Lecture 4

Topics covered in this lecture session

1. Trigonometric functions.
2. More about Trigonometric functions.
3. Solving Trigonometric equations.

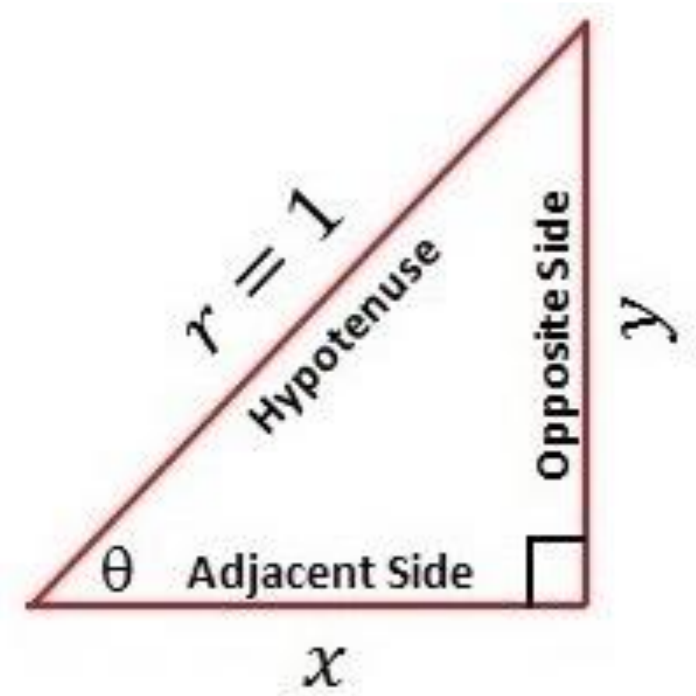
Trigonometric functions

$$\cos\theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{x}{r} = \frac{x}{1} = x$$

$$\sin\theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{y}{r} = \frac{y}{1} = y$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$



$$\sec\theta = \frac{1}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$

Trigonometric identities

The basic trigonometric identities are:

$$\cos^2\theta + \sin^2\theta = 1 \qquad (1)$$

$$1 + \tan^2\theta = \sec^2\theta \quad ; \quad \cos\theta \neq 0$$

obtained by dividing (1) by $\cos^2\theta$

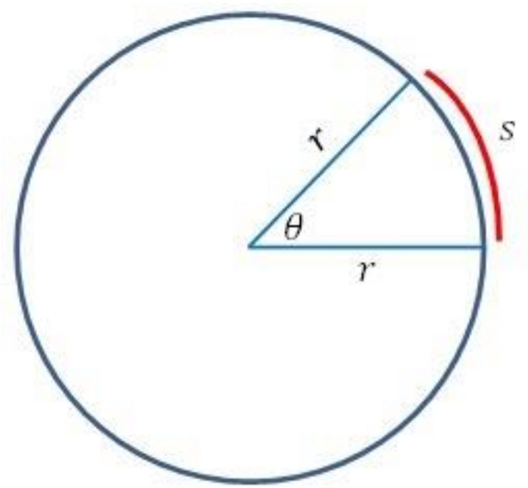
$$1 + \cot^2\theta = \mathrm{cosec}^2\theta \quad ; \quad \sin\theta \neq 0$$

obtained by dividing (1) by $\sin^2\theta$

Conversion (degree $\longleftrightarrow$ radians)

By definition, the length of the enclosed arc (s) is equal to the radius (r) multiplied by the magnitude of the angle (θ) in radians.

$$s = r\,\theta \quad \Rightarrow \quad \theta = \frac{s}{r}$$



$\therefore$ For one complete revolution (360^o), the magnitude in radians is

$$360^\circ = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

$$\Rightarrow \boxed{\pi = 180^\circ}$$

An important relation to convert degrees to radians and vice-versa.

Conversion (degree $\longleftrightarrow$ radians)

$$\boxed{\text{angle in radians} = \text{angle in degrees} \times \left(\frac{\pi}{180^\circ}\right)}$$

$$\boxed{\text{angle in degrees} = \text{angle in radians} \times \left(\frac{180^\circ}{\pi}\right)}$$

$$45^\circ = 45^\circ \times \left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{4} \text{ radians}$$

$$270^\circ = 270^\circ \times \left(\frac{\pi}{180^\circ}\right) = \frac{3\pi}{2} \text{ radians}$$

$$\frac{\pi}{6} \text{ radians} = \left(\frac{180^\circ}{\pi}\right) \times \frac{\pi}{6} = 30^\circ$$

$$\frac{5\pi}{12} \text{ radians} = \left(\frac{180^\circ}{\pi}\right) \times \frac{5\pi}{12} = 75^\circ$$

Periodic functions

If $f(x + p) = f(x)$, the function f is called periodic and p is defined as its period. The smallest positive value of p is called the Principal period of f.

