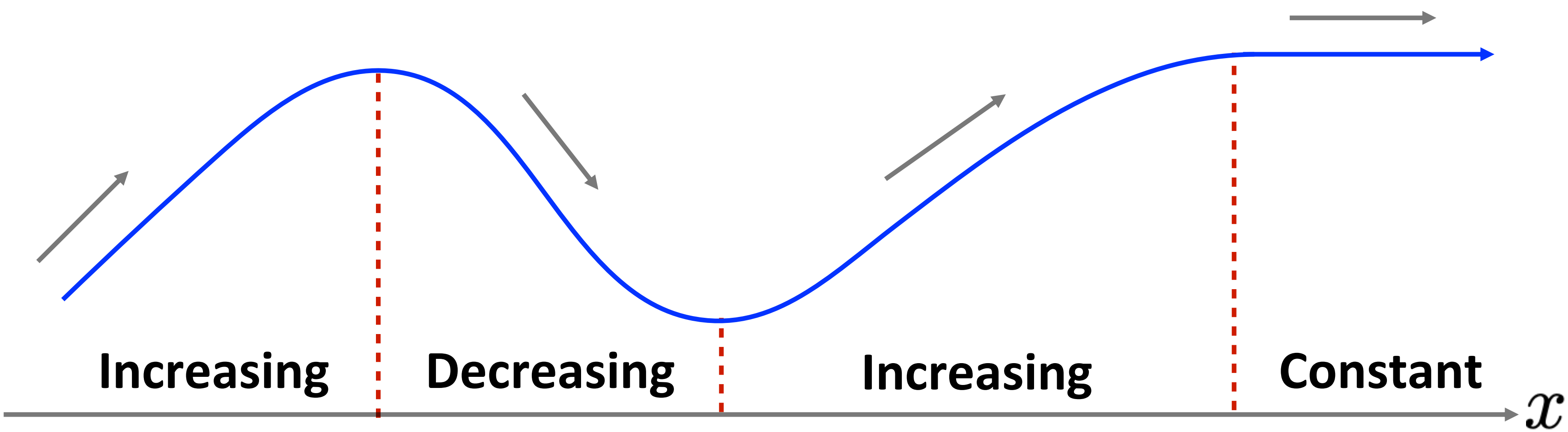
Lecture 03

Structure of lecture

1. Increasing, decreasing and constant functions
2. Critical points, stationary points, inflection points
3. Relative maximum and minimum
4. First derivative test, second derivative test
5. Absolute extrema on a closed interval
6. Optimisation Problem
7. Newton-Raphson Method

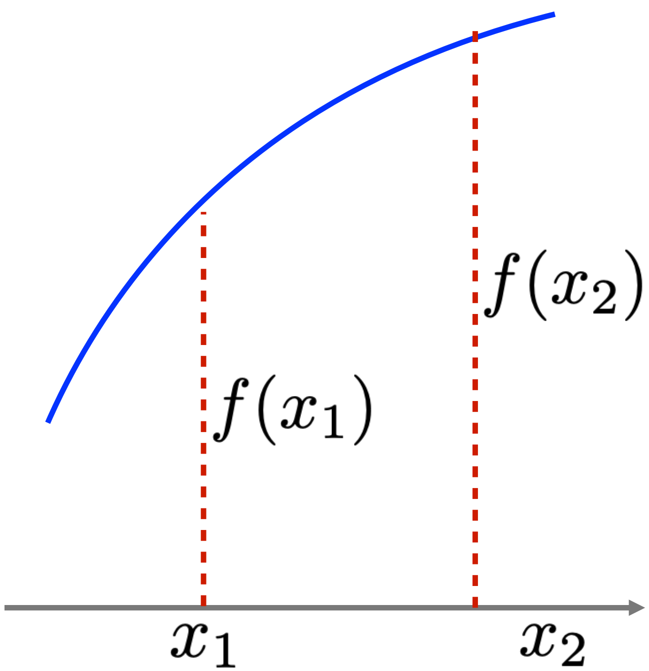
Increasing, Decreasing and Constant Functions

The terms increasing, decreasing, and constant are used to describe the behaviour of a function as we travel from left to right along its graph.



Increasing, Decreasing and Constant Functions

Increasing function



If for a function f

$$x_2 > x_1 \;\Rightarrow\; f(x_2) > f(x_1)$$

then f is increasing ($\uparrow$) function.

$$\therefore\; f'(x) \approx \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Thus, $\boxed{f \text{ is } \uparrow \text{ if } f'(x) > 0}$

Similarly, f is $\downarrow$ if $f'(x) < 0$

and f is constant if $f'(x) = 0$

Increasing, Decreasing and Constant Functions

Example 1: Show that $f(x) = 3e^x + x^3 - 20$ is always increasing.

$$f'(x) = 3e^x + 3x^2$$

$$\because e^x > 0 \text{ and } x^2 \geq 0$$

$$\therefore f'(x) = 3e^x + 3x^2 > 0, \forall x \in \mathbb{R}$$

$$\therefore f(x) \text{ is always increasing.}$$

Increasing, Decreasing and Constant Functions

Example 2: Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and the intervals on which it is decreasing.

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2)$$

$$\therefore\ f'(x) = 0 \ \Rightarrow\ 12x(x^2 + x - 2) = 0$$

$$\Rightarrow\ x(x+2)(x-1) = 0$$

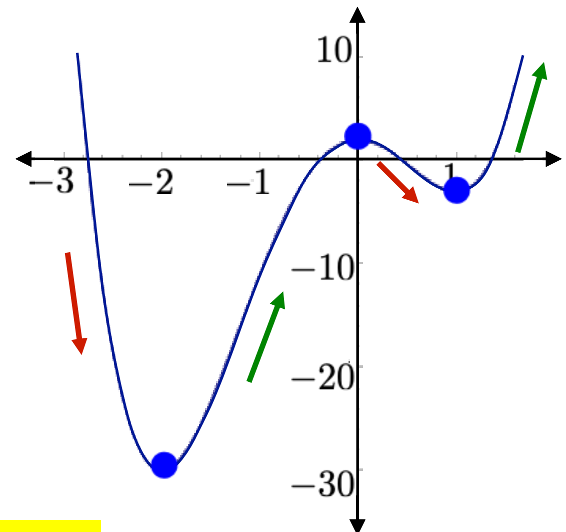
$$\Rightarrow\ x = 0, 1, -2$$

Increasing, Decreasing and Constant Functions

$$x(x+2)(x-1)$$

$f'(x) = 12x^3 + 12x^2 - 24x$		
$x < -2$	$f'(x) < 0$	$f \downarrow$ in $(-\infty, -2)$
$-2 < x < 0$	$f'(x) > 0$	$f \uparrow$ in $(-2, 0)$
$0 < x < 1$	$f'(x) < 0$	$f \downarrow$ in $(0, 1)$
$x > 1$	$f'(x) > 0$	$f \uparrow$ in $(1, \infty)$

