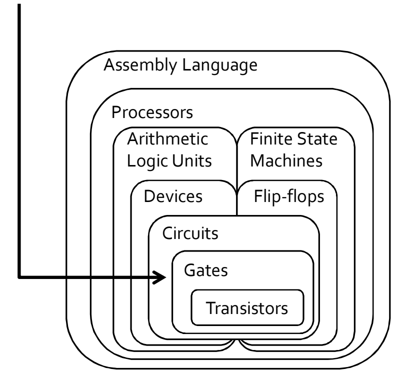
Circuit Creation

You are here



Making boolean expressions

- So how would you represent boolean expressions using logic gates?

```
Y = (A or B) and (not A or not B) or C
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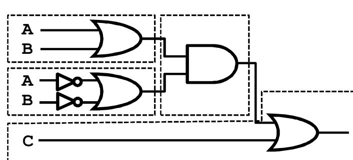
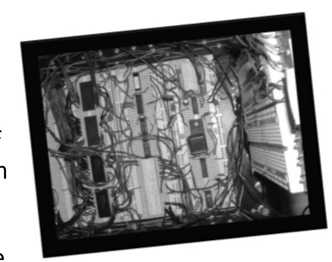
- Like so:



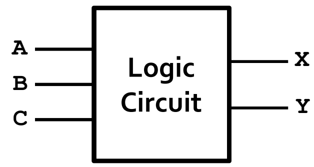
Creating complex circuits

- What do we do in the case of more complex circuits, with several inputs and more than one output?
  - If you're lucky, a truth table is provided to express the circuit.
  - Usually the behaviour of the circuit is expressed in words, and the first step involves creating a truth table that represents the described behaviour.



Circuit example

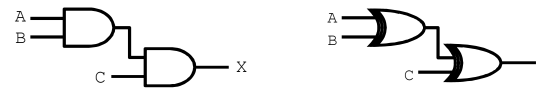
- The circuit on the right has three inputs (A, B and C) and two outputs (X and Y).



- *What logic is needed to set X high when all three inputs are high?*
- *What logic is needed to set Y high when the number of high inputs is odd?*

Combinational circuits

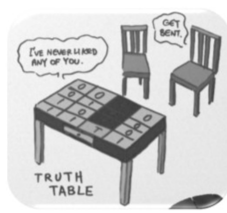
- Small problems can be solved easily.



- Larger problems require a more systematic approach.
  - Example: Given three inputs A, B, and C, make output Y high in the case where all of the inputs are low, or when A and B are low and C is high, or when A and C are low but B is high, or when A is low and B and C are high.

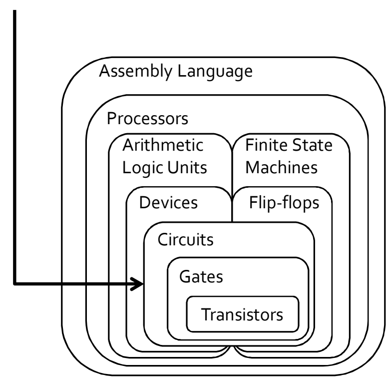
Creating complex logic

- How do we approach problems like these (and circuit problems in general)?



- Basic steps:
  1. Create truth tables.
  2. Express as boolean expression.
  3. Convert to gates.
- The key to an efficient design?
  - Spending extra time on Step #2.

Now you are here



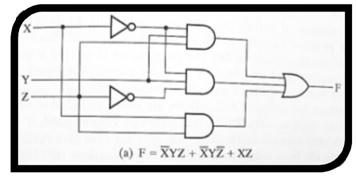
Lecture Goals

- After this lecture, you should be able to:
  - Create a truth table that represents the behaviour of a circuit you want to create.
  - Translate the minterms from a truth table into gates that implement that circuit.
  - Use Karnaugh maps to reduce the circuit to the minimal number of gates.