<table>
<tr><th>Trigonometric function</th><th>Principal Period</th></tr>
<tr><td>cos</td><td rowspan="4">2π</td></tr>
<tr><td>sin</td></tr>
<tr><td>sec</td></tr>
<tr><td>cosec</td></tr>
<tr><td>tan</td><td rowspan="2">π</td></tr>
<tr><td>cot</td></tr>
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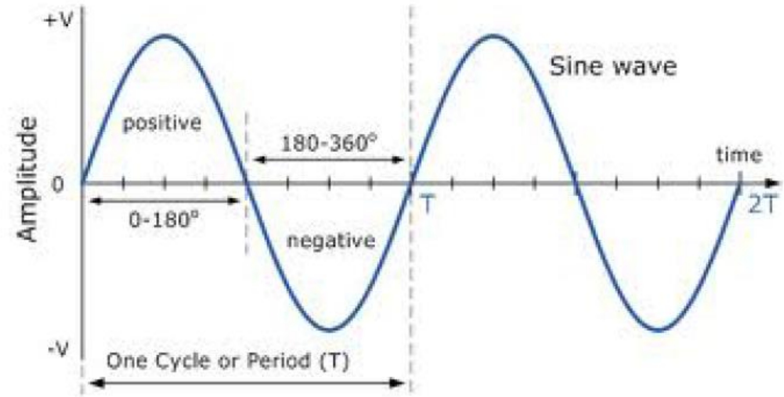

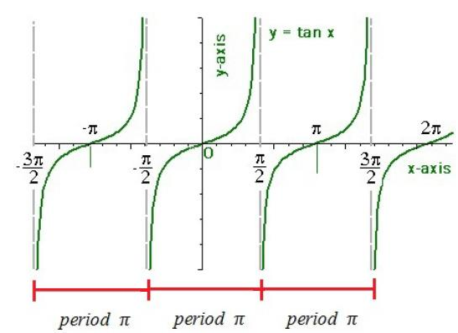

Periods of Trigonometric functions

where T_1 is the trig function

sin, cos, sec or cosec.

Principal period of $a\,T_1(bx+c)+d$ is $\dfrac{\boldsymbol{2\pi}}{|\,\boldsymbol{b}\,|}$

e.g. principal period of : $2\cos\,(4x-5)+6 \;=\; \dfrac{2\pi}{4}$

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where T_2 is the trig function

tan or cot.

Principal period of $a\,T_2(bx+c)+d$ is $\dfrac{\boldsymbol{\pi}}{|\,\boldsymbol{b}\,|}$

e.g. principal period of : $3\tan\,(4-5x)+2 = \dfrac{\pi}{|\,(-5)\,|} \;=\; \dfrac{\pi}{5}$

