Magic square

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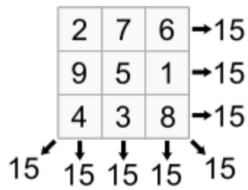
In recreational mathematics, a **magic square** is an arrangement of distinct numbers (i.e., each number is used once), usually integers, in a square grid, where the numbers in each row, and in each column, and the numbers in the main and secondary diagonals, all add up to the same number, called the **“magic constant.”** A magic square has the same number of rows as it has columns, and in conventional math notation, "n" stands for the number of rows (and columns) it has. Thus, a magic square always contains n^2 numbers, and its size (the number of rows [and columns] it has) is described as being "of order n."[1] A magic square that contains the integers from 1 to n^2 is called a *normal* magic square. (The term "magic square" is also sometimes used to refer to any of various types of word squares.)

Normal magic squares of all sizes except 2 × 2 (that is, where $n = 2$) can be constructed. The 1 × 1 magic square, with only one cell containing the number 1, is trivial. The smallest (and unique up to rotation and reflection) non-trivial case, 3 × 3, is shown below.



Any magic square can be rotated and reflected to produce 8 trivially distinct squares. In magic square theory, all of these are generally deemed equivalent and the eight such squares are said to make up a single equivalence class.[2]

The constant that is the sum of every row, column and diagonal is called the magic constant or magic sum, M. Every normal magic square has a constant dependent on n, calculated by the formula $M = [n(n^2 + 1)] / 2$. For normal magic squares of order n = 3, 4, 5, 6, 7, and 8, the magic constants are, respectively: 15, 34, 65, 111, 175, and 260 (sequence A006003 in the OEIS).

Magic squares have a long history, dating back to 650 BC in China. At various times they have acquired magical or mythical significance, and have appeared as symbols in works of art. In modern times they have been generalized a number of ways, including using extra or different constraints, multiplying instead of adding cells, using alternate shapes or more than two dimensions, and replacing numbers with shapes and addition with geometric operations.

Contents

History

Magic squares were known to Chinese mathematicians as early as 650 BC, and explicitly given since 570 AD,[3] and to Islamic mathematicians possibly as early as the seventh century AD. The first magic squares of order 5 and 6 appear in an encyclopedia from Baghdad *circa* 983, the *Encyclopedia of the Brethren of Purity* (*Rasa'il Ihkwan al-Safa*); simpler magic squares were known to several earlier Arab mathematicians.[3] Some of these squares were later used in conjunction with magic letters, as in Shams Al-ma'arif, to assist Arab illusionists and magicians.[4]

Iron plate with an order 6 magic square in Arabic numbers from China, dating to the Yuan Dynasty (1271–1368).

Lo Shu square (3×3 magic square)

Chinese literature dating from as early as 650 BC tells the legend of Lo Shu (洛書) or "scroll of the river Lo".[3] Early records are ambiguous references to a "river map", but clearly refer to a magic square by 80 AD, and explicitly give one since 570 AD.[3] According to the legend, there was at one time in ancient China a huge flood. While the great king Yu (禹) was trying to channel the water out to sea, a turtle emerged from it with a curious figure / pattern on its shell: a 3×3 grid in which circular dots of numbers were arranged, such that the sum of the numbers in each row, column and diagonal was the same: 15, which is also the number of days in each of the 24 cycles of the Chinese solar year. According to the legend, thereafter people were able to use this pattern in a certain way to control the river and protect themselves from floods.

4	9	2
3	5	7
8	1	6

The Lo Shu Square, as the magic square on the turtle shell is called, is the unique normal magic square of order three in which 1 is at the bottom and 2 is in the upper right corner. Every normal magic square of order three is obtained from the Lo Shu by rotation or reflection.

The Square of Lo Shu is also referred to as the Magic Square of Saturn.

Persia

Although the early history of magic squares in Persia is not known, it has been suggested that they were known in pre-Islamic times.[5] It is clear, however, that the study of magic squares was common in medieval Islam in Persia, and it was thought to have begun after the introduction of chess into the region.[6] The 10th-century Persian mathematician Buzjani, for example, left a manuscript that on page 33 contains a series of magic squares, filled by numbers in arithmetical progression, in such a way that the sums of each row, column and diagonal are equal.[7]

Original script from the *Shams al-Ma'arif*.

Arabia

Magic squares were known to Islamic mathematicians in Arabia as early as the seventh century. They may have learned about them when the Arabs came into contact with Indian culture and learned Indian astronomy and mathematics – including other aspects of combinatorial mathematics. Alternatively, the idea may have come to them from China. The first magic squares of order 5 and 6 known to have been devised by Arab mathematicians appear in an encyclopedia from Baghdad *circa* 983, the Rasa'il Ikhwan al-Safa (the Encyclopedia of the Brethren of Purity); simpler magic squares were known to several earlier Arab mathematicians.[3]

Printed version of the previous manuscript. Eastern Arabic numerals were used.

