 e^{-t/\tau}$$

Where again the time constant is $\tau = \frac{L}{R}$.

- The current decays exponentially to zero as shown above.

LR Circuits



To conclude:

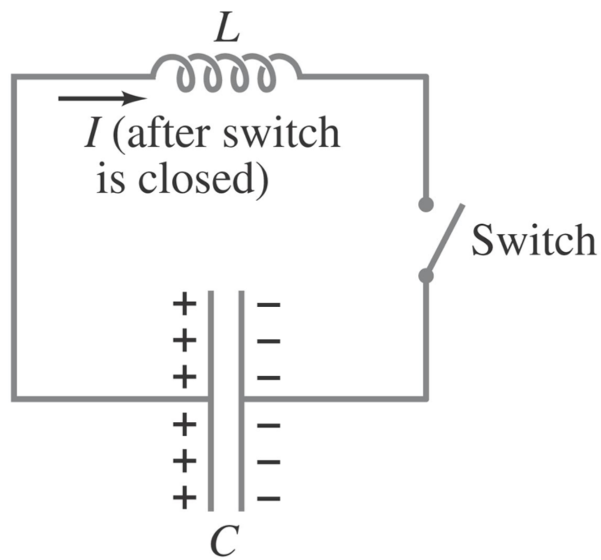
- Analysing LR circuits shows that for electromagnets, there is always some ‘reaction time’ when the circuit is turned on or off.
- We can also see that an LR circuit has properties similar to an RC circuit.
- Unlike the capacitor case, however, the time constant is inversely proportional to R.

Surge Protection

- An inductor can act as a 'surge protector' for sensitive electronic equipment that can be damaged by high currents.
- If equipment is plugged into a standard wall plug, a sudden 'surge', or increase, in voltage will normally cause a corresponding large change in current and damage the electronics.
- However, if there is an inductor in series with the voltage to the device, the sudden change in current produces an opposing emf preventing the current from reaching dangerous levels.

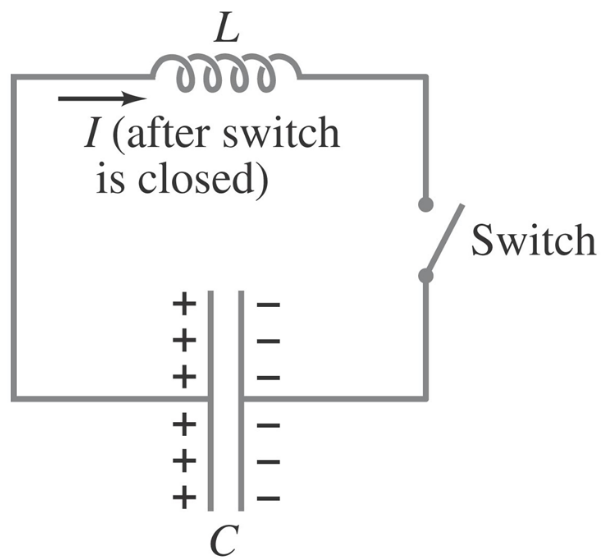
5. LC Circuits and Electromagnetic Oscillations

LC Circuits and Electromagnetic Oscillations



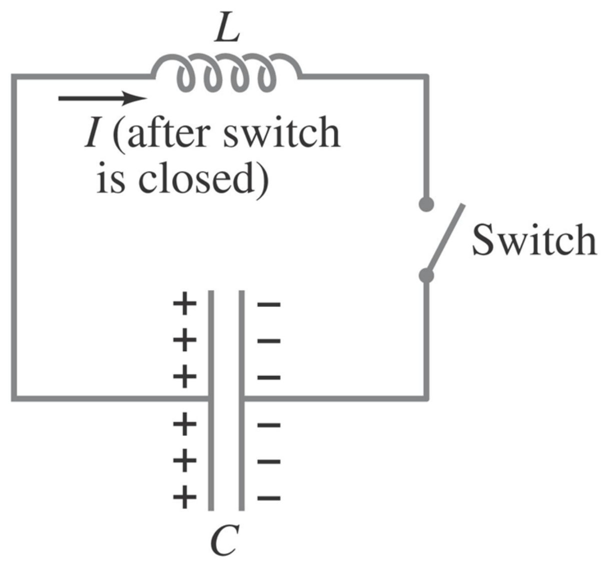
- In any electric circuit, there can be three basic components: resistance, capacitance, and inductance, in addition to a source of emf.
- There can also be more complex components, such as diodes or transistors, but we will not consider these components.
- We've previously discussed both RC and LR circuits. Now we are going to look at an LC circuit, one that contains only a capacitance C and an inductance, L.

