SUN YAT-SEN UNIVERSITY

Lecture 11
Search

Algorithm Design and Analysis

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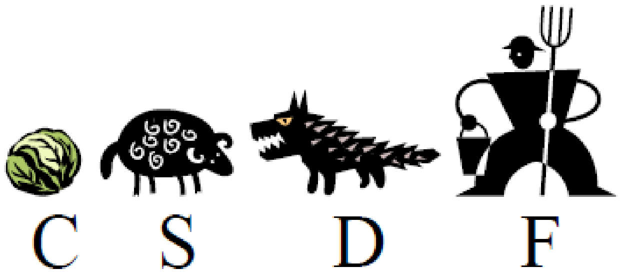
The search example

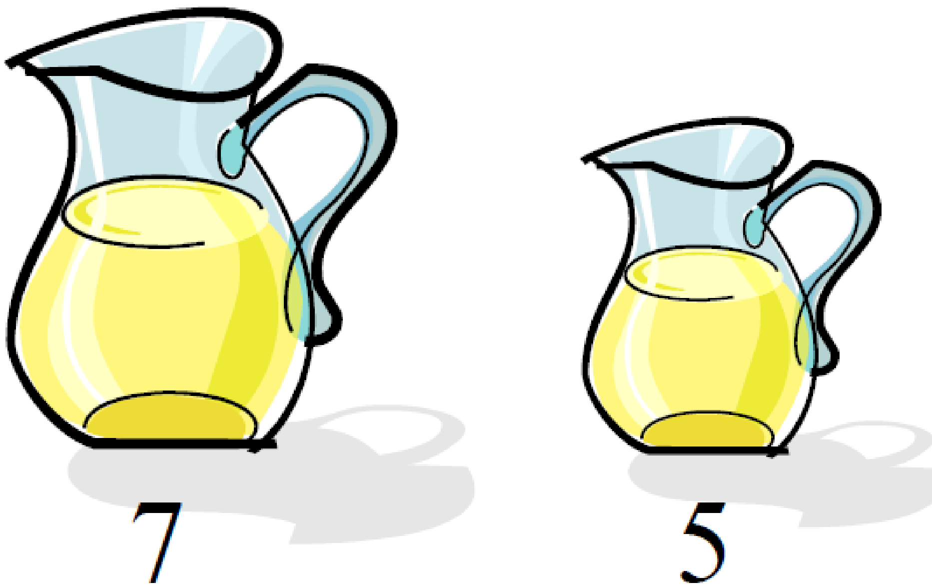
- State space ***S***: all valid configurations
- Initial states (nodes) ***I* = *{(CSDF,)}*** ⊆ ***S***
  - Where's the boat?
- Goal states ***G* = *{(,CSDF)}*** ⊆ ***S***



- Successor function ***succs(s)*** ⊆ ***S***: states reachable in one step (one arc) from ***s***
  - ***succs((CSDF,)) = {(CD, SF)}***
  - ***succs((CDF,S)) = {(CD,FS), (D,CFS), (C, DFS)}***
- cost(s,s') = 1 for all arcs. (weighted later)
- The search problem: find a solution path from a state in ***I*** to a state in ***G.***
  - Optionally minimize the cost of the solution.

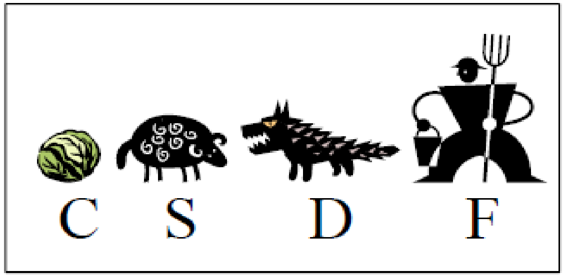
The search example

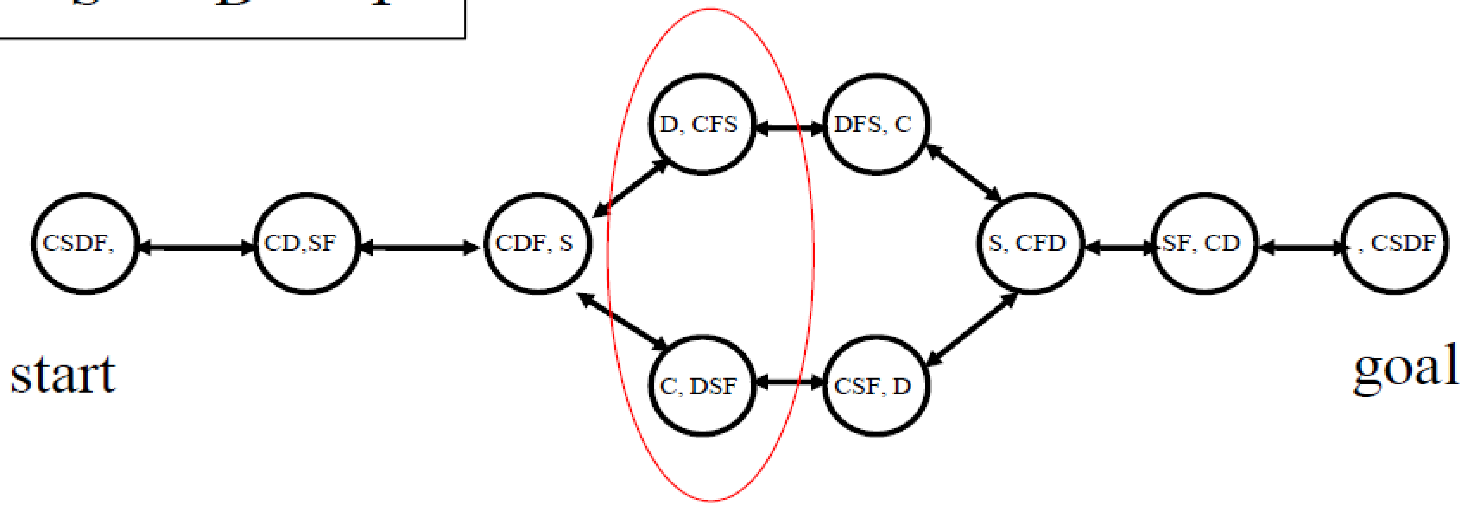
- Water jugs: how to get 1?



