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Foundation Calculus and Mathematical Techniques CELEN037

Practice Problems SET-10 Sample Solution

Type 1: Ordinary Differential Equations

1. Solve the following ODEs: $(xii) \quad (1+x^2)e^{\sqrt{3}y}\dfrac{dy}{dx} = 2x$

   Solution:

$$e^{\sqrt{3}y}dy = \frac{2x}{1+x^2}dx$$

$$\int e^{\sqrt{3}y}dy = \int \frac{2x}{1+x^2}dx$$

$$\frac{\sqrt{3}}{3}e^{\sqrt{3}y} = \ln(1+x^2) + C$$

   Therefore the general solution is: $\dfrac{\sqrt{3}}{3}e^{\sqrt{3}y} = \ln(1+x^2) + C$

2. Solve the ODE: $\dfrac{dy}{dx} = \dfrac{x-y+5}{x-y+8}$, using variable separable method. (Hint: let $u = x - y$, therefore $du = dx - dy$)

   Solution:

$$u = x - y \implies \frac{du}{dx} = 1 - \frac{dy}{dx} \implies du = dx - dy \implies dy = dx - du$$

$$\therefore \frac{dy}{dx} = \frac{dx-du}{dx} = \frac{x-y+5}{x-y+8} = \frac{u+5}{u+8}$$

$$\therefore 1 - \frac{du}{dx} = \frac{u+8-3}{u+8} = 1 - \frac{3}{u+8}$$

$$\therefore \frac{du}{dx} = \frac{3}{u+8}$$

$$\int (u+8)\ du = \int 3\ dx$$

$$\frac{u^2}{2} + 8u = 3x + C$$

$$\therefore \frac{(x-y)^2}{2} + 8x - 8y = 3x + C \implies (x-y)^2 + 16x - 16y = 6x + C$$

$$\therefore x^2 - 2xy + y^2 - 16y + 10x + C = 0$$

3. Solve the ODE: $\frac{dy}{dx} = \frac{x^3}{y} + xy + \frac{x}{y}$, using variable separable method. (Hint: let $u = x^2 + y^2 + 1$) Solution:

$$u = x^2 + y^2 + 1 \implies \frac{du}{dx} = 2x + 2y\frac{dy}{dx} \implies du = 2xdx + 2ydy \implies dy = \frac{du - 2xdx}{2y}$$

$$\therefore \frac{dy}{dx} = \frac{du - 2xdx}{2ydx} = \frac{x^3}{y} + xy + \frac{x}{y} = \frac{x^3}{y} + \frac{x^2y}{y} + \frac{x}{y} = \frac{x}{y}(x^2 + y^2 + 1) = \frac{x}{y}u$$

$$\therefore \frac{du - 2xdx}{dx} = 2xu \implies \frac{du}{dx} - 2x = 2xu \implies \frac{du}{dx} = 2x(u + 1)$$

$$\therefore \int \frac{1}{u+1}\ du = \int 2x\ dx$$

$$\therefore \ln(u + 1) = x^2 + C$$

$$u + 1 = e^{x^2+C} \implies x^2 + y^2 + 2 = Ce^{x^2}$$

Type 2: Initial Value Problems

4. Use the method of separation of variables to solve the following IVPs: (iv) $\quad \dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}}$

$$\frac{\theta}{\sec\theta}d\theta = \frac{t}{e^{t^2}}dt$$

$$\int \frac{\theta}{\sec\theta}d\theta = \int \frac{t}{e^{t^2}}dt$$

$$\int \theta\cos\theta d\theta = \int \frac{t}{e^{t^2}}dt$$

Let $g = \theta,\ \dfrac{dh}{d\theta} = \cos\theta$

Therefore $\dfrac{dg}{d\theta} = 1,\ h = \sin\theta$

$$\implies \theta\sin\theta - \int (\sin\theta)\ d\theta = \int \frac{t}{e^{t^2}}dt$$

$$\implies \theta\sin\theta + \cos\theta = \int \frac{t}{e^{t^2}}dt$$

Let $u = t^2,\ \implies \dfrac{du}{dt} = 2t,\ \implies \dfrac{1}{2}du = tdt$

Therefore $\theta\sin\theta + \cos\theta = \dfrac{1}{2}\displaystyle\int \frac{1}{e^u}du$

$$\theta\sin\theta + \cos\theta = \frac{1}{2}\int e^{-u}du$$

$$\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-u} + C$$

$$\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C$$

Substitute $t = 0,\ \theta = 0,\ \implies\ 0\sin 0 + \cos 0 = -\dfrac{1}{2}e^0 + C$

$$1 = -\frac{1}{2} + C$$

$$C = \frac{3}{2}$$

Therefore the particular solution is: $\theta\sin\theta + \cos\theta = -\dfrac{1}{2}e^{-t^2} + \dfrac{3}{2}$

Type 3: Application of Differential Equations

5. A bacteria culture grows exponentially so that the initial number has doubled in 2 hours. How many times the initial number will be present after 8 hours?

Solution:

Let the population of bacteria $= P$ changes with time t, therefore:

$$\frac{dP}{dt} = kP$$

$$\implies \frac{1}{P}dP = kdt$$

$$\implies \int \frac{1}{P}dP = \int kdt$$

$$\implies \ln P = kt + C$$

Let initial number $= P_0$

$$\implies \ln P_0 = C$$

$$\implies \ln P = kt + \ln P_0$$

$$\implies P = P_0 e^{kt}$$

As $t = 2,\ P = 2P_0$

$$\implies \ln(2P_0) = 2k + \ln P_0$$

$$\implies k = \frac{\ln 2}{2}$$

Therefore, when $t = 8,\ P = P_0 e^{\frac{\ln 2}{2} \cdot 8} = 16P_0$