Lecture Goals

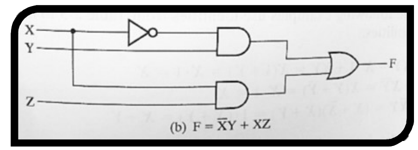
- Which implementation do you prefer? Why?

A.



(a) $F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$

B.



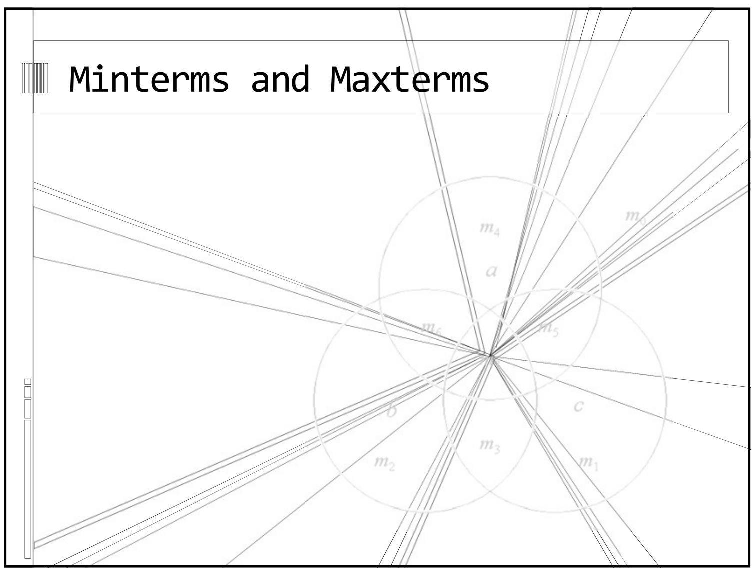
(b) $F = \overline{X}Y + XZ$

Example truth table

- Consider the following example:
  - *"Given three inputs* A*,* B*, and* C*, make output* Y *high wherever any of the inputs are low, except when all three are low or when* A *and* C *are high."*
- This leads to the truth table on the right.
  - Is there a better way to describe the cases when the circuit's output is high?

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Minterms and Maxterms



Minterms

- An easier way to express circuit behaviour is to assume the standard truth table format, and then list which input rows cause high output.
  - These rows are referred to as minterms.

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Minterm	Y
m_0	0
m_1	1
m_2	1
m_3	1
m_4	1
m_5	0
m_6	1
m_7	0

Minterms and maxterms

- A more formal description:
  - Minterm = an AND expression with every input present in true or complemented form.
  - Maxterm = an OR expression with every input present in true or complemented form.
  - For example, given four inputs (A, B, C, D):
    - Valid minterms:
      - $A \cdot \overline{B} \cdot C \cdot D$, $\overline{A} \cdot B \cdot \overline{C} \cdot D$, $A \cdot B \cdot C \cdot D$
    - Valid maxterms:
      - $A+\overline{B}+C+D$, $\overline{A}+B+\overline{C}+D$, $A+B+C+D$
    - Neither minterm nor maxterm:
      - $A \cdot B+C \cdot D$, $A \cdot B \cdot D$, $A+B$

Creating boolean expressions

- While we're talking about notation…
  - AND operations are denoted in these expressions by the multiplication symbol.
    - e.g. $A \cdot B \cdot C$ or $A*B*C \approx A \wedge B \wedge C$
  - OR operations are denoted by the addition symbol.
    - e.g. $A+B+C \approx A \vee B \vee C$
  - NOT is denoted by multiple symbols.
    - e.g. $\neg A$ or A' or $\overline{A}$
  - XOR occurs rarely in circuit expressions.
    - e.g. $A \oplus B$

Back to minterms

- Circuits are often described using minterms or maxterms, as a form of logic shorthand.
  - Given n inputs, there are 2^n minterms and maxterms possible (same as rows in a truth table).
