Topic 1: Increasing and Decreasing Functions

Illustration: Show that $f(x) = e^x + x^3 + 1$ is always increasing

$$f'(x) = e^x + 3x^2$$

Since $f'(x) > 0$ for all $x \in \mathbb{R}$, $f(x)$ is always increasing

1. Show that $f(x) = e^{-2x} + 2$ is decreasing $\forall\ x\ \in D_f$.

Answer:

2. Show that $f(x) = \cos x$ is increasing in the fourth quadrant.

Answer:

Topic 1: Increasing and Decreasing Functions

Illustration: Given $f(x) = x^3 + 3x^2 + 9$. Determine the intervals in which the function $f(x)$ is increasing and the intervals in which it is decreasing.

$f'(x) = 3x^2 + 6x$ For stationary points $f'(x) = 0$

$\therefore\ 3x^2 + 6x = 0 \ \Rightarrow\ x(x+2) = 0$

$\Rightarrow f(x)$ has stationary points at $x = -2,\ 0$

$f(x) = x^3 + 3x^2 + 9$			
x	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
$f'(x)$	> 0	< 0	> 0
$f(x)$	increasing	decreasing	increasing

1. Given $f(x) = 4x^3 - 3x^2 - 6x$. Determine the intervals in which $f(x)$ is increasing and the intervals in which it is decreasing.

Answer:

2. Given $f(x) = 2x^3 + 9x^2 + 12x - 1$. Determine the intervals in which $f(x)$ is increasing and the intervals in which it is decreasing.

Answer:

Topic 2: Classification of Stationary points

Second Derivative Test:

- If $f'(x) = 0$ for some $x = x_0$, and $f''(x)|_{x=x_0} < 0$, then f has a maximum value at $x = x_0$.
- If $f'(x) = 0$ for some $x = x_0$, and $f''(x)|_{x=x_0} > 0$, then f has a minimum value at $x = x_0$.

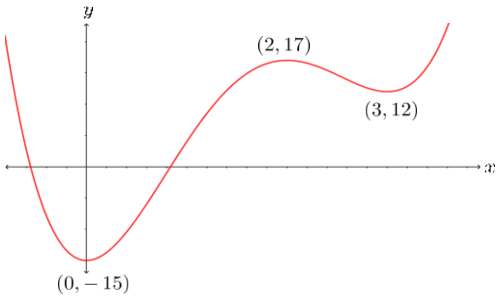
Illustration: Find the stationary points for the function:
$f(x) = 3x^4 - 20x^3 + 36x^2 - 15.$
Use the second derivative test to classify the stationary points as points of maximum and minimum, and plot the curve of $y = f(x)$

$f'(x) = 12x^3 - 60x^2 + 72x$

For stationary points $f'(x) = 0$

$\therefore\ 12x^3 - 60x^2 + 72x = 0$

$\Rightarrow\ x(x-2)(x-3) = 0$



$\Rightarrow\ f(x)$ has stationary points at $x = 0,\ 2,\ 3$

$f''(x) = 36x^2 - 120x + 72$

$f''(x)|_{x=0} = 72 > 0$ $\quad\boxed{\therefore\ f(x) \text{ has a minimum value at } x = 0}$

Also $f(0) = -15$

$\therefore (0, -15)$ is a point of minimum value

$f''(x)|_{x=2} = -24 < 0$ $\quad\boxed{\therefore\ f(x) \text{ has a maximum value at } x = 2}$

Also $f(2) = 17$

$\therefore (2, 17)$ is a point of maximum value

$f''(x)|_{x=3} = 36 > 0$ $\quad\boxed{\therefore\ f(x) \text{ has a minimum value at } x = 3}$

Also $f(3) = 12$

$\therefore (3, 12)$ is a point of minimum value

For the following functions, find and classify the stationary point. Sketch the graph of $y = f(x)$.

1. $f(x) = x^3 - 3x^2 - 9x + 10$

Answer:

2. $f(x) = 8x^3 - 9x^2 + 3x$

Answer:

For the following functions, find and classify the stationary point. Sketch the graph of $y = f(x)$.

