Lecture 5

Topics covered in this lecture session

1. Formulae for addition, factor and multi-angle.
2. Inverse Trigonometric functions.
3. Expressing $a\cos x + b\sin x$ in the form $r\cos(\theta - x)$.

Addition and factor formulae

Note: $x(A+B) = xA + xB$, but $\sin(A+B) \neq \sin A + \sin B$.

$$\boxed{\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\sin(A-B) &= \sin A\cos B - \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\cos(A-B) &= \cos A\cos B + \sin A\sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A\tan B}
\end{aligned}}$$

Addition and factor formulae

Example

$$\sin 75^\circ = \sin\left(45^\circ + 30^\circ\right)$$

$$\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Addition and Factor formulae

$$\sin(A+B) = \sin A\cos B + \cancel{\cos A\sin B}$$

$$\sin(A-B) = \sin A\cos B - \cancel{\cos A\sin B}$$

Adding

$$\boxed{\sin(A+B)+\sin(A-B) = 2\sin A\cos B}$$

$$\sin(A+B) = \cancel{\sin A\cos B} + \cos A\sin B$$

$$-\sin(A-B) = -\cancel{\sin A\cos B} + \cos A\sin B$$

Subtracting

$$\boxed{\sin(A+B)-\sin(A-B) = 2\cos A\sin B}$$

Similarly, it can be proved that

$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B$$

$$\cos(A+B)-\cos(A-B) = -2\sin A\sin B$$

Addition and Factor formulae

Writing $A+B=C$ and $A-B=D \Rightarrow A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

$$\begin{aligned}
\sin(A+B)+\sin(A-B) &= 2\sin A\cos B\\
\sin(A+B)-\sin(A-B) &= 2\cos A\sin B\\
\cos(A+B)+\cos(A-B) &= 2\cos A\cos B\\
\cos(A+B)-\cos(A-B) &= -2\sin A\sin B
\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}
\sin C+\sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\
\sin C-\sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\cos C+\cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\
\cos C-\cos D &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)
\end{aligned}$$

Example

Prove that $\sin 50^\circ+\sin 10^\circ=\sin 70^\circ$

Multi-angle formulae

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\boxed{\begin{aligned} &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta &= \frac{1}{2}(1+\cos 2\theta) \end{aligned}}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

With $t = \tan\left(\frac{\theta}{2}\right)$,

useful formulae in Calculus

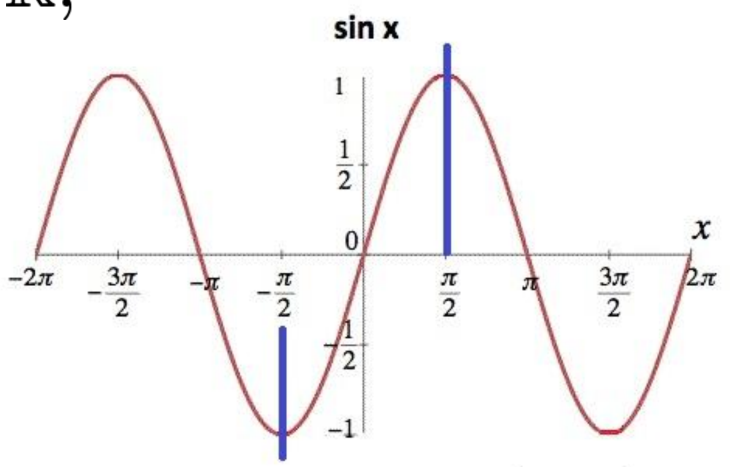
$\sin\theta = \dfrac{2t}{1+t^2}$	$\cos\theta = \dfrac{1-t^2}{1+t^2}$	$\tan\theta = \dfrac{2t}{1-t^2}$

Worked Example

Prove that $\dfrac{\sin 3\theta}{1+2\cos 2\theta} = \sin\theta$. Hence deduce the value of $\sin 15^o$.

Inverse Trigonometric Functions

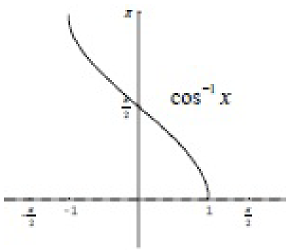
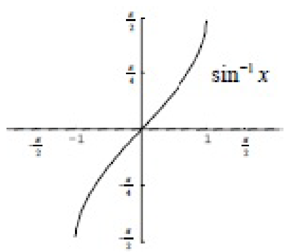
The graph of the sine function over $\mathbb{R}$, indicates that it is not one-one however, if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then sine function is one-one and its inverse exists.