LC Circuits and Electromagnetic Oscillations



- This LC circuit is an idealised circuit, one in which we assume there is no resistance.
- Let us suppose the capacitor in the above figure is initially charged so that one plate has charge Q_0 and the other plate has charge $-Q_0$, and the potential difference across it is $V = Q/C$.
- Suppose that at $t = 0$, the switch is closed.

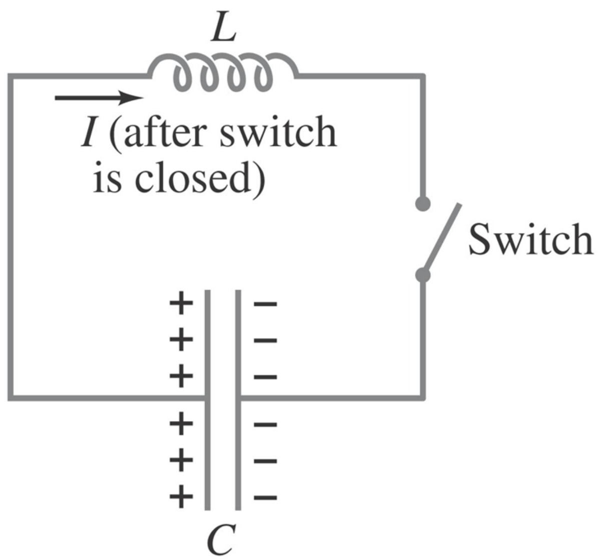
LC Circuits and Electromagnetic Oscillations



- The capacitor immediately begins to discharge. As it does so, the current I through the inductor increases.
- We can now apply Kirchhoff's loop rule (sum of potential changes around a loop is zero):

$$-L\frac{dI}{dt}+\frac{Q}{C}=0$$

LC Circuits and Electromagnetic Oscillations



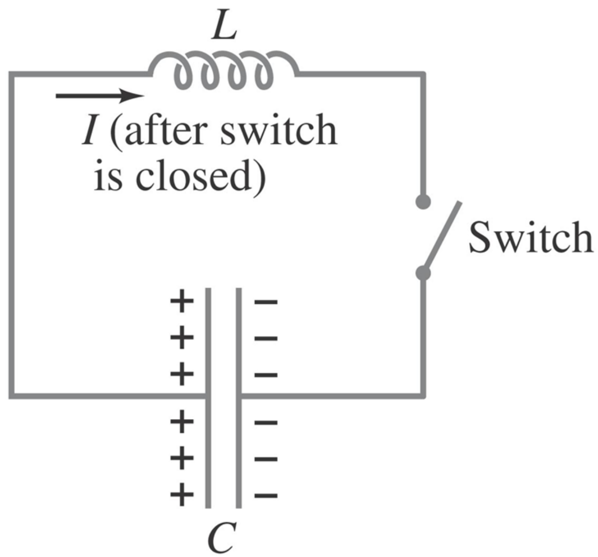
$$-L\frac{dI}{dt} + \frac{Q}{C} = 0$$

- Because charge leaves the positive plate on the capacitor to produce the current I as shown above, the charge Q on the (positive) plate of the capacitor is decreasing, so $I = -\frac{dQ}{dt}$.

- We can then rewrite the above equation as

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

LC Circuits and Electromagnetic Oscillations



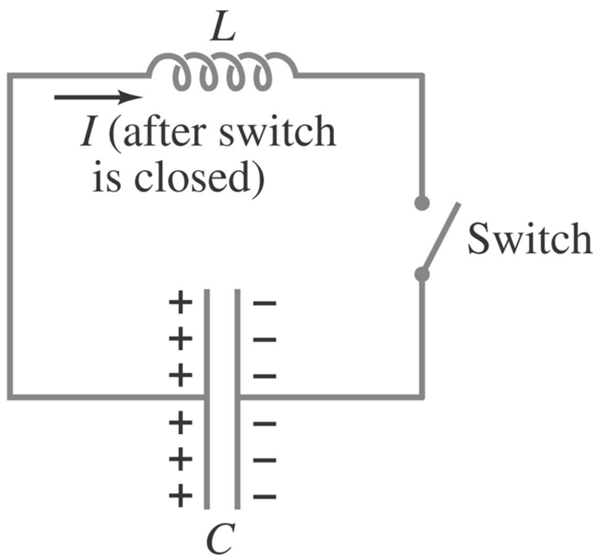
$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