- Goal? (How many goal states?)
- Successor function: fill up (from tap or other jug), empty (to ground or other jug)

A directed graph in state space





- In general there will be many generated, but un-expanded states at any given time
- One has to choose which one to expand next
- Deep or shallow?

Uninformed search

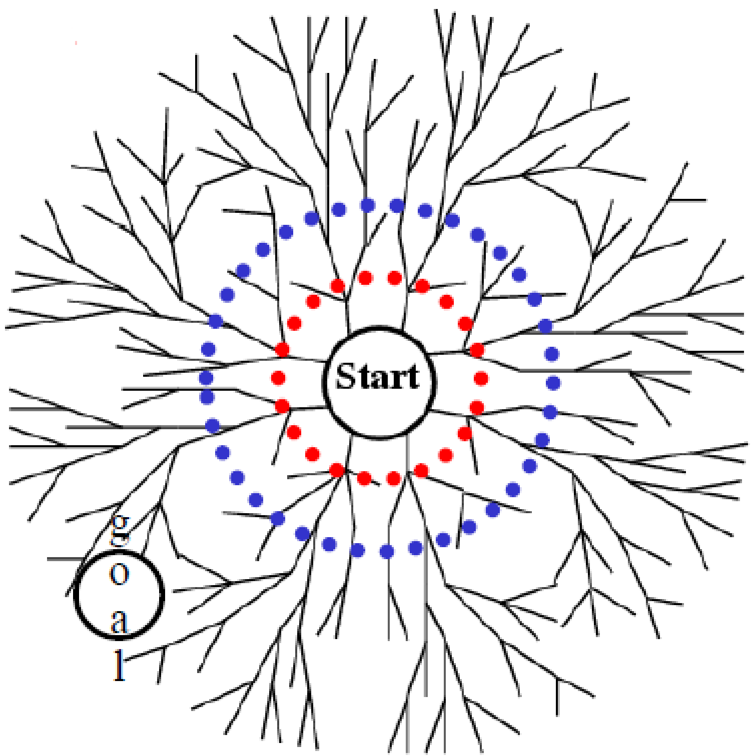
- Uninformed means we only know:
  - The goal test
  - The ***succs()*** function
- But not which non-goal states are better: that would be informed search.
- For now, we also assume ***succs()*** graph is a tree.
  - Won’t encounter repeated states.
  - We will discuss it later.
- Search strategies: BFS, UCS, DFS, IDS.

Breadth-first search (BFS)

- Use a queue (First-in First-out)

```
en_queue(Initial states)
While (queue not empty)
    s = de_queue()
    if (s==goal) success!
    T = succs(s)
    for t in T: t.prev=s
    en_queue(T)
endWhile
```

We need back pointers to recover the solution path.



Performance of BFS

- Assume:
  - the graph may be infinite.
  - Goal(s) exists and is only finite steps away.
- Will BFS find at least one goal?
- Will BFS find the least cost goal?
- Time complexity?
  - Number of states generated
  - Goal: ***d*** edges away
  - Branching factor: ***b***
- Space complexity?
  - Number of states stored

Performance of BFS

- Completeness: yes, BFS will find a goal.
- Optimality: yes, if edges cost 1 (more generally positive non-decreasing in depth), no otherwise.
- Time complexity (worst case): goal is the last node at radius ***d.***
  - Have to generate all nodes at radius ***d.***
  - $b + b^2 + \dots + b^d \sim O(b^d)$
- Space complexity: $O(b^d)$

Uniform-cost search

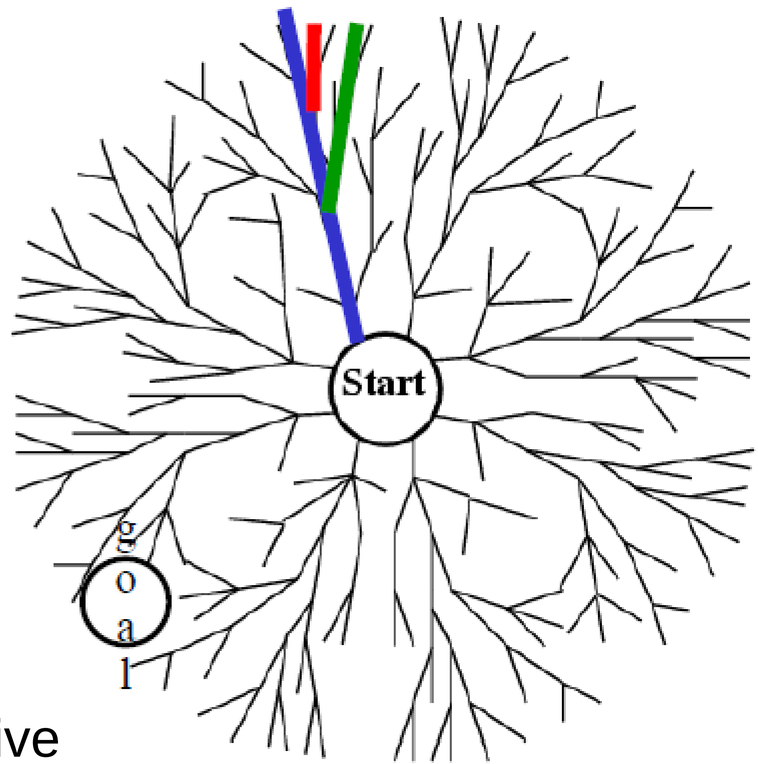
- Find the least-cost goal
- Each node has a path cost from start (= sum of edge costs along the path). Expand the least cost node first.
- Use a priority queue instead of a normal queue
  - Always take out the least cost item
  - Remember *heap?* time ***O(log(number of items in heap))***
- Complete and optimal (if edge costs >= ε > 0)
- Time and space: can be much worse than BFS
  - Let ***C**** be the cost of the least-cost goal
  - $O(b^{C^*/\varepsilon})$, possibly $C^*/\varepsilon \gg d$

Depth-first search

- Use a stack (First-in Last-out)

```
push(Initial states)
While (stack not empty)
        s = pop()
        if (s==goal) success!
        T = succs(s)
        push(T)
endWhile
```

This is non-recursive implementation of DFS, recursive implementation is more common

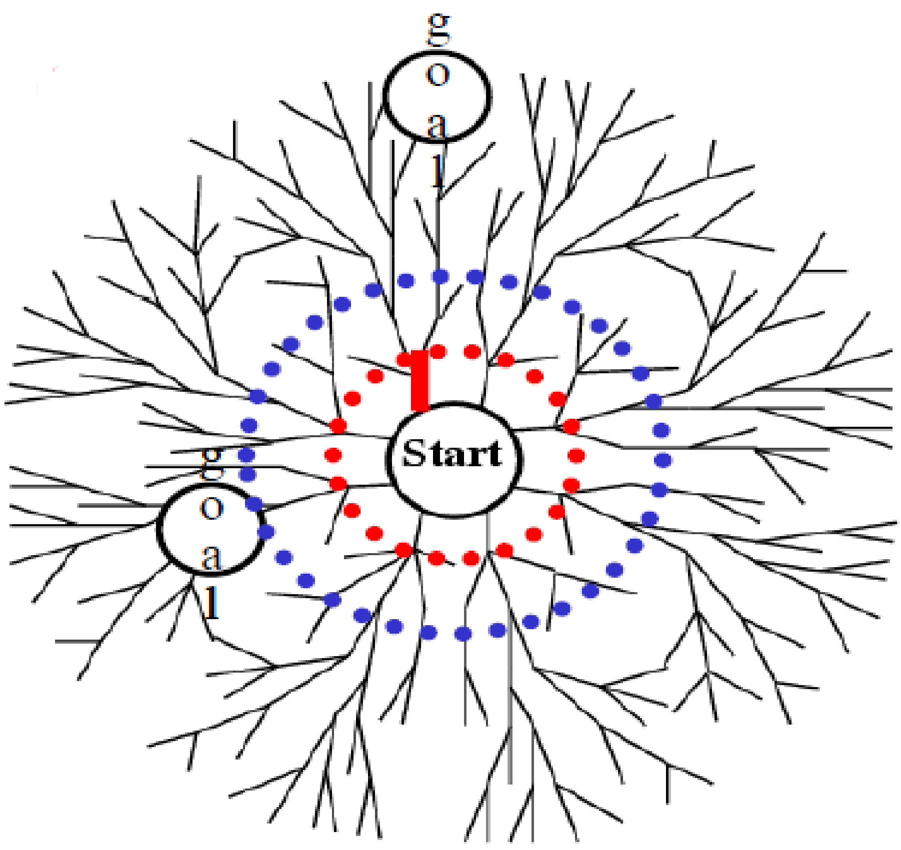


Performance of DFS

- ***m*** = maximum depth of graph from start
- Space complexity: $O(mb)$
- Infinite tree: may not find goal (incomplete)
- May not be optimal
- Finite tree: may visit almost all nodes, time complexity $O(b^m)$

Iterative deepening search

1. DFS, but stop if path length > 1.
2. If goal not found, repeat DFS, stop if path length >2.
3. And so on…

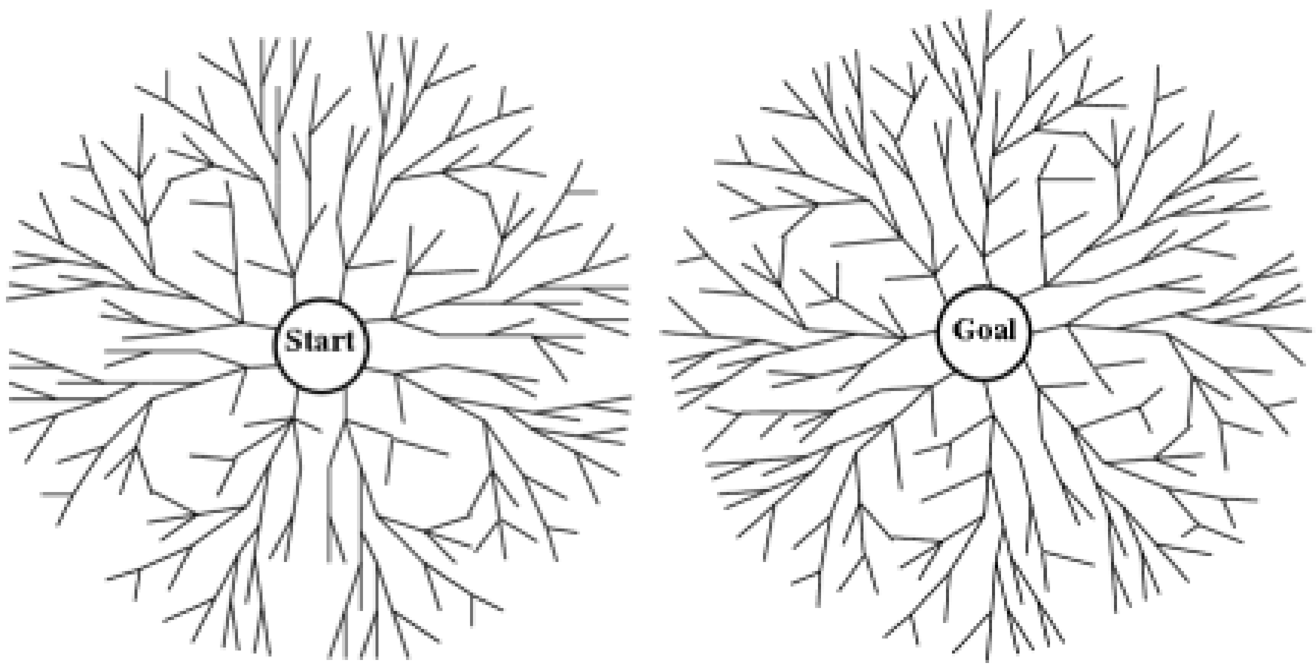


Iterative deepening search

- BFS + DFS
  - Complete, optimal like BFS
  - Small space complexity like DFS
- •A huge waste?
  - Each deepening repeats DFS from the beginning
  - No! $db+(d-1)b^2+(d-2)b^3+\ldots+b^d \sim O(b^d)$
  - Time complexity like BFS

Bidirectional search

- Breadth-first search from both start and goal
- Stop when fringes meet
- The fringes(边缘) are $O(b^{d/2})$
- Generates $O(b^{d/2})$ instead of $O(b^{d})$ nodes



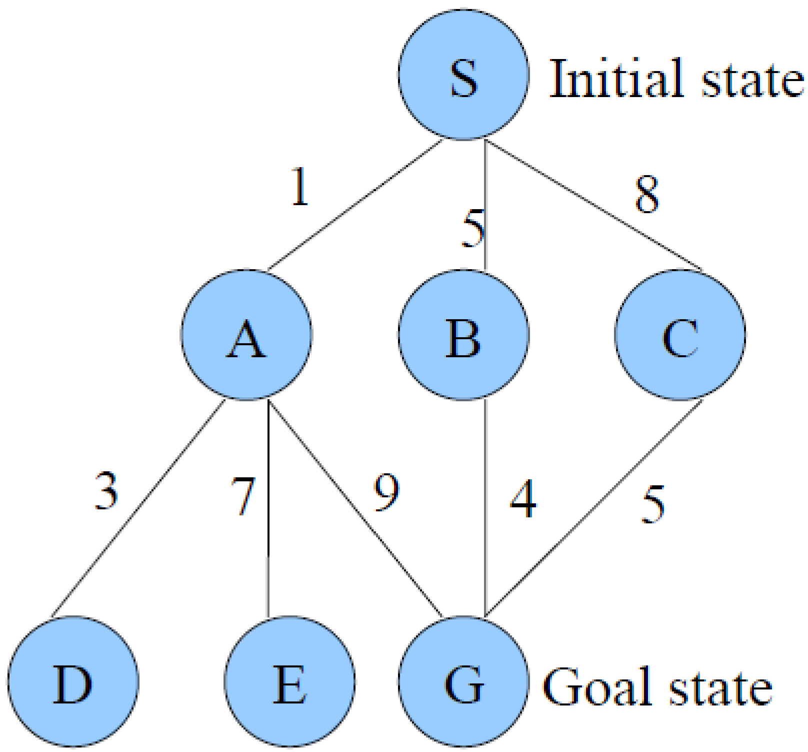
Performance of search algorithms

b: branching factor (assume finite) d: goal depth m: graph depth

	Complete	optimal	time	space
Breadth-first search	Y	Y, if [1]	$O(b^d)$	$O(b^d)$
Uniform-cost search	Y	Y	$O(b^{C^*/\varepsilon})$	$O(b^{C^*/\varepsilon})$
Depth-first search	N	N	$O(b^m)$	$O(bm)$
Iterative deepening	Y	Y, if [1]	$O(b^d)$	$O(bd)$
Bidirectional search	Y	Y, if [1]	$O(b^{d/2})$	$O(b^{d/2})$