Q1

Find the Principal Period of $\operatorname{cosec}\left(\frac{1}{3}x-\pi\right)$

A $\frac{3\pi}{2}$

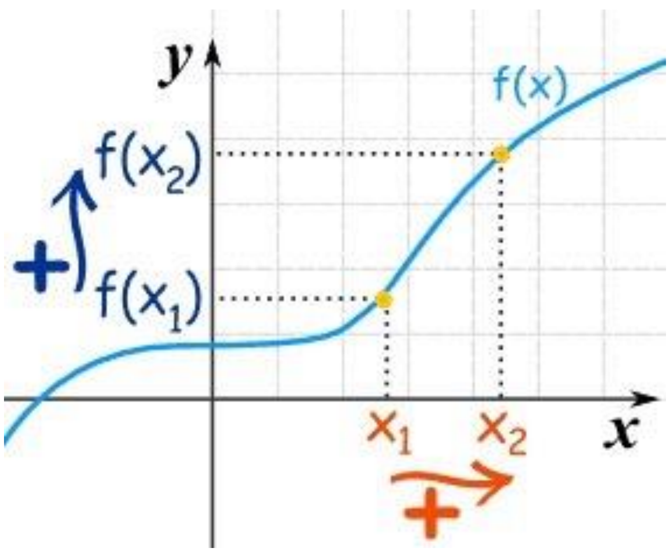
B 4π

$\boxed{\text{C} \quad 6\pi}$

Increasing and Decreasing functions

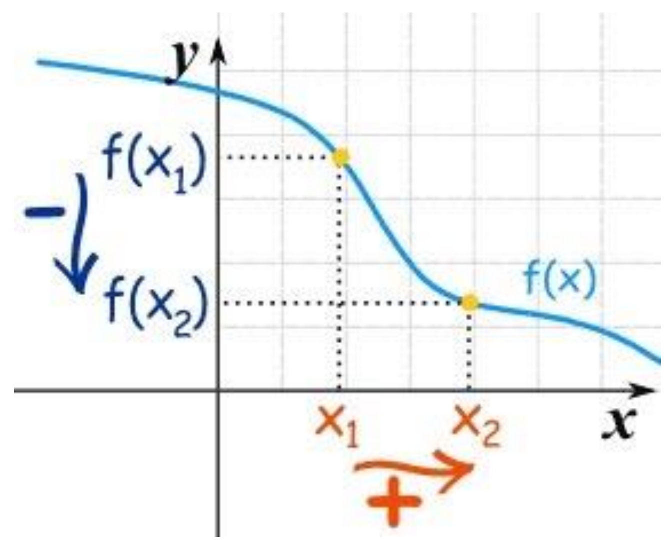
If $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$,

then the function f is said to be an increasing (↑) function.



If $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$,

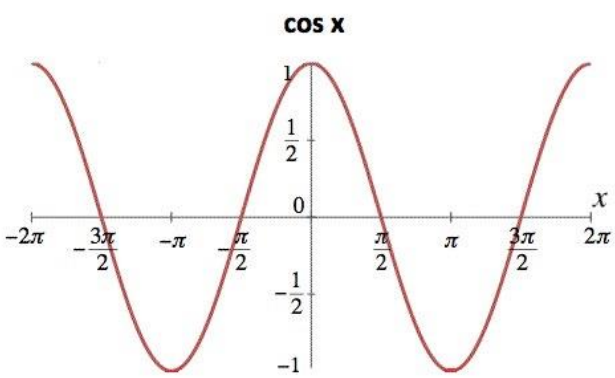
then the function f is said to be a decreasing (↓) function.