- This is a familiar differential equation. It has the same form as the equation for simple harmonic motion.
- Using integration, we can show that the solution for the above equation can be written as

$$Q = Q_0 \cos(\omega t + \phi)$$

where Q_0 and ϕ are constants that depend on the initial conditions.

LC Circuits and Electromagnetic Oscillations

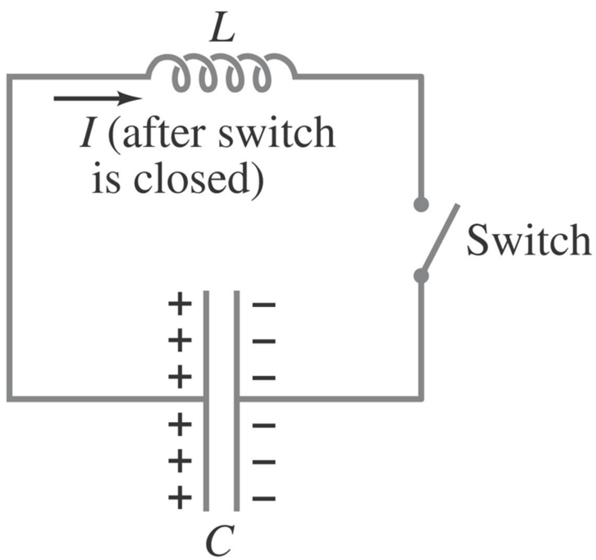


$$Q = Q_0 \cos(\omega t + \phi)$$

- We can insert the above equation into $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$, noting that $\frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi)$; thus

$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{1}{LC} Q_0 \cos(\omega t + \phi) = 0$$

LC Circuits and Electromagnetic Oscillations

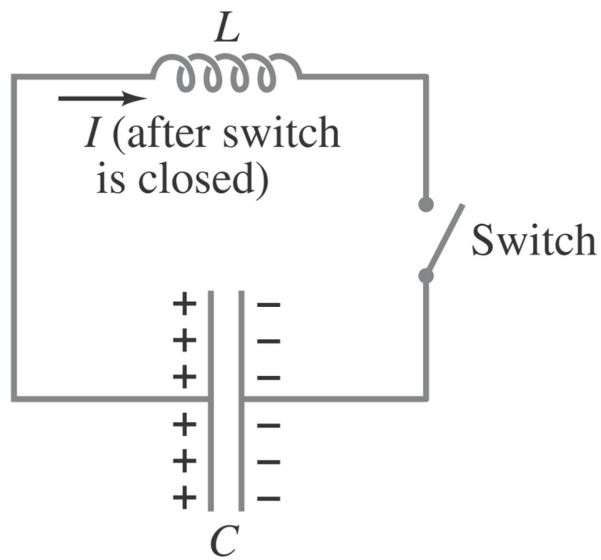


$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{1}{LC} Q_0 \cos(\omega t + \phi) = 0$$