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$



Note:

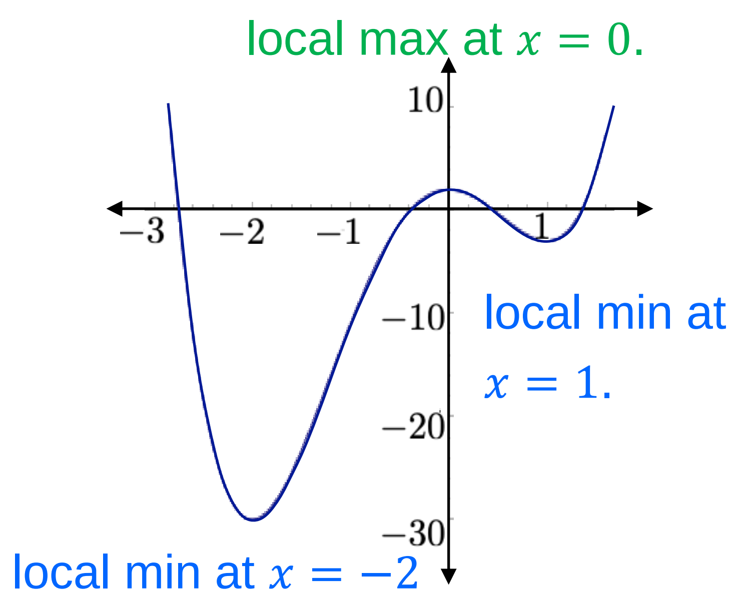
$$f'(-2) = f'(0) = f'(1) = 0$$

Relative Extrema

A function f is said to have a relative/local maximum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value.

Similarly, f is said to have a relative/local minimum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value.

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$



Stationary and Critical Points

A stationary point is a point on the graph of a function where the derivative is zero.

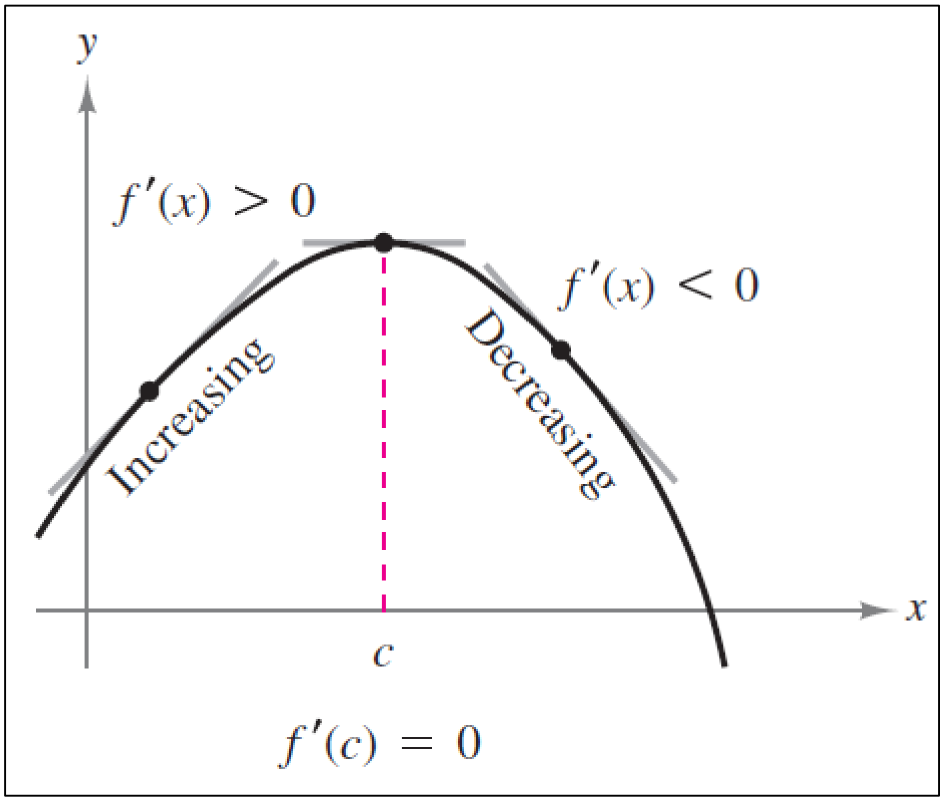
Thus, for a stationary point x, $f'(x) = 0$.

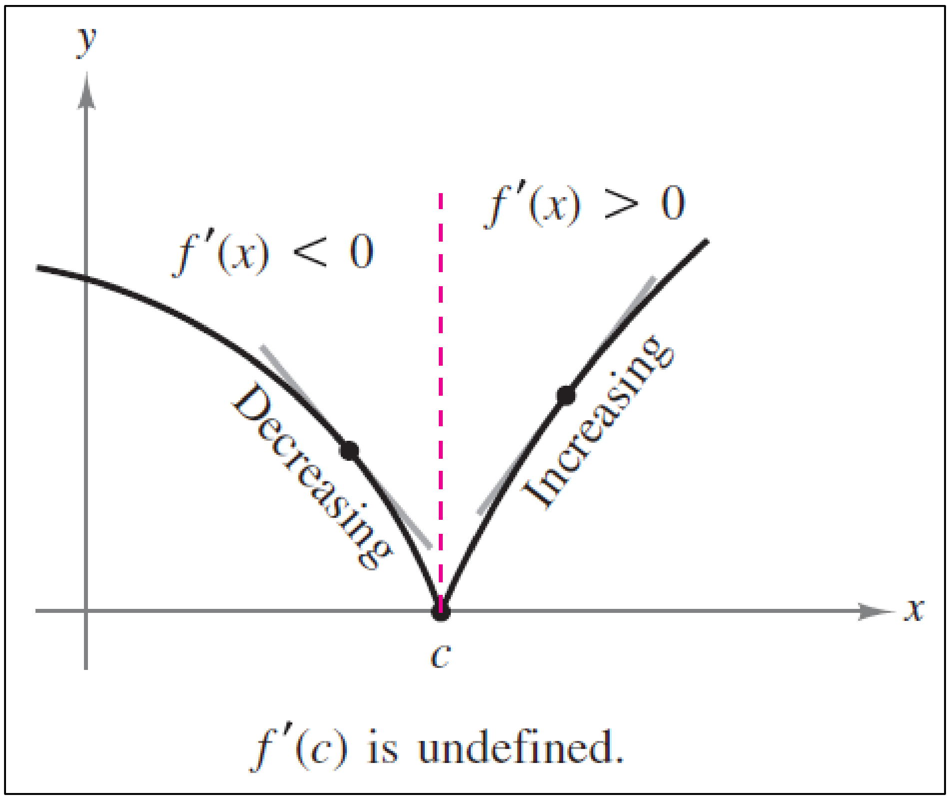
A critical point of a function is any value in the domain where either the function is not differentiable, or its derivative is zero.