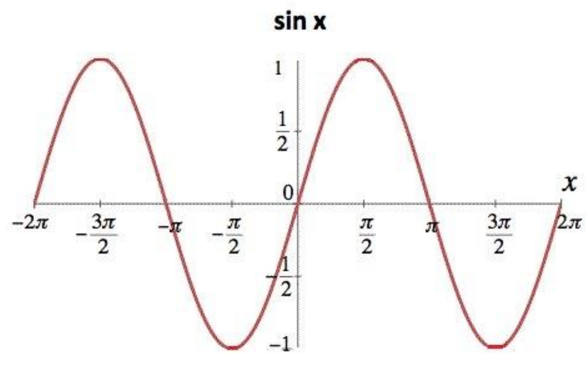


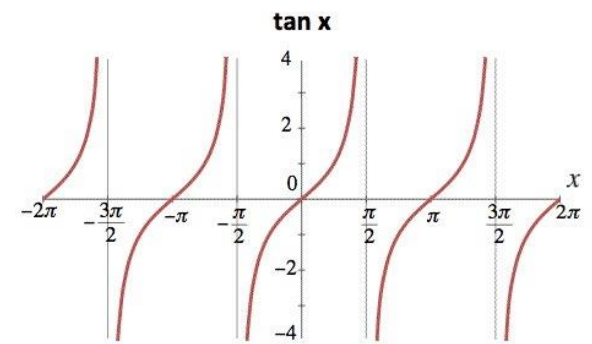
Increasing and Decreasing functions

Quadrant	1	2	3	4
cos	↓	↓	↑	↑
sin	↑	↓	↓	↑
tan	↑	↑	↑	↑

Quadrant	1	2	3	4
sec	↑	↑	↓	↓
cosec	↓	↑	↑	↓
cot	↓	↓	↓	↓





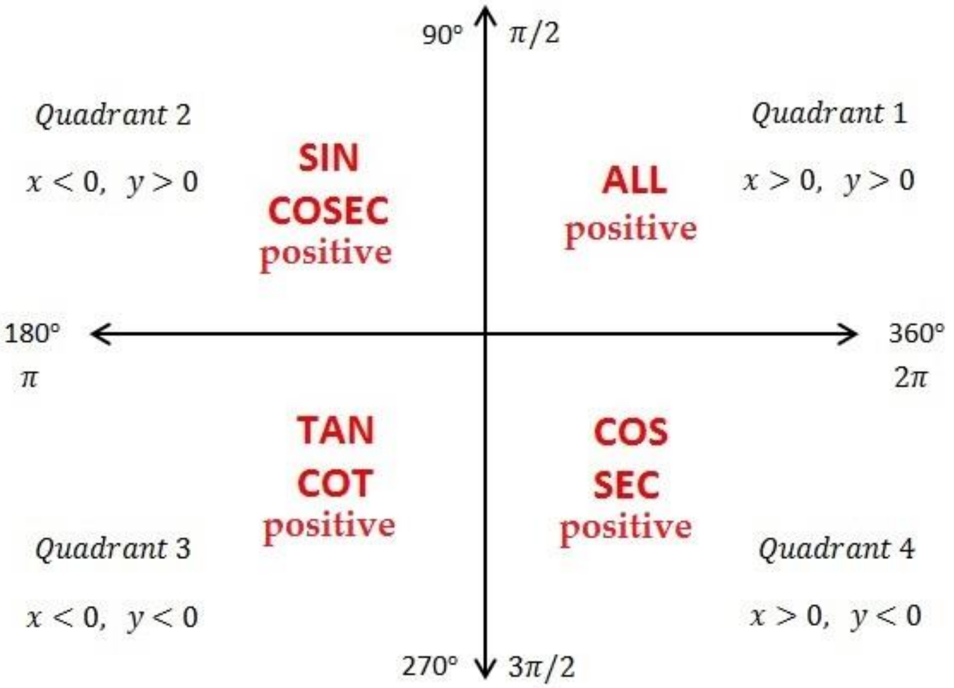


Signs of Trigonometric functions in the quadrants

$$\cos\theta = \frac{x}{r}$$

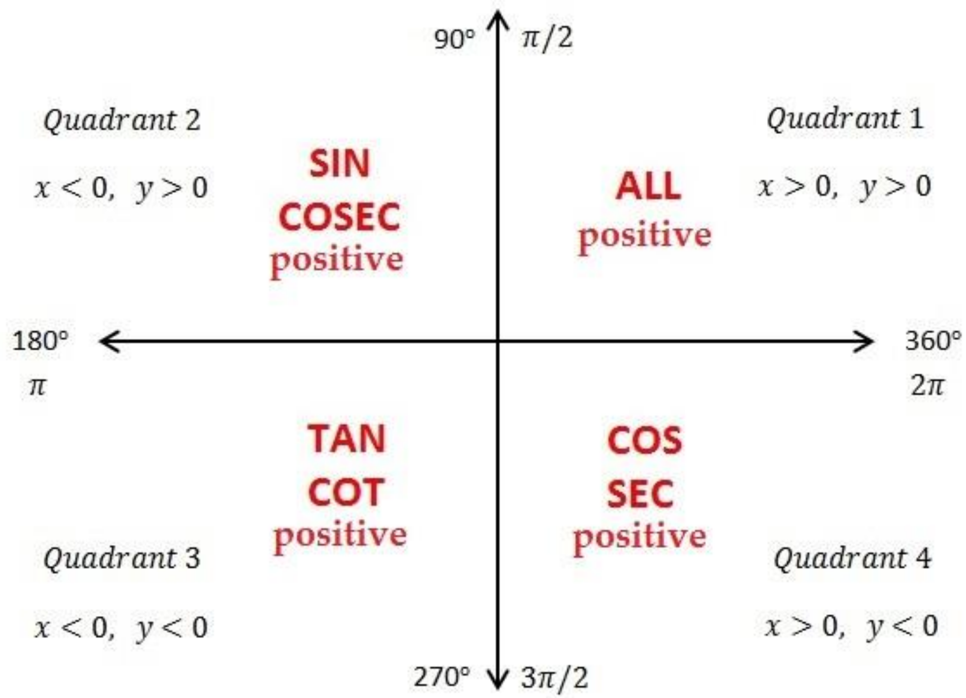
$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}$$



For unit circle $(r = 1)$ $\cos\theta = x$ & $\sin\theta = y$

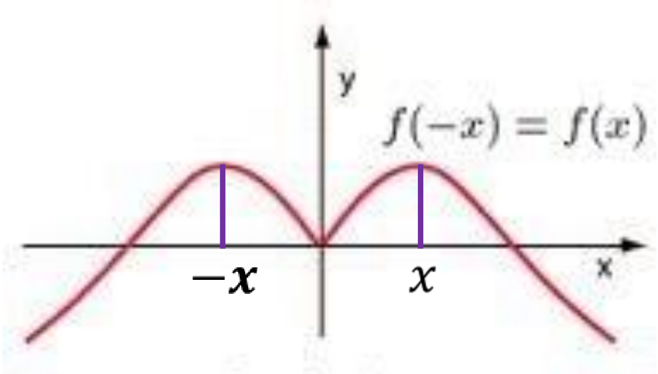
Signs of Trigonometric functions in the quadrants



Example If $\tan\theta = \dfrac{-3}{4}$; $\dfrac{3\pi}{2} \le \theta \le 2\pi$, find $\cos\theta$ and $\sin\theta$.