or

$$\left(-\omega^2 + \frac{1}{LC}\right) \cos(\omega t + \phi) = 0$$

LC Circuits and Electromagnetic Oscillations

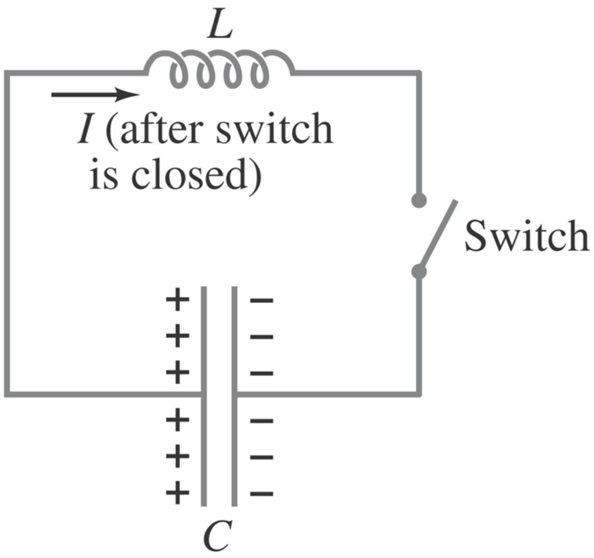


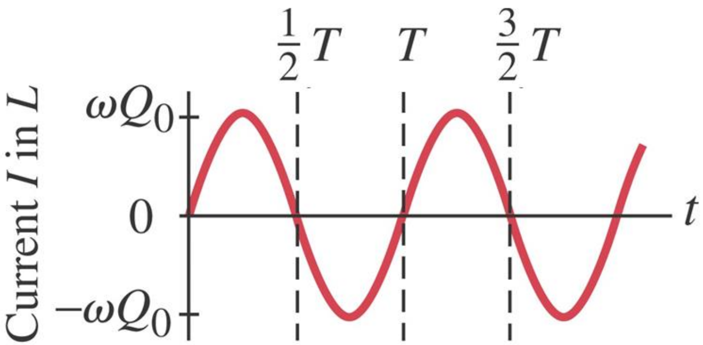
$$\left(-\omega^2 + \frac{1}{LC}\right)\cos(\omega t + \phi) = 0$$

- The above relation can be true for all times t only if $(-\omega^2 + 1/LC) = 0$, which tells us that

$$\omega = 2\pi f = \sqrt{\frac{1}{LC}}$$

LC Circuits and Electromagnetic Oscillations



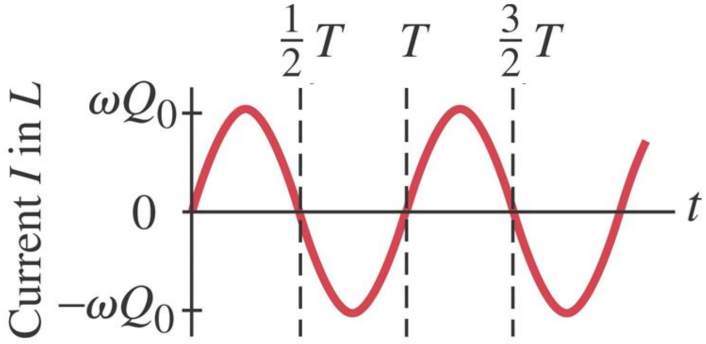


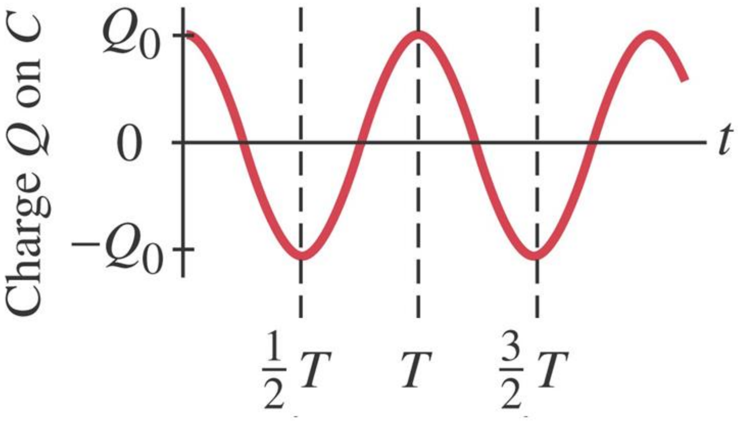
$$Q = Q_0 \cos(\omega t + \phi)$$

- The above equation shows that the charge on the capacitor in an LC circuit oscillates sinusoidally. The current in the inductor is given by the following, and is also sinusoidal.

