Seminar 03

In this seminar you will study:

- The Remainder and Factor Theorem
- The method of synthetic division
    - Finding quotients and remainders
    - Solving polynomial equations
    - The method of synthetic division
- Partial Fractions

The Remainder theorem and the Factor theorem

The Remainder theorem

If a polynomial $p(x)$ is divided by $(x - c)$, then the remainder is $p(c)$.

The Factor theorem

A polynomial $p(x)$ has a factor $(x - c)$ if and only if $p(c) = 0$.

The method of Synthetic division

Example: Use the method of synthetic division to find the quotient $q(x)$ and the remainder r if $p(x) = 2x^4 + 5x^3 - 2x - 8$ is divided by $(x+3)$.

$+ 0x^2$

Here $s(x) = x + 3 = x - c$

$\Rightarrow \boxed{c = -3}$

$$
\begin{array}{c|ccccc}
 & 2 & 5 & 0 & -2 & -8 \\
\boxed{-3} & \downarrow & -6 & 3 & -9 & 33 \\
\hline
 & 2 & -1 & 3 & -11 & \boxed{25}
\end{array}
$$

Quotient $q(x) \boxed{= 2x^3 - x^2 + 3x - 11}$ Remainder $r = \boxed{25}$

Solving polynomial equations

Result

Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients (i.e. $c_i \in \mathbb{Z}$, for $i = 0, 1, \cdots, n$). If m is an integer zero of $p(x)$, then m is a divisor of the constant term c_0.

For example, if $p(x) = x^3 - 27$, 3 is an integer zero of $p(x)$, (i.e. $m = 3$), and 3 is a divisor of -27, (since $c_0 = -27$).

Solving polynomial equations

Example: Solve $p(x) = x^3 - 2x^2 - 9x + 18 = 0$

Solution:

Possible zeros are $\pm 1, \pm 2 \pm 3, \pm 6, \pm 9, \pm 18$.

Try $\pm 1, \pm 2 \pm 3$.

Note: exam question on solving cubic equation will have at least one of these integers as a zero of $p(x)$.

$p(1) = 8 \Rightarrow p(x) \neq 0 \;\therefore\; 1$ is not a zero of $p(x)$

$p(-1) = 24 \Rightarrow p(x) \neq 0 \;\therefore\; -1$ is not a zero of $p(x)$

$p(2) = 0 \Rightarrow p(x) = 0 \;\therefore\; 2$ is a zero of $p(x)$

$\Rightarrow (x - 2)$ is one of the factors of $p(x)$

Use the method of synthetic division to find the other factor.

Solving polynomial equations

Example: Solve $p(x) = x^3 - 2x^2 - 9x + 18 = 0$

Here $s(x) = x - 2 = x - c$

$\Rightarrow \boxed{c = 2}$

$$
\begin{array}{c|cccc}
 & 1 & -2 & -9 & 18 \\
\boxed{2} & \downarrow & 2 & 0 & -18 \\
\hline
 & \boxed{1 \quad 0 \quad -9} & & & \boxed{0}
\end{array}
$$

Note: remainder $r = 0 \because s(x) = x - 2$ is a factor of $p(x) = x^3 - 2x^2 - 9x + 18$.

Thus, the other factor is $\boxed{x^2 - 9}$

$$\therefore\ p(x) = (x-2)\cdot(x^2-9)$$

$$= (x-2)\cdot(x-3)\cdot(x+3)$$

$$\therefore\ p(x) = 0 \Rightarrow (x-2)\cdot(x-3)\cdot(x+3) = 0$$

$$\Rightarrow\ x = 2 \text{ or } x = 3 \text{ or } x = -3$$

The method of partial fractions

Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Non-repeated quadratic factors

$$\frac{p(x)}{(x^2+a)(x+b)} = \frac{Ax+B}{x^2+a} + \frac{C}{x+b}$$

Repeated linear factors

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

In all these types, the constants A and B or A, B and C are to be determined.

Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\Rightarrow p(x) = A(x+b) + B(x+a)$$

Example: Express $\frac{14}{(x-9)(x+5)}$ as a sum of partial fractions.

Solution:

$$\frac{14}{(x-9)(x+5)} = \frac{A}{x-9} + \frac{B}{x+5}$$

$$\therefore \quad 14 = A(x+5) + B(x-9)$$

Let $x = 9$	Let $x = -5$
$\Rightarrow 14 = 14A$	$\Rightarrow 14 = -14B$
$\Rightarrow A = 1$	$\Rightarrow B = -1$

$$\therefore \frac{14}{(x-9)(x+5)} = \frac{1}{x-9} - \frac{1}{x+5}$$

Repeated linear factors

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{(x+b)} + \frac{B}{(x+a)} + \frac{C}{(x+a)^2}$$

$$\Rightarrow p(x) = A(x+a)^2 + B(x+a)(x+b) + C(x+b)$$

Example: Separate $\frac{2x+5}{(x-1)(x+2)^2}$ into partial fractions.

Solution:

$$\frac{2x+5}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\therefore \quad 2x+5 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

Let $x = 1$	Let $x = -2$	Let $x = 0 \Rightarrow 5 = 4A - 2B - C$
$\Rightarrow 7 = 9A$	$\Rightarrow 1 = -3C$	$\therefore$ Substituting the values of A and B
$\Rightarrow A = 7/9$	$\Rightarrow C = -1/3$	$\Rightarrow B = -7/9$

$$\therefore \frac{2x+5}{(x-1)(x+2)^2} = \frac{7}{9(x-1)} - \frac{7}{9(x+2)} - \frac{1}{3(x+2)^2}$$

Non-repeated quadratic factors

$$\frac{p(x)}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$

$$\Rightarrow p(x) = (Ax+B)(x+b) + C(x^2+a)$$

Example: Express $\frac{20}{(x-4)(x^2+4)}$ as a sum of partial fractions.

Solution:

$$\frac{20}{(x-4)(x^2+4)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+4}$$

$$\therefore \quad 20 = A(x^2+4) + (Bx+C)(x-4) = (A+B)x^2 + (C-4B)x + (4A-4C)$$

Let $x = 4$

$\Rightarrow A(4^2+4) = 20$

$\Rightarrow A = 1$

Equating constant

$\Rightarrow 4A - 4C = 20$

$\Rightarrow C = -4$

Equating coefficient of x

$\Rightarrow C - 4B = 0$

$\Rightarrow B = -1$

$$\therefore \frac{20}{(x-4)(x^2+4)} = \frac{1}{x-4} + \frac{-x-4}{x^2+4}$$