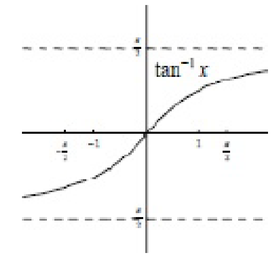
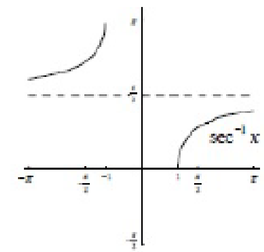
It is denoted by $\sin^{-1}$ **or** $\arcsin$ and is defined by

$$y = \sin x \quad \Leftrightarrow \quad x = \sin^{-1} y \quad ; \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

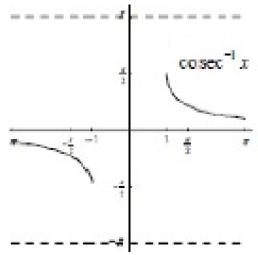
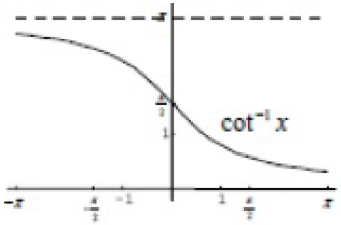
Inverse Trigonometric Functions

Inverse Function	Domain of Inverse function $\equiv$ Range of Trigonometric function	Range of Inverse function *i.e.* Restricted Domain for Trigonometric function	Graph of Inverse Trigonometric function
$\cos^{-1} x$ or $\arccos$	$[-1, 1]$	$[0, \pi]$	
$\sin^{-1} x$ or $\arcsin$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	

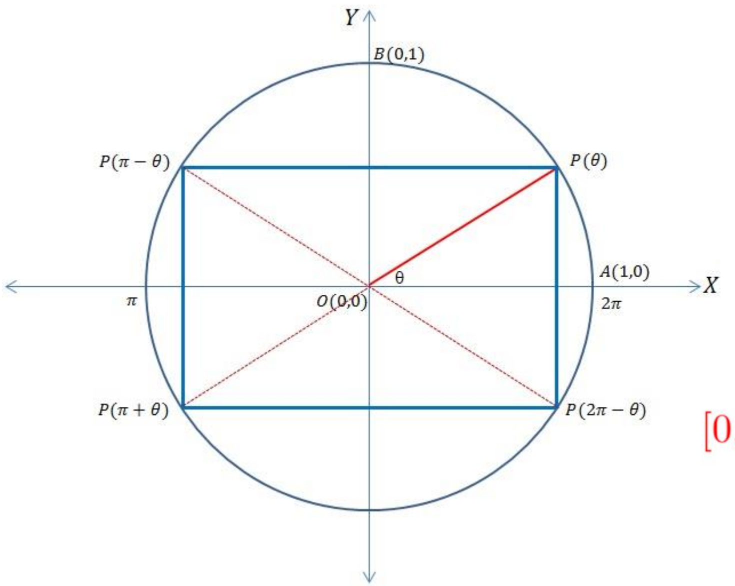
Inverse Trigonometric Functions

Inverse Function	Domain of Inverse function $\equiv$ Range of Trigonometric function	Range of Inverse function *i.e.* Restricted Domain for Trigonometric function	Graph of Inverse Trigonometric function
$\tan^{-1} x$ or $\arctan$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
$\sec^{-1} x$ or arcsec	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	

Inverse Trigonometric Functions

Inverse Function	**Domain of Inverse function** $\equiv$ Range of Trigonometric function	**Range of Inverse function** *i.e.* Restricted Domain for Trigonometric function	**Graph of Inverse Trigonometric function**
$\operatorname{cosec}^{-1}x$ **or** $\operatorname{arccosec}$	$\mathbb{R}-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	
$\cot^{-1}x$ **or** arccot	$\mathbb{R}$	$(0,\pi)$	

