Topic 1: Simple Integration

Illustration 1: Evaluate $\displaystyle\int \frac{x^4-1}{x^2}\,dx.$

$$
\begin{aligned}
I &= \int \frac{x^4-1}{x^2}\,dx \\
&= \int \frac{x^4}{x^2}\,dx - \int \frac{1}{x^2}\,dx \\
&= \int x^2\,dx - \int x^{-2}\,dx \\
&= \frac{x^{2+1}}{2+1} - \frac{x^{-2+1}}{-2+1} + C \\
&= \frac{x^3}{3} + \frac{1}{x} + C
\end{aligned}
$$

1. $\displaystyle\int \left(e^x - \frac{2}{\sqrt{x}}\right) dx$ **Answer:**	2. $\displaystyle\int \left(\frac{3}{x} + 2e^x + \frac{1}{1+x^2}\right) dx$ **Answer:**

Topic 1: Simple Integration

Illustration 2: Evaluate $\displaystyle\int \left(\frac{x^4-1}{x^2-1}\right) dx$.

$$
\begin{aligned}
I &= \int \left(\frac{x^4-1}{x^2-1}\right) dx \\
&= \int \frac{(x^2+1)(x^2-1)}{x^2-1} dx \\
&= \int (x^2+1) dx \\
&= \int x^2\, dx + \int (1)\, dx \\
&= \frac{x^3}{3} + x + C
\end{aligned}
$$

1. $\displaystyle\int \left(x^7 - \frac{1}{x^5} + \sqrt{x}\right) dx$ **Answer:**	2. $\displaystyle\int (a^x - x^a)\, dx \; ; \; a > 1$ **Answer:**

Topic 1: Simple Integration

Illustration 3: Evaluate $\displaystyle\int \left(\frac{1}{\cos^2 x \sin^2 x}\right) dx.$

$$
\begin{aligned}
I &= \int \left(\frac{1}{\cos^2 x \sin^2 x}\right) dx \\
&= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} dx \\
&= \int \frac{\cos^2 x}{\cos^2 x \sin^2 x} dx + \int \frac{\sin^2 x}{\cos^2 x \sin^2 x} dx \\
&= \int \operatorname{cosec}^2 x \, dx + \int \sec^2 x \, dx \\
&= -\cot x + \tan x + C
\end{aligned}
$$

1. $\displaystyle\int \left(2\sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right) - \cot^2 x\right) dx$

Answer:

1. $\displaystyle\int \frac{\sin x}{\sin^2 x - 1} dx$

Answer:

2. $\displaystyle\int \left(\frac{1 + \cos^2 x}{\cos^2 x} \right) dx$

Answer:

3. $\displaystyle\int \left(\frac{5 \sin x + 2}{\cos^2 x} \right) dx$

Answer:

Topic 2: Integration by Substitution

Some useful substitutions

Integral	Substitution	Integral	Substitution
$\int f\ (g(x))\ g'(x)\ dx$	$g(x) = t$	$\int f\ (\tan x)\ \sec^2 x\ dx$	$\tan x = t$
$\int f\ (x^n)\ x^{n-1}\ dx$	$x^n = t$	$\int f\ (\ln x)\ \frac{1}{x}\ dx$	$\ln x = t$
$\int f\ (x^3)\ x^2\ dx$	$x^3 = t$	$\int f\ (\sqrt{x})\ \frac{1}{\sqrt{x}}\ dx$	$\sqrt{x} = t$
$\int f\ (\sin x)\ \cos x\ dx$	$\sin x = t$	$\int f\ (\tan^{-1} x)\ \frac{1}{1+x^2}\ dx$	$\tan^{-1} x = t$

Illustration 1: Evaluate $\displaystyle\int \frac{\cos x}{(1+\sin x)^2}\, dx$

by using the substitution: $1 + \sin x = t$.

$$I = \int \frac{\cos x}{(1+\sin x)^2}\, dx$$

$$\text{Let } 1 + \sin x = t \quad \Rightarrow \quad \cos x\, dx = dt$$

$$\begin{aligned}
\therefore \quad I &= \int \frac{1}{t^2}\, dt \\
&= \frac{t^{-1}}{(-1)} + C \\
&= -\frac{1}{t} + C \\
&= -\frac{1}{1+\sin x} + C
\end{aligned}$$

Note: make sure your final integral is expressed in terms of x.

1. $\displaystyle\int \frac{\sec^2 x}{1+2\tan x}dx$, use: $1+2\tan x = t$. **Answer:**	2. $\displaystyle\int x^2\cos\left(1-x^3\right)dx$, use: $1-x^3=t$. **Answer:**
3. $\displaystyle\int x^3\left(1-x^4\right)^7 dx$, use: $1-x^4=t$. **Answer:**	4. $\displaystyle\int \left[\frac{\sin\left(\frac{1}{x}\right)}{3x^2}\right]dx$, use: $\dfrac{1}{x}=t$. **Answer:**