e.g. $x = 0$ is a critical point of $f(x) = |x|$.

Note: Stationary points are critical points.

Stationary and Critical Points





c is a critical point (also stationary point)

c is a critical point, but not a stationary point

Critical Points

Example 3: Find all the critical points of the function:

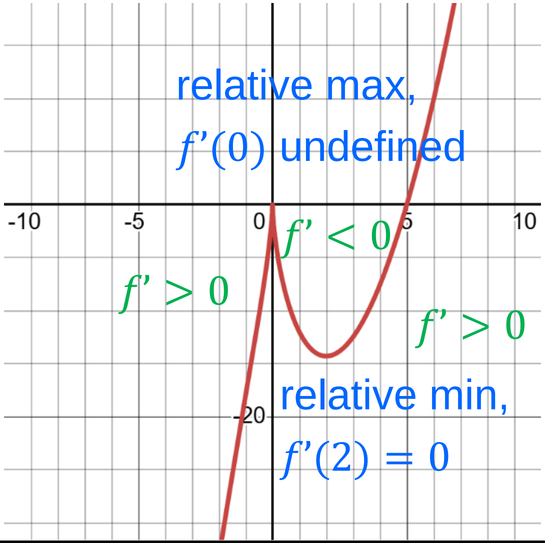
$$f(x) = 3x^{5/3} - 15x^{2/3}$$

$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2) = \frac{5(x-2)}{x^{\frac{1}{3}}}$$

Critical points:

$x = 2$ (stationary point);

$x = 0$ (f is not differentiable because $f'(0)$ is undefined).



First Derivative Test

A function f has a relative extremum at critical points where <u>f’ changes sign</u>.

Suppose that f is continuous at a critical point c.

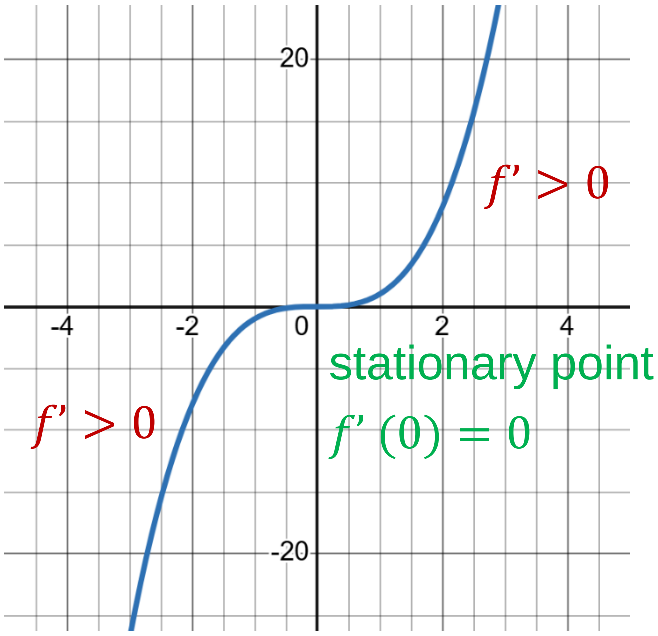
a) If $f'(x) > 0$ on an open interval extending left from c and $f'(x) < 0$ on an open interval extending right from c, then f has a *relative maximum* at c.

b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from c, then f has a *relative minimum* at c.

First Derivative Test

Note: If $f'(x)$ does not change sign at c, then f has no relative extrema at c.

e.g. $f(x) = x^3$ has a stationary point at $x = 0$, but it does *not* have a relative maximum or minimum at $x = 0$ because $f'(x) > 0$ on both $(-\infty, 0)$ and $(0, \infty)$.



Second-order derivative

The derivative of $\dfrac{dy}{dx}$ is called the second order derivative of $y = f(x)$ and is denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$.

Thus,

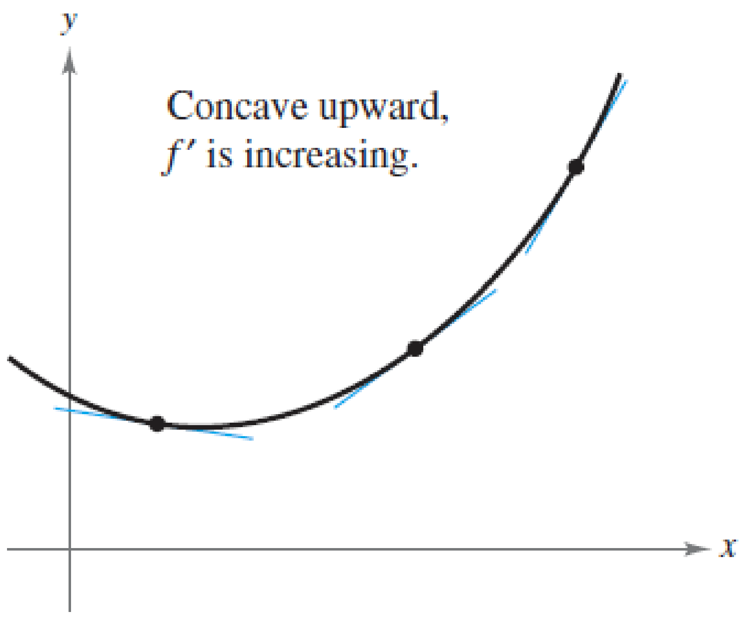
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) = \frac{d^2y}{dx^2}$$