Even and Odd Trigonometric functions

The function f is said to be an even function if $f(-x) = f(x)$

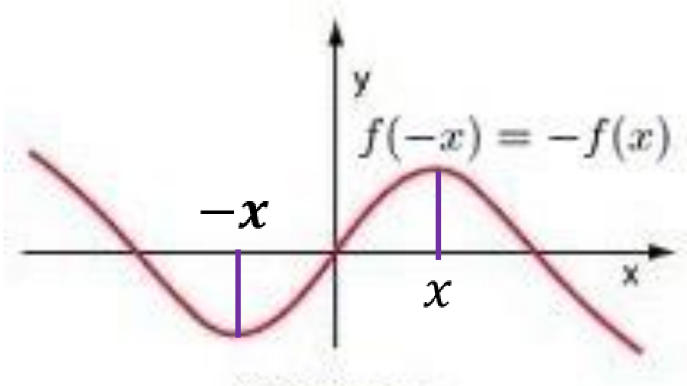


Even function

e.g. $\cos(-\theta) = \cos\theta$

$\Rightarrow$ $\cos$ is an even function.

The function f is said to be an odd function if $f(-x) = -f(x)$



Odd function

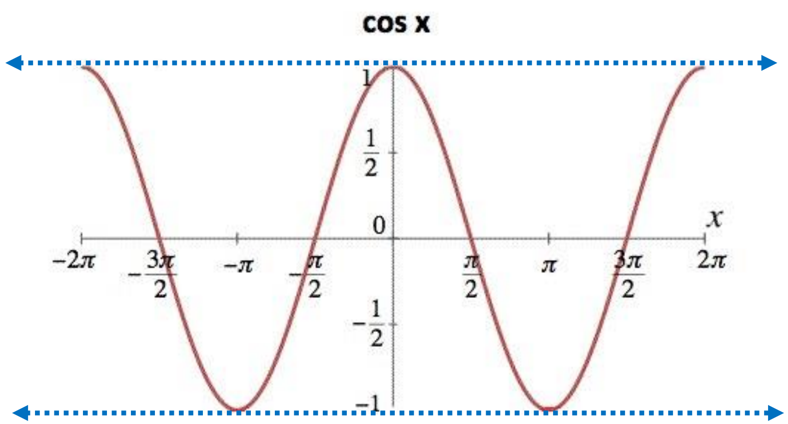
e.g. $\sin(-\theta) = -\sin\theta$

$\Rightarrow$ $\sin$ is an odd function.