The magic square of order three was described as a child-bearing charm[8] since its first literary appearances in the works of Jābir ibn Hayyān (fl. c. 721– c. 815)[9] and al-Ghazālī (1058–1111)[10] and it was preserved in the tradition of the planetary tables, known from H.C.Agrippa's work,[11] too.

The Arab mathematician Ahmad al-Buni, who worked on magic squares around 1250, attributed mystical properties to them, although no details of these supposed properties are known. There are also references to the use of magic squares in astrological calculations, a practice that seems to have originated with the Arabs.[3]

India

The 3×3 magic square has been a part of rituals in India since Vedic times, and still is today. The Ganesh yantra is a 3×3 magic square. There is a well-known 10th-century 4×4 magic square on display in the Parshvanath Jain temple in Khajuraho, India.[12]

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

This is known as the Chautisa Yantra. Each row, column, and diagonal, as well as each 2×2 sub-square, the corners of each 3×3 and 4×4 square, the corners of each 2x4 and 4x2 rectangle, and the offset diagonals (12+8+5+9, 1+11+16+6, 2+12+15+5, 14+2+3+15 and 7+11+10+6, 12+2+5+15, 1+13+16+4) sum to 34.

In this square, every second diagonal number adds to 17 (the same applies to offset diagonals). In addition to squares and rectangles, there are eight trapeziums – two in one direction, and the others at a rotation of 90 degrees, such as (12, 1, 16, 5) and (13, 8, 9, 4).

These characteristics (which identify it as one of the three 4x4 pandiagonal magic squares and as a most-perfect magic square) mean that the rows or columns can be rotated and maintain the same characteristics - for example:

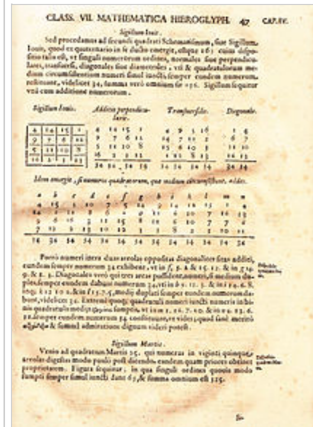
12	1	14	7
13	8	11	2
3	10	5	16
6	15	4	9

The Kubera-Kolam, a magic square of order three, is commonly painted on floors in India. It is essentially the same as the Lo Shu Square, but with 19 added to each number, giving a magic constant of 72.

23	28	21
22	24	26
27	20	25

Europe

In 1300, building on the work of the Arab Al-Buni, Greek Byzantine scholar Manuel Moschopoulos wrote a mathematical treatise on the subject of magic squares, leaving out the mysticism of his predecessors.[13] Moschopoulos was essentially unknown to the Latin west. He was not, either, the first Westerner to have written on magic squares. They appear in a Spanish manuscript written in the 1280s, presently in the Biblioteca Vaticana (cod. Reg. Lat. 1283a) due to Alfonso X of Castille.[14] In that text, each magic square is assigned to the respective planet, as in the Islamic literature.[15] Magic squares surface again in Italy in the 14th century, and specifically in Florence. In fact, a 6×6 and a 9×9 square are exhibited in a manuscript of the *Trattato d'Abbaco* (Treatise of the Abacus) by Paolo dell'Abbaco, aka Paolo Dagomari, a mathematician, astronomer and astrologer who was, among other things, in close contact with Jacopo Alighieri, a son of Dante. The squares can be seen on folios 20 and 21 of MS. 2433, at the Biblioteca Universitaria of Bologna. They also appear on folio 69rv of Plimpton 167, a manuscript copy of the *Trattato dell'Abbaco* from the 15th century in the Library of Columbia University.[16] It is interesting to observe that Paolo Dagomari, like Pacioli after him, refers to the squares as a useful basis for inventing mathematical questions and games, and does not mention any magical use. Incidentally, though, he also refers to them as being respectively the Sun's and the Moon's squares, and mentions that they enter astrological calculations that are not better specified. As said, the same point of view seems to motivate the fellow Florentine Luca Pacioli, who describes 3×3 to 9×9 squares in his work *De Viribus Quantitatis*.[17] Pacioli states: *A lastronomia summamente hanno mostrato li supremi di quella commo Ptolomeo, al bumasar ali, al fragano, Geber et gli altri tutti La forza et virtu de numeri eserli necessaria* (Masters of astronomy, such as Ptolemy, Albumasar, Alfraganus, Jabir and all the others, have shown that the force and the virtue of numbers are necessary to that science) and then goes on to describe the seven planetary squares, with no mention of magical applications.