Concavity and Point of Inflection

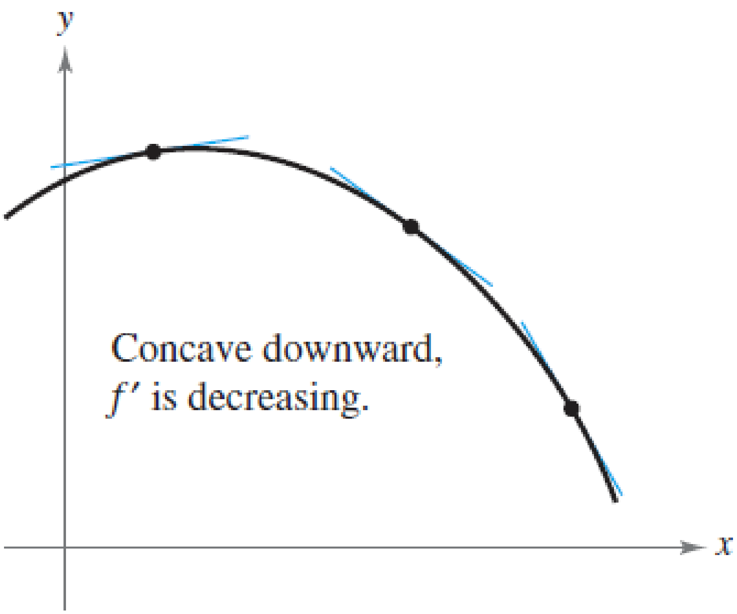
If f is differentiable on an open interval, then f is said to be

- Concave up on the interval if f' is increasing on that interval. *slope of the curve is increasing*
- Concave down on the interval if f' is decreasing on that interval. *slope of the curve is decreasing*

Concavity and Point of Inflection



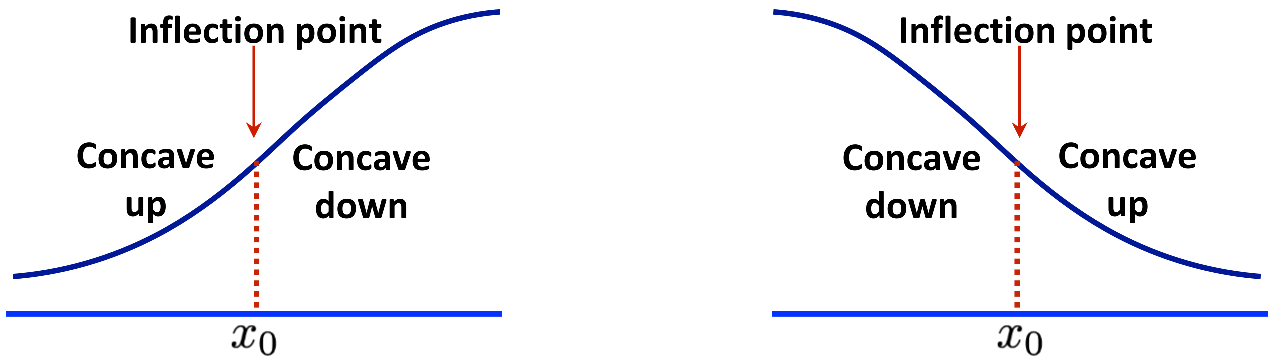
The graph of f lies above its tangent lines.



The graph of f lies below its tangent lines.

Concavity and Point of Inflection

An inflection point is a point on a curve at which it changes concavity.



The curve changes from being concave upward (positive curvature) to concave downward (negative curvature), or vice versa.

Second Derivative Test

There is another test for relative extrema that is based on the geometric observation:

A function f has a relative maximum at a stationary point if the graph of f is concave down (f' is decreasing $\Rightarrow f'' < 0$) on an open interval containing that point;

And f has a relative minimum at the point if the graph is concave up (f' is increasing $\Rightarrow f'' > 0$) .

Second Derivative Test

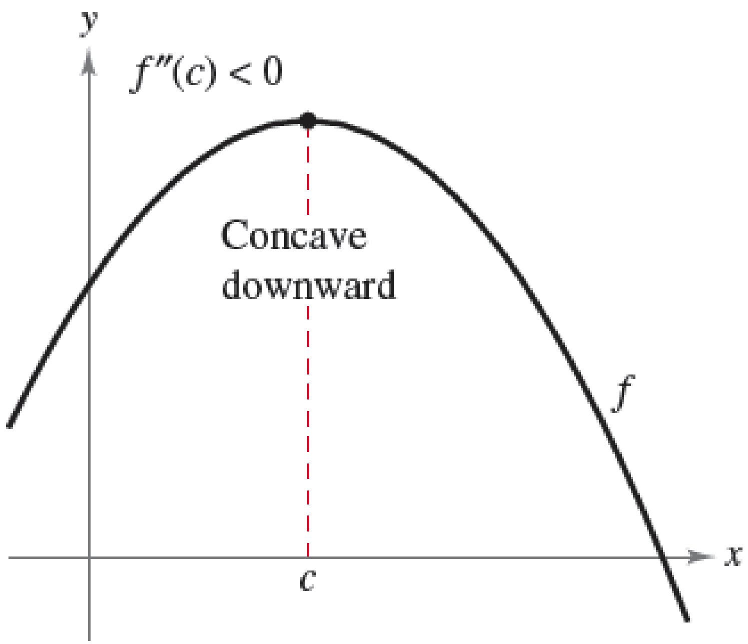
Suppose that f is has a stationary point at c. That is, $f'(c) = 0$.

a) If $f''(c) < 0$, then f has a *relative maximum* at c.

b) If $f''(c) > 0$, then f has a *relative minimum* at c.

