Seminar 4

In this seminar you will study:

- Trigonometric Identities
- Converting angles: from degrees to radians and vice-versa
- Finding range and period of trigonometric functions
- Finding values of trigonometric function
- Solving trigonometric equations

Trigonometric functions

$$\cos\theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{x}{r} = \frac{x}{1} = x$$

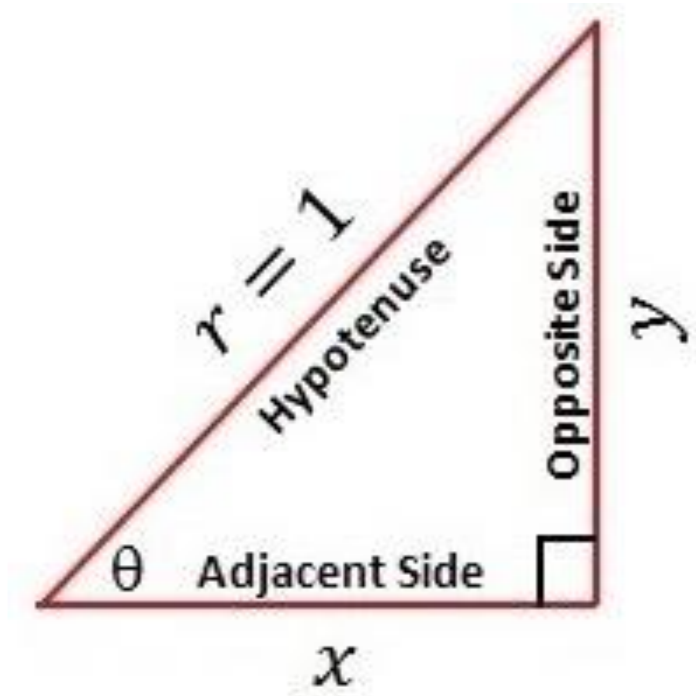
$$\sin\theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{y}{r} = \frac{y}{1} = y$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$



Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta \quad ; \quad \cos\theta \neq 0$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad ; \quad \sin\theta \neq 0$$

Trigonometric identities

Example: Prove that $\dfrac{1+\cot^2\theta}{\operatorname{cosec}^2\theta - 1} = \sec^2\theta$

Solution:

$$\text{LHS} = \frac{1+\cot^2\theta}{\operatorname{cosec}^2\theta - 1}$$

$$= \frac{1 + \dfrac{\cos^2\theta}{\sin^2\theta}}{\dfrac{1}{\sin^2\theta} - 1}$$

$$= \frac{\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}}{\dfrac{1-\sin^2\theta}{\sin^2\theta}}$$

$$= \frac{1}{(1-\sin^2\theta)} \quad = \frac{1}{\cos^2\theta}$$

$$= \sec^2\theta = \text{RHS}$$

Alternative method

$$\text{LHS} = \frac{1+\cot^2\theta}{\operatorname{cosec}^2\theta - 1}$$

$$= \frac{\operatorname{cosec}^2\theta}{\cot^2\theta}$$

$$= \frac{\dfrac{1}{\sin^2\theta}}{\dfrac{\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{1}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} = \sec^2\theta = \text{RHS}$$

Conversion Formulae

- Degrees to Radians

$$\text{angle in radians} = \text{angle in degrees} \times \left(\frac{\pi}{180^\circ}\right)$$

- Radians to Degree

$$\text{angle in degrees} = \text{angle in radians} \times \left(\frac{180^\circ}{\pi}\right)$$

The range of Trigonometric functions

- The range of $\sin$ and $\cos$ functions is: $[-1, 1]$.

i.e. $\quad -1 \leq \cos\theta \leq 1 \quad$ and $\quad -1 \leq \sin\theta \leq 1, \quad \theta \in \mathbb{R}$

- The range of $\sec$ and cosec functions is: $\mathbb{R} - (-1, 1)$.

i.e. $\quad \sec\theta \leq -1 \quad$ or $\quad \sec\theta \geq 1, \quad \theta \neq (2k+1)\dfrac{\pi}{2}, \; k \in \mathbb{Z}$

and $\quad \mathrm{cosec}\,\theta \leq -1 \quad$ or $\quad \mathrm{cosec}\,\theta \geq 1, \quad \theta \neq k\pi, \; k \in \mathbb{Z}$

- The range of $\tan$ and $\cot$ functions is: $\mathbb{R}$.

i.e. $\quad \tan\theta \in (-\infty, +\infty), \quad \theta \neq (2k+1)\dfrac{\pi}{2}, \; k \in \mathbb{Z}$

and $\quad \cot\theta \in (-\infty, +\infty), \quad \theta \neq k\pi, \; k \in \mathbb{Z}$

The range of Trigonometric functions

Example: Find the range of $f(x) = 5 - 3\sin(4x - 7)$

Solution:

For any $\theta \in \mathbb{R}$, $-1 \leq \sin\theta \leq 1$.

For $f(x) = 5 - 3\sin(4x - 7)$, the angle θ is $4x - 7$.

$\Rightarrow \quad -1 \leq \sin(4x - 7) \leq 1$

$\Rightarrow \quad -1 \times (-3) \leq \sin(4x - 7) \times (-3) \leq 1 \times (-3)$ Multiply the inequality through by (-3)

$\Rightarrow \quad 3 \geq -3\sin(4x - 7) \geq -3$

$\Rightarrow \quad -3 \leq -3\sin(4x - 7) \leq 3$

$\Rightarrow \quad -3 + (5) \leq -3\sin(4x - 7) + (5) \leq 3 + (5)$ Add (5) to the inequality

$\Rightarrow \quad 2 \leq 5 - 3\sin(4x - 7) \leq 8$

$\Rightarrow \quad 2 \leq f(x) \leq 8 \quad \Rightarrow$ The range of f : $R_f = [2, 8]$

The period of Trigonometric functions

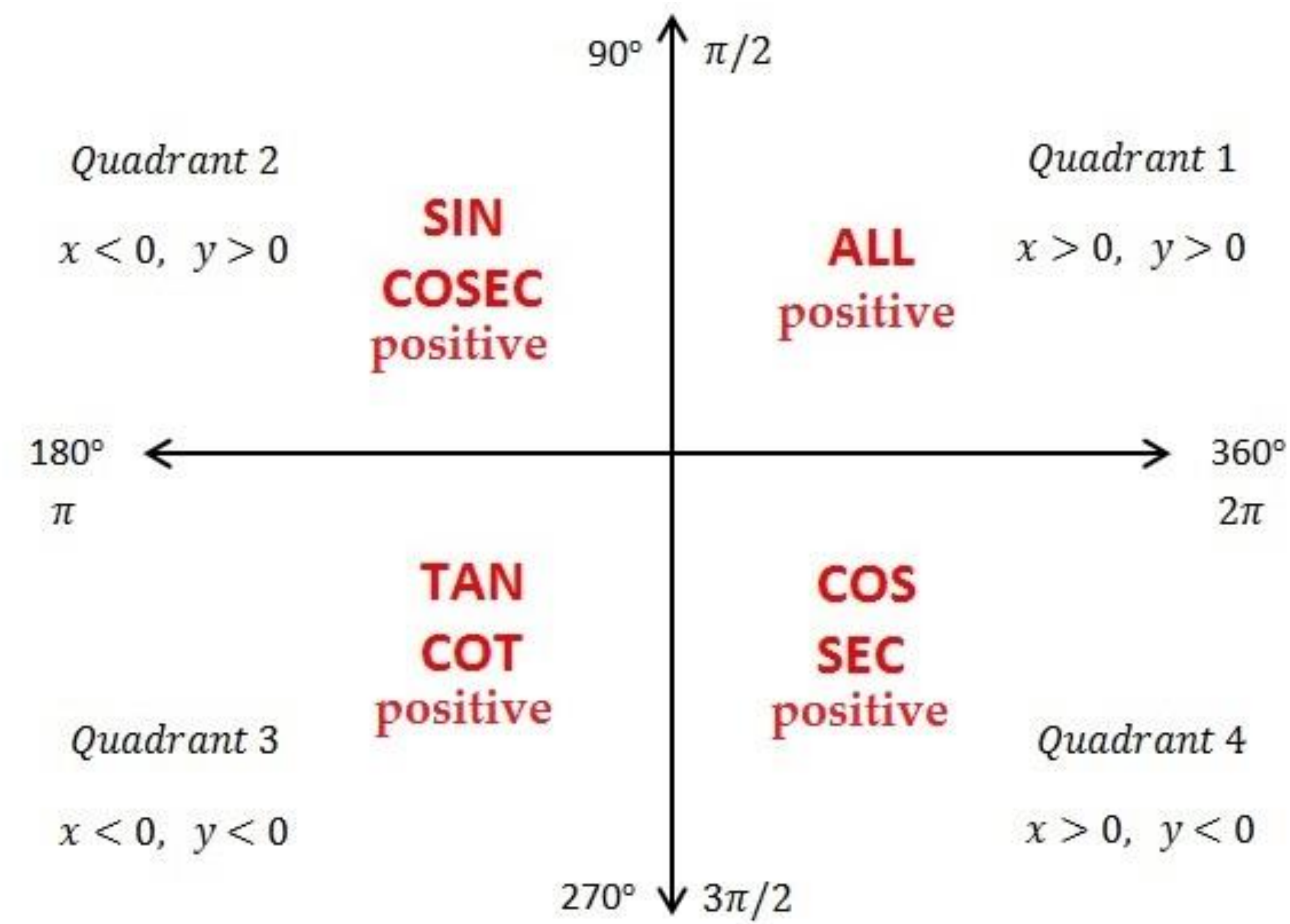
- The period (principal period) of $aT_1(bx+c)+d$ is $\dfrac{2\pi}{|b|}$,

where T_1 is the trigonometric function: $\sin$, $\cos$, cosec, or $\sec$.

- The period (principal period) of $aT_2(bx+c)+d$ is $\dfrac{\pi}{|b|}$,

where T_2 is the trigonometric function: $\tan$ or $\cot$.

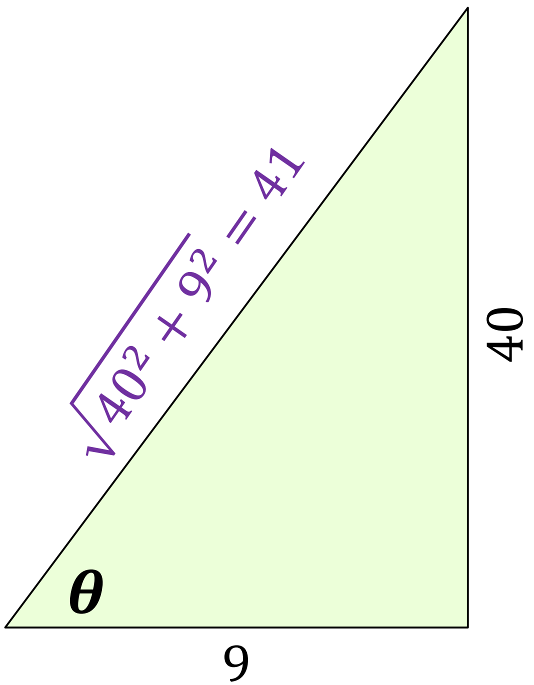
Signs of Trigonometric functions in the quadrants



Finding values of Trigonometric functions

Example: If $\cot\theta = -\dfrac{9}{40}$, find $\cos\theta + \sin\theta$, where $\dfrac{3\pi}{2} < \theta < 2\pi$.

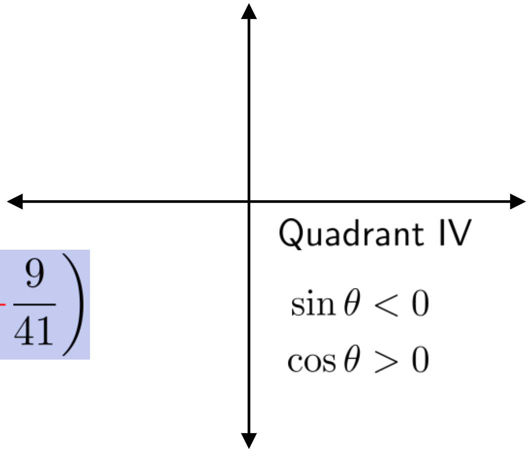
Solution:



$$\cot\theta = -\frac{9}{40} \quad \Rightarrow \quad \tan\theta = -\frac{40}{9}$$

Since $\dfrac{3\pi}{2} < \theta < 2\pi$

θ is in Quadrant IV



$$\sin\theta + \cos\theta = \left(-\frac{40}{41}\right) + \left(+\frac{9}{41}\right)$$

$$= -\frac{31}{41}$$

Solving Trigonometric equations

Example 1: Solve $\sin\theta = \frac{1}{2}, \quad \theta \in [0, \pi].$

Solution:

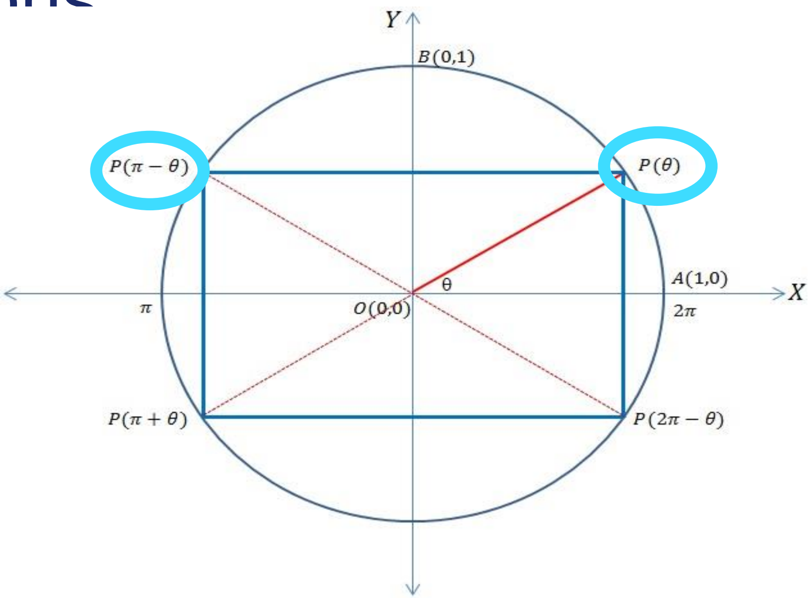
$$\sin\theta = \frac{1}{2}$$

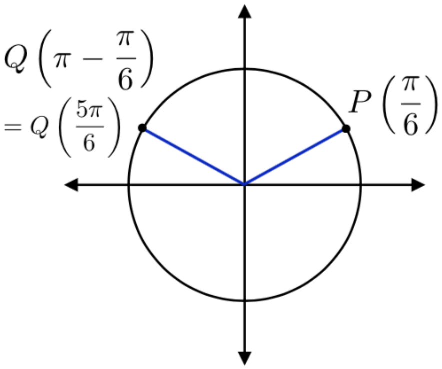
$$\therefore \text{ reference angle in quadrant I is } \alpha = \frac{\pi}{6}$$

$$\text{But } \theta \in [0, \pi]$$

$$\therefore \quad \theta = \begin{cases} \dfrac{\pi}{6} \\ \\ \dfrac{5\pi}{6} \end{cases}$$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1





Solving Trigonometric equations

Example 2: Solve for $\theta \in [0, 2\pi]$, $\sin^2\theta + 2\sin\theta - 3 = 0$.

Solution:

Let $\sin\theta = t$

$\therefore\ t^2 + 2t - 3 = 0$

$\Rightarrow (t+3)(t-1) = 0$

$\Rightarrow t = -3 \text{ or } t = 1$

But $\sin\theta \in [-1, 1]$

$\therefore\ \sin\theta \neq -3$

$\Rightarrow\ \sin\theta = 1$

$\therefore$ reference angle in quadrant I is $\alpha = \dfrac{\pi}{2}$

since $\theta \in [0, 2\pi]$

$\therefore\ \theta = \dfrac{\pi}{2}$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

THANKS FOR YOUR ATTENTION