3. $f(x) = 3x^4 - 10x^3 + 6x^2 + 5$

Answer:

4. $f(x) = x^4 + 4x^3 + 4x^2 + 1$

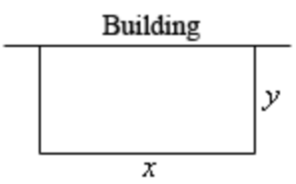
Answer:

Topic 3: Optimisation Problems

Key Concepts:

- List the known and unknown functions
- Identify what is to be optimised
- It is useful to draw a diagram
- Assign symbols for all quantities (e.g. A for area, r for radius)
- Express the quantity to be optimised as a function of others e.g. $V = \pi r^2 h$ for the volume of a cylinder
- Reduce the expression to only one variable e.g. $V = f(r)$
- Apply the second derivative test

Illustration 1: We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.



A $= xy$ needs to be maximised

Condition: $x + 2y = 500$ ft

$\therefore x = 500 - 27 \Rightarrow A(y) = (500 - 2y) \cdot y = -2y^2 + 500y$

For maximum/minimum value $A'(y) = 0$

$\Rightarrow A'(y) = -4y + 500 = 0 \quad \Rightarrow y = 125$

Now $A''(y) = -4 < 0$

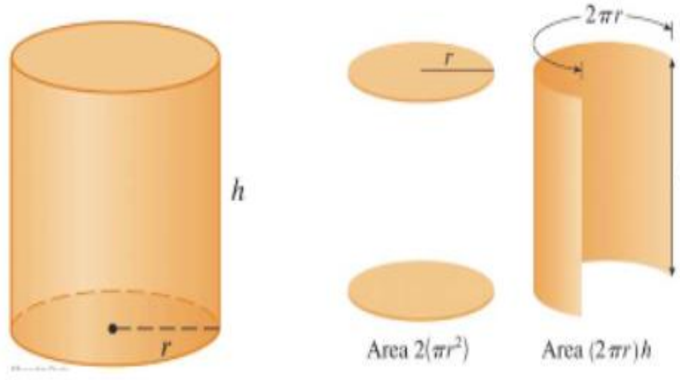
$\therefore$ A has a maximum point when $y = 125$

$\therefore$ The largest area A $= (500 - 2 \cdot 125) \cdot 125 = 31250$ ft^2

Topic 3: Optimisation Problems

Illustration 2: A cylindrical can must be made to hold 1 L of liquid. Find the dimensions that minimize the cost of metal used to make it.

© Media.cheg



$$1\text{L} = 1000 \ \ \text{cm}^3$$

$$V = \pi r^2 h \Rightarrow 1000 = \pi r^2 h \qquad (1)$$

To minimise cost of metal,
the total surface area S must be minimised

$$S = 2\pi r h + 2\pi r^2$$

From (1) $h = \dfrac{1000}{\pi r^2}$ $\quad \therefore S = f(r) = \dfrac{2000}{r} + 2\pi r^2$

For maximum/minimum value $f'(r) = 0$

$$\Rightarrow f'(r) = 4\pi r - \frac{2000}{r^2} = 0$$

$$\Rightarrow r^3 = \frac{500}{\pi} \ \therefore \ \ r = \sqrt[3]{\frac{500}{\pi}}$$

Now $f''(r) = 4\pi + \dfrac{4000}{r^3}$ $\quad \Rightarrow f''(r)\Big|_{r=\sqrt[3]{\frac{500}{\pi}}} = 12\pi > 0$

$\therefore f$ has minimum value at $r = \sqrt[3]{\dfrac{500}{\pi}}$

$\therefore S$ is minimised when: $r = \sqrt[3]{\dfrac{500}{\pi}}$ cm

and $h = \dfrac{1000}{\pi r^2} = \dfrac{1000}{\pi} \cdot \left(\dfrac{\pi}{500}\right)^{\frac{2}{3}}$ cm $= \sqrt[3]{\dfrac{4000}{\pi}}$ cm

1. An 80 cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other into a rectangle with one side 4 times the length of the other side. Determine where the wire should be cut to minimise the area enclosed by both the triangle and the rectangle.