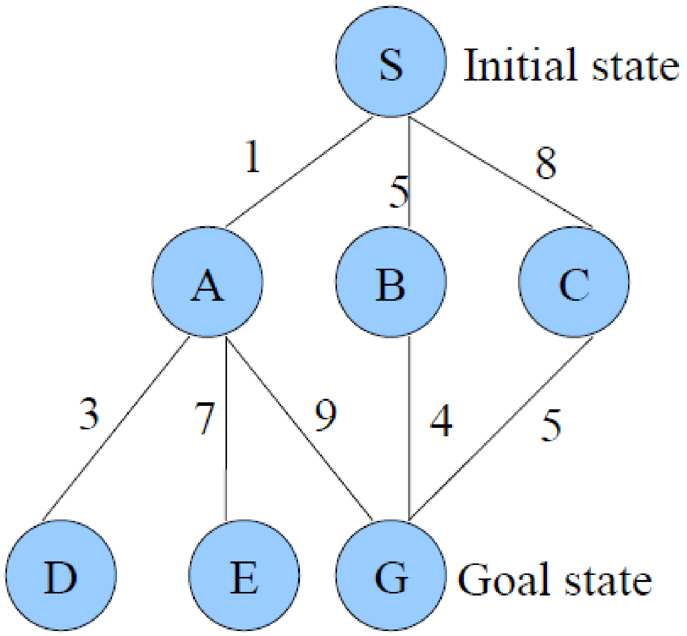
1. edge cost constant, or positive non-decreasing in depth

Search example



- All edges are directed, pointing downwards

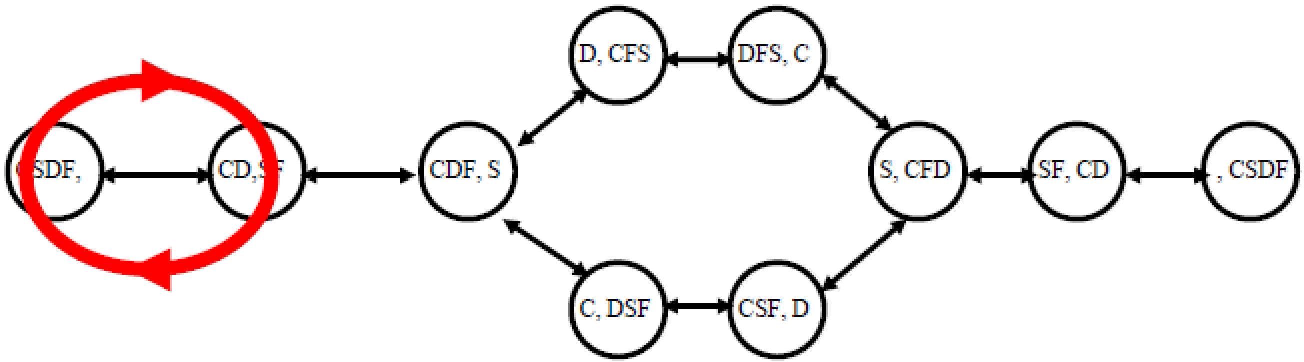
Node expansion



- Depth-First Search:
  - S A D E G
  - Solution found: S A G
- Breadth-First Search:
  - S A B C D E G
  - Solution found: S A G
- Uniform-Cost Search:
  - S A D B C E G
  - Solution found: S B G (This is the only uninformed search that worries about costs.)
- Iterative-Deepening Search:
  - S A B C S A D E G
  - Solution found: S A G

General graph search

- The problem: repeated states



- We have to remember already-expanded states (CLOSED).
- When we take out a state from the fringe (OPEN), check whether it is in CLOSED (already expanded).
  - If yes, throw it away.
  - If no, expand it (add successors to OPEN), and move it to CLOSED.

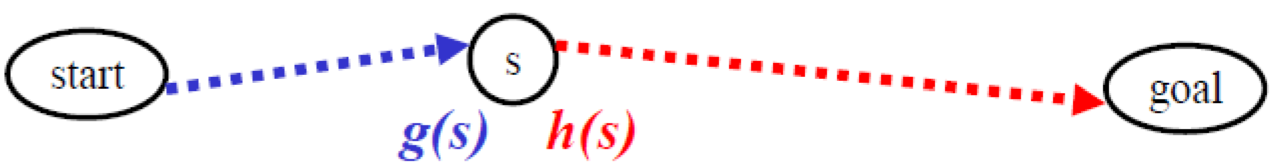
General graph search

- BFS:
  - Still ***O(b^d)*** space complexity
- DFS:
  - Memorizing DFS (MEMDFS): memorize every expanded states
  - Path Check DFS (PCDFS): remember only expanded states on current path (from start to the current node)

Informed search

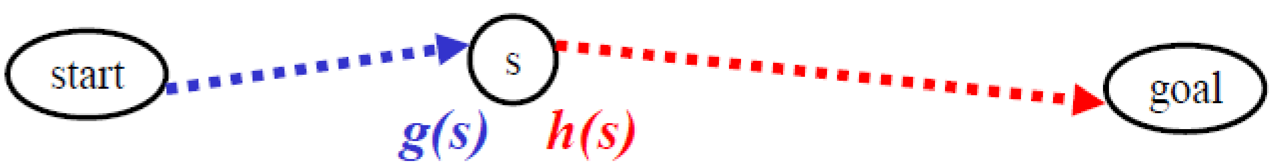
- Uninformed search
  - Knows the actual path cost ***g(s)*** from the start to a node ***s.***

- Informed search
  - Also has a heuristic ***h(s)*** of the cost from ***s*** to goal.
  - Can be much faster than uninformed search.



