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Foundation Calculus and Mathematical Techniques — CELEN037

Practice Problems SET-9 Sample Solution

Type 1: Area Calculation using Definite Integrals

1. Find the area of the region bounded the curve $y = (x-1)^3$, the lines $x = 0$, $x = 2$, and the X-axis.

$$A = \int_0^2 |(x-1)^3| \; dx$$

$$= -\int_0^1 (x-1)^3 \; dx + \int_1^2 (x-1)^3 \; dx$$

$$= -\left[\frac{(x-1)^4}{4}\right]_0^1 + \left[\frac{(x-1)^4}{4}\right]_1^2$$

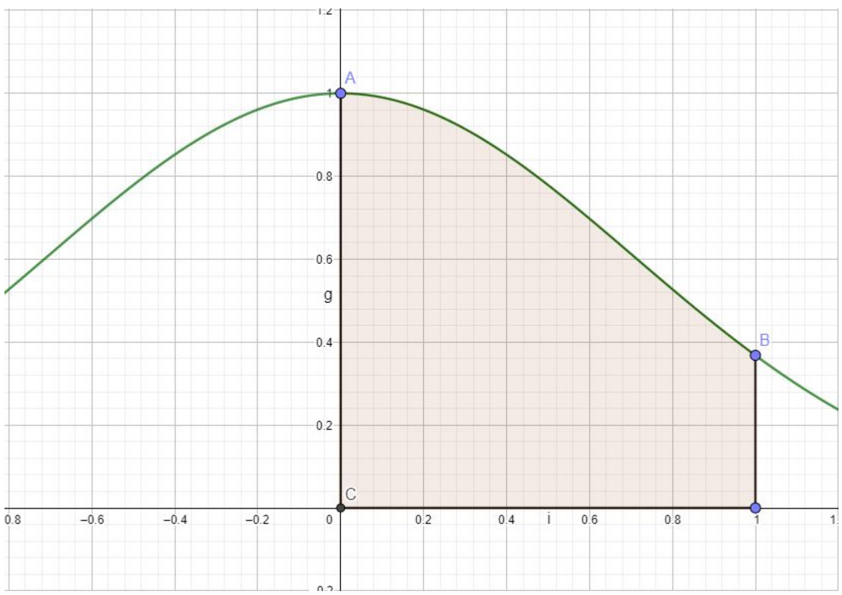
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Type 2: Volume Calculation using Definite Integrals

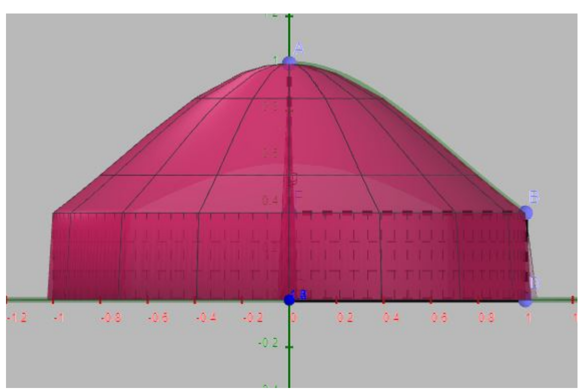
26. The region bounded by $y = e^{-x^2}$, lines $x = 0$, $x = 1$ and the X-axis is revolved about the Y-axis.

Solution:

The area bounded is shown as below:



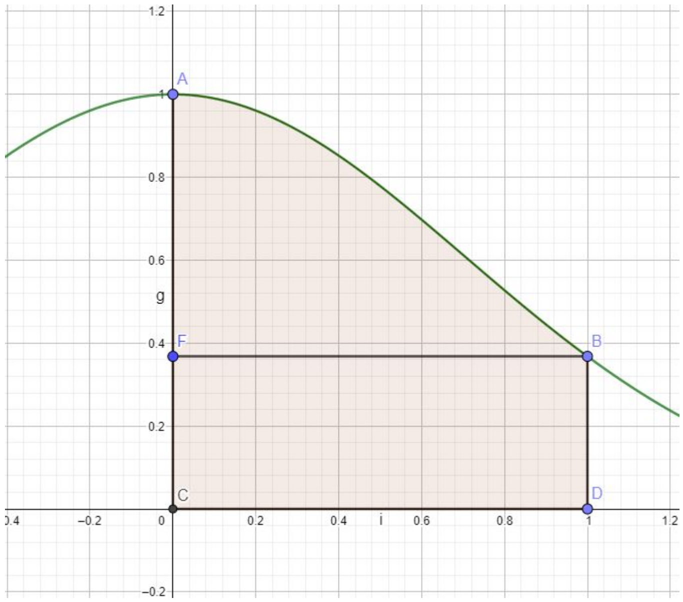
The volume generated by revolving this area about Y-axis is shown as below:



Therefore, the volume calculation needs to be separated into two parts:

Volume from region revolution of ABF and region revolution of CDBF.

The volume from region revolution of CDBF can be easily calculated as:



When $x = 0,\ y = 1$ and $x = 1,\ y = \dfrac{1}{e}$

Therefore $V_{CDBF} = \pi \displaystyle\int_0^{\frac{1}{e}} 1^2\ dy = \pi \frac{1}{e}$

The volume from region revolution of ABF can be easily calculated as:

$y = e^{-x^2} \implies \ln y = -x^2 \implies x^2 = -\ln y$

Therefore $V_{ABF} = \pi \displaystyle\int_{\frac{1}{e}}^{1} x^2\ dy \implies \pi \int_{\frac{1}{e}}^{1} -\ln y\ dy$

$= \pi \left[y - y \ln y\right]_{\frac{1}{e}}^{1}$

$= \pi \left((1 - 1 \ln 1) - \left(\dfrac{1}{e} - \dfrac{1}{e} \ln \dfrac{1}{e} \right) \right)$

$= \pi \left(1 - \left(\dfrac{1}{e} + \dfrac{1}{e} \ln e \right) \right)$

$= \pi \left(1 - \dfrac{2}{e} \right)$

Therefore, the total volume $V = V_{CDBF} + V_{ABF} = \pi \dfrac{1}{e} + \pi \left(1 - \dfrac{2}{e} \right) = \pi \left(1 - \dfrac{1}{e} \right)$

Type 3: Numerical Integration

28. Evaluate $\int_1^2 x^3\sqrt{x^5+2x^2-1}\,dx$ by using:

    (a) Trapezoidal rule with 4 sub-intervals of equal width;

    (b) Simpson's rule with 4 sub-intervals of equal width.

Give approximation to 6 d.p.

Solution:

$$h = \frac{2-1}{4} = 0.25$$

x	f_n	$f(x) = x^3\sqrt{x^5+2x^2-1}$
1	f_0	1.414214
1.25	f_1	4.443846
1.5	f_2	11.241208
1.75	f_3	24.872400
2	f_4	49.959984

$$(a)\quad I = \int_1^2 x^3\sqrt{x^5+2x^2-1}\,dx \approx \frac{h}{2}\left[f_0 + 2\left(f_1+f_2+f_3\right) + f_4\right] = 16.561138$$

$$(b)\quad I = \int_1^2 x^3\sqrt{x^5+2x^2-1}\,dx \approx \frac{h}{3}\left[f_0 + 2\left(f_2\right) + 4\left(f_1+f_3\right) + f_4\right] = 15.926800$$