Range of Trigonometric functions

From the graph of cosine function, it is clear that

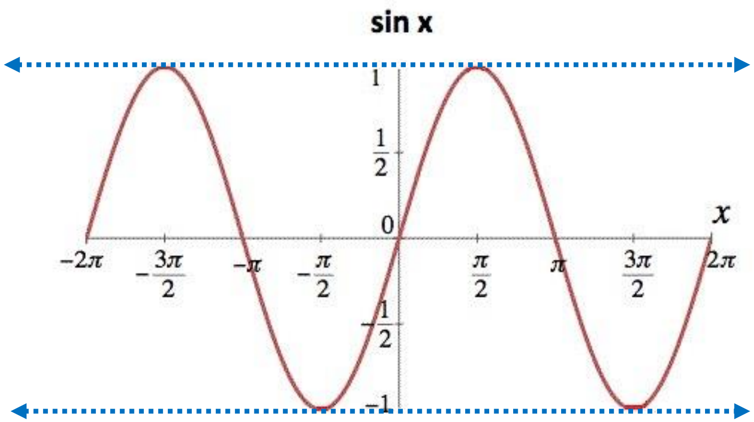
$$-1 \leq \cos\theta \leq 1$$



$\therefore$ Range of cos function is $[-1, 1]$

Similarly,

$$-1 \leq \sin\theta \leq 1$$

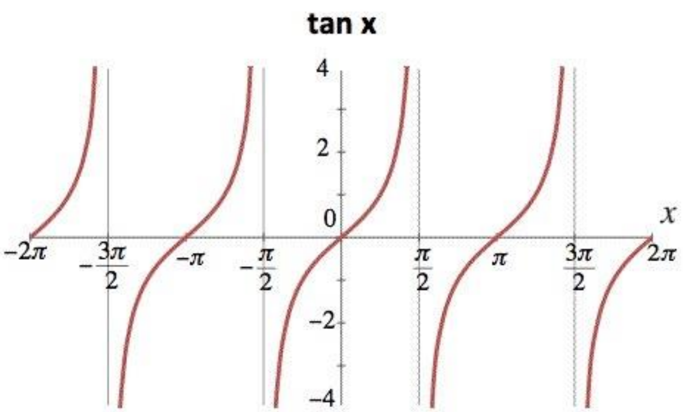


$\therefore$ Range of sin function is $[-1, 1]$

Range of Trigonometric functions

Also, $\tan\theta \in \mathbb{R}$

$\cot\theta \in \mathbb{R}$

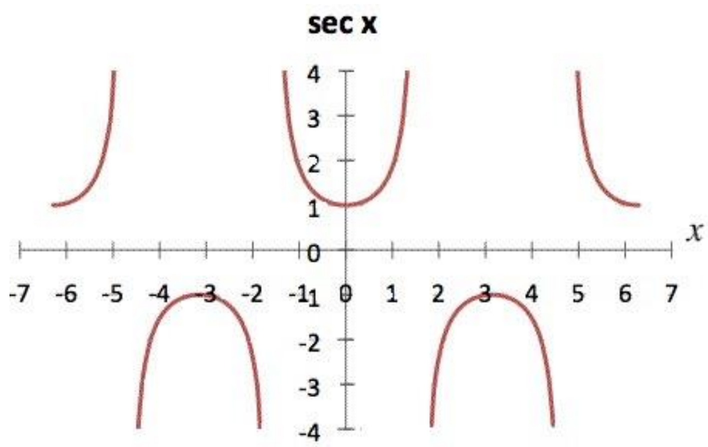


$\therefore$ Range of $\tan$ function is $\mathbb{R}$.

Range of $\cot$ function is $\mathbb{R}$.

And, $\sec\theta \le -1$ **or** $\sec\theta \ge 1$

$\operatorname{cosec}\theta \le -1$ **or** $\operatorname{cosec}\theta \ge 1$



$\therefore$ Range of $\sec$ function is $\mathbb{R} - (-1, 1)$.

Range of cosec function is $\mathbb{R} - (-1, 1)$.

Range of Trigonometric functions

Trigonometric function	Range
sin and cos	$[-1, 1]$
sec and cosec	$\mathbb{R} - (-1, 1)$
tan and cot	$\mathbb{R}$

Q2

Find the Range of $y = 3\cos x + 2$

A $[-1, 1]$

B $\boxed{[-1, 5]}$

C $[-3, 2]$

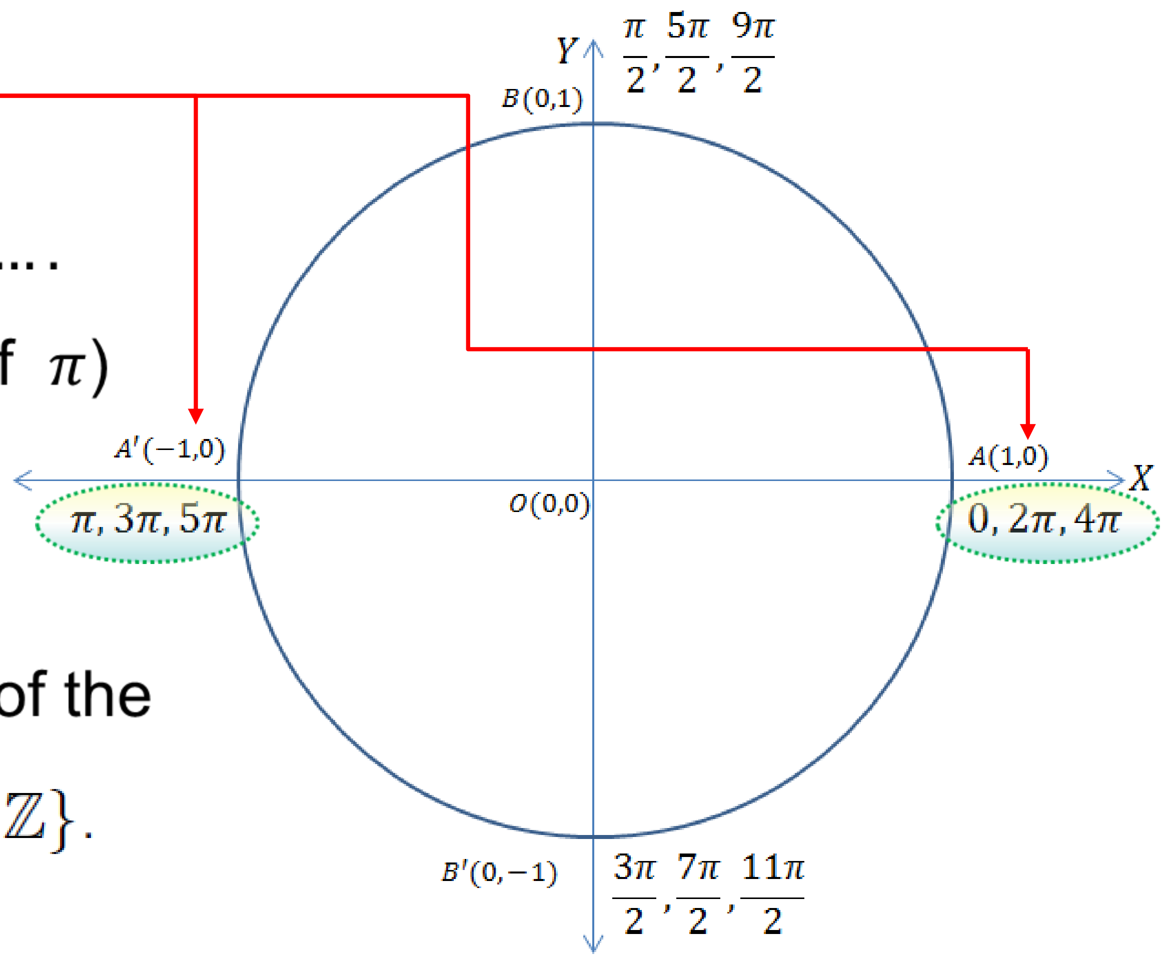
Sets of Zeros of Trigonometric functions