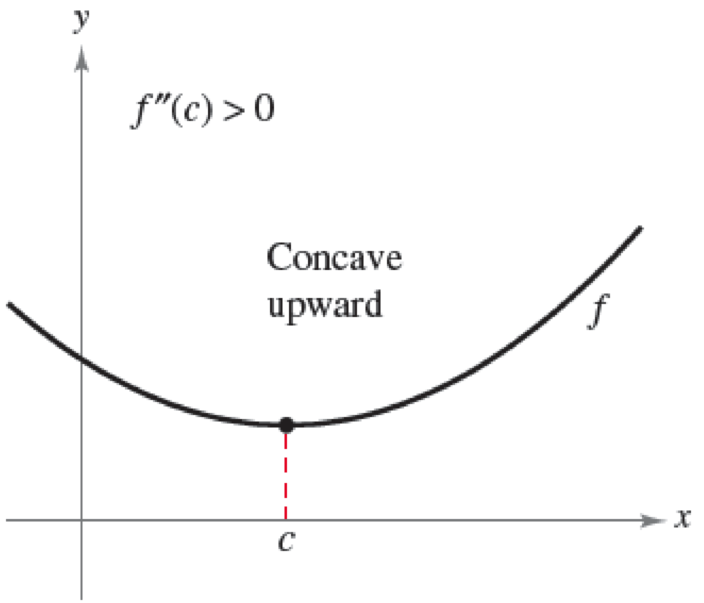
c) If $f''(c) = 0$, then the test is <u>inconclusive</u>. Further discussion is required.

Second Derivative Test



If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

a)



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.

b)

Classifying stationary points

Example 4: Given $f(x) = 2x^3 - 7x^2 + 4x - 5.$

Find the stationary points of f. Apply the **second derivative test** to classify the stationary points.

Also, sketch the graph of $y = f(x)$.

$$\because f'(x) = 6x^2 - 14x + 4$$

$$\therefore f'(x) = 0 \Rightarrow 6x^2 - 14x + 4 = 0$$

$$\Rightarrow 2(3x-1)(x-2) = 0$$

$$\Rightarrow x = \frac{1}{3} \text{ or } 2$$

Classifying stationary points

$$\because f(x) = 2x^3 - 7x^2 + 4x - 5$$

$$\therefore \text{When } x = \frac{1}{3}$$

$$f\left(\frac{1}{3}\right) = 2 \cdot \left(\frac{1}{3}\right)^3 - 7 \cdot \left(\frac{1}{3}\right)^2 + 4 \cdot \frac{1}{3} - 5 = -\frac{118}{27}$$

$$\text{When } x = 2$$

$$f(2) = 2 \cdot 2^3 - 7 \cdot 2^2 + 4 \cdot 2 - 5 = -9$$

$$\therefore \text{The stationary points are } \left(\frac{1}{3}, -\frac{118}{27}\right) \text{ and } (2, -9).$$

Classifying stationary points

Using the Second Derivative Test

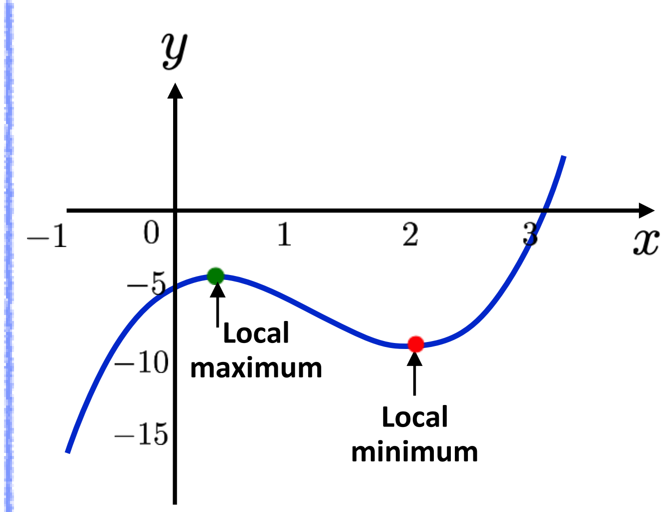
$$f'(x) = 6x^2 - 14x + 4 \Rightarrow \; f''(x) = 12x - 14$$

$$\therefore f''(x)\big|_{x=\frac{1}{3}} = 12 \cdot \frac{1}{3} - 14 = -10 < 0$$

$$\Rightarrow \left(\frac{1}{3}, -\frac{118}{27}\right) \text{ is a point of local maximum.}$$

$$\therefore f''(x)\big|_{x=2} = 12 \cdot 2 - 14 = 10 > 0$$

$$\Rightarrow (2, -9) \text{ is a point of local minimum.}$$



Classifying stationary points

Example 5: Given $f(x) = x^4$. Find and classify the stationary point(s).

$$\because f'(x) = 4x^3$$

$$\therefore f'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0 \Rightarrow f(0) = 0$$

$$\therefore \text{The stationary point is } (0, 0).$$

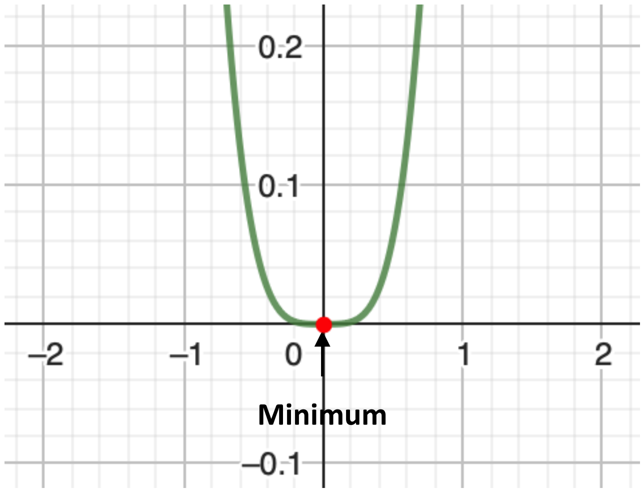
$$\text{If we use the Second Derivative Test } \Rightarrow f''(x) = 12x^2$$

$$\therefore f''(x)|_{x=0} = 12 \times 0^2 = 0 \Rightarrow \text{ Further discussion is required.}$$

Classifying stationary points

$f'(x) = 4x^3$		
$x < 0$	$f'(x) < 0$	f is $(\downarrow)$ in $(-\infty, 0)$
$x > 0$	$f'(x) > 0$	f is $(\uparrow)$ in $(0, +\infty)$