$$I = -\frac{dQ}{dt} = \omega Q_0 \sin(\omega t + \phi)$$

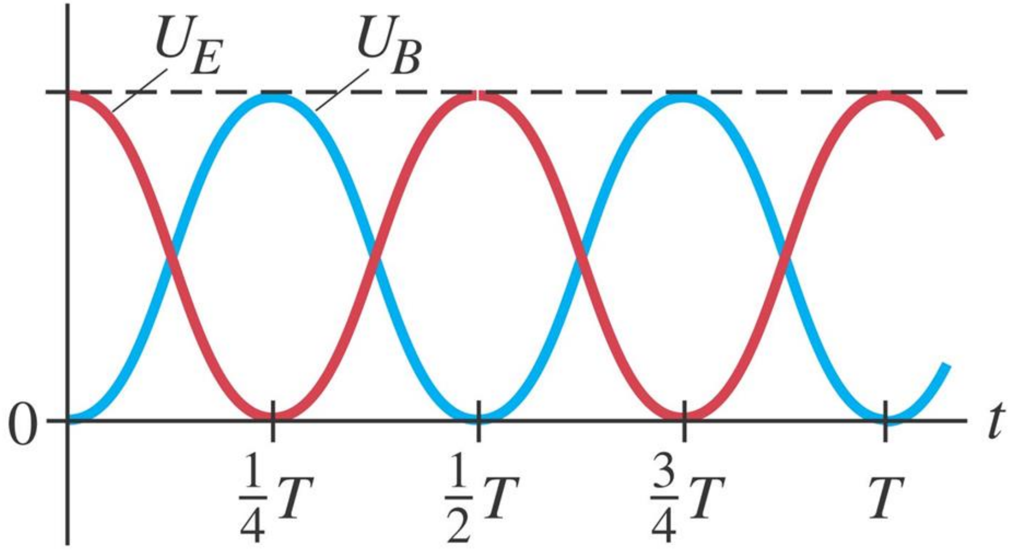
LC Circuits and Electromagnetic Oscillations





N.B. The above two graphs are for Q and I when $\phi = 0$.

LC Oscillations from the Point of View of Energy



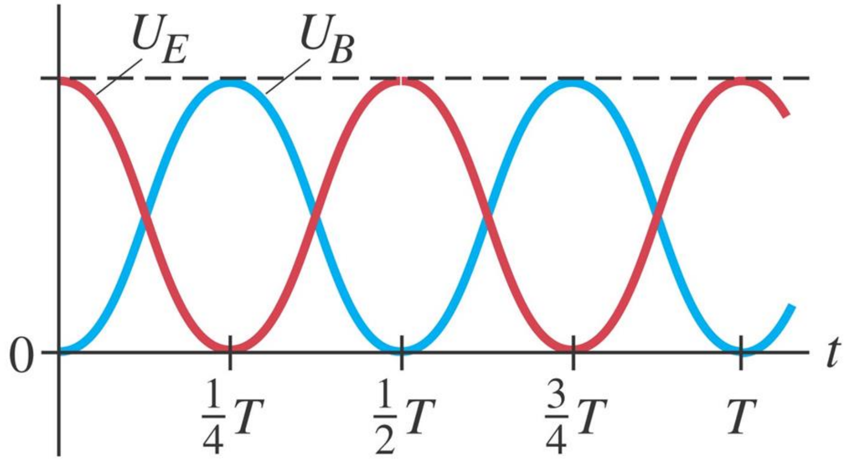
The energy stored in the electric field of the capacitor at any time t is:

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{{Q_0}^2}{2C}cos^2(\omega t + \phi)$$

The energy stored in the magnetic field of the inductor at the same instant is:

$$U_B = \frac{1}{2}LI^2 = \frac{L\omega^2 {Q_0}^2}{2C}sin^2(\omega t + \phi) = \frac{{Q_0}^2}{2C}sin^2(\omega t + \phi)$$