This page from Athanasius Kircher's *Oedipus Aegyptiacus* (1653) belongs to a treatise on magic squares and shows the *Sigillum Iovis* associated with Jupiter

Magic squares of order 3 through 9, assigned to the seven planets, and described as means to attract the influence of planets and their angels (or demons) during magical practices, can be found in several manuscripts all around Europe starting at least since the 15th century. Among the best known, the *Liber de Angelis*, a magical handbook written around 1440, is included in Cambridge Univ. Lib. MS Dd.xi.45.[18] The text of the *Liber de Angelis* is very close to that of *De septem quadraturis planetarum seu quadrati magici*, another handbook of planetary image magic contained in the Codex 793 of the Biblioteka Jagiellońska (Ms BJ 793).[19] The magical operations involve engraving the appropriate square on a plate made with the metal assigned to the corresponding planet,[20] as well as performing a variety of rituals. For instance, the 3×3 square, that belongs to Saturn, has to be inscribed on a lead plate. It will, in particular, help women during a difficult childbirth.

In 1514 Albrecht Dürer immortalized a 4×4 square, of order four, in his famous engraving *Melencolia I*. It is described in more detail below.

In about 1510 Heinrich Cornelius Agrippa wrote *De Occulta Philosophia*, drawing on the Hermetic and magical works of Marsilio Ficino and Pico della Mirandola. In its 1531 edition, he expounded on the magical virtues of the seven magical squares of orders 3 to 9, each associated with one of the astrological planets, much in the same way as the older texts did. This book was very influential throughout Europe until the counter-reformation, and Agrippa's magic squares, sometimes called kameas, continue to be used within modern ceremonial magic in much the same way as he first prescribed.[3][21]

Saturn=15		
4	9	2
3	5	7
8	1	6

Jupiter=34			
4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

Mars=65				
11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

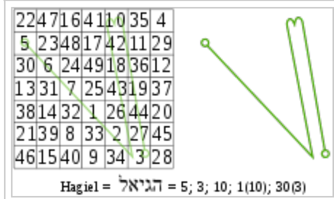
Sol=111					
6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

Venus=175						
22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Mercury=260							
8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

Luna=369								
37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

The most common use for these kameas is to provide a pattern upon which to construct the sigils of spirits, angels or demons; the letters of the entity's name are converted into numbers, and lines are traced through the pattern that these successive numbers make on the kamea. In a magical context, the term *magic square* is also applied to a variety of word squares or number squares found in magical grimoires, including some that do not follow any obvious pattern, and even those with differing numbers of rows and columns. They are generally intended for use as talismans. For instance the following squares are: The Sator square, one of the most famous magic squares found in a number of grimoires including the *Key of Solomon*; a square "to overcome envy", from *The Book of Power*;[22] and two squares from *The Book of the Sacred Magic of Abramelin the Mage*, the first to cause the illusion of a superb palace to appear, and the second to be worn on the head of a child during an angelic invocation:



The derivation of the sigil of Hagiel, the planetary intelligence of Venus, drawn on the magic square of Venus. Each Hebrew letter provides a numerical value, giving the vertices of the sigil.

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

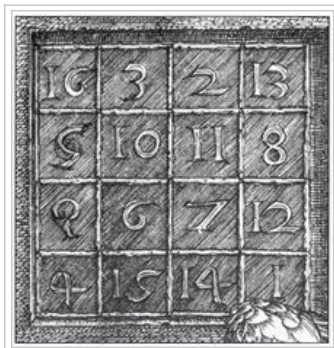
6	66	848	938
8	11	544	839
1	11	383	839
2	73	774	447

H	E	S	E	B
E	Q	A	L	
S				
E		G		
B				

A	D	A	M
D	A	R	A
A	R	A	D
M	A	D	A

Albrecht Dürer's magic square

The order-4 magic square Albrecht Dürer immortalized in his 1514 engraving *Melencolia I,* referred to above, is believed to be the first seen in European art. It is very similar to Yang Hui's square, which was created in China about 250 years before Dürer's time. The sum 34 can be found in the rows, columns, diagonals, each of the quadrants, the center four squares, and the corner squares (of the 4×4 as well as the four contained 3×3 grids). This sum can also be found in the four outer numbers clockwise from the corners (3+8+14+9) and likewise the four counter-clockwise (the locations of four queens in the two solutions of the 4 queens puzzle[23]), the two sets of four symmetrical numbers (2+8+9+15 and 3+5+12+14), the sum of the middle two entries of the two outer columns and rows (5+9+8+12 and 3+2+15+14), and in four kite or cross shaped quartets (3+5+11+15, 2+10+8+14, 3+9+7+15, and 2+6+12+14). The two numbers in the middle of the bottom row give the date of the engraving: 1514. The numbers 1 and 4 at either side of the date correspond respectively to the letters “A” and “D,” which are the initials of the artist.


Detail of *Melencolia I*

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer's magic square can also be extended to a magic cube.[24]

Dürer's magic square and his *Melencolia I* both also played large roles in Dan Brown's 2009 novel, *The Lost Symbol*.

Sagrada Família magic square

The Passion façade of the Sagrada Família church in Barcelona, conceptualized by Antoni Gaudí and designed by sculptor Josep Subirachs, features a 4×4 magic square:

A magic square on the Sagrada Família church façade

The magic constant of the square is 33, the age of Jesus at the time of the Passion. Structurally, it is very similar to the Melancholia magic square, but it has had the numbers in four of the cells reduced by 1.

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

While having the same pattern of summation, this is not a *normal* magic square as above, as two numbers (10 and 14) are duplicated and two (12 and 16) are absent, failing the $1 \rightarrow n^2$ rule.

Similarly to Dürer's magic square, the Sagrada Familia's magic square can also be extended to a magic cube.[25]

Srinivasa Ramanujan's magic square

The Indian mathematician Srinivasa Ramanujan created a square where - in addition to several groups of four squares - the first row shows his date of birth, Dec. 22nd, 1887.

This magic square has 24 groups of four fields with the sum of 139 and in the first row - shown at bottom-right - Ramanujan's date of birth.

Types of construction

There are many ways to construct magic squares, but the standard (and most simple) way is to follow certain configurations/formulas which generate regular patterns. Magic squares exist for all values of n, with only one exception: it is impossible to construct a magic square of order 2. Magic squares can be classified into three types: odd, doubly even (n divisible by four) and singly even (n even, but not divisible by four). Odd and doubly even magic squares are easy to generate; the construction of singly even magic squares is more difficult but several methods exist, including the LUX method for magic squares (due to John Horton Conway) and the Strachey method for magic squares.

Group theory was also used for constructing new magic squares of a given order from one of them.[26]

Unsolved problem in mathematics:

How many n×n magic squares, and how many magic tori of order n, for n>5?

(more unsolved problems in mathematics)

The numbers of different $n \times n$ magic squares for n from 1 to 5, not counting rotations and reflections are: 1, 0, 1, 880, 275305224 (sequence A006052 in OEIS). The number for $n = 6$ has been estimated to be $(1.7745 \pm 0.0016) \times 10^{19}$.[27][28]

Cross-referenced to the above sequence, a new classification enumerates the magic tori that display these magic squares. The numbers of magic tori of order n from 1 to 5, are: 1, 0, 1, 255, 251449712 (sequence A270876 in OEIS).

Method for constructing a magic square of order 3

In the 19th century, Édouard Lucas devised the general formula for order 3 magic squares. Consider the following table made up of positive integers a, b and c:

$c - b$	$c + (a + b)$	$c - a$
$c - (a - b)$	c	$c + (a - b)$
$c + a$	$c - (a + b)$	$c + b$

These 9 numbers will be distinct positive integers forming a magic square so long as $0 < a < b < c - a$ and $b \neq 2a$. Moreover, every 3 x 3 square of distinct positive integers is of this form.

Method for constructing a magic square of odd order

A method for constructing magic squares of odd order was published by the French diplomat de la Loubère in his book, *A new historical relation of the kingdom of Siam* (Du Royaume de Siam, 1693), in the chapter entitled *The problem of the magical square according to the Indians*.[29] The method operates as follows:

The method prescribes starting in the central column of the first row with the number 1. After that, the fundamental movement for filling the squares is diagonally up and right, one step at a time. If a filled square is encountered, one moves vertically down one square instead, then continues as before. When an "up and to the right" move would leave the square, it is wrapped around to the last row or first column, respectively.

step 1		
	1	

step 2		
	1	
		2

step 3		
	1	
3		
		2

step 4		
	1	
3		
4		2

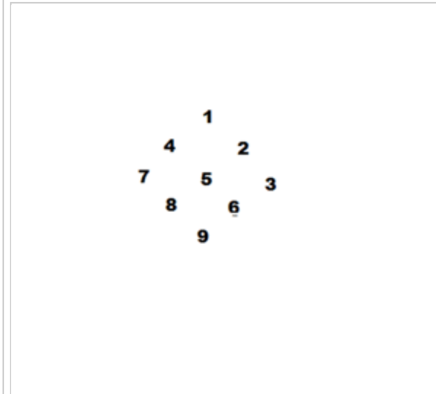
step 5		
	1	
3	5	
4		2

step 6		
	1	6
3	5	
4		2

step 7		
	1	6
3	5	7
4		2

step 8		
8	1	6
3	5	7
4		2

step 9		
8	1	6
3	5	7
4	9	2



Yang Hui's construction method

Starting from other squares rather than the central column of the first row is possible, but then only the row and column sums will be identical and result in a magic sum, whereas the diagonal sums will differ. The result will thus be a semimagic square and not a true magic square. Moving in directions other than north east can also result in magic squares.

Order 3		
8	1	6
3	5	7
4	9	2

Order 5				
17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Order 9								
47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

The following formulae help construct magic squares of odd order

Order n				
Squares (n)	Last no.	Middle no.	Sum (M)	I_{th} row and J_{th} column no.
n	n^2	$\frac{n^2+1}{2}$	$\left(\frac{n^2+1}{2}\right)n$	$n((I+J-1+\lfloor\frac{n}{2}\rfloor) \bmod n)+((I+2J-2) \bmod n)+1$

Example:

Order 5			
Squares (n)	Last no.	Middle no.	Sum (M)
5	25	13	65

The "*middle number*" is always in the diagonal bottom left to top right.
The "*last number*" is always opposite the number **1** in an outside column or row.

A method of constructing a magic square of doubly even order

Doubly even means that n is an even multiple of an even integer; or $4p$ (e.g. 4, 8, 12), where p is an integer.

Generic pattern All the numbers are written in order from left to right across each row in turn, starting from the top left hand corner. The resulting square is also known as a mystic square. Numbers are then either retained in the same place or interchanged with their diametrically opposite numbers in a certain regular pattern. In the magic square of order four, the numbers in the four central squares and one square at each corner are retained in the same place and the others are interchanged with their diametrically opposite numbers.

A construction of a magic square of order 4 (This is reflection of Albrecht Dürer's square.) Go left to right through the square counting and filling in on the diagonals only. Then continue by going left to right from the top left of the table and fill in counting down from 16 to 1. As shown below.

***M* = Order 4**			
1			4
	6	7	
	10	11	
13			16

***M* = Order 4**			
1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

An extension of the above example for Orders 8 and 12 First generate a "truth" table, where a '1' indicates selecting from the square where the numbers are written in order 1 to n^2 (left-to-right, top-to-bottom), and a '0' indicates selecting from the square where the numbers are written in reverse order n^2 to 1. For $M = 4$, the "truth" table is as shown below, (third matrix from left.)

***M* = Order 4**			
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

***M* = Order 4**			
16	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1

***M* = Order 4**			
1	0	0	1
0	1	1	0
0	1	1	0
1	0	0	1

***M* = Order 4**			
1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Note that a) there are equal number of '1's and '0's; b) each row and each column are "palindromic"; c) the left- and right-halves are mirror images; and d) the top- and bottom-halves are mirror images (c & d imply b.) The truth table can be denoted as (9, 6, 6, 9) for simplicity (1-nibble per row, 4 rows.) Similarly, for M=8, two choices for the truth table are (A5, 5A, A5, 5A, 5A, A5, 5A, A5) or (99, 66, 66, 99, 99, 66, 66, 99) (2-nibbles per row, 8 rows.) For M=12, the truth table (E07, E07, E07, 1F8, 1F8, 1F8, 1F8, 1F8, 1F8, E07, E07, E07) yields a magic square (3-nibbles per row, 12 rows.) It is possible to count the number of choices one has based on the truth table, taking rotational symmetries into account.

Medjig-method of constructing magic squares of even number of rows

This method is based on a 2006 published mathematical game called medjig (author: Willem Barink, editor: Philos-Spiele). The pieces of the medjig puzzle are squares divided in four quadrants on which the numbers 0, 1, 2 and 3 are dotted in all sequences. There are 18 squares, with each sequence occurring 3 times. The aim of the puzzle is to take 9 squares out of the collection and arrange them in a 3 × 3 "medjig-square" in such a way that each row and column formed by the quadrants sums to 9, along with the two long diagonals.

The medjig method of constructing a magic square of order 6 is as follows:

- Construct any 3 × 3 medjig-square (ignoring the original game's limit on the number of times that a given sequence is used).
- Take the 3 × 3 magic square and divide each of its squares into four quadrants.
- Fill these quadrants with the four numbers from 1 to 36 that equal the original number modulo 9, i.e. $x+9y$ where x is the original number and y is a number from 0 to 3, following the pattern of the medjig-square.

Example:

Order 3		
8	1	6
3	5	7
4	9	2

Medjig 3 × 3					
2	3	0	2	0	2
1	0	3	1	3	1
3	1	1	2	2	0
0	2	0	3	3	1
3	2	2	0	0	2
0	1	3	1	1	3

Order 6					
26	35	1	19	6	24
17	8	28	10	33	15
30	12	14	23	25	7
3	21	5	32	34	16
31	22	27	9	2	20
4	13	36	18	11	29

Similarly, for any larger integer N, a magic square of order $2N$ can be constructed from any $N \times N$ medjig-square with each row, column, and long diagonal summing to $3N$, and any $N \times N$ magic square (using the four numbers from 1 to $4N^2$ that equal the original number modulo N^2).

Construction of panmagic squares

Any number p in the order-n square can be uniquely written in the form $p = an + r$, with r chosen from $\{1,...,n\}$. Note that due to this restriction, a and r are *not* the usual quotient and remainder of dividing p by n. Consequently, the problem of constructing can be split in two problems easier to solve. So, construct two matching square grids of order n satisfying panmagic properties, one for the a-numbers (0,..., n–1), and one for the r-numbers (1,...,n). This requires a lot of puzzling, but can be done. When successful, combine them into one panmagic square. Van den Essen and many others supposed this was also the way Benjamin Franklin (1706–1790) constructed his famous Franklin squares. Three panmagic squares are shown below. The first two squares have been constructed April 2007 by Barink, the third one is some years older, and comes from Donald Morris, who used, as he supposes, the Franklin way of construction.

Order 8, sum 260							
62	4	13	51	46	20	29	35
5	59	54	12	21	43	38	28
52	14	3	61	36	30	19	45
11	53	60	6	27	37	44	22
64	2	15	49	48	18	31	33
7	57	56	10	23	41	40	26
50	16	1	63	34	32	17	47
9	55	58	8	25	39	42	24

Order 12, sum 870											
138	8	17	127	114	32	41	103	90	56	65	79
19	125	140	6	43	101	116	30	67	77	92	54
128	18	7	137	104	42	31	113	80	66	55	89
5	139	126	20	29	115	102	44	53	91	78	68
136	10	15	129	112	34	39	105	88	58	63	81
21	123	142	4	45	99	118	28	69	75	94	52
130	16	9	135	106	40	33	111	82	64	57	87
3	141	124	22	27	117	100	46	51	93	76	70
134	12	13	131	110	36	37	107	86	60	61	83
23	121	144	2	47	97	120	26	71	73	96	50
132	14	11	133	108	38	35	109	84	62	59	85
1	143	122	24	25	119	98	48	49	95	74	72

Order 12, sum 870											
1	120	121	48	85	72	73	60	97	24	25	144
142	27	22	99	58	75	70	87	46	123	118	3
11	110	131	38	95	62	83	50	107	14	35	134
136	33	16	105	52	81	64	93	40	129	112	9
8	113	128	41	92	65	80	53	104	17	32	137
138	31	18	103	54	79	66	91	42	127	114	7
5	116	125	44	89	68	77	56	101	20	29	140
139	30	19	102	55	78	67	90	43	126	115	6
12	109	132	37	96	61	84	49	108	13	36	133
135	34	15	106	51	82	63	94	39	130	111	10
2	119	122	47	86	71	74	59	98	23	26	143
141	28	21	100	57	76	69	88	45	124	117	4

The order 8 square satisfies all panmagic properties, including the Franklin ones. It consists of 4 perfectly panmagic 4×4 units. Note that both order 12 squares show the property that any row or column can be divided in three parts having a sum of 290 (= 1/3 of the total sum of a row or column). This property compensates the absence of the more standard panmagic Franklin property that any 1/2 row or column shows the sum of 1/2 of the total. For the rest the order 12 squares differ a lot. The Barink 12×12 square is composed of 9 perfectly panmagic 4×4 units, moreover any 4 consecutive numbers starting on any odd place in a row or column show a sum of 290. The Morris 12×12 square lacks these properties, but on the contrary shows constant Franklin diagonals. For a better understanding of the constructing decompose the squares as described above, and see how it was done. And note the difference between the Barink constructions on the one hand, and the Morris/Franklin construction on the other hand.

In the book *Mathematics* in the Time-Life Science Library Series, magic squares by Euler and Franklin are shown. Franklin designed this one so that any four-square subset (any four contiguous squares that form a larger square, or any four squares equidistant from the center) total 130. In Euler's square, the rows and columns each total 260, and halfway they total 130 – and a chess knight, making its L-shaped moves on the square, can touch all 64 boxes in consecutive numerical order.

Construction similar to the Kronecker Product

There is a method reminiscent of the Kronecker product of two matrices, that builds an $nm \times nm$ magic square from an $n \times n$ magic square and an $m \times m$ magic square.[30]

The construction of a magic square using genetic algorithms

A magic square can be constructed using genetic algorithms.[31] In this process an initial population of squares with random values is generated. The *fitness* scores of these individual squares are calculated based on the degree of deviation in the sums of the rows, columns, and diagonals. The population of squares *reproduce* by exchanging values, together with some random mutations. Those squares with a higher fitness score are more likely to reproduce. The fitness scores of the next generation squares are calculated, and this process continues until a magic square is found or a time limit is reached.

Solving partially completed magic squares

Similar to the Sudoku and KenKen puzzles, solving partially completed has become a popular mathematical puzzle. Puzzle solving centers on analyzing the initial given values and possible values of the empty squares. One or more solution arises as the participant uses logic and permutation group theory to rule out all unsuitable number combinations.

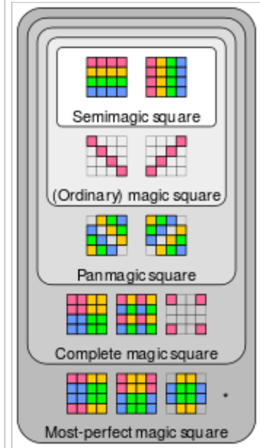
Variations of the magic square

Extra constraints

Certain extra restrictions can be imposed on magic squares. If not only the main diagonals but also the broken diagonals sum to the magic constant, the result is a panmagic square.

If raising each number to the *n*th power yields another magic square, the result is a bimagic (n = 2), a trimagic (n = 3), or, in general, a multimagic square.

A magic square in which the number of letters in the name of each number in the square generates another magic square is called an alphamagic square.



Euler diagram of requirements of some types of 4×4 magic squares. Cells of the same colour sum to the magic constant. * In 4×4 most-perfect magic squares, any 2 cells that are 2 cells diagonally apart (including wraparound) sum to half the magic constant, hence any 2 such pairs also sum to the magic constant.

Different constraints

Sometimes the rules for magic squares are relaxed, so that only the rows and columns but not necessarily the diagonals sum to the magic constant (this is usually called a **semimagic square**).

In heterosquares and antimagic squares, the $2n + 2$ sums must all be *different*.

Multiplicative magic squares

Instead of *adding* the numbers in each row, column and diagonal, one can apply some other operation. For example, a multiplicative magic square has a constant *product* of numbers. A multiplicative magic square can be derived from an additive magic square by raising 2 (or any other integer) to the power of each element, because the logarithm of the product of 2 numbers is the sum of logarithm of each. Alternatively, if any 3 numbers in a line are 2^a, 2^b and 2^c, their product is 2^{a+b+c}, which is constant if $a+b+c$ is constant, as they would be if a, b and c were taken from ordinary (additive) magic square.[32] For example, the original Lo-Shu magic square becomes:

***M* = 32768**		
16	512	4
8	32	128
256	2	64

Other examples of multiplicative magic squares include:

***M* = 216**		
2	9	12
36	6	1
3	4	18

***M* = 6720**			
1	6	20	56
40	28	2	3
14	5	24	4
12	8	7	10

***M* = 6,227,020,800**						
27	50	66	84	13	2	32
24	52	3	40	54	70	11
56	9	20	44	36	65	6
55	72	91	1	16	36	30
4	24	45	60	77	12	26
10	22	48	39	5	48	63
78	7	8	18	40	33	60

Multiplicative magic squares of complex numbers

Still using Ali Skalli's non iterative method, it is possible to produce an infinity of multiplicative magic squares of complex numbers[33] belonging to $\mathbb{C}$ set. On the example below, the real and imaginary parts are integer numbers, but they can also belong to the entire set of real numbers $\mathbb{R}$. The product is: **−352,507,340,640 − 400,599,719,520 *i***.

Skalli multiplicative 7 × 7 of complex numbers						
21+14i	−70+30i	−93−9i	−105−217i	16+50i	4−14i	14−8i
63−35i	28+114i	−14i	2+6i	3−11i	211+357i	−123−87i
31−15i	13−13i	−103+69i	−261−213i	49−49i	−46+2i	−6+2i
102−84i	−28−14i	43+247i	−10−2i	5+9i	31−27i	−77+91i
−22−6i	7+7i	8+14i	50+20i	−525−492i	−28−42i	−73+17i
54+68i	138−165i	−56−98i	−63+35i	4−8i	2−4i	70−53i
24+22i	−46−16i	6−4i	17+20i	110+160i	84−189i	42−14i

Additive-multiplicative magic and semimagic squares

Additive-multiplicative magic squares and semimagic squares satisfy properties of both ordinary and multiplicative magic squares and semimagic squares, respectively.[34]

First known additive-multiplicative magic square 8×8 found by W. W. Horner in 1955 Sum = 840 Product = 2 058 068 231 856 000							
162	207	51	26	133	120	116	25
105	152	100	29	138	243	39	34
92	27	91	136	45	38	150	261
57	30	174	225	108	23	119	104
58	75	171	90	17	52	216	161
13	68	184	189	50	87	135	114
200	203	15	76	117	102	46	81
153	78	54	69	232	175	19	60

Smallest known additive-multiplicative semimagic square 4×4 found by L. Morgenstern in 2007 Sum = 247 Product = 3 369 600			
156	18	48	25
30	144	60	13
16	20	130	81
45	65	9	128

It is unknown if any additive-multiplicative magic squares smaller than 8×8 exist, but it has been proven that no 3×3 or 4×4 additive-multiplicative magic squares and no 3×3 additive-multiplicative semimagic squares exist.[35]

Other magic shapes

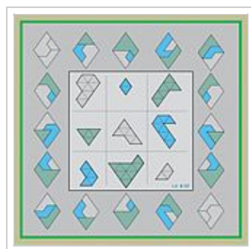
Other shapes than squares can be considered. The general case is to consider a design with *N* parts to be magic if the *N* parts are labeled with the numbers 1 through *N* and a number of identical sub-designs give the same sum. Examples include magic dodecahedrons, magic triangles[36] magic stars, and magic hexagons. Going up in dimension results in magic cubes and other magic hypercubes.

Edward Shineman has developed yet another design in the shape of magic diamonds.

Possible magic shapes are constrained by the number of equal-sized, equal-sum subsets of the chosen set of labels. For example, if one proposes to form a magic shape labeling the parts with $\{1, 2, 3, 4\}$, the sub-designs will have to be labeled with {1,4} and {2,3}.[36]

Other component elements

Magic squares may be constructed which contain geometric shapes instead of numbers. Such squares, known as geometric magic squares, were invented and named by Lee Sallows in 2001.[37]



A geometric magic square.

Combined extensions

One can combine two or more of the above extensions, resulting in such objects as *multiplicative multimagic hypercubes*. Little seems to be known about this subject.

Related problems

Over the years, many mathematicians, including Euler, Cayley and Benjamin Franklin have worked on magic squares, and discovered fascinating relations.

Magic square of primes

Rudolf Ondrejka (1928–2001) discovered the following 3×3 magic square of primes, in this case nine Chen primes:

17	89	71
113	59	5
47	29	101

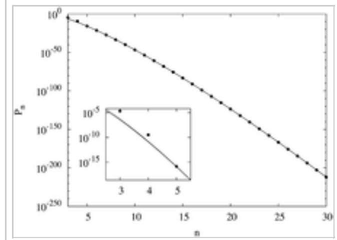
The Green–Tao theorem implies that there are arbitrarily large magic squares consisting of primes.

n-Queens problem

In 1992, Demirörs, Rafraf, and Tanik published a method for converting some magic squares into *n*-queens solutions, and vice versa.[38]

Enumeration of magic squares

As mentioned above, the set of normal squares of order three constitutes a single equivalence class-all equivalent to the Lo Shu square. Thus there is basically just one normal magic square of order 3. But the number of distinct normal magic squares rapidly increases for higher orders.[39] There are 880 distinct magic squares of order 4 and 275,305,224 of order 5.[40] These squares are respectively displayed on 255 magic tori of order 4, and 251,449,712 of order 5.[41] The number of magic tori and distinct normal squares is not yet known for any higher order.[42]



Semi-log plot of Pn, the probability of magic squares of dimension n

Algorithms tend to only generate magic squares of a certain type or classification, making counting all possible magic squares quite difficult. Traditional counting methods have proven unsuccessful, statistical analysis using the Monte Carlo method has been applied. The basic principle applied to magic squares is to randomly generate n × n matrices of elements 1 to n^2 and check if the result is a magic square. The probability that a randomly generated matrix of numbers is a magic square is directly proportional to the number of magic squares.[43]

More intricate versions of the Monte Carlo method, such as the exchange Monte Carlo, and Monte Carlo Backtracking have produced even more accurate estimations. Using these methods it has been shown that the probability of magic squares decreases rapidly as n increases. Using fitting functions give the curves seen to the right.

Magic squares in popular culture

On October 9, 2014 the post office of Macao in the People's Republic of China issued a series of stamps based on magic squares.[44] The figure below shows the stamps featuring the nine magic squares chosen to be in this collection.[45]


Macau stamps featuring magic squares

See also

- Arithmetic sequence
- Combinatorial design
- Freudenthal magic square
- John R. Hendricks
- Hexagonal tortoise problem
- Latin square
- Magic circle
- Magic cube classes
- Magic series
- Most-perfect magic square
- Nasik magic hypercube
- Prime reciprocal magic square
- Room square
- Square matrices
- Sriramachakra
- Sudoku
- Unsolved problems in mathematics
- Vedic square

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11. This tradition about a series of magic squares from order three to nine, which are associated with the seven planets, survives in Greek, Arabic, and Latin versions. The Latin version is Liber de septem figuris septem planetarum figurarum Geberi regis Indorum. This treatise is the identified source of Dürer and Heinrich Cornelius Agrippa von Nettesheim. Cf. Peter, J. Barta, The Seal-Ring of Proportion and the magic rings (2016), pp. 8-9, n. 10
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13. Manuel Moschopoulos – Mathematics and the Liberal Arts (http://mtcs.truman.edu/~thammond/history/ManuelMoschopoulos.html)
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16. In a 1981 article ("Zur Frühgeschichte der magischen Quadrate in Westeuropa" i.e. "Prehistory of Magic Squares in Western Europe", Sudhoffs Archiv Kiel (1981) vol. 65, pp. 313–338) German scholar Menso Folkerts lists several manuscripts in which the "Trattato d'Abbaco" by Dagomari contains the two magic square. Folkerts quotes a 1923 article by Amedeo Agostini in the Bollettino dell'Unione Matematica Italiana: "A. Agostini in der Handschrift Bologna, Biblioteca Universitaria, Ms. 2433, f. 20v-21r; siehe Bollettino della Unione Matematica Italiana 2 (1923), 77f. Agostini bemerkte nicht, dass die Quadrate zur Abhandlung des Paolo dell'Abbaco gehören und auch in anderen Handschriften dieses Werks vorkommen, z. B. New York, Columbia University, Plimpton 167, f. 69rv; Paris, BN, ital. 946, f. 37v-38r; Florenz, Bibl. Naz., II. IX. 57, f. 86r, und Targioni 9, f. 77r; Florenz, Bibl. Riccard., Ms. 1169, f. 94-95."
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