Topic 1: Chain Rule for Differentiation

Key Formula: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ | $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$ | $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dw} \cdot \dfrac{dw}{dx}$

Illustration: Given $y = \ln(\sec x)$, find $\dfrac{dy}{dx}$ using Chain Rule.

Let $u = \sec x \Rightarrow y = \ln u$.

Then $\dfrac{du}{dx} = \sec x \cdot \tan x$, and $\dfrac{dy}{du} = \dfrac{1}{u}$.

$$\text{Hence } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \sec x \cdot \tan x$$
$$= \frac{1}{\sec x} \cdot \sec x \cdot \tan x$$
$$= \tan x$$

1. $y = \sin(\cos x)$ **Answer:**	2. $y = \cos(\sin x)$ **Answer:**

Find dy/dx of the following functions.

1. $y = e^{5x}$ **Answer:**	2. $y = 5^{ex}$ **Answer:**
3. $y = \tan(\ln x)$ **Answer:**	4. $y = \sec(3x^2)$ **Answer:**

Find dy/dx of the following functions.

1. $y = 2^{\cot x}$

Answer:

2. $y = \sin e^{x^3}$

Answer:

Topic 2: The Fast-Track Chain Rule Method

Key Formula: $$\boxed{\frac{d}{dx}[f(u)] = f'(u) \cdot \frac{du}{dx}}$$

Illustration: Given $y = \sqrt{\sin(e^{\cos x})}$, find $\frac{dy}{dx}$ using Chain Rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2} \cdot \frac{1}{\sqrt{\sin(e^{\cos x})}} \cdot \frac{d}{dx}\left(\sin(e^{\cos x})\right) \\
&= \frac{1}{2\sqrt{\sin(e^{\cos x})}} \cdot \cos\left(e^{\cos x}\right) \cdot \frac{d}{dx}\left(e^{\cos x}\right) \\
&= \frac{\cos\left(e^{\cos x}\right)}{2\sqrt{\sin(e^{\cos x})}} \cdot e^{\cos x} \cdot \frac{d}{dx}(\cos x) \\
&= -\frac{\cos\left(e^{\cos x}\right) \cdot e^{\cos x} \cdot \sin x}{2\sqrt{\sin(e^{\cos x})}}
\end{aligned}$$

1. $y = \sin(\cos(\ln x))$ **Answer:**	2. $y = \sin(\ln(\cos x))$ **Answer:**

Find dy/dx of the following functions.

1. $y = \cos(e^{-2x})$ **Answer:**	2. $y = e^{-\cos(x^2)}$ **Answer:**
3. $y = \tan\left(\cos\left(\sqrt{x}\right)\right)$ **Answer:**	4. $y = \ln\left(\sin\left(e^{x}\right)\right)$ **Answer:**

Topic 3: Implicit differentiation

Some Results: $\boxed{\dfrac{d}{dx}\left(y^2\right) = 2\,y\,\dfrac{dy}{dx}}$ $\boxed{\dfrac{d}{dx}\left(xy^2\right) = x\,(2\,y)\,\dfrac{dy}{dx} + y^2}$

Illustration: Given $\ln(x+y) = \ln(xy) + 1$, show that $\dfrac{dy}{dx} + \dfrac{y^2}{x^2} = 0$.

Differentiate both sides w.r.t. x

$$\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = \frac{1}{xy}\left(y + x \cdot \frac{dy}{dx}\right) + 0$$

$$\frac{dy}{dx}\left(\frac{1}{x+y} - \frac{1}{y}\right) = \frac{1}{x} - \frac{1}{x+y}$$

$$\frac{dy}{dx} \cdot \frac{-x}{y(x+y)} = \frac{y}{x(x+y)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y^2}{x^2}$$

$$\therefore \quad \frac{dy}{dx} + \frac{y^2}{x^2} = 0$$

1. $\sin xy = \sqrt{x+y}$ **Answer:**	2. $xy = e^{x+y}$ **Answer:**

Find dy/dx of the following functions.

1. $(1+x-y)^3 = (1-x+y)^2$ **Answer:**	2. $e^{xy} + y \cdot \ln x = \cos 2x$ **Answer:**
3. $e^{2x+3y} = x^2 - \ln(xy^3)$ **Answer:**	4. $\ln(xy) = x + y$ **Answer:**

Find dy/dx of the following functions.

1. Given $2y^3 + 4x^2 - y = x^6$, show that $(6y^2 - 1) \cdot \dfrac{dy}{dx} = (6x^5 - 8x)$.

Answer:

2. Given $\tan\left(x^2 y^4\right) = 3x + y^2$, show that $\dfrac{dy}{dx} = \dfrac{3 - 2x \cdot y^4 \cdot \sec^2\left(x^2 y^4\right)}{4x^2 \cdot y^3 \cdot \sec^2\left(x^2 y^4\right) - 2y}$

Answer:

Topic 4: Logarithmic Differentiation

Illustration: Given $y = (\tan x)^{\sin x}$, find $\dfrac{dy}{dx}$

$$\ln y = \ln\left((\tan x)^{\sin x}\right)$$

$$= \sin x \cdot \ln(\tan x)$$

Then $$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \cdot \ln(\tan x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \ln(\tan x) \cdot \cos x$$

$$= \sec x + \ln(\tan x) \cdot \cos x$$

Hence $$\frac{dy}{dx} = y \cdot (\sec x + \ln(\tan x) \cdot \cos x)$$

$$= (\tan x)^{\sin x} \cdot (\sec x + \ln(\tan x) \cdot \cos x)$$

1. Given $y = (\cos x)^{\sin x}$ find $\frac{dy}{dx}$.	2. Given $y = (\sin x)^{\cos x}$ find $\frac{dy}{dx}$.
Answer:	**Answer:**

Find dy/dx of the following functions.