LC Oscillations from the Point of View of Energy



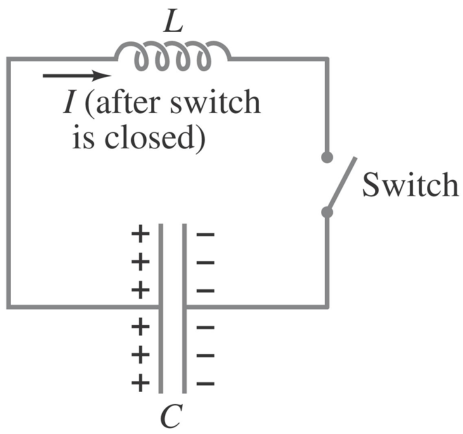
- At any time t, the total energy is

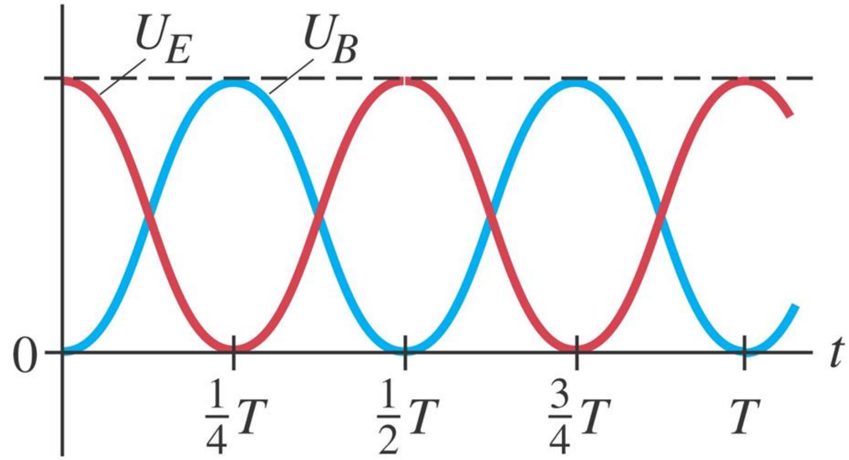
$$U = U_E + U_B = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$$

$$= \frac{{Q_0}^2}{2C}[cos^2(\omega t + \phi) + sin^2(\omega t + \phi)] = \frac{{Q_0}^2}{2C}$$

- Hence the total energy is constant, and energy is conserved.

LC Oscillations from the Point of View of Energy

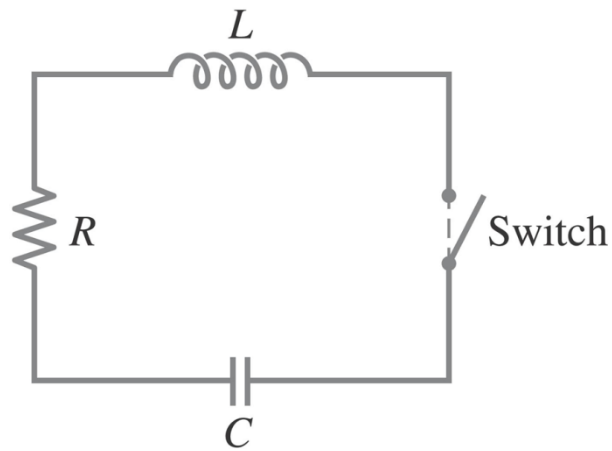




- What we have in an LC circuit is an LC oscillator or electromagnetic oscillation.
- The charge Q oscillates back-and-forth, from one plate of the capacitor to the other, and repeats this continuously.
- Likewise, the current oscillates back-and-forth as well.

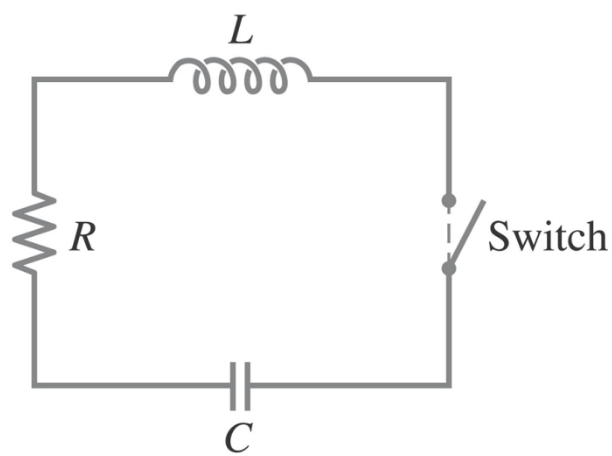
6. LC Oscillations with Resistance (an LRC Circuit)

LC Oscillations with Resistance (*LRC* Circuit)



- The LC circuit discussed in the previous section is an idealisation.
- There is always some resistance R in any circuit, and so now we discuss such a simple LRC circuit.
- Again, the capacitor is initially given a charge Q_0 and the battery or other source is then removed from the circuit.