  - Naming scheme:
    - Minterms are labeled as m_x, maxterms are labeled as M_x
      - The x subscript indicates the row in the truth table.
      - x starts at 0 (when all inputs are low), and ends with 2^n-1.
  - <u>Example:</u> Given 3 inputs –
    - Minterms are m_0 ($\overline{A} \cdot \overline{B} \cdot \overline{C}$) to m_7 ($A \cdot B \cdot C$)
    - Maxterms are M_0 ($A+B+C$) to M_7 ($\overline{A}+\overline{B}+\overline{C}$)

Quick Exercises

- Given 4 inputs A, B, C and D write:
  - m_9
  - m_{15}
  - m_{16}
  - M_2
- Which minterm is this?
  - $\overline{A} \cdot B \cdot \overline{C} \cdot \overline{D}$
- Which maxterm is this?
  - $A+B+C+\overline{D}$

Using minterms and maxterms

- What are minterms used for?
  - A single minterm indicates a set of inputs that will make the output go high.
  - Example: m_2
    - Output only goes high in third line of truth table.

A	B	C	D	m_2
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Using minterms and maxterms

- What happens when you OR two minterms?
  - Result is output that goes high in both minterm cases.
  - For m_2+m_8, both third and ninth lines of truth table result in high output.

A	B	C	D	m_2	m_8	m_2+m_8
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	1	0	1
0	0	1	1	0	0	0
0	1	0	0	0	0	0
0	1	0	1	0	0	0
0	1	1	0	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	1	1
1	0	0	1	0	0	0
1	0	1	0	0	0	0
1	0	1	1	0	0	0
1	1	0	0	0	0	0
1	1	0	1	0	0	0
1	1	1	0	0	0	0
1	1	1	1	0	0	0

Creating boolean expressions

- Two canonical forms of boolean expressions:
  - Sum-of-Minterms (SOM):
    - Since each minterm corresponds to a single high output in the truth table, the combined high outputs are a union of these minterm expressions.
    - Expressed in “Sum-of-Products” form.
  - Product-of-Maxterms (POM):
    - Since each maxterm only produces a single low output in the truth table, the combined low outputs are an intersection of these maxterm expressions.
    - Expressed in “Product-of-Sums” form.

$Y = m_2 + m_6 + m_7 + m_{10}$ (SOM)

A	B	C	D	m_2	m_6	m_7	m_{10}	Y
0	0	0	0					
0	0	0	1					
0	0	1	0					
0	0	1	1					
0	1	0	0					
0	1	0	1					
0	1	1	0					
0	1	1	1					
1	0	0	0					
1	0	0	1					
1	0	1	0					
1	0	1	1					
1	1	0	0					
1	1	0	1					
1	1	1	0					
1	1	1	1					

$Y = m_2 + m_6 + m_7 + m_{10}$ (SOM)

A	B	C	D	m_2	m_6	m_7	m_{10}	Y
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	1	0	1	0	0	0	1
0	0	1	1	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0
0	1	1	0	0	1	0	0	1
0	1	1	1	0	0	1	0	1
1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0
1	0	1	0	0	0	0	1	1
1	0	1	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0
1	1	1	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0

Using Sum-of-Minterms

- Sum-of-Minterms is a way of expressing which inputs cause the output to go high.
  - Assumes that the truth table columns list the inputs according to some logical or natural order.
- Minterm and maxterm expressions are used for efficiency reasons:
  - More compact that displaying entire truth tables.
  - Sum-of-minterms are useful in cases with very few input combinations that produce high output.