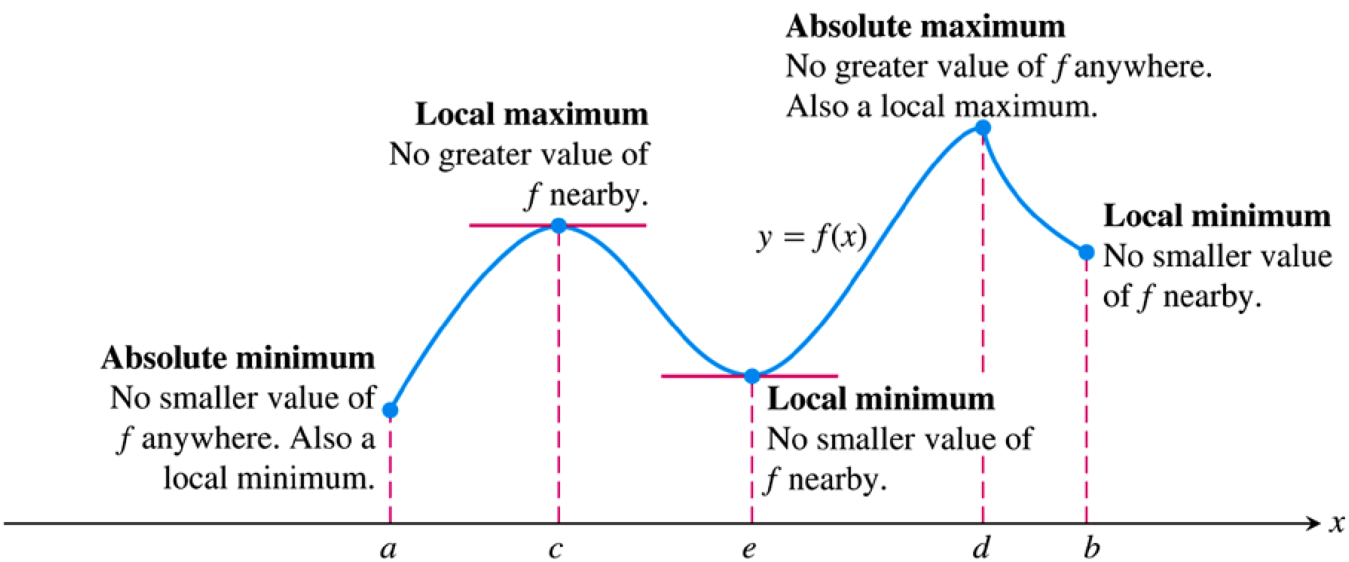
First Derivative Test



$\therefore (0,0)$ is a point of minimum.

Absolute Extrema

- If $f(x) \le f(x_0)$ for all x in an interval I, then f has an *absolute maximum* at x_0.
- If $f(x) \ge f(x_0)$ for all x in an interval I, then f has an *absolute minimum* at x_0.



Identifying types of maxima and minima for a function with domain $a \le x \le b$

Finding the absolute extrema of f on $[a, b]$

If a function f is continuous on a closed interval $[a, b]$, then the absolute extrema occur either at the endpoints of the interval or at critical points inside the interval.

Example 6: Find the absolute extrema of $f(x) = 3x^{5/3} - 15x^{2/3}$ on the interval $[-1, 3]$.

	x	$f(x)$	
End point	-1	$\mathbf{-18}$	**Absolute Min.**
Critical point	0	$\mathbf{0}$	**Absolute Max.**
Critical point	2	-14.29	
End point	3	-12.48	

Example 7

A factory that produces vehicles wants to maximize its profit. Suppose that $R(Q) = 10Q - Q^2/1000$, for all $Q \in [0, 7000]$ gives the total revenue generated in a certain period by producing and selling Q units. And $C(Q)$ $= 5000 + 2Q$ $(Q \geq 0)$ gives the associated total productions cost. **Find the value Q that maximizes profits.**

Solution

The profit function is

$$P(Q) = R(Q) - C(Q) = -\frac{Q^2}{1000} + 8Q - 5000$$

$$\Rightarrow P'(Q) = -\frac{Q}{500} + 8$$

$\Rightarrow Q = 4000$ is the only critical point.

Find the absolute maximum of P on interval $[0, 7000]$:

Q	$P(Q)$
0	$-5{,}000$
7000	$2{,}000$
$\mathbf{4000}$	$11{,}000$

Solving Optimization (maximum/minimum) Problems

- List the known and unknown quantities
- Identify what is to be optimised
- It is useful to draw a diagram
- Assign symbols for all quantities (e.g. A for area, r for radius)
- Express the quantity to be optimised as a function of others

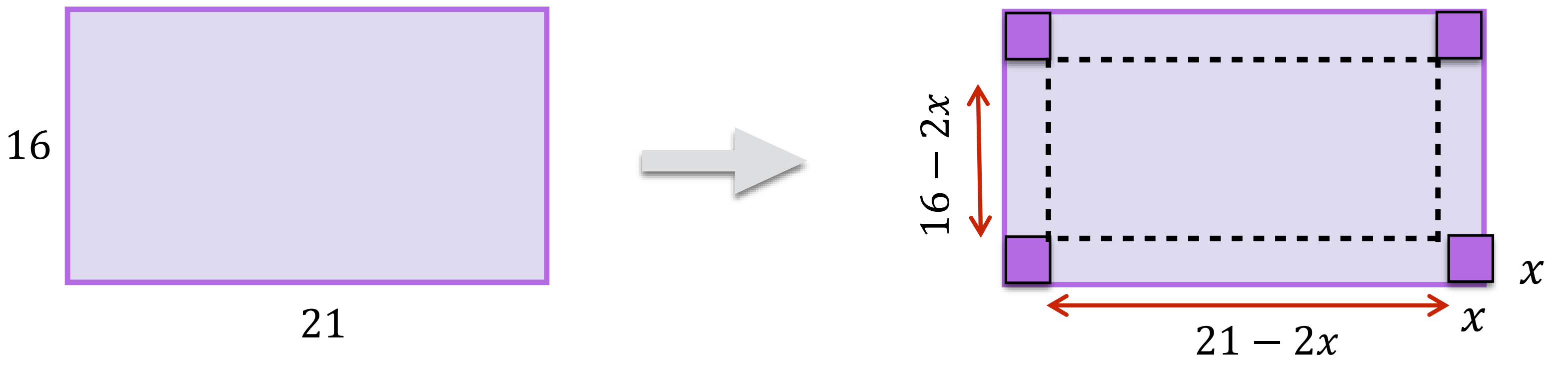
  e.g. $V = \pi r^2 h$ for the volume of a cylinder
- Reduce the expression to only one variable e.g. $V = \pi r^2 f(r)$
- Apply the second derivative test