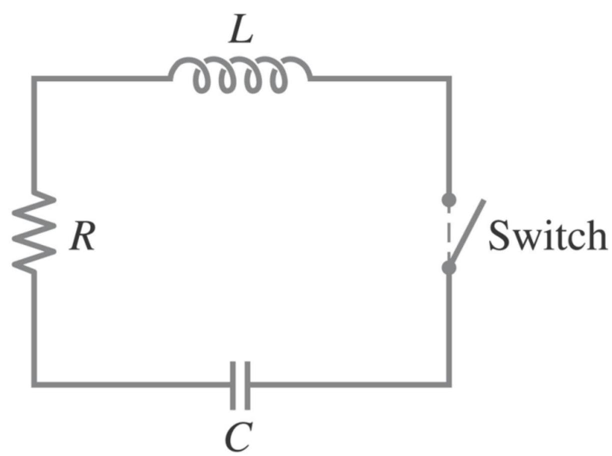
LC Oscillations with Resistance (*LRC* Circuit)



- The switch is closed at $t = 0$.
- Since there is now a resistance in the circuit, we expect some of the energy to be converted to thermal energy, and so we don't expect undamped oscillations as in a pure LC circuit.
- Indeed, if we use Kirchhoff's loop rule around this circuit, we obtain

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0$$

LC Oscillations with Resistance (*LRC* Circuit)



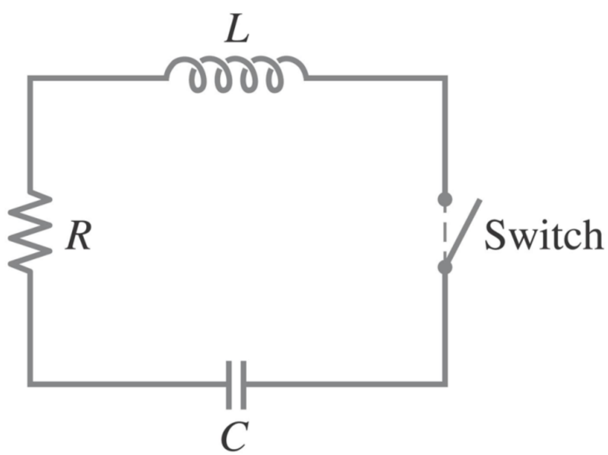
$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0$$

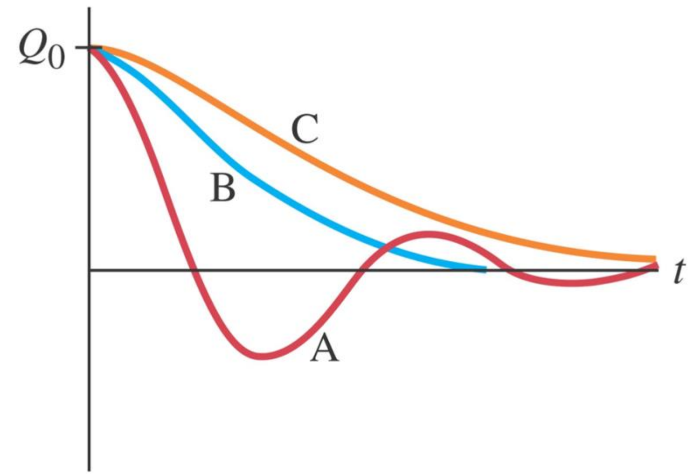
- Since $I = -dQ/dt$, the above equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

- This is what we call a second-order differential equation with respect to the variable Q.

LC Oscillations with Resistance (*LRC* Circuit)





$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=0$$

- The type of damping depends on the relative values of R, L, and C.
- Because some resistance is always present, electrical oscillators generally need a periodic input of power to compensate for the energy converted to thermal energy in the resistance.
- We will see later that oscillators are an important element in many electronic devices: radios and television sets use them for tuning, for example.

Summary of today's Lecture

1. Mutual Inductance
2. Self-Inductance
3. Energy Stored in a Magnetic Field
4. LR Circuits
5. **LC Circuits and Electromagnetic Oscillations**
6. **LC Oscillations with Resistance (an LRC Circuit)**

Lecture 28: Optional Reading

The University of Nottingham
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- **Ch. 30.1**, Mutual Inductance; p.908-909.
- **Ch. 30.2**, Self-Inductance; p.910-911.
- **Ch. 30.3**, Energy Stored in a Magnetic Field; p.912.
- **Ch. 30.4**, LR Circuits; p.912-914.

Home Work

Do not forget to attempt the **Additional Problems** for this lecture before logging in to **Mastering Physics** to complete your assignments.