Consider $\sin\theta = 0$

Then, $\theta = 0\,, \pi, 2\pi, 3\pi, \ldots$.

(i.e. multiples of π)

Thus, the set of zeros of the sin function $\{k\pi \,/\, k \in \mathbb{Z}\}$.

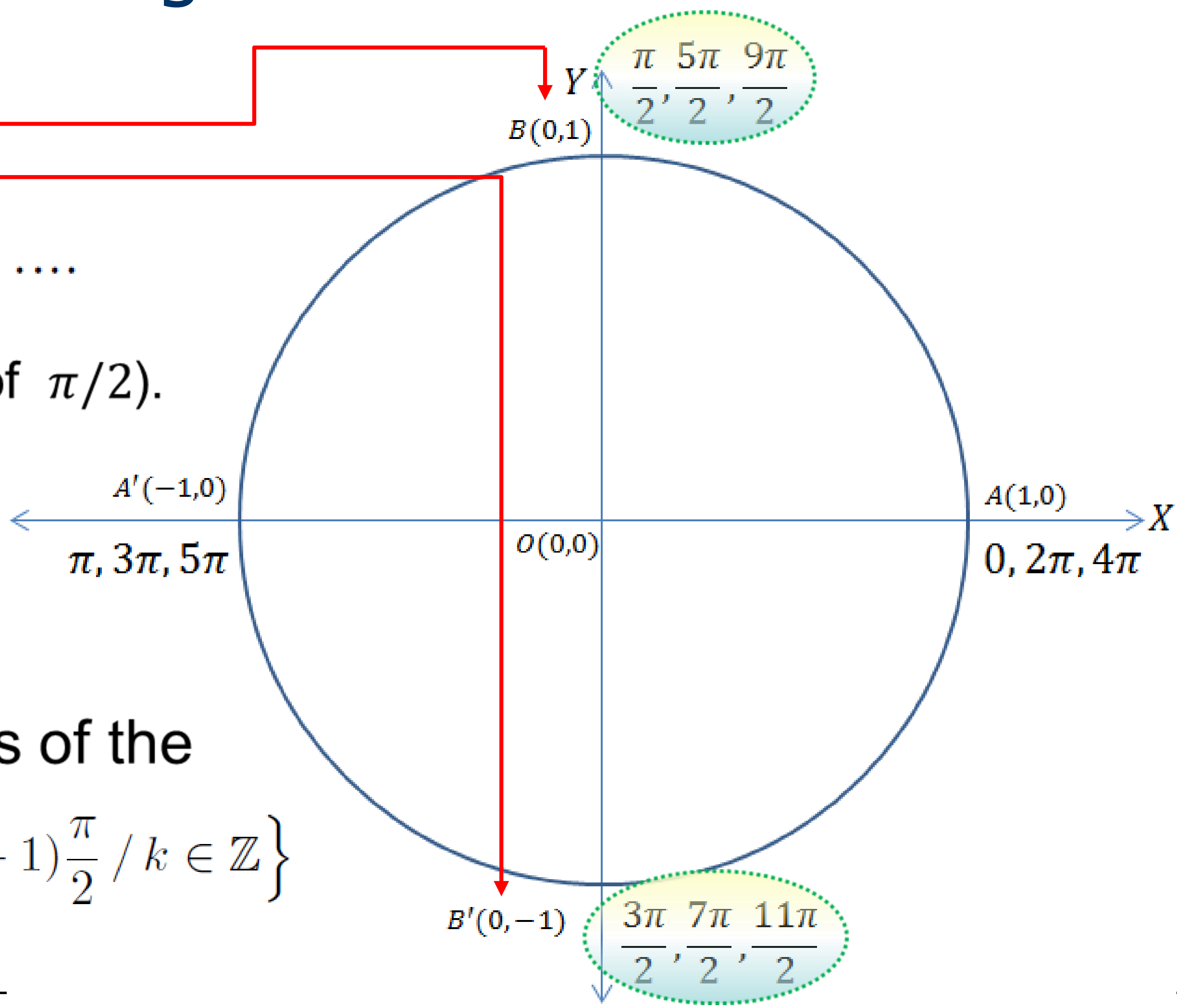


Sets of Zeros of Trigonometric functions

Similarly, $\cos\theta = 0$

$$\Rightarrow \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

(i.e. odd multiples of $\pi/2$).



Thus, the set of zeros of the

$\cos$ function is $\left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$

Note...

Function	Domain	Range	Set of zeros	Period
$\cos$	$\mathbb{R}$	$[-1,1]$	$\left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$	2π
$\sin$	$\mathbb{R}$	$[-1,1]$	$\{k\pi \,/\, k \in \mathbb{Z}\}$	2π
$\tan$	$\mathbb{R} - \left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$	$\mathbb{R}$	$\{k\pi \,/\, k \in \mathbb{Z}\}$	π
$\sec$	$\mathbb{R} - \left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$	$\mathbb{R} - (-1,1)$	ϕ	2π
cosec	$\mathbb{R} - \{k\pi \,/\, k \in \mathbb{Z}\}$	$\mathbb{R} - (-1,1)$	ϕ	2π
$\cot$	$\mathbb{R} - \{k\pi \,/\, k \in \mathbb{Z}\}$	$\mathbb{R}$	$\left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$	π

Solving Trigonometric equations

A trigonometric equation is an equation containing one or more trigonometric functions of the variable, say θ.

Solving for θ means finding the values of θ (in given interval) which makes the trigonometric equation true.

e.g. The solution of $\cos\theta = \frac{1}{2}$ in $\left(0, \frac{\pi}{2}\right)$ is $\frac{\pi}{3}$ radian

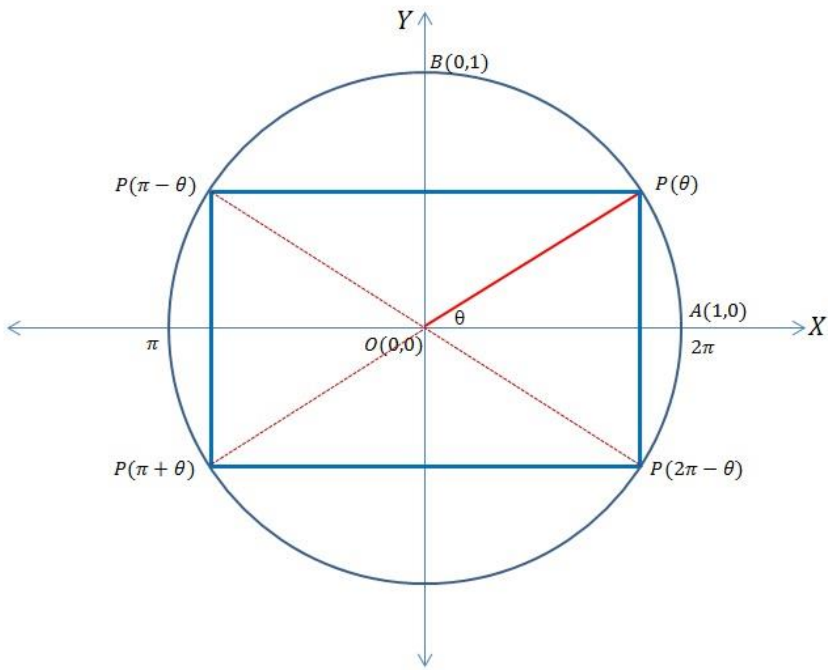
whereas its solution in $(0, 2\pi)$ is

$$\frac{\pi}{3} \quad \text{or} \quad \left(2\pi - \frac{\pi}{3}\right) = \frac{5\pi}{3} \text{ radians.}$$

Worked Examples

1. Solve: $\tan^2\theta - 2\sec\theta + 1 = 0 \quad ; \quad 0 \leq \theta \leq \pi.$

2. Solve: $2\cot^2\theta = 7\operatorname{cosec}\theta - 8 \quad ; \quad 0 < \theta < 180°.$



Further Reading (click on links)

Foundation Algebra by P. Gajjar.

Chapter 5, and Chapter 6 (Sections 6.1 to 6.7)

Foundations of Mathematics by P. Brown.

Chapter 4 (Sections 4.1 to 4.12)

THANKS FOR YOUR ATTENTION