Solving Optimization (maximum/minimum) Problems

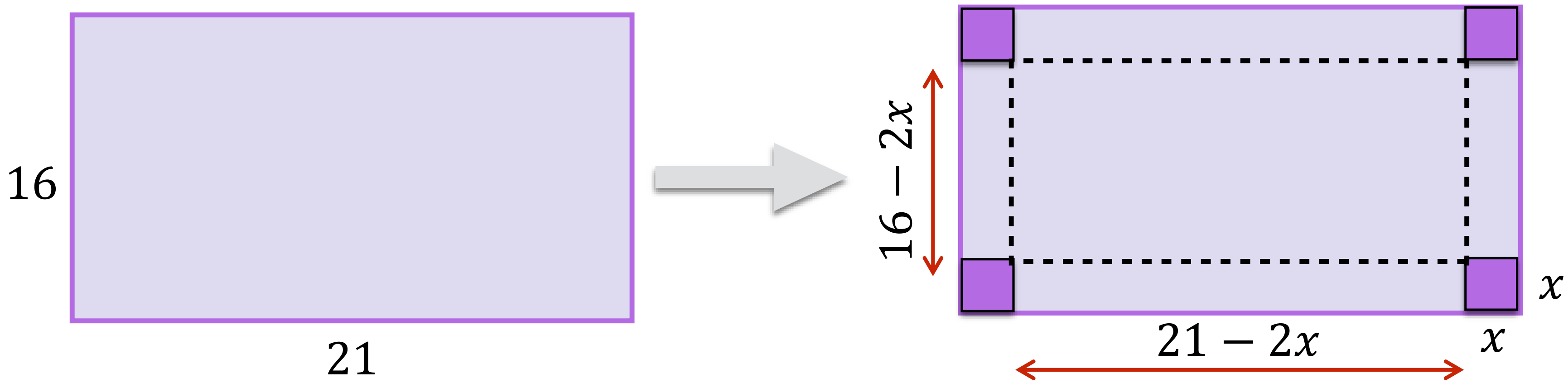
Example 8

An open-top box with a rectangular base is to be constructed from a rectangular sheet of metal with dimensions 16 x 21 inches, by cutting the same size square from each corners and then bending up the resulting sides. **Find the size of the corner square to be removed so as to maximize the volume of the box so formed.**

Solution



Solving Optimization (maximum/minimum) Problems



$f(x) = \text{Volume}$

$= x(21 - 2x)(16 - 2x)$ is to be maximized.

$$f(x) = x(21 - 2x)(16 - 2x)$$
$$= x(336 - 74x + 4x^2)$$
$$= 4x^3 - 74x^2 + 336x$$

Solving Optimization (maximum/minimum) Problems

$$\Rightarrow \quad f'(x) = 12x^2 - 148x + 336$$

$$\therefore \quad f'(x) = 0 \Rightarrow x = 3 \text{ or } \frac{28}{3}$$

$$\therefore \quad x = 3 \text{ (as } \frac{28}{3} \text{ is not possible.)}$$

$x > 0, 21 - 2x > 0$ and $16 - 2x > 0$

and $f''(x)\,|_{x=3} = 24x - 148\,|_{x=3} = -76 < 0$

$\therefore$ V is maximum, when $x = 3$.

Newton-Raphson Method

Recap :: Methods for solving equations

Linear equation: $ax + b = 0$ $\Rightarrow \; x = -\dfrac{a}{b}$

Quadratic equation: $ax^2 + bx + c = \; 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic equation: Cardano's method

Quartic equation: Ferrari's method

Equation of degree : ≥ 5

analytical solution impossible.

Abel-Ruffini Theorem

Numerical Methods:

Bisection method

Fixed-point iteration method

(You learnt last semester)

Newton Raphson Method: uses the concept of derivatives to find roots

Newton-Raphson Method

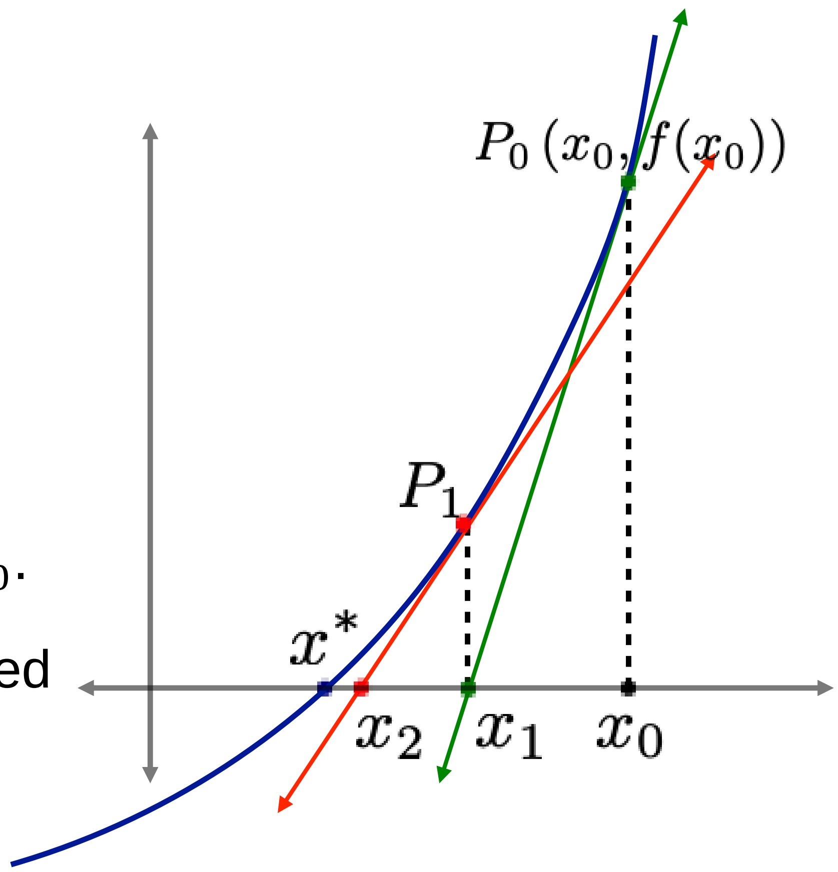
Let x_0 be the initial approximation.

