Seminar 10

In this seminar you will study:

- Arithmetic/Geometric progressions
- Arithmetic/Geometric series
- Sum of Powers
- Method of difference

Arithmetic progressions (AP)

An arithmetic sequence (arithmetic progression) is given by

$$
\begin{array}{cccccc}
a, & a+d, & a+2d, & a+3d, & \dots\dots\dots\;, & a+(n-1)d \\
\downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
1^{st} & 2^{nd} & 3^{rd} & 4^{th} & & n^{th} \\
\text{Term} & \text{Term} & \text{Term} & \text{Term} & & \text{Term}
\end{array}
$$

$\therefore$ the n^{th} term of an A.P. is $\boxed{a_n = a + (n-1)d}$

a is the first term of an A.P.

d is the common difference

Arithmetic progressions (AP)

Example: For an AP, the third term is 8 and the sixteenth term is 47.
Find the first term a and the common difference d. Hence, write the first seven terms of the AP.

Solution:

$$\text{Third term is } 8 \quad \Rightarrow \quad a + 2d = 8 \qquad (1)$$

$$\text{Sixteenth term is } 47 \quad \Rightarrow \quad a + 15d = 47 \qquad (2)$$

$$(2) - (1) \quad \text{gives } 39 = 13d \ \therefore \ d = 3$$

$$\text{From} \quad (1) \quad a + 2 \times 3 = 8 \ \therefore \ a = 2$$

$$\text{Thus,} \quad a = 2, \quad \text{and} \quad d = 3.$$

$$\Rightarrow \quad \text{The first seven terms of the AP are:} \quad 2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20 \ldots$$

Geometric progressions (GP)

A geometric sequence (Geometric progression) is given by

$a,$	$a\,r,$	$a\,r^2,$	$a\,r^3,$	$\ldots\ \ldots\ \ldots\ ,$	$a\,r^{n-1}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$
1^{st} Term	2^{nd} Term	3^{rd} Term	4^{th} Term		n^{th} Term

$\therefore$ the n^{th} term of the G.P. is $\boxed{a_n = a\,r^{n-1}}$

a is the first term of the G.P.

r is the common ratio.

Geometric progressions (GP)

Example: For a GP, the third term is 400 and the seventh term is $250,000$. Find the first term a and the common ratio r. Hence, write the first seven terms of the GP.

Solution:

$$\text{Third term is } 400 \quad \Rightarrow \quad ar^2 = 400 \qquad (1)$$

$$\text{Seventh term is } 250,000 \quad \Rightarrow \quad ar^6 = 250,000 \qquad (2)$$

$$(2) \div (1) \quad \text{gives } r^4 = 625 \;\Rightarrow\; (r^2)^2 = (25)^2$$

$$\Rightarrow \; r^2 = 25$$

$$\therefore \; r = \pm 5$$

$$\text{From} \quad (1) \quad ar^2 = 400 \;\Rightarrow\; a \times 25 = 400$$

$$\therefore \; a = 16$$

$$\text{Thus,} \quad a = 16, \quad \text{and} \quad r = \pm 5.$$

Geometric progressions (GP)

Example: For a GP, the third term is 400 and the seventh term is $250,000$. Find the first term a and the common ratio r. Hence, write the first seven terms of the GP.

Solution:

$\Rightarrow$ with $r = 5$ the first seven terms of the GP are:

$$16,\ 80,\ 400,\ 2000,\ 10,000,\ 50,000,\ 250,000\ldots$$

$\Rightarrow$ with $r = -5$ the first seven terms of the GP are:

$$16,\ -80,\ 400,\ -2000,\ 10,000,\ -50,000,\ 250,000\ldots$$

Arithmetic series

The sum of the first n terms of an A.P. is :

$$S_n = \frac{n}{2}[\,2a + (n-1)d\,]$$

where a is the first term and d is the common difference.

The sum of the first n terms of an A.P. is also given by

$$S_n = \frac{n}{2}[\,a + l\,]$$

where $l = a + (n-1)d =$ last term given in A.P.

Arithmetic series

Example: The sixth term of an A.P. is 23 and its twenty-second term is 39. Find the sixteenth term of the sequence and sum of its first 19 terms.