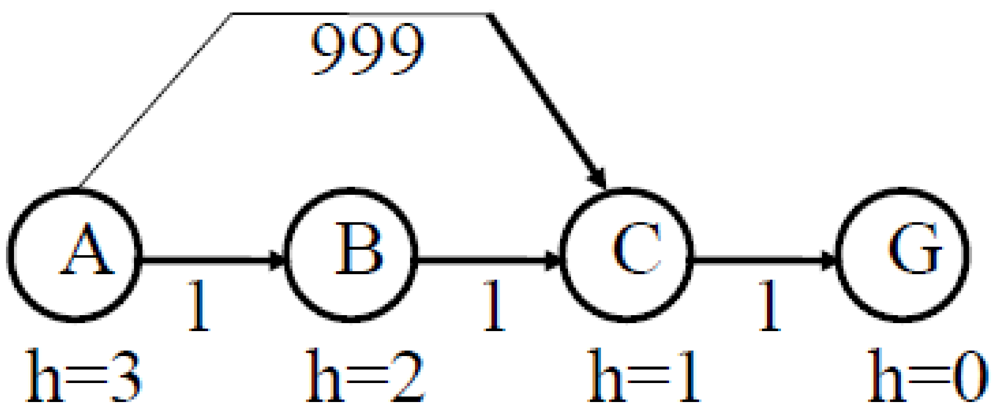
Recall: Uniform-cost search

- Uniform-cost search: uninformed search when edge costs are not the same.
- Complete (will find a goal). Optimal (will find the least-cost goal).
- Always expand the node with the least ***g(s)***
- Use a priority queue:
  - Push in states with their first-half-cost ***g(s)***
  - Pop out the state with the least ***g(s)*** first
- Now we have an estimate of the second-half-cost ***h(s),*** how to use it?



Best-first greedy search

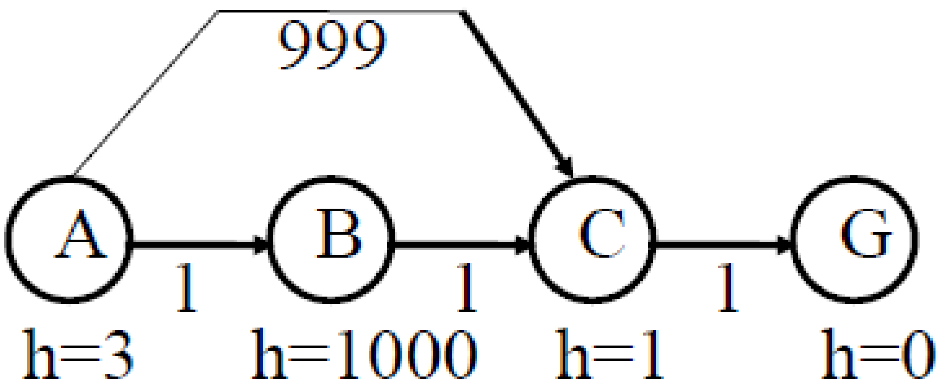
- Use ***h(s)*** instead of ***g(s)***
- Always expand the node with the least ***h(s)***
- Not optimal



