The University of Nottingham Ningbo China

Centre for English Language Education

SAMPLE EXAM

Foundation Calculus and Mathematical Techniques

Time allowed: 1 hour 30 minutes

Candidates may complete the front cover of their answer book and sign their attendance card but must NOT write anything else until the start of the examination period is announced.

This paper contains SEVEN questions which carry equal marks.

Answer all questions.

An indication is given of the weighting of each subsection of a question by means of a figure enclosed by square brackets, e.g. [3], immediately following that subsection.

Only a CELE approved calculator $(fx - 82 \text{ series})$ **is allowed during this exam.**

Dictionaries are not allowed with one exception.Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do not turn examination paper over until instructed to do so.

ADDITIONAL MATERIAL: *Formula Sheet (attached to the back of the question paper).*

INFORMATION FOR INVIGILATORS:

1. *Please give a 15-minute warning before the end of the exam.*
2. *Please collect Answer Booklets, Question Papers and Formula Sheet at the end of the exam.*

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1. (a) Given $y = \sqrt{x-2}$, use the definition of the derivative to find $\dfrac{dy}{dx}$.

[2]

(b) (i) Given $y = \ln(2x-1) \cdot \cos 3x$, use the product rule of derivatives to find $\dfrac{dy}{dx}$.

(ii) Given $y = \dfrac{e^{-x}}{x^3+8}$, use the quotient rule of derivatives to find $\dfrac{dy}{dx}$.

[4]

(c) (i) Given $\sin(xy) = x - y$, use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

(ii) Given $y = \cos^{-1}(\sqrt{x}\,)$, find $\left.\dfrac{dy}{dx}\right|_{x=1/4}$.

[4]

2. (a) Given $y = (\sin x)^{2x}$, use the method of logarithmic differentiation to find $\dfrac{dy}{dx}$.

[3]

(b) The parametric equations of a curve are given by the following:

$x = \dfrac{1}{t}$ and $y = 2t + 1 \;\;;\;\; t \neq 0.$

Find:

(i) $\left.\dfrac{dy}{dx}\right|_{t=-0.5}$

(ii) the equation of the tangent line to the curve at the point $(-2,\ 0)$.

(ii) the equation of the normal line to the curve at the point $(-2,\ 0)$.

[4]

(c) (i) A spherical balloon is being inflated at a rate of 50π m^3/min. How fast is the balloon's radius changing at the instant when the radius reaches 3 m?

(ii) Find the interval on which the function $f(x) = -x^3 + 2x^2 + 23$ is increasing.

[3]

3. (a) Given the cubic polynomial function $f(x) = x^3 - 4x^2 + 2$,

   (i) Find any stationary points of f.

   (ii) Use the Second Derivative Test to classify the stationary points obtained in $3(a)(i)$ as the points of maximum or minimum values.

   (iii) Sketch the graph of the function $y = f(x)$. Label the maximum and minimum points on your graph.

[4]

(b) The Newton-Raphson iteration formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

   (i) Show that the Newton-Raphson iteration formula to approximate the solution to the equation $x^3 - 4x^2 + 2 = 0$ is

$$x_{n+1} = \frac{2x_n^3 - 4x_n^2 - 2}{3x_n^2 - 8x_n} \quad ; \quad n = 0, 1, 2, 3, \ldots \tag{3.1}$$

   (ii) Starting with $x_0 = -0.5$, apply formula (3.1) to approximate the real root of $f(x) = 0$ on the interval $(-1,\ 0)$, correct to 4 decimal places.

   Prepare a table of all x_n values until the sequence of approximation converges.

[4]

(c) Given $y = e^{2x} + \sin 3x$. Find $\left.\dfrac{d^2y}{dx^2}\right|_{x=\pi/2}$. [2]

4. (a) Given $f(x) = \sqrt[3]{1-x} \quad ; \quad -1 < x < 1$.

   (i) Obtain the Maclaurin's series expansion of $f(x)$ up to the terms with x^2.

   (ii) Use the series expansion obtained in $4(a)(i)$ to approximate the value of $\sqrt[3]{0.9}$. Round your final answer to 3 decimal places.

[4]

(b) (i) Evaluate $\displaystyle\int 3x^{10} - \frac{\sqrt{x}}{x^2} + 4\,dx$.

   (ii) Evaluate $\displaystyle\int \frac{x}{\sqrt{4+x^2}}\,dx$, use the substitution $4 + x^2 = t$.

   (iii) Evaluate $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$, use the substitution $\ln(x) = t$.

[6]

5. (a) Evaluate the following integrals

   (i) $\int e^{-3\cos x} \sin x \, dx$, by using appropriate substitution.

   (ii) $\int \frac{1}{x\sqrt{\ln x}} \, dx$, by using the result $\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)} + C$.

[4]

   (b) Evaluate $\int \frac{2}{x^2 - 4x + 8} \, dx$ by completing the square in the denominator.

[2]

   (c) (i) Evaluate $\int \cos 4x \sin 5x \, dx$ by using appropriate trigonometric formulae.

   (ii) Use the $t-$substitution, i.e. $\tan\left(\frac{x}{2}\right) = t$, to evaluate $\int \frac{1}{2 + \cos x} \, dx$.

[4]

6. (a) Evaluate $\int \frac{x+8}{(x-1)(x+2)} \, dx$ using the method of partial fractions for integration.

[2]

   (b) Evaluate the following definite integrals

   (i) $\int_{-4}^{-1} x^2(3 - 4x) \, dx$

   (ii) $\int_{0}^{\frac{\pi}{2}} 7\sin x - 2\cos x \, dx$

[4]

   (c) Evaluate $\int_{\frac{1}{e}}^{1} \ln x \, dx$ using the method of integration by parts.

[2]

   (d) Find the area bounded by the curves $y = (x-1)^2 + 1$ and $y = x + 2$.

[2]

7. (a) (i) What is the order and degree of the differential equation:

$$\sqrt{\left(\frac{dy}{dx}\right) - 3\left(\frac{d^3y}{dx^3}\right)} = \left(\frac{d^2y}{dx^2}\right)^2.$$

(ii) Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation

$y' - 3y = 6x + 4.$

[2]

(b) (i) Solve the variable-separable ordinary differential equation (ODE):

$$\frac{dy}{dx} = (2x + 3)(y^2 - 4).$$

(ii) Solve the initial value problem of the variable-separable ODE:

$$2\frac{dy}{dx} - 4xy = 2x \ ; \quad y(0) = 0.$$

[5]

(c) The differential equation model for the decay of radioactive substance is defined by

$$\frac{dM}{dt} = -kM, \text{ where } k > 0 \text{ is constant, and } t \text{ is time.} \qquad (7.1)$$

(i) Show that the general solution of (7.1) is given by

$M = M_0 \cdot e^{-kt}$, where $M_0 = M(0) \equiv \text{initial mass.}$

(ii) If the half-life of the radioactive substance is 10 days and there are 25 milligrams initially, how much is present after 8 days?

[3]