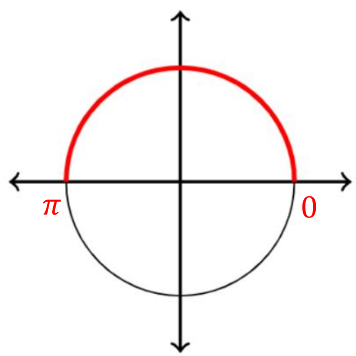
Inverse Trigonometric Functions



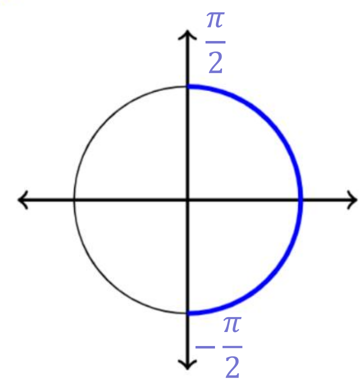
Reference Angle $\theta \in (0, 2\pi)$

Restricted domain for Inverse Trigonometric functions

$[0, \pi]$, used for $\cos^{-1}$



$[-\frac{\pi}{2}, \frac{\pi}{2}]$, used for $\sin^{-1}$



$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, used for $\tan^{-1}$

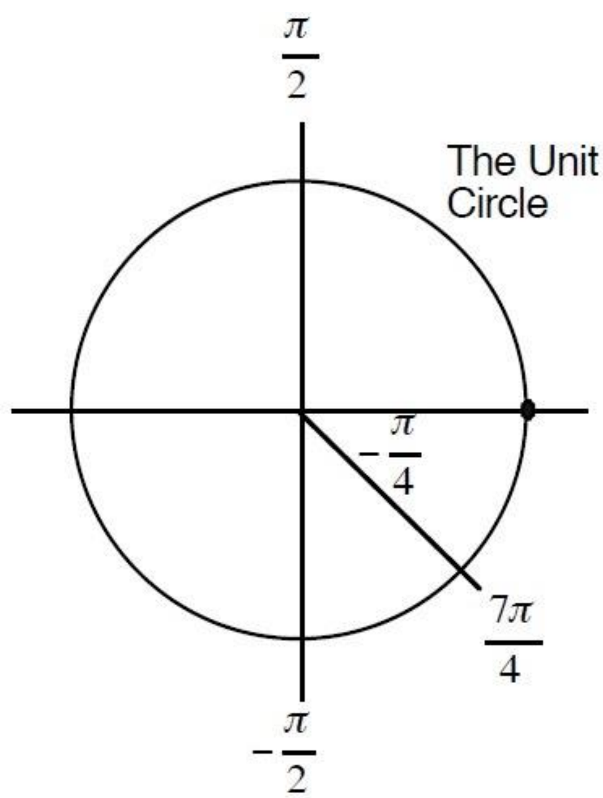
Inverse Trigonometric Functions

Find the values of:

$$(i) \quad \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$(ii) \quad \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$(iii) \quad \sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$$



Inverse Trigonometric Functions

Question 4-N36-Q1

Solve $\cos^{-1}\left(\cos\left(\frac{14\pi}{3}\right)\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

A $\frac{\pi}{2}$

$\boxed{\text{B} \quad \pi}$

C 2π

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$

Sometimes it is important to express

$f(x) = a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$,

so as to

- determine the range of f;
- find the period of f;
- sketch the graph of the function f.

Expressing $a\cos x + b\sin x$ in the form $r\cos(\theta - x)$

The Method: Let $a = r\cos\theta$

$$b = r\sin\theta$$

where θ and r are to be determined.

- Squaring and adding $\Rightarrow \quad a^2 + b^2 = r^2 \quad \Rightarrow \quad r = \sqrt{a^2 + b^2}$

- Dividing the second equation by the first equation, gives

$$\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a} \quad \Rightarrow \quad \tan\theta = \frac{b}{a}$$

(from which θ can be found)

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$

Thus, $f(x) = a\ \cos x + b\ \sin x$

$$= r\cos\theta\cos x + r\sin\theta\sin x$$

$$= r\ \left[\cos\theta\cos x + \sin\theta\sin x\right]$$

$$= r\cos(\theta - x) \quad \textbf{or} \quad r\cos(x - \theta)$$

because, $\cos(-\theta) = \cos\theta$

Expressing $a\cos x + b\sin x$ in the form $r\,\cos(\theta - x)$

1. Prove that $\cos 2x - \sqrt{3}\sin 2x = 2\cos\left(2x + \frac{\pi}{3}\right)$.

2. Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \pi/2$.

   Hence (i) sketch the graph of $y = f(x) = \sin x - \sqrt{3}\cos x$;

   (ii) find the range of f;

   (iii) find the period of f.

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$

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   (ii) find the range of f;

   (iii) find the period of f.

Inverse Trigonometric Functions

Question 4-N36-Q2

Express $4\sin x - 3\cos x$ in the form $r\sin(x-\alpha)$

A $5\sin(x + 45.00^o)$

B $5\sin(x - 30.00^o)$

C $\boxed{5\sin(x - 36.87^o)}$

Suggested Reading

Foundation Algebra by P. Gajjar.

(Chapter 6)

THANKS FOR YOUR ATTENTION