It will follow the path *A -> C -> G*

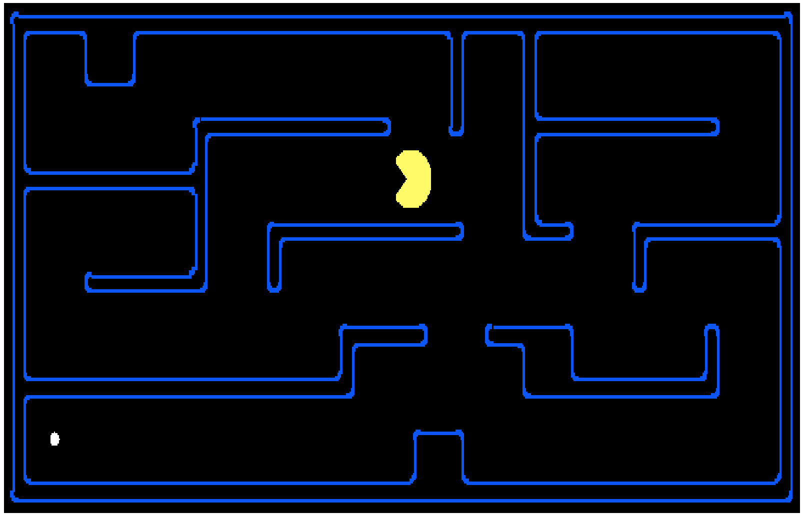
A search

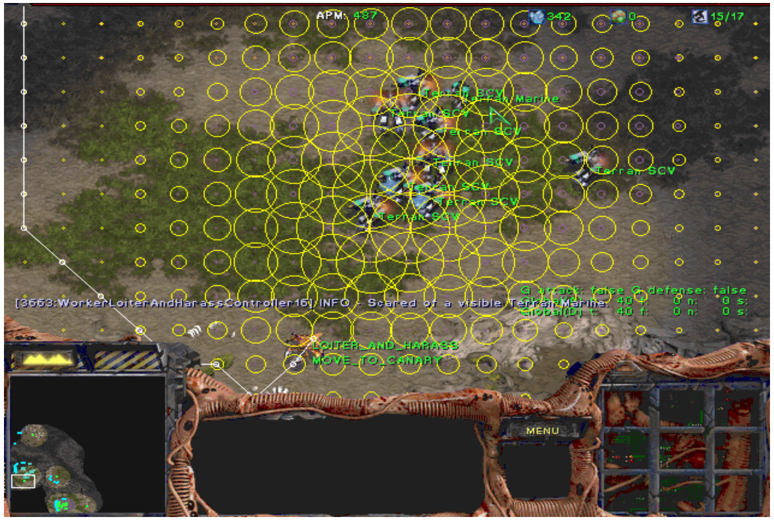
- Use ***f(s)=g(s)+h(s)***
- Always expand the node with the least ***g(s)+h(s)***
- A search is not always optimal



A* search

- Same as A search, but the heuristic function ***h()*** has to satisfy ***h(s)<=h*(s),*** where ***h*(s)*** is the true cost from node s to the goal.
- Such heuristic function ***h()*** is called ***admissible.***
- An admissible heuristic never over-estimates
- A search with admissible ***h()*** is called ***A* search.***





Admissible heuristic functions *h*

- 8-puzzle example

Example State

1		5
2	6	3
7	4	8

Goal State

1	2	3
4	5	6
7	8	

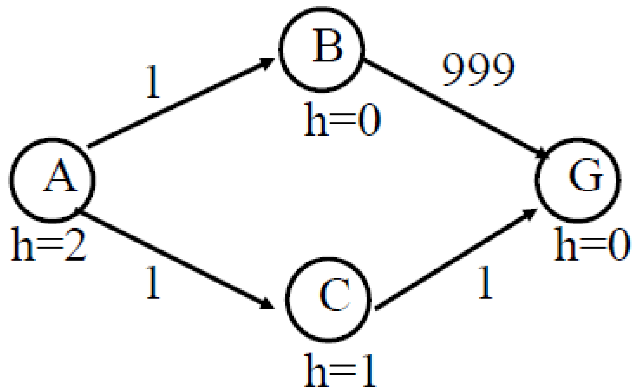
- Which of the following are admissible heuristics?
  - h(n)=number of tiles in wrong position
  - h(n)=0
  - h(n)=1
  - h(n)=sum of Manhattan distance between each tile and its goal location

Admissible heuristics

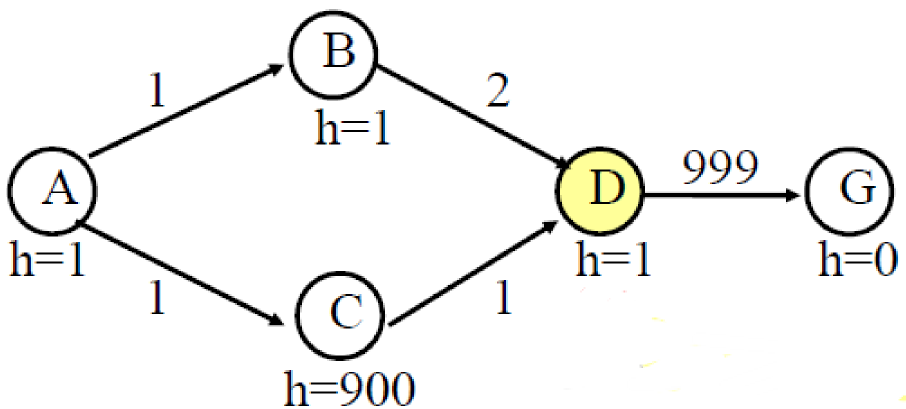
- A heuristic function h_2 **dominates** h_1 if for all s
  $h_1(s) <= h_2(s) <= h^*(s)$
- d = 14,
  - $A^*(h_1)$ = 539 nodes
  - $A^*(h_2)$ = 113 nodes
- d = 24,
  - $A^*(h_1)$ = 39,135 nodes
  - $A^*(h_2)$ = 1,641 nodes
- We prefer heuristic functions as close to ***h**** as possible, but not over ***h****.
- Good heuristic function might need complex computation
- Time may be better spent, if we use a faster, simpler heuristic function and expand more nodes.

Some tricks

- A* should terminate only when a goal is popped from the priority queue



- A* can revisit an expanded state, and discover a shorter path



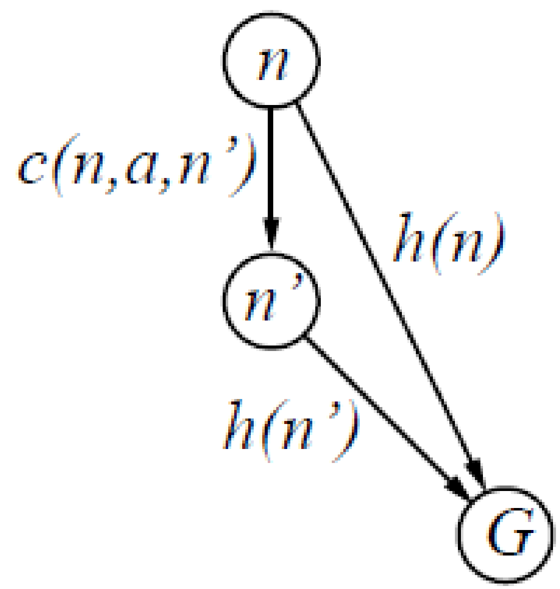
The A* algorithm

1. Put the start node ***S*** on the priority queue, called **OPEN**
2. If **OPEN** is empty, exit with failure
3. Remove from **OPEN** and place on **CLOSED** a node ***n*** for which ***f(n)*** is minimum
4. If ***n*** is a goal node, exit (trace back pointers from ***n*** to ***S***)
5. Expand ***n***, generating all its successors and attach to them pointers back to ***n***. For each successor ***n'*** of ***n*** not on **CLOSED**
   1. If n' is not already on **OPEN**, estimate ***h(n')***,***g(n')=g(n)+c(n,n')***, ***f(n')=g(n')+h(n')***, and place it on **OPEN**.
   2. If ***n'*** is already on **OPEN**, then check if ***g(n')*** is lower for the new version of ***n'***. If so, then:
      - Redirect pointers backward from ***n'*** along path yielding lower ***g(n')***.
      - Put ***n'*** on **OPEN**.
      - If ***g(n')*** is not lower for the new version, do nothing.
6. Goto 2.

Consistent heuristics

- Consistency is analogous to the triangle inequality from Euclidian geometry.
- A heuristic is consistent if

  $h(n) <= c(n, a, n') + h(n')$



- If h is consistent, then for

  every child n' of n, we have:

  $f(n') = g(n') + h(n')$

  $= g(n) + c(n, a, n') + h(n')$

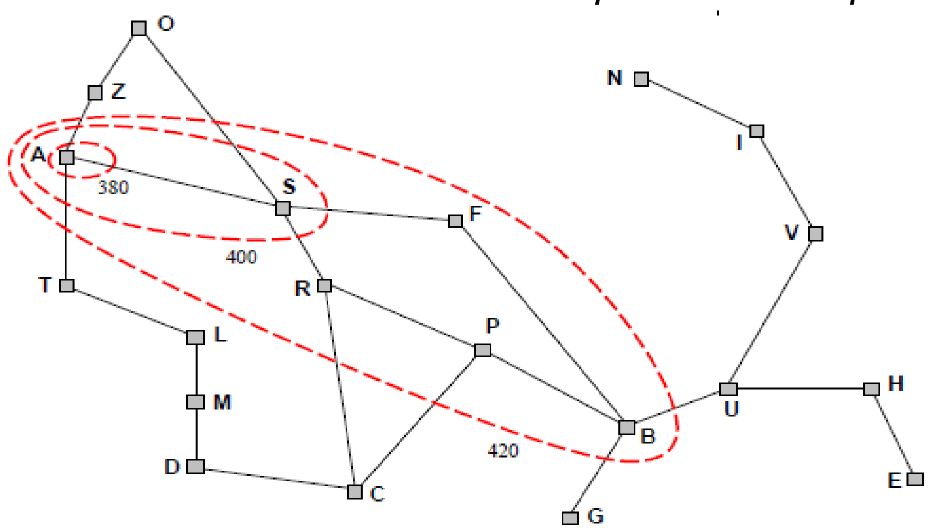
  $>= g(n) + h(n)$

  $= f(n)$

- That is, $f(n)$ is non-decreasing along any path.

Behavior of A* with consistent heuristic

- If h is consistent, then A* expands nodes in order of increasing *f* value. In such a case, A* can be implemented more efficiently — no node needs to be processed more than once.
- Gradually adds "*f*-contours" of nodes (breadth-first adds layers)
- Contour *i* has all nodes with $f = f_i$, where $f_i < f_{i+1}$

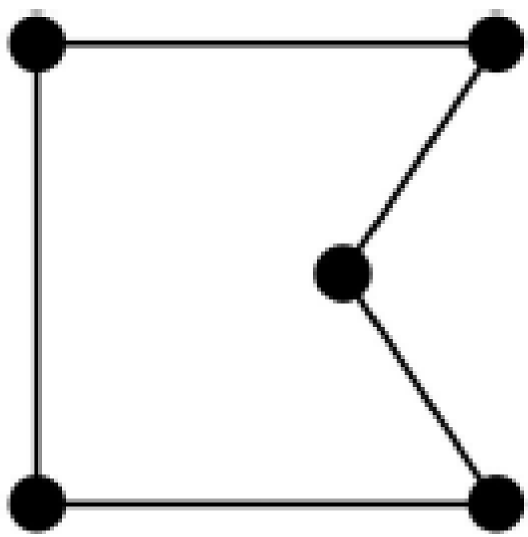


Properties of A*

- Complete? Yes
- Time? *O(entire state space)* in worst case, *O(d)* in best case
- Space? Keeps all nodes in memory
- Optimal? Yes

Traveling salesman problem

- For a node ***s***
  f(s)=g(s)+h(s)
- What is ***g(s)***, ***h(s)***?
- A* v.s. branch-and-bound



Iterative-Deepening A*

```
function IDA*(problem) returns a solution
    inputs: problem, a problem

    f_0 ← h(initial state)
    for i ← 0 to ∞ do
        result ← COST-LIMITED-SEARCH(problem, f_i)
        if result is a solution then return result
        else f_{i+1} ← result
    end

function COST-LIMITED-SEARCH(problem, fmax) returns solution or number

    depth-first search, backtracking at every node n such that f(n) > fmax
    if the search finds a solution then
        return the solution
    else
        return min{f(n) | the search backtracked at n}
```

Thank you!