Topic 2: Integration by Substitution

Illustration 2: Evaluate $\displaystyle\int \frac{x}{\sqrt{x-2}}\,dx.$

$$I = \int \frac{x}{\sqrt{x-2}}\,dx \qquad \text{Let } \sqrt{x-2} = t \quad \Rightarrow \quad x-2 = t^2,\ t>0$$

$$\Rightarrow \quad 1 = 2t\,\frac{dt}{dx} \quad \Rightarrow \quad dx = 2t\,dt$$

$$\therefore I = \int \frac{(t^2+2)}{t}\,2t\,dt$$

$$= 2\int (t^2+2)\,dt$$

$$= 2\left[\frac{t^3}{3} + 2t\right] + C$$

$$= 2\left[\frac{(x-2)^{\frac{3}{2}}}{3} + 2\sqrt{x-2}\right] + C$$

1. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Answer:

2. $\displaystyle\int \frac{x}{\sqrt{1+2x}}\,dx$

Answer:

1. $\displaystyle\int \frac{1}{x \cdot (\ln x + 1)^2}\, dx$

Answer:

2. $\displaystyle\int x \cdot e^{x^2}\, dx$

Answer:

3. $\displaystyle\int x^2 \cdot \cos(x^3)\, dx$

Answer:

1. $\int (2x+7)(x^2+7x+3)^{\frac{4}{5}}\, dx$

Answer:

2. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\, dx$

Answer:

3. $\int x \cdot \sqrt{1-x}\, dx$

Answer:

Topic 3: Integration by Trigonometric Substitution

Integrand	Trigonometric substitution
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$x = a\,\sin\theta$
$\dfrac{1}{x^2 + a^2}$	$x = a\,\tan\theta$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$x = a\,\sec\theta$

Illustration: Evaluate $\displaystyle\int \frac{x}{\sqrt{9 - x^2}}\,dx$, by using trigonometric substitution.

$$I = \int \frac{x}{\sqrt{9 - x^2}}\,dx$$

$$\text{Let } x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta d\theta$$

$$\therefore\ I = \int \frac{3\sin\theta}{\sqrt{9 - 9\sin^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int \frac{3\sin\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= 3\int \sin\theta\ d\theta \ = -3\cos\theta$$

$$\because \cos\theta = \sqrt{1 - \sin^2\theta} = \frac{\sqrt{9 - x^2}}{3}$$

$$\therefore\ 3\cos\theta = \sqrt{9 - x^2}$$

$$\Rightarrow \quad I = -\sqrt{9 - x^2} + C$$

1. $\int \frac{x}{\sqrt{x^2 - 9}} \, dx$

Answer:

2. $\int \frac{1}{x\,(1 + x^2)} \, dx$

Answer:

Topic 4: Integral of the Form $\int f(ax+b)\,dx$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C, \text{ where } F \text{ is the antiderivative of } f.$$

Illustration: Evaluate $\displaystyle\int \sec^2(4x+1)dx.$

$$\begin{aligned} I &= \int \sec^2(4x+1)dx \qquad \text{as} \qquad \int \sec^2 x\,dx = \tan x + C \\ &= \frac{1}{4}\tan(4x+1) + C \end{aligned}$$

1. $\displaystyle\int \sin(2x+5)\,dx$ **Answer:**	2. $\displaystyle\int e^{3x+7}\,dx$ **Answer:**
3. $\displaystyle\int e^{-2x}\,dx$ **Answer:**	4. $\displaystyle\int \frac{1}{3x+5}\,dx$ **Answer:**

Integration by Substitution (Additional Example)

Illustration:

Evaluate $\int \sec^3 x \tan x \, dx$, by using appropriate substitution.

$$I = \int \sec^3 x \tan x \, dx$$
$$= \int \sec^2 x \cdot \sec x \tan x \, dx$$

Let $\sec x = t \quad \Rightarrow \quad \sec x \tan x \, dx = dt$

$$\therefore \quad I = \int t^2 \, dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{\sec^3 x}{3} + C$$

1. $\int \cot^4 x \; dx$	2. $\int \tan^2 x \sec^4 x \, dx$
Hint: $\mathrm{I} = \int \cot^2 x \, (\mathrm{cosec}^2 x \; - 1) \, dx$ $= \int \cot^2 x \, \mathrm{cosec}^2 x \; dx - \int (\mathrm{cosec}^2 x \; - 1) \; dx$	**Hint**: $\mathrm{I} = \int \tan^2 x \, (1 + \tan^2 x) \sec^2 x \, dx$
Answer:	**Answer:**

Additional Exercises:

1. $\displaystyle\int x^3 \cdot \sqrt{5+x^4}\,dx$ **Answer:**	2. $\displaystyle\int \frac{1}{x \cdot (\ln x)^2}\,dx$ **Answer:**
3. $\displaystyle\int \frac{1}{x} \cdot (\ln x)^n\,dx,\ n \in \mathbb{N}$ **Answer:**	4. $\displaystyle\int e^{\tan x} \cdot \sec^2 x\,dx$ **Answer:**

Additional Exercises:

1. $\displaystyle\int \frac{\sec x \tan x}{\sqrt{4-\sec^2 x}}\,dx$

Answer:

2. $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

Answer:

3. $\displaystyle\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

Answer:

4. $\displaystyle\int \frac{1}{x + x\cdot(\ln x)^2}\,dx$

Answer: