



中山大學
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Lecture 7

Dynamic Programming

Part II

Algorithm Design and Analysis

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Longest Non-Decreasing Subsequence

- The longest non-decreasing subsequence (**LNDS**) problem is to find a subsequence of a given sequence in which the subsequence's elements are in sorted order, lowest to highest, and in which the subsequence is as long as possible. This subsequence is not necessarily contiguous, or unique.
- Example: Consider the following sequence
[1, 2, 5, 2, 8, 6, 3, 6, 9, 7]
[1, 5, 8, 9] forms a non-decreasing subsequence, so does [1, 2, 2, 6, 6, 7] but it is longer.

Longest Non-Decreasing Subsequence

- Solve subproblem on s_1, \dots, s_{n-1} and then try to extend using s_n .
- Two cases:
 - s_n is not used, answer is the same answer as on s_1, \dots, s_{n-1} .
 - s_n is used, answer is s_n preceded by the longest increasing subsequence in s_1, \dots, s_{n-1} that ends in a number smaller than s_n .
- Recurrence:
 - Let $L[i]$ be the length of longest non-decreasing subsequence in s_1, \dots, s_n that ends in s_i .
 - $L[j] = 1 + \max\{L[i] : i < j \text{ and } s_i \leq s_j\}$
 - $L[0] = 0$
 - Length of longest increasing subsequence:
 $\max\{L[i] : 1 \leq i \leq n\}$

Longest Non-Decreasing Subsequence

- We also maintain $P[j]$ to be the value of i that achieved the $\max L[j]$.
 - This will be the index of the predecessor of s_j in a longest increasing subsequence that ends in s_j .
 - By following the $P[j]$ values we can reconstruct the whole sequence in linear time.
- Implementation: $O(n^2)$

```

for (j = 1; j <= n; j++) {
    L[j] = 1;
    P[j] = 0;
    for (i = 1; i < j; i++)
    if (s[i] <= s[j] && L[i] + 1 > L[j]) {
        P[j] = i;
        L[j] = L[i] + 1;
    }
}

```

Longest Non-Decreasing Subsequence

- Exercise

index	1	2	3	4	5	6	7	8	9	10
sequence	1	2	5	2	8	6	3	6	9	7
L[i]										
P[i]										

Longest Non-Decreasing Subsequence

- Improvement

L[i]	index set	element set	min. element
0	\emptyset	\emptyset	-
1	{1}	{1}	1
2	{2}	{2}	2
3	{3, 4}	{2, 5}	2
4	{5, 6, 7}	{3, 6, 8}	3
5	{8}	{6}	6
6	{9, 10}	{7, 9}	7



**Non-decreasing
Order**

- Maintain the min.element array, which is a sorted array
- Use binary search to update the table
- $O(n \log n)$

Matrix-Chain Multiplication

- Let A be an n by m matrix, let B be an m by p matrix, then $C = AB$ is an n by p matrix.
- $C = AB$ can be computed in $O(nmp)$ time, using traditional matrix multiplication.
- Suppose we want to compute $A_1A_2A_3A_4$.
- Matrix Multiplication is associative, so we can do the multiplication in several different orders.
- Given n matrices, the size of the matrix A_i is $p_{i-1} \times p_i$, find the minimum multiplication operations.

Matrix-Chain Multiplication

- Example:
 A_1 is 10 by 100 matrix
 A_2 is 100 by 5 matrix
 A_3 is 5 by 50 matrix
 A_4 is 50 by 1 matrix
 $A_1A_2A_3A_4$ is a 10 by 1 matrix
- 5 different orderings = 5 different parenthesizations
 $(A_1(A_2(A_3A_4)))$
 $((A_1A_2)(A_3A_4))$
 $((A_1A_2)A_3)A_4$
 $((A_1(A_2A_3))A_4)$
 $(A_1((A_2A_3)A_4))$

Matrix-Chain Multiplication

- $(A_1(A_2(A_3A_4)))$
 - $A_{34} = A_3A_4$, 250 mults, result is 5 by 1
 - $A_{24} = A_2A_{34}$, 500 mults, result is 100 by 1
 - $A_{14} = A_1A_{24}$, 1000 mults, result is 10 by 1
 - Total is 1750
- $((A_1A_2)(A_3A_4))$
 - $A_{12} = A_1A_2$, 5000 mults, result is 10 by 5
 - $A_{34} = A_3A_4$, 250 mults, result is 5 by 1
 - $A_{14} = A_{12}A_{34}$, 50 mults, result is 10 by 1
 - Total is 5300
- $((A_1A_2)A_3)A_4$
 - $A_{12} = A_1A_2$, 5000 mults, result is 10 by 5
 - $A_{13} = A_{12}A_3$, 2500 mults, result is 10 by 50
 - $A_{14} = A_{13}A_4$, 500 mults, results is 10 by 1
 - Total is 8000

Matrix-Chain Multiplication

- $((A_1(A_2A_3))A_4)$
 - $A_{23} = A_2A_3$, 25000 mults, result is 100 by 50
 - $A_{13} = A_1A_{23}$, 50000 mults, result is 10 by 50
 - $A_{14} = A_{13}A_4$, 500 mults, results is 10 by 1
 - Total is 75500
- $(A_1((A_2A_3)A_4))$
 - $A_{23} = A_2A_3$, 25000 mults, result is 100 by 50
 - $A_{24} = A_{23}A_4$, 5000 mults, result is 100 by 1
 - $A_{14} = A_1A_{24}$, 1000 mults, result is 10 by 1
 - Total is 31000
- Conclusion: Order of operations makes a huge difference.
How do we compute the minimum?

Matrix-Chain Multiplication

- **Parenthesization:** A product of matrices is fully parenthesized if it is either
 - a single matrix, or
 - a product of two fully parenthesized matrices, surrounded by parentheses
- Each parenthesization defines a set of $n-1$ matrix multiplications. We just need to pick the parenthesization that corresponds to the best ordering.
- Question: How many parenthesizations are there?

Matrix-Chain Multiplication

- Let $P(n)$ be the number of ways to parenthesize n matrices.

$$P(n) = \begin{cases} \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

- This recurrence is related to the Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

- Asymptotically, the Catalan numbers grow as

$$C_n \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}$$

Matrix-Chain Multiplication

- Structure of an optimal solution: If the outermost parenthesization is
$$((A_1 A_2 \cdot \cdot \cdot A_i)(A_{i+1} \cdot \cdot \cdot A_n))$$
then the optimal solution consists of solving $A_{1,i}$ and $A_{i+1,n}$ optimally and then combining the solutions.
- Overlapping subproblems: In the enumeration of the $P(n) = \Omega(4^n/n^{3/2})$ subproblems, how many unique subproblems are there?

Recursive solution

- A subproblem is of the form A_{ij} with $1 \leq i \leq j \leq n$, so there are $O(n^2)$ subproblems.
- Let A_i be p_{i-1} by p_i .
- Let $m[i, j]$ be the cost of computing A_{ij}
- If the final multiplication for A_{ij} is $A_{ij} = A_{ik}A_{k+1,j}$ then $m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$.
- We don't know k a priori, so we take the minimum

$$m[i, j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- Direct recursion on this does not work! We must use the fact that there are at most $O(n^2)$ different calls. What is the order?

0-1 Knapsack Problem

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ and has value $v_i > 0$.
- Knapsack has a capacity of W .
- Goal: fill knapsack so as to maximize total value.
- Example:
 - $\{ 1, 2, 5 \}$ has value 35.
 - $\{ 3, 4 \}$ has value 40.
 - $\{ 3, 5 \}$ has value 46,
(but exceeds weight limit).

i	v_i	w_i
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

knapsack instance
(weight limit $W = 11$)

0-1 Knapsack Problem

- Let $OPT(i) = \text{max profit subset of items } 1, \dots, i$.
- Case 1. OPT does not select item i .
 - OPT selects best of $\{1, 2, \dots, i-1\}$.
- Case 2. OPT selects item i .
 - Selecting item i does not immediately imply that we will have to reject other items.
 - Without knowing what other items were selected before i , we don't even know if we have enough room for i .
- Conclusion: Need more subproblems!

0-1 Knapsack Problem

- Let $OPT(i, w)$ = max profit subset of items $1, \dots, i$ with weight limit w .
- Case 1. OPT does not select item i .
 - OPT selects best of $\{1, 2, \dots, i-1\}$ using weight limit w .
- Case 2. OPT selects item i .
 - New weight limit = $w - w_i$.
 - OPT selects best of $\{1, 2, \dots, i-1\}$ using this new weight limit.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{OPT(i-1, w), v_i + OPT(i-1, w - w_i)\} & \text{otherwise} \end{cases}$$

0-1 Knapsack Problem

- Implementation: $O(nW)$

```

for ( $w = 0$ ;  $w \leq W$ ;  $w++$ )
     $M[0, w] = 0$ ;
for ( $i = 1$ ;  $i \leq n$ ;  $i++$ )
    for ( $w = 1$ ;  $w \leq W$ ;  $w++$ )
        if ( $wt[i] > w$ )  $M[i, w] = M[i - 1, w]$ ;
        else  $M[i, w] = \max \{ M[i - 1, w], v[i] + M[i - 1, w - wt[i]] \}$ ;
    return  $M[n, W]$ ;

