# Foundation Calculus and Mathematical Techniques CELEN037

## Practice Problems SET-7 Sample Solution

### Type 1: The method of $t$-substitution

1. Evaluate the following integrals: (v) $\displaystyle\int \frac{\sin x + \cos x}{2\cos 2x}\,dx$

Solution:

$$\int \frac{\sin x + \cos x}{2\cos 2x}\,dx = \frac{1}{2}\int \frac{\sin x + \cos x}{\cos^2 x - \sin^2 x}\,dx = \frac{1}{2}\int \frac{1}{\cos x - \sin x}\,dx$$

$$\tan\left(\frac{x}{2}\right) = t \implies dx = \frac{2\ dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

$$I = \frac{1}{2}\int \frac{1}{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$= \int \frac{1}{1-t^2-2t}dt = \int \frac{1}{(\sqrt{2})^2 - (t+1)^2}dt = \frac{1}{2\sqrt{2}}\ln\left|\frac{t+1+\sqrt{2}}{t+1-\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)+1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)+1-\sqrt{2}}\right| + C$$

### Type 2: Integrals of the form $\displaystyle\int \frac{1}{a\cos^2 x + b\sin^2 x + c}\,dx$

1. Evaluate the following integrals: (iii) $\displaystyle\int \frac{1}{2+2\sin^2 x}\,dx$

Solution:

$$I = \int \frac{1}{2+2\sin^2 x}\,dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{2}{\cos^2 x} + \frac{2\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int \frac{\sec^2 x}{2\sec^2 x + 2\tan^2 x}\,dx = \int \frac{\sec^2 x}{2+4\tan^2 x}\,dx$$

Let $\tan x = t, \quad \implies \sec^2 x dx = dt$

$$I = \int \frac{1}{2+4t^2}\,dt = \int \frac{1}{(\sqrt{2})^2 + (2t)^2}\,dt$$

Use the formula: $\displaystyle\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$

$$I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2t}{\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{2}}\right) + C$$

**Type 3: Integration using Partial Fractions: Non-repeated linear factors**

2. Evaluate the following integrals (Non-repeated linear factors): $(i)$ $\displaystyle\int \frac{x}{(x+2)(x+3)}\, dx$

Solution:

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\implies \quad x = A(x+3) + B(x+2)$$

Let $x = -3 \implies B = 3$

Let $x = -2 \implies A = -2$

$$\int \frac{x}{(x+2)(x+3)}\, dx = \int \frac{-2}{x+3}\, dx + \int \frac{3}{x+2}\, dx$$

$$= 3\ln(|x+3|) - 2\ln(|x+2|) + C$$

**Type 4: Integration using Partial Fractions: Non-repeated quadratic factors**

3. Evaluate the following integrals (Non-repeated quadratic factors): $(i)$ $\displaystyle\int \frac{x^2-x+4}{\big(2x-1\big)\big(x^2+1\big)}dx$

$$\frac{x^2-x+4}{\big(2x-1\big)\big(x^2+1\big)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$$

$$\implies \quad x^2 - x + 4 = A(x^2+1) + (Bx+C)(2x-1)$$

Let $x = \dfrac{1}{2} \implies A = 3$

Let $x = 0,\ A = 3 \implies C = -1$

Let $x = 1,\ A = 3,\ C = -1 \implies B = -1$

$$\int \frac{x^2-x+4}{\big(2x-1\big)\big(x^2+1\big)}\, dx = \int \Big(\frac{3}{2x-1} + \frac{-1x-1}{x^2+1}\Big)dx$$

$$= \int \Big(\frac{3}{2x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\Big)dx$$

$$= \frac{3\ln|2x-1|}{2} - \frac{\ln(x^2+1)}{2} - \tan^{-1}(x) + C$$

**Type 5: Integration using Partial Fractions: Repeated linear factors**

4. Evaluate the following integrals (Repeated linear factors): $(i)$ $\displaystyle\int \frac{3x-2}{x^3+x^2}\,dx$

Solution:

$$\frac{3x-2}{x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\implies \quad 3x-2 = Ax(x+1) + B(x+1) + C(x^2)$$

Let $x=-1 \implies C=-5$

Let $x=0 \implies B=-2$

Let $x=1$, $C=-5$, $B=-2 \implies A=5$

$$\int \frac{3x-2}{x^3+x^2}\,dx = \int \left(\frac{5}{x} + \frac{-2}{x^2} + \frac{-5}{x+1}\right)dx$$

$$= 5\ln(|x|) - 5\ln(|x+1|) + \frac{2}{x} + C$$

**Type 6: Integration by Parts**

5. Evaluate the following integrals: $(i)$ $\displaystyle\int x\cdot e^{-3x}\,dx$

Solution:

$$u = x,\ \frac{dv}{dx} = e^{-3x}$$

$$\implies \quad \frac{du}{dx} = 1,\ v = -\frac{e^{-3x}}{3}$$

$$I = -\frac{xe^{-3x}}{3} - \int -\frac{e^{-3x}}{3}\,dx$$

$$= -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9} + C$$

6. Evaluate the following integrals by appropriate substitution and integration by parts:

   $(i) \quad \int \sqrt{x} \cdot e^{\sqrt{x}}\ dx$

$$t = \sqrt{x} \implies x = t^2$$

$$\frac{dx}{dt} = 2t \implies dx = 2t \cdot dt$$

$$I = 2\int t^2 e^t\ dt$$

$$u = t^2,\ \frac{dv}{dt} = e^t \implies \frac{du}{dt} = 2t,\ v = e^t$$

$$I = 2t^2e^t - 4\int te^t\ dt$$

$$g = t,\ \frac{dh}{dt} = e^t \implies \frac{dg}{dt} = 1,\ h = e^t$$

$$I = 2t^2e^t - 4te^t + 4\int e^t\ dt$$

$$I = 2t^2e^t - 4te^t + 4e^t + C$$

$$I = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

7. Integration by parts for multiple times: $(i) \quad \int e^{-x} \cdot \cos x\ dx$

   Solution:

$$u = \cos x,\ \frac{dv}{dx} = e^{-x} \implies \frac{du}{dx} = -\sin x,\ v = -e^{-x}$$

$$\int e^{-x}\cos x\ dx = -e^{-x}\cos x - \int \sin x\, e^{-x}\ dt$$

$$g = \sin x,\ \frac{dh}{dx} = e^{-x} \implies \frac{dg}{dx} = \cos x,\ h = -e^{-x}$$

$$\int e^{-x}\cos x\ dx = -e^{-x}\cos x - \left(-\sin x\, e^{-x} + \int e^{-x}\cos x\ dx\right)$$

$$\int e^{-x}\cos x\ dx = -e^{-x}\cos x + \sin x\, e^{-x} - \int e^{-x}\cos x\ dx$$

$$2\int e^{-x}\cos x\ dx = \sin x\, e^{-x} - e^{-x}\cos x$$

$$\int e^{-x}\cos x\ dx = \frac{e^{-x}(\sin x - \cos x)}{2} + C$$