1. Given $y = x^{x^2+1}$ find $\frac{dy}{dx}$. **Answer:**	2. Given $y = (\ln x)^{\ln x}$ find $\frac{dy}{dx}$. **Answer:**
3. Given $y = x^{\cos x}$ find $\frac{dy}{dx}$. **Answer:**	4. Given $y = (\cos x)^{x}$ find $\frac{dy}{dx}$. **Answer:**

Topic 4: Logarithmic Differentiation

Illustration: Given $y = \dfrac{\sqrt{x}\cdot\tan x}{(3x^2-1)^2}$ find, $\dfrac{dy}{dx}$

$$\ln y = \ln\left(\frac{\sqrt{x}\cdot\tan x}{(3x^2-1)^2}\right)$$

$$\ln y = \frac{1}{2}\ln x + \ln(\tan x) - 2\ln(3x^2-1)$$

$$\text{Then}\qquad \frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{1}{2}\ln x + \ln(\tan x) - 2\ln(3x^2-1)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2x} + \frac{\sec^2 x}{\tan x} - \frac{12x}{3x^2-1}$$

$$\text{Hence}\qquad \frac{dy}{dx} = \frac{\sqrt{x}\cdot\tan x}{(3x^2-1)^2}\cdot\left(\frac{1}{2x} + \frac{\sec^2 x}{\tan x} - \frac{12x}{3x^2-1}\right)$$

1. $y = (2x - e^{8x})^{\sin(2x)}$

Answer:

2. $y = \dfrac{\sin(3x+x^2)}{(6-x^4)^3}$

Answer:

Topic 5: Derivatives of inverse trigonometric functions

$\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$

Illustration: Given $y = \cos^{-1}\left(x^4\right)$, find $\dfrac{dy}{dx}$; $|\,x\,| < 1$.

Using $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$, we obtain

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^4)^2}} \cdot \frac{d}{dx}\left(x^4\right)$$
$$= \frac{-1}{\sqrt{1-x^8}} \cdot \left(4x^3\right)$$
$$= -\frac{4x^3}{\sqrt{1-x^8}}$$

1. Given $y = \sin^{-1}\left(\frac{1}{x}\right)$ find $\frac{dy}{dx}$ if $x < -1$. **Answer:**	2. Given $y = \tan^{-1}(\sqrt{x})$ find $\frac{dy}{dx}$. **Answer:**

Topic 5: Derivatives of inverse trigonometric functions

$\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$

Illustration: Given $y = \sec^{-1}\left(\sqrt{1+x^2}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$, $x > 0$, find $\dfrac{dy}{dx}$

Using $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$, we obtain

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\dfrac{1}{\sqrt{1+x^2}}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{-1}{\sqrt{1-\dfrac{1}{1+x^2}}} \cdot \left(-\frac{1}{2}\right)\frac{1}{\sqrt{(1+x^2)^3}} \cdot 2x$$

$$= \frac{x}{\sqrt{\dfrac{x^2}{1+x^2}} \cdot \sqrt{(1+x^2)^3}}$$

$$\underbrace{= \frac{x}{\sqrt{x^2(1+x^2)^2}} = \frac{x}{x(1+x^2)}}_{x>0}$$

$$= \frac{1}{1+x^2}$$

2. What is the relationship between $y = \sec^{-1} x$ and $y = \cos^{-1}\left(\frac{1}{x}\right)$?

Answer:

4. Given $y = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ find $\frac{dy}{dx}$; $0 < x < 1$.

Answer:

Find dy/dx of the following functions.

1. Given $y = \sin^{-1}\left(\frac{1-x}{1+x}\right)$ find $\frac{dy}{dx}$ if $x > 0$. **Answer:**	2. Given $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ find $\frac{dy}{dx}$ if $x > 0$. **Answer:**
3. Given $y = \tan^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{dy}{dx}$. **Answer:**	4. Given $y = \cot^{-1}\left(\frac{2x}{1+x^2}\right)$ find $\frac{dy}{dx}$. **Answer:**

Extended Topic: Logarithmic Differentiation

Example: Given $y = (\sin x)^x + (x)^{\sin x}$, find $\frac{dy}{dx}$.

Let $u = (\sin x)^x$ and $v = (x)^{\sin x} \Rightarrow y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ (1)

$$\ln u = x \ln(\sin x)$$

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \ln(\sin x) \cdot (1)$$

$$\frac{du}{dx} = (\sin x)^x \left[x \cdot \cot x + \ln(\sin x) \right]$$

$$\ln v = \sin x \ln(x)$$

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{x} + \ln(x) \cdot (\cos x)$$

$$\frac{dv}{dx} = (x)^{\sin x} \left[\frac{\sin x}{x} + \ln(x) \cos x \right]$$

$$\therefore (1) \Rightarrow \frac{dy}{dx} = (\sin x)^x \left[x \cdot \cot x + \ln(\sin x) \right] + (x)^{\sin x} \left[\frac{\sin x}{x} + \ln(x) \cos x \right]$$

1. Given $y = (\cos x)^x + \tan(x^x)$ find $\frac{dy}{dx}$.

Answer:

2. Given $y = (\ln x)^x + (x)^{\ln x}$ find $\frac{dy}{dx}$.

Answer: