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CELEN037 Foundation Calculus and Mathematical Techniques (Chenyang Xue)

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Lecture 10

Lecture Content

- Differential equations: Introduction
- Formation of ordinary differential equations (ODEs)
- Solution of differential equations
- Solving variable separable form ODEs
- Solving ODEs of variable-separable form: Initial Value Problem (IVP)
- Applications of ODEs

Differential Equations: Introduction

What is a differential equation?

An equation involving independent variable and dependent variable along with its derivatives is called a differential equation.

1. $x + y \cdot \dfrac{dy}{dx} = 0$
2. $e^x + e^y = \dfrac{dy}{dx}$
3. $(x + y)^2 \cdot \dfrac{dy}{dx} + \dfrac{d^2y}{dx^2} + xy = 0$ are differential equations.

Differential Equations: Introduction

Differential equations are of two main types:

Ordinary Differential Equations (ODEs)

An equation involving derivatives of one dependent variable with respect to one independent variable.

Partial Differential Equations (PDEs)

An equation involving derivatives of more than one dependent variable with respect to one or more independent variable.

We shall focus only on <u>ODEs</u> in this module.

Order and Degree of an ODE

Order of a differential equation

The order of the highest derivative in a differential equation is called the **order of the differential equation**.

e.g. the order of the following differential equation is 2.

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xy = 0$$

Order and Degree

Degree of a differential equation

When the differential equation is in the polynomial from of derivatives, then the power of the highest derivative in a DE is called the degree of the differential equation.

e.g. the degree of the following differential equation is 3.

$$\sqrt[3]{1+\frac{dy}{dx}}=\frac{d^2y}{dx^2}$$

Order and Degree

ODEs	Order	Degree
$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$	2	2
$4\sin\left(\frac{dy}{dx}\right) + 5y = 9$	1	**Not defined**
$\left(\frac{d^2y}{dx^2}\right)^2 + 4\left(\frac{dy}{dx}\right)^3 + 3x = 0$	2	2
$x + \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$	1	1

Order and Degree

Practice Problems

Find the order and degree of the following differential equations.

(i) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + xy = 0$

(ii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + \ln y = 0$

(iii) $\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$

(iv) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}}$

$Ans: (i) \quad 2, 1 \quad (ii) \quad 2, 2 \quad (iii) \quad 2, 2 \quad (iv) \quad 2, 3$

Order and Degree

Differential equations are generated when we model the change or process which occurs between two quantities.

e.g.

- velocity (distance and time)
- acceleration (velocity and time)
- pressure (force and area)
- charge (current and time).

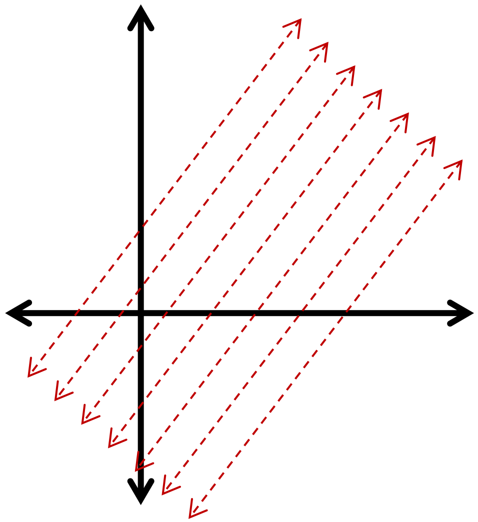
Formation of Differential equation

Consider $y = 3x + c$, where c is arbitrary constant.

It represents the family of parallel lines as shown in the figure.

Differentiating $y = 3x + c$ with respect to x, we obtain

$$\frac{dy}{dx} = 3$$



There is no arbitrary constant in the above equation.

$\therefore$ It is the differential equation corresponding to the family of lines $y = 3x + c$ (where c is a arbitrary constant).

Formation of Differential equation

Now, consider

$$y = mx + c \qquad (i)$$

where m and c are arbitrary constants.

It represents the family of parallel lines with slope m and making intercept c on Y −axis.

Differentiate two times w.r.t x

$$\frac{dy}{dx} = m \qquad (ii)$$

$$\frac{d^2y}{dx^2} = 0 \qquad (iii)$$

There is no arbitrary constant in equation (iii).

$\therefore$ It is the differential equation corresponding to $y = mx + c$.

If the given family of curves has n arbitrary constants, then differentiate n times w. r. t. x and eliminate the constants to find differential equation.

Formation of Differential equation

Example: Find the differential equation corresponding to the family of curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are arbitrary constants.

Given family of curves:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (i)$$

Differentiate w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \qquad (ii)$$

Again, differentiate w.r.t x

$$\therefore \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y(1) - x \cdot \frac{dy}{dx}}{y^2}\right]$$

Formation of Differential equation

From equation (ii),

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} \Rightarrow -\frac{b^2}{a^2} = \frac{y}{x} \cdot \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y}{x} \cdot \frac{dy}{dx} \left[\frac{1}{y} - \frac{x}{y^2} \cdot \frac{dy}{dx} \right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x} \cdot \frac{dy}{dx} - \frac{1}{y} \left(\frac{dy}{dx}\right)^2$$

$$\therefore xy \cdot \frac{d^2y}{dx^2} = y \cdot \frac{dy}{dx} - x \left(\frac{dy}{dx}\right)^2$$

$$\therefore xy \cdot \frac{d^2y}{dx^2} - y \cdot \frac{dy}{dx} + x \left(\frac{dy}{dx}\right)^2 = 0 \quad (iii)$$

$\therefore$ Equation (iii) is the differential equation corresponding to (i).

Formation of Differential equation

Practice Problems

Obtain differential equations for the following family of curves

(i) $x^2 + y^2 = a^2$, where a is arbitrary constant.

(ii) $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are arbitrary constants.

$$Ans:\ (i)\ y\frac{dy}{dx} + x = 0\ \ (ii)\ \frac{d^2y}{dx^2} = 0$$

Solution of an ODE

Solution of a differential equation

A function $y = f(x)$ is said to be a solution of ODE if y along with its derivatives satisfy the ODE.

e.g. $y = \frac{x^4}{4} + C$ is a solution of the ODE. $\frac{dy}{dx} = x^3$

$y = A\sin mx + B\cos mx$ is a solution of the ODE $\frac{d^2y}{dx^2} + m^2y = 0.$

Solution an ODE

Example:

Show that $y = \sin x$ is a solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

Solution:

We have, $y = \sin x \quad (i)$

Differentiate w. r. t. x

$$\therefore \frac{dy}{dx} = \cos x$$

Again, differentiate w. r. t. x

$$\therefore \frac{d^2y}{dx^2} = -\sin x = -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

$\therefore y = \sin x$ is a solution of differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution of an ODE

Example: Show that $y = a\cos(\ln x) + b\sin(\ln x)$ is a solution of differential equation $x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + y = 0$, where a and b are arbitrary constants.

Solution:

$$y = a\cos(\ln x) + b\sin(\ln x)$$

$$\therefore \quad \frac{dy}{dx} = -a\sin(\ln x)\left(\frac{1}{x}\right) + b\cos(\ln x)\left(\frac{1}{x}\right)$$

$$\therefore \quad x \cdot \frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

Solution of an ODE

$$\therefore \quad x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}\,(1) = -a\cos(\ln x)\left(\frac{1}{x}\right) - b\sin(\ln x)\left(\frac{1}{x}\right)$$

$$\therefore \quad x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} = -[\,a\cos(\ln x) + b\sin(\ln x)]$$

$$\therefore \quad x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} = -y$$

$\therefore \quad x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + y = 0$ is a solution of given differential equation.

Solution of an ODE

Practice Problems

(i) Show that $y = C_1 e^x + C_2 e^{-x}$ is a solution of differential equation

$$\frac{d^2y}{dx^2} - y = 0$$

where C_1 and C_2 are arbitrary constants.

(ii) Show that $y = \frac{a}{x} + b$ is a solution of differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} = 0$$

where a and b are arbitrary constants.

Solving ODEs of variable-separable form

The differential equation in variable-separable form can be written in the form: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

$$\Rightarrow \; g(y)\,dy = f(x)\,dx$$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$$

$$\Rightarrow \; G(y) = F(x) + C$$

Where $G(y)$ and $F(x)$ are integrals of $g(y)$ and $f(x)$ respectively.

Solving ODEs of variable-separable form

Example: Solve the ODE: $\frac{dy}{dx} = \frac{6x^2}{2y + cos\, y}$.

$$\Rightarrow (2y + \cos y)\, dy = 6x^2\, dx$$

$$\Rightarrow \int (2y + \cos y)\, dy = \int 6x^2 dx \; + \; C$$

$\Rightarrow \; y^2 + \sin y = 2x^3 + C$, where C is an arbitrary constant, and this expression is the general solution (GS).

Solving ODEs of variable-separable form

Practice Problem

Solve the following differential equations of variable-separable form.

(1) $\dfrac{dy}{dx} = \dfrac{y}{x}$ $\quad$ $Ans\colon\ y = cx$

(2) $\dfrac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$ $\quad$ $Ans\colon tan^{-1}\, y = x + \dfrac{x^3}{3} + c$

(3) $\dfrac{dy}{dx} = x \cdot e^y$ $\quad$ $Ans\colon -e^{-y} = \dfrac{x^2}{2} + c$

(4) $\dfrac{dy}{dx} = e^{-y}\left(e^x + x^2\right)$ $\quad$ $Ans\colon e^y = e^x + \dfrac{x^3}{3} + c$

Solving ODEs of variable-separable form: Initial Value Problem (IVP)

Example: Solve the IVP: $\frac{dy}{dx} = \frac{2x}{y^2}$; $y(0) = 3$.

$$\Rightarrow\ y^2 dy = 2x\ dx$$

$$\Rightarrow \int y^2\ dy = \int 2x\ dx$$

$$\Rightarrow \frac{y^3}{3} = x^2 + C'$$

General Solution

$$\Rightarrow y^3 = 3x^2 + C$$

Now, $y(0) = 3$

i.e. $y = 3$ when $x = 0$

$$\Rightarrow 27 = 0 + C$$

$$\Rightarrow C = 27$$

Particular Solution

$$\Rightarrow y^3 = 3x^2 + 27$$

Solving ODEs of variable-separable form: Initial Value Problem (IVP)

Example: Solve the IVP: $\frac{dy}{dx} = \frac{3x^2}{\cos y - 4y}$; $y(0) = 0.$

$$\Rightarrow (\cos y - 4y)\, dy = 3x^2\, dx$$

$$\Rightarrow \int (\cos y - 4y)\, dy = \int 3x^2 dx$$

General Solution

$$\Rightarrow \sin y - 2y^2 = x^3 + C$$

Now, $y(0) = 0$

i.e. $y = 0$ when $x = 0$

$$\Rightarrow C = 0$$

Particular Solution

$$\Rightarrow \sin y - 2y^2 = x^3$$

Solving ODEs of variable-separable form: Initial Value Problem (IVP)

Practice Problems

(1) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$; $y(0) = 2$

General solution : $y^3 = x^3 + c$;

Particular solution: $y^3 = x^3 + 8$

(2) $xy\dfrac{dy}{dx} = y + 2$; $y(2) = 0$

G.S. $y = \ln[\,x(y+2)^2\,] + c$;

P.S. $y = \ln\left[\dfrac{x(y+2)^2}{8}\right]$

(3) $\left(1 + y^2\right) dy = y \sin x \, dx$; $y(0) = 1$

G.S. $\ln y + \dfrac{y^2}{2} + \cos x = c$;

P.S. $\ln(y^2) + y^2 + 2\cos x = 3$

(4) $\dfrac{dy}{dx} = e^{x+y}$; $y(0) = -1$

G.S. $-e^{-y} = e^x + c$;

P.S. $e^x + e^{-y} = 1 + e$

Application of ODEs

1. Exponential growth model

Assume that the rate of change of population is proportional to the population at time t.

Let $t \equiv$ time (independent variable)

$P \equiv$ population (dependent variable)

$$\therefore \frac{dP}{dt} \propto P \quad \Rightarrow \quad \frac{dP}{dt} = kP$$

Application of ODEs

t	0	10	30
P	P_0	$2P_0$	$?$

Example: Population P of a city increases at a rate proportional to the total population at that time. ($t = 0 \Rightarrow P = P_0$).

(i) Show that $P = P_0\, e^{kt}$ $(k > 0)$.

(ii) If the population doubles in 10 years, estimate the population after 30 years.

$$\therefore \frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt \Rightarrow \int \frac{dP}{P} = k \int dt$$

Application of ODEs

t	0	10	30
P	P_0	$2P_0$	?

$$\Rightarrow \ln P = kt + C$$

When $t = 0,\ P = P_0$

$$\Rightarrow \ln P_0 = k(0) + C \qquad \Rightarrow\ C = \ln P_0$$

$$\Rightarrow \ln P = kt + \ln P_0$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = kt$$

When $t = 10,\ P = 2P_0$

$$\Rightarrow \ln\left(\frac{2P_0}{P_0}\right) = k(10)$$

$$\therefore k = \tfrac{1}{10}\ln 2$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \tfrac{t}{10}\ln 2$$

Application of ODEs

t	0	10	30
P	P_0	$2P_0$	?

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \tfrac{t}{10}\ln 2$$

When $t = 30$, $\ln\left(\frac{P}{P_0}\right) = \tfrac{30}{10}\ln 2 = \ln 8$

$$\Rightarrow P = 8\,P_0$$

$\therefore$ The population will be 8 times the initial population after 30 years.

Application of ODEs

2. Exponential decay model

Suppose we want to estimate the amount m of a radioactive substance present at any later time of a given amount.

Based on experiments it is found that at any instant, the radioactive substance decomposes at a rate proportional to the amount present.

$$\therefore \ \frac{dm}{dt} \propto m \qquad \Rightarrow \ \frac{dm}{dt} = -km$$

Where $k > 0$ is the proportionality constant.

Application of ODEs

Example: Radium decays at a rate proportional to the amount m present at time t.

If it takes 1600 years for half the original amount to decay, find the percentage of the original amount that remains after 200 years.

$$\therefore \quad \frac{dm}{dt} \propto m$$

$$\Rightarrow \quad \frac{dm}{dt} = -km$$

$$\Rightarrow \int \frac{dm}{m} = -k \int dt$$

$$\Rightarrow \ln m = -kt + C$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

Application of ODEs

$$\Rightarrow \ln m = -kt + C$$

When $t = 0,\ m = m_0$

$$\Rightarrow \ln m_0 = -k(0) + C$$

$$\Rightarrow \ C = \ln m_0$$

$$\Rightarrow \ln m = -kt + \ln m_0$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = -kt$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	?

When $t = 1600,\ m = \frac{m_0}{2}$

$$\Rightarrow \ln\left(\frac{m_0}{2m_0}\right) = -k(1600)$$

$$\therefore k = \frac{1}{1600}\ln 2$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = -\frac{t}{1600}\ln 2$$

Applications of ODEs

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \ln 2 - \frac{t}{1600}$$

When $t = 200$,

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \ln 2 - \frac{200}{1600}$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \ln\left(\frac{1}{2}\right)^{1/8}$$

$$\Rightarrow \frac{m}{m_0} = \left(\frac{1}{2}\right)^{1/8}$$

$$\Rightarrow m \approx 0.917\, m_0$$

$\therefore$ After 200 years, 91.7% of the initial amount of the substance is remaining.

Application of ODEs

3. Newton's law of cooling

Example: According to Newton's law of cooling, the rate of decrease of temperature of a body is proportional to the difference between the temperature of the body and the surrounding temperature.

A room is at a constant temperature of 20^{o}C. An object with temp. 80^{o}C is brought into the room and 5 minutes later, its temperature falls to 65^{o}C. What will its temperature be after a further interval of 5 minutes?

t	0	5	10
T	80	65	?

Application of ODEs

$$\frac{dT}{dt} \propto (T - T_s)$$

$$\Rightarrow \frac{dT}{dt} = k(T - T_s)$$

$$\Rightarrow \frac{dT}{T - T_s} = k\,dt$$

$$\Rightarrow \int \frac{dT}{T - T_s} = k \int dt$$

$$\Rightarrow \int \frac{dT}{T - 20} = k \int dt \qquad (\because T_s = 20)$$

$$\Rightarrow \ln(T - 20) = kt + C$$

t	0	5	10
T	80	65	?

$$\Rightarrow \ln(80 - 20) = k(0) + C$$

$$\Rightarrow C = \ln 60$$

Application of ODEs

$$\Rightarrow \ \ln\,(T-20) = kt + \ln 60$$

$$\therefore \ \ln\left(\frac{T-20}{60}\right) = kt$$

When $t = 5, T = 65$

$$\therefore \ \ln\left(\frac{65-20}{60}\right) = k(5)$$

$$\Rightarrow k = \frac{1}{5}\ln\left(\frac{3}{4}\right)$$

$$\therefore \ \ln\left(\frac{T-20}{60}\right) = \frac{t}{5}\ln\left(\frac{3}{4}\right)$$

When $t = 10$

$$\ln\left(\frac{T-20}{60}\right) = \ln\left(\frac{3}{4}\right)^2$$

$$\Rightarrow T - 20 = \frac{60\cdot 9}{16}$$

i.e. $T = 56^0\ C$ after 10 min.

Summary on Simple models with ODE

Exponential Models

Exponential Growth model

$$\frac{dy}{dt} = kt \qquad (i)$$

Solution

$$y(t) = y_0 \cdot e^{kt}$$

Exponential Decay model

$$\frac{dy}{dt} = -kx \qquad (ii)$$

Solution

$$y(t) = y_0 \cdot e^{-kt}$$

where;

$y(0) = y_0$: Quantity/amount of y at $t = 0$

(initial quantity/amount)

$y(t)$: Quantity/amount of y at some t

k: Constant of proportionality

Note:

Take $k > 0$ When deriving the solution to either the growth or decay models.

Summary on Simple models with ODE

Newton's law of cooling

Differential equation model

$$\frac{dT}{dt} = k(T - T_s) \qquad (i)$$

Solution to Differential equation

$$T(t) - T_s = (T_0 - T_s) \cdot e^{kt} \qquad (ii)$$

where;

$T(0) = T_0$: Temperature at $t = 0$

(initial temperature of body)

$T(t)$: Temperature at time t

T_s: Temperature of the surrounding

k: Constant of proportionality

Application of ODEs

Practice Problems

1. The population P of a city increases at a rate proportional to the population of the city at that time.
   (i) Formulate a differential equation model to show that $P = P_0 e^{kt}$ $(k > 0)$, where P_0 is the initial population.
   (ii) If the population doubles in 15 years, find the constant k of proportionality.

2. The rate of decay of a radioactive substance is proportional to the mass m of the substance present at time t. Given that its initial mass $100\ gm$ is reduced to half in 30 years. Express mass as m a function of time t.

Answers

1. $\dfrac{1}{15}\ln 2$ 2. $m = 100\, e^{-\frac{t}{30}\ln 2}$

Thank You!