Seminar 6

In this seminar you will study:

- Solving quadratic equations with negative discriminant ($\Delta < 0$)
- Simplification of expression involving i
- Algebra of complex numbers and their cartesian form: $z = x + iy$
- Properties of modulus
- Polar form of a complex number: $z = r(\cos\theta + i\sin\theta)$
- Algebraic operations with polar form of complex numbers

Complex numbers

Complex number: Cartesian Form

$$z = x + i\,y$$

where $x,\ y \in \mathbb{R}$, and the imaginary number $i = \sqrt{-1} \quad \Rightarrow \quad i^2 = -1.$

Real part of z: $\mathrm{Re}(z) = x$

Imaginary part of z: $\mathrm{Im}(z) = y$

Conjugate of a Complex number: Cartesian Form

If $z = x + i\,y$ is a complex number then its conjugate is defined and denoted by:

$$\overline{z} = x - i\,y$$

Real part of $\overline{z}$: $\mathrm{Re}(\overline{z}) = x$

Imaginary part of $\overline{z}$: $\mathrm{Im}(\overline{z}) = -y$

Complex numbers

Example: Solve the quadratic equation $2x^2 - 10x + 17 = 0.$

Solution: On comparing $2x^2 - 10x + 17 = 0$ with $ax^2 + bx + c = 0$

$$a = 2, \qquad b = -10, \qquad c = 17$$

$$\Delta = b^2 - 4ac = (-10)^2 - 4(2)(17) = -36 < 0 \Rightarrow \Delta = 36i^2$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-10) \pm \sqrt{36i^2}}{2(2)} = \frac{10 \pm 6i}{4} = \frac{5 \pm 3i}{2}$$

$x = \left(\frac{5}{2}\right) + i\left(\frac{3}{2}\right)$ or $x = \left(\frac{5}{2}\right) - i\left(\frac{3}{2}\right)$

Simplification of expressions involving *i*

$$i=\sqrt{-1} \qquad i^2=-1 \qquad i^3=-i \qquad i^4=-i^2=1 \qquad i^5=i$$

Example: Simplify: $(i^2+i+2)^{10}+(i^2-i+2)^{10}$.

Solution: $(i^2+i+2)^{10}+(i^2-i+2)^{10}$

$$=(-1+i+2)^{10}+(-1-i+2)^{10}$$

$$=(1+i)^{10}+(1-i)^{10}$$

$$(1+i)^{10}=[\,(1+i)^2\,]^5$$

$$=[1+2i+i^2]^5=[1+2i-1]^5$$

$$\therefore (1+i)^{10}=[2i]^5=32i^5=32i$$

Similarly,

$$(1-i)^{10}=[-2i]^5=-32i$$

$$\therefore (i^2+i+2)^{10}+(i^2-i+2)^{10}$$

$$=32i-32i=0$$

Property of Modulus

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad |\overline{z}| = |z|$$

Example: Given $z_1 = 3 + 4i$ and $z_2 = 12 + 5i$, find $|\overline{z_1} \cdot z_2|$ and $\left|\frac{z_1}{z_2}\right|$.

Solution:

$$|z_1| = \sqrt{3^2 + 4^2} = 5$$

$$|z_2| = \sqrt{12^2 + 5^2} = 13$$

$$\therefore\ |\overline{z_1} \cdot z_2| = |\overline{z_1}| \cdot |z_2| = |z_1| \cdot |z_2| = 5 \cdot 13 = 65$$

$$\therefore\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{5}{13}$$

Polar form of Complex numbers

Cartesian form

$z = x + i\,y$

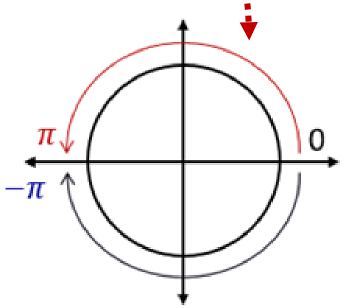
where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

Polar form

$z = r\,(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \leq \pi$.

$$r = |z| = \sqrt{x^2 + y^2}$$



and $\theta = \arg(z)$ is

obtained from

$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$

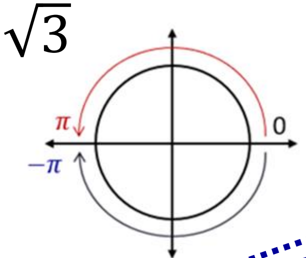
(Not given in the formula sheet)

Polar form of Complex numbers

Example: Express the complex number $z = 1 + \sqrt{3}\, i$ in the polar form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

Solution: $z = 1 + \sqrt{3}\, i = x + iy$

$$\therefore \quad x = 1 \quad \text{and} \quad y = \sqrt{3}$$



$$\Rightarrow r = \sqrt{x^2 + y^2} = 2$$

$$\text{and} \quad \theta = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right|$$

$$= \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$

$$\therefore \quad z = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

Algebraic operations with Polar form of Complex numbers

Given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

$$z_1 \cdot z_2 = r_1 \cdot r_2 \left[\, \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \,\right].$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\, \cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2) \,\right]$$

Note: In the above results, $(\theta_1 \pm \theta_2)$ only represent the arguments of $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$ respectively.

The principal argument can be obtained by using $(\theta_1 \pm \theta_2) \pm 2\pi$

Algebraic operations with Polar form of Complex numbers

Example: Given $z_1 = 1+\sqrt{3}\,i$ and $z_2 = \sqrt{3}+i$. Find the polar form of

$z_1 \cdot z_2$ and $\dfrac{z_1}{z_2}$.

Solution: $z_1 = 1+\sqrt{3}\,i = 2\left[\left(\frac{1}{2}\right)+i\left(\frac{\sqrt{3}}{2}\right)\right] = 2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$

$$z_2 = \sqrt{3}+i = 2\left[\left(\frac{\sqrt{3}}{2}\right)+i\left(\frac{1}{2}\right)\right] = 2\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$$

$$z_1 \cdot z_2 = 2\times 2\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right] = 4\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{2}{2}\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right] = \cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)$$

THANKS FOR YOUR ATTENTION

Monday	1 pm to 2 pm	YFB 412
Tuesday	9 am to 10 am	YFB 412
Tuesday	11 am to 12 noon	Trent 437
Tuesday	1 pm to 2 pm	YFB 412
Tuesday	3 pm to 4 pm	PB 205
Wednesday	10 am to 11 am	PMB 449
Wednesday	11 am to 12 noon	YFB 412
Wednesday	12 noon to 1 pm	YFB 412
Wednesday	1 pm to 2 pm	YFB 104
Friday	4 pm to 5 pm	YFB 219
Friday	1 pm to 2 pm	PB 102