```

i	v_i	w_i
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

knapsack instance
(weight limit $W = 11$)

subset of items
1, ..., i

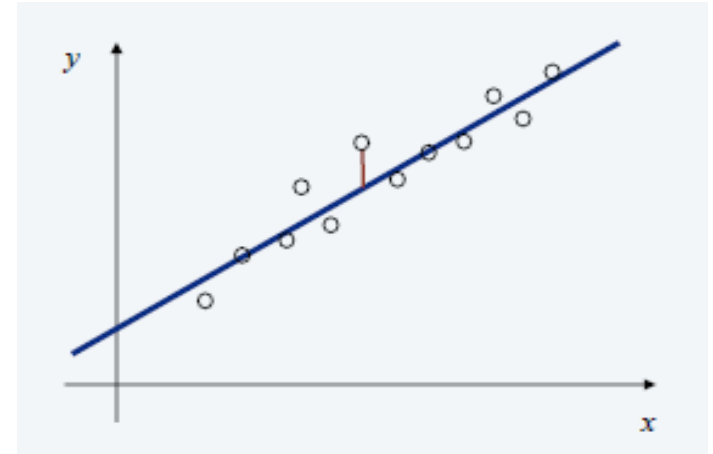
	weight limit w											
	0	1	2	3	4	5	6	7	8	9	10	11
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT(i, w) = max profit subset of items 1, ..., i with weight limit w.

Segmented Least Squares

- Least squares. Foundational problem in statistics.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$

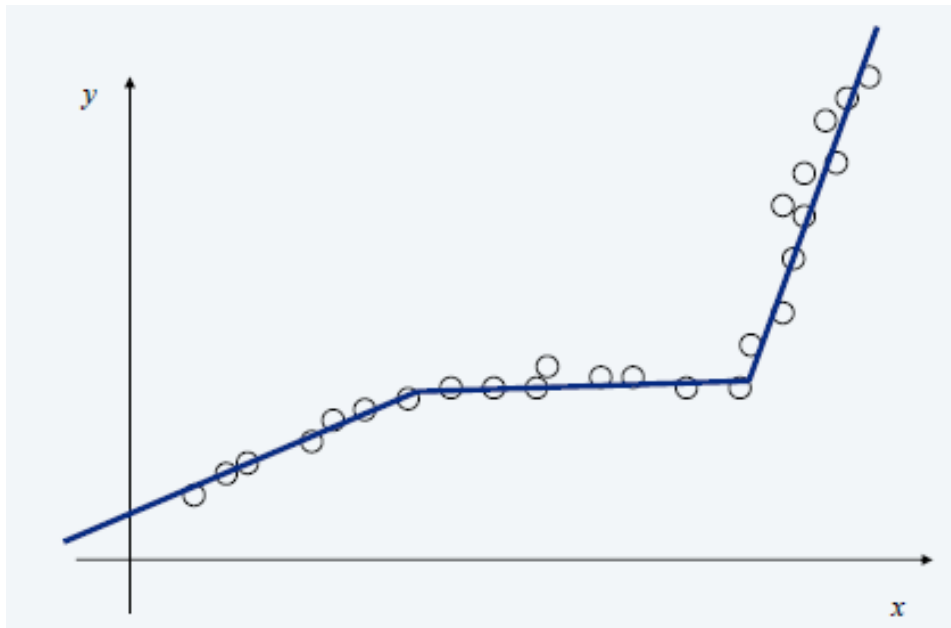


- Solution. Calculus \Rightarrow min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i) (\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

- Segmented least squares: Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.



Segmented Least Squares

- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, and a constant $c > 0$, find a sequence of lines that minimizes $f(x) = E + c L$:
- $E =$ the sum of the sums of the squared errors in each segment.
- $L =$ the number of lines.

Segmented Least Squares

- Notation:
 - $OPT(j)$ = minimum cost for points p_1, p_2, \dots, p_j .
 - $e(i, j)$ = minimum sum of squares for points p_i, p_{i+1}, \dots, p_j .
- To compute $OPT(j)$:
 - Last segment uses points p_i, p_{i+1}, \dots, p_j for some i .
 - Cost = $e(i, j) + c + OPT(i - 1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e(i, j) + c + OPT(i - 1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares

SEGMENTED-LEAST-SQUARES (n, p_1, \dots, p_n, c)

FOR $j = 1$ TO n

 FOR $i = 1$ TO j

 Compute the least squares $e(i, j)$ for the segment p_i, p_{i+1}, \dots, p_j .

$M[0] \leftarrow 0$.

FOR $j = 1$ TO n

$M[j] \leftarrow \min_{1 \leq i \leq j} \{ e_{ij} + c + M[i-1] \}$.

RETURN $M[n]$.

Segmented Least Squares

- The dynamic programming algorithm solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.
- Bottleneck: computing $e(i, j)$ for $O(n^2)$ pairs.
- $O(n)$ per pair using formula.

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i) (\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

- Can be improved to $O(n^2)$ time and $O(n)$ space by precomputing various statistics.

Thank you!