Answer:

2. An open tank is to be constructed with a square base and vertical sides so as to contain $500\ \mathrm{m}^3$ of water. Determine the dimension of the tank if the area of the metal sheet used in its construction is to be minimised.

Answer:

3. Find the point on the curve $y = x^2$ that is nearest to the point $(18, 0)$.

Hint: $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Answer:

Topic 4: Newton-Raphson Method

Key Formula: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 0, 1, 2, 3, \cdots)$$

Illustration: Given $f(x) = x^4 - \sin x - 1$.

Use the Newton-Raphson formula to show that:

$$x_{n+1} = \frac{3x_n^4 - x_n \cos x_n + \sin x_n + 1}{4x_n^3 + \cos x_n} \qquad (1).$$

Use Equation (1) above to approximate the root of $f(x) = 0$ correct to 5 d.p., take $x_0 = 1.5$.

$$f'(x) = 4x^3 - \cos x$$

$$x_{n+1} = x_n - \frac{x_n^4 - \sin x_n - 1}{4x_n^3 - \cos x_n}$$

$$x_{n+1} = \frac{3x_n^4 - x_n \cos x_n + \sin x_n + 1}{4x_n^3 - \cos x_n}$$

Step 1: Set calculator to RADIAN mode: **Shift Mode 4**

Step 2: Fix calculator to 5 d. p.: **Shift Mode 6 5**

Step 3: Set up Iterative formula on calculator **On calculator:**

Start with: $x_0 = 1.5$ Enter 1.5 and press "="

Step 4: Enter the Newton-Raphson formula (replace x_n with ANS)

$$x_{n+1} = \frac{3x_n^4 - x_n \cos x_n + \sin x_n + 1}{4x_n^3 - \cos x_n}$$

On calculator: $$\frac{(3(\mathtt{ANS})^4 - (\mathtt{ANS})\cos(\mathtt{ANS}) + \sin(\mathtt{ANS}) + 1)}{(4(\mathtt{ANS})^3 - \cos(\mathtt{ANS}))}$$

Topic 4: Newton-Raphson Method

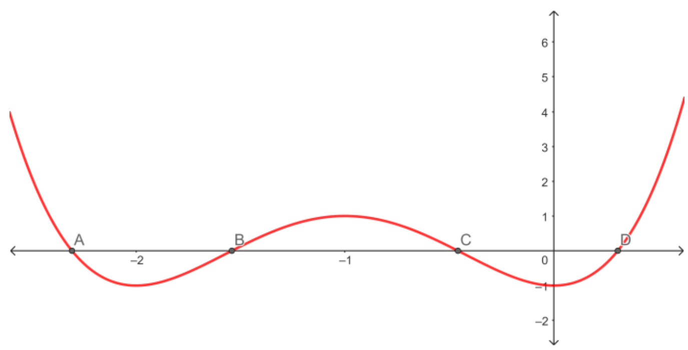
Step 5: Write down succesive approximations

n	x_n
0	1.5
1	1.27177
2	1.18853
3	1.17787
4	1.17770
5	1.17770

Note: All approximations and the final result must be given with the required d.p.

Note: The desired root is obtained when succesive approximations are equal

$\therefore$ the desired root $x^* = 1.17770$



Use the Newton-Raphson formula in the following equation to approximate the root at the following points for $f(x) = 0$, give your answer correct to 5 d.p.

1. Point **A** take $x_0 = -2.6$.
2. Point **B** take $x_0 = -1.8$.
3. Point **C** take $x_0 = -0.6$.
4. Point **D** take $x_0 = 0.1$.

$$x_{n+1} = \frac{6x_n^4 + 16x_n^3 + 8x_n^2 + 1}{8x_n^3 + 24x_n^2 + 16x_n}$$

Answer:

1. $2\sin x - x = 0,\ x_0 = 2.$ **Answer:**	2. $2\cos x - x^2 = 0,\ x_0 = 1.$ **Answer:**
3. $x^4 - x^2 = 1,\ \ x_0 = -1.5.$ **Answer:**	4. $x^3 - 2x - 5 = 0,\ x_0 = 2.$ **Answer:**