As a first step, we approximate the curve by a tangent line at $P_0(x_0, f(x_0))$.

Let the tangent line at P_0 intersect the X-axis at x_1.

Then, x_1 is a better approximation than x_0.

Repeat the process until the root of desired accuracy is obtained.



Newton-Raphson Method

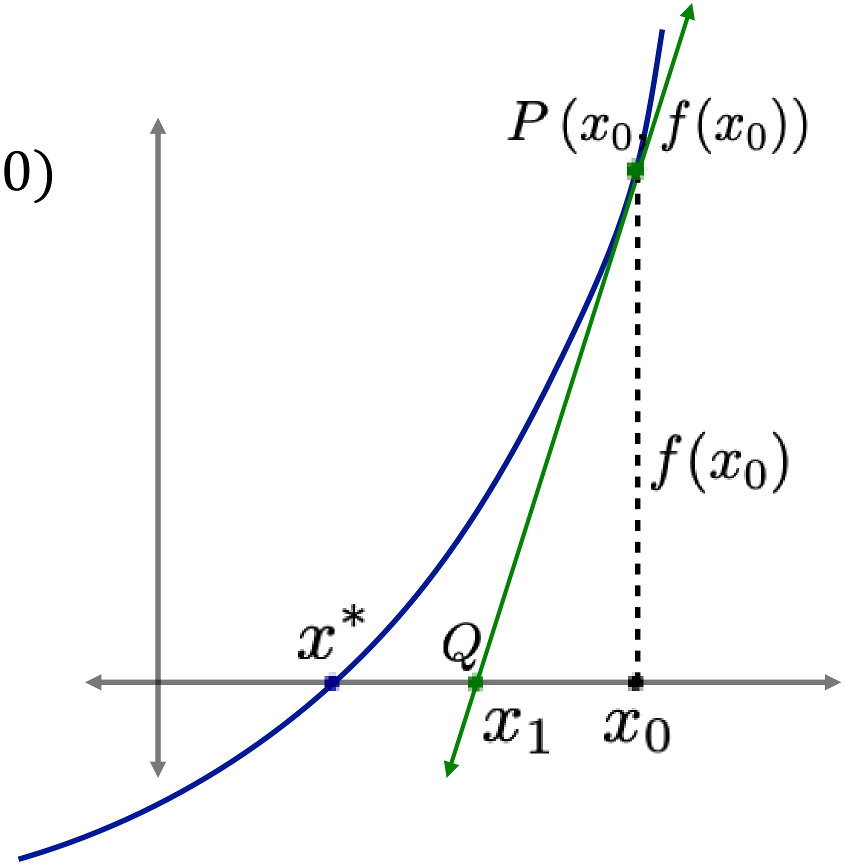
Formulation

Here, $P \leftrightarrow (x_0, f(x_0))$ and $Q \leftrightarrow (x_1, 0)$

By definition,

Slope of $\overleftrightarrow{PQ} \;=\; \frac{d}{dx} f(x) \Big|_{x=x_0}$

$$\Rightarrow \frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$



Newton-Raphson Method

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0) \quad \Rightarrow \quad x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Generalising the result, we have

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \quad \text{where } n = 0, 1, 2, 3, \ldots\ldots}$$

Newton-Raphson Method

Example 9

Apply the Newton-Raphson method to approximate $\sqrt{2}$ correct to 6 d.p.

$$x = \sqrt{2}$$
$$\Rightarrow x^2 - 2 = 0$$

$\therefore$ We have to solve
$f(x) = x^2 - 2 = 0$

Differentiating $f(x)$,
we have $f'(x) = 2x$

The Newton-Raphson formula is:

$$x_{n+1}$$
$$= x_n - \frac{f(x_n)}{f'(x_n)}$$
$$\Rightarrow x_n - \frac{x_n^2 - 2}{2x_n} \Rightarrow \frac{x_n^2 + 2}{2x_n}$$

Newton-Raphson Method

Example 9

Apply the Newton-Raphson method to approximate $\sqrt{2}$ correct to 6 d.p.

Starting with $x_0 = 1$, successive approximations are as shown in the adjacent table.

$$\therefore \quad \sqrt{2} \; = \; \mathbf{1.414214} \quad \text{(6 d.p.)}$$

n	x_n
0	1.000000
1	1.500000
2	1.416667
3	1.414216
4	1.414214
5	1.414214

Example (From Examination Paper)

3 (a) A function f is defined by $f(x) = 2x^3 - 9x^2 - 24x + 60$.

(*i*) Find the stationary points of f by solving the equation $f\,'(x) = 0$.

(*ii*) Use the second derivative test to classify the stationary points obtained in 3(a)(*i*) as the points of maximum or minimum values.

(*iii*) Sketch the curve $y = f(x)$.

(*iv*) Given that one of the roots of the equation $f(x) = 0$ lies in the interval $(5, 6)$, use the Newton-Raphson iteration formula

$$x_{n+1} = \frac{4\,x_n^3 - 9\,x_n^2 - 60}{6\,x_n^2 - 18\,x_n - 24} \quad ; \quad n = 0,\ 1,\ 2,\ 3,\ \ldots .$$

with $x_0 = 5.5$ to approximate this root correct to 6 decimal places.

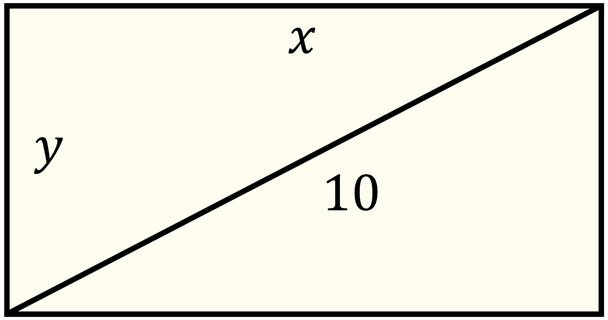
Write all of the x_n values until the sequence of approximations converges. [7]

Practice Problems

Example

Find the dimensions of the rectangle with the largest area whose diagonal is $10m$.

Ans: $x = y = 5\sqrt{2}\,m$



Area $= A = xy$ is to be maximized.

From figure, $x^2 + y^2 = 100$

$$\Rightarrow \quad y = \sqrt{100 - x^2}$$

i.e. $f(x) = x\sqrt{100 - x^2}$ is to be maximized.

Thank you!