Solution:

$$\text{Sixth term is } 23 \quad \Rightarrow \quad a + 5d = 23 \qquad (1)$$

$$\text{Twenty-second term is } 39 \quad \Rightarrow \quad a + 21d = 39 \qquad (2)$$

$$(2) - (1) \ \text{ gives } 16d = 16 \ \therefore \ d = 1$$

$$\text{From} \quad (1) \ \ a + 5 \times 1 = 23 \ \therefore \ a = 18$$

$$\Rightarrow \ \text{Sixteenth term is} : a_{16} = a + 15d$$

$$\therefore a_{16} = 18 + 15 \times 1 = 33$$

$$\text{Sum of first nineteen terms is} : S_{19} = \frac{19}{2}\left[2 \times 18 + (18) \times 1\right]$$

$$= 19[27] = 513$$

Geometric series

The sum of the first n terms of a G.P. is :

$$S_n = \begin{cases} na & ; \quad r = 1 \\ a\left(\dfrac{1-r^n}{1-r}\right) & ; \quad r \neq 1 \end{cases}$$

where a is the first term and r is the common ratio.

The sum of infinite terms of a G.P. is $S = \dfrac{a}{1-r}$ (if $-1 < r < 1$)

Geometric series

Example 1: For a G.P. the second term is 6 and the fifth term is 48, find the sum of the first 10 terms.

Solution:

$$\text{Second term is } 6 \quad \Rightarrow \quad ar = 6 \qquad (1)$$

$$\text{Fifth term is } 48 \quad \Rightarrow \quad ar^4 = 48 \qquad (2)$$

$$(2) \div (1) \quad \text{gives } r^3 = 8 \;\Rightarrow\; r = 2$$

$$\text{From} \quad (1) \quad a \times 2 = 6 \;\Rightarrow\; a = 3$$

$$\text{Sum of first ten terms is : } S_{10} = \frac{3(2^{10} - 1)}{2 - 1}$$

$$\boxed{\because S_n = \frac{a(r^n - 1)}{r - 1}}$$

$$= 3(1023) = 3069$$

Geometric series

Example 2: Express $3.123123\overline{123}$ as a vulgar fraction.

Solution:

$$3.123123\overline{123} = 3 + \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \dots$$

$$3.123123\overline{123} = 3 + \underbrace{\frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \dots}_{\text{sum of infinite terms of a GP}}$$

The first term of the GP is: $a = \dfrac{123}{10^3}$

The common ratio of the GP is: $r = \left(\dfrac{123}{10^6}\right) \div \left(\dfrac{123}{10^3}\right) = \dfrac{1}{10^3}$

$$\therefore \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \dots = \frac{\frac{123}{10^3}}{1 - \frac{1}{10^3}} = \frac{41}{333}$$

$$\because S = \frac{a}{1-r} \text{ if } |r| < 1$$

$$\Rightarrow \quad 3.123123\overline{123} = 3 + \frac{41}{333} = \frac{1040}{333}$$

The formulae for the sum of power series

$$\underbrace{1+1+1+\ldots+1}_{n\text{ times}} = \sum_{1}^{n} 1 = n$$

$$1+2+3+\ldots+n = \sum_{1}^{n} k = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\ldots n^2 = \sum_{1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+\ldots n^3 = \sum_{1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

The formulae for the sum of power series

Example: Prove that $\sum_{1}^{n}(6n^2+4n-1) = n(n+2)(2n+1)$

Solution: $\sum_{1}^{n}(6n^2+4n-1) = 6\sum_{1}^{n} n^2 + 4\sum_{1}^{n} n - \sum_{1}^{n} 1$

$$= 6 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} - n$$

$$= n(n+1)(2n+1) + 2n(n+1) - n$$

$$= n\,[\,(n+1)(2n+1) + 2(n+1) - 1\,]$$

$$= n\,[\,(n+1)(\underline{2n+1}) + (\underline{2n+1})\,]$$

$$= n(2n+1)[\,\underline{(n+1)+1}\,]$$

$$= n(2n+1)(n+2)$$

The Method of differences

Example:

(i). Use the method of partial fractions to show that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$

(ii). Hence use the method of differences to show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

Solution:

(i). Let $\dfrac{1}{k(k+1)} = \dfrac{A}{k} + \dfrac{B}{k+1}$

$$\Rightarrow \quad 1 = A(k+1) + B(k)$$

Put $k = 0 \Rightarrow \; 1 = A(1)$

$$\therefore \; A = 1$$

Put $k = -1 \Rightarrow \; 1 = B(-1)$

$$\therefore \; B = -1$$

$$\Rightarrow \quad \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

The Method of differences

Solution:

$$(ii). \quad \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\cdots} + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

THE MATHS TEAM AT CELE WISHES YOU

THE BEST OF LUCK

IN YOUR EXAMINATIONS