    - Product-of-maxterms useful when expressing truth tables that have very few low output cases…

$Y = M_3 \cdot M_5 \cdot M_7 \cdot M_{10} \cdot M_{14}$ (POM)

A	B	C	D	M_3	M_5	M_7	M_{10}	M_{14}	Y
0	0	0	0						
0	0	0	1						
0	0	1	0						
0	0	1	1						
0	1	0	0						
0	1	0	1						
0	1	1	0						
0	1	1	1						
1	0	0	0						
1	0	0	1						
1	0	1	0						
1	0	1	1						
1	1	0	0						
1	1	0	1						
1	1	1	0						
1	1	1	1						

$Z = M_3 \cdot M_5 \cdot M_7 \cdot M_{10} \cdot M_{14}$ (POM)

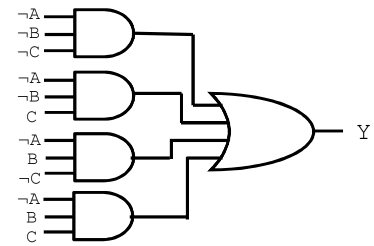
A	B	C	D	M_3	M_5	M_7	M_{10}	M_{14}	Z
0	0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1	1
0	0	1	0	1	1	1	1	1	1
0	0	1	1	0	1	1	1	1	0
0	1	0	0	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1	0
0	1	1	0	1	1	1	1	1	1
0	1	1	1	1	1	0	1	1	0
1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	1
1	0	1	0	1	1	1	0	1	0
1	0	1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1
1	1	1	0	1	1	1	1	0	0
1	1	1	1	1	1	1	1	1	1

Converting SOM to gates

- Once you have a Sum-of-Minterms expression, it is easy to convert this to the equivalent combination of gates:

$m_0 + m_1 + m_2 + m_3 =$

$\overline{A} \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot B \cdot C =$



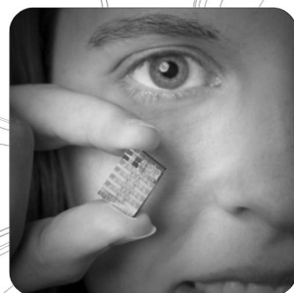
2-input XOR gate (SOM,POM)

- $m_x = M_x'$
  - Minterm x is the complement of maxterm x.
  - e.g., $m_0 = A'B'$ while $M_0 = A + B$
- 2-input XOR gate in SOM and POM form.
  - Sum-Of-Minterms: $F = m_1 + m_2$
  - Product-Of-Maxterms : $F = M_O \cdot M_3$
- Write F’ in Sum-Of-Minterms form:
  - We need to include the minterms not present in F.
  - F’ = mo + m3

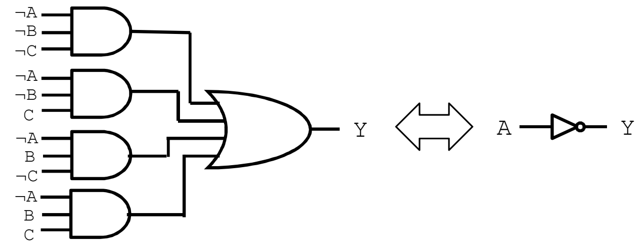
2-input XOR gate (SOM,POM)-cont’d

- Write F’ in Sum-Of-Minterms form:
  - We need to include the minterms not present in F.
  - $F' = m_0 + m_3$
- Now let’s take the complement of F’.
  - $(F')' = F = (m_0 + m_3)' = m_0'm_3'$
  - But m_0' is M_0 and m_3' is M_3
  - Therefore, $F = M_0 \cdot M_3$
- The canonical representations SOM and POM for a given function are equivalent! ☺

Reducing circuits



Reasons for reducing circuits



- Note example of Sum-of-Minterms circuit design.
- To minimize the number of gates, we want to reduce the boolean expression as much as possible from a collection of minterms to something smaller.
- This is where CSC165 skills come in handy ☺

Boolean algebra review

- Axioms:

$0 \cdot 0 = 0$	$0 \cdot 1 = 1 \cdot 0 = 0$
$1 \cdot 1 = 1$	if $x = 1$, $\overline{x} = 0$

- From this, we can extrapolate:

$x \cdot 0 =$	$x+1 =$
$x \cdot 1 =$	$x+0 =$
$x \cdot x =$	$x+x =$
$x \cdot \overline{x} =$	$x+\overline{x} =$
$\overline{\overline{x}} =$	

If one input of a 2-input AND gate is 1, then the output is whatever value the other input is.

If one input of a 2-input OR gate is 0, then the output is whatever value the other input is.

Boolean algebra review

- Axioms:

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$x \cdot 0 = 0$	$x+1 = 1$
$x \cdot 1 = x$	$x+0 = x$
$x \cdot x = x$	$x+x = x$
$x \cdot \overline{x} = 0$	$x+\overline{x} = 1$
$\overline{\overline{x}} = x$	

Other Boolean identities

- Commutative Law:

$$x \cdot y = y \cdot x \qquad x+y = y+x$$

- Associative Law:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x+(y+z) = (x+y)+z$$

- Distributive Law:

$$x \cdot (y+z) = x \cdot y + x \cdot z$$
$$x+(y \cdot z) = (x+y) \cdot (x+z)$$

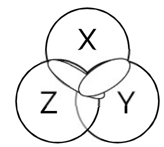
Does this hold in conventional algebra?

Consensus Law Proof -Venn diagram

- Consensus Law:

$$x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$$

- Proof by Venn diagram:
  - $x \cdot y$
  - $\overline{x} \cdot z$
  - $y \cdot z$
    - Already covered!

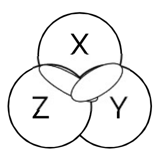


Consensus Law Proof -Venn diagram

- Consensus Law:

$$x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$$

- Proof by Venn diagram:
  - $x \cdot y$
  - $\overline{x} \cdot z$
  - $y \cdot z$
    - Already covered!



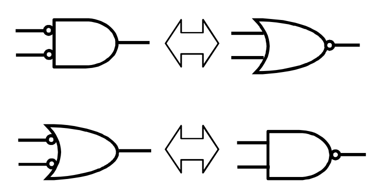
Other boolean identities

- Absorption Law:

$$x \cdot (x+y) = x \qquad x+(x \cdot y) = x$$

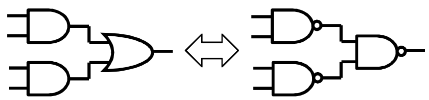
- De Morgan's Laws:

$$\overline{x} \cdot \overline{y} = \overline{x+y}$$
$$\overline{x}+\overline{y} = \overline{x \cdot y}$$

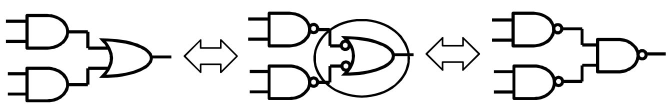


Converting to NAND gates

- De Morgan's Law is important because out of all the gates, NANDs are the cheapest to fabricate.
  - a Sum-of-Products circuit could be converted into an equivalent circuit of NAND gates:



- This is all based on de Morgan's Law:



Reducing boolean expressions

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

- Assuming logic specs at left, we get the following:

$$\boxed{Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} + A\cdot B\cdot\overline{C} + A\cdot B\cdot C}$$

- Now start combining terms, like the last two:

$$\boxed{Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} + \mathbf{A\cdot B}}$$

Reducing boolean expressions

- Different final expressions possible, depending on what terms you combine.
- For instance, given the previous example:

$$\boxed{Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} + A\cdot B\cdot\overline{C} + A\cdot B\cdot C}$$

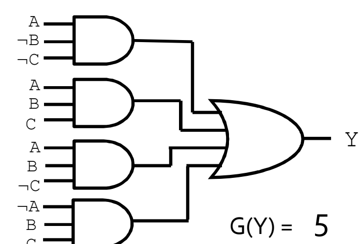
- If you combine the end and middle terms…

$$\boxed{Y = B\cdot C + A\cdot\overline{C}}$$

- Which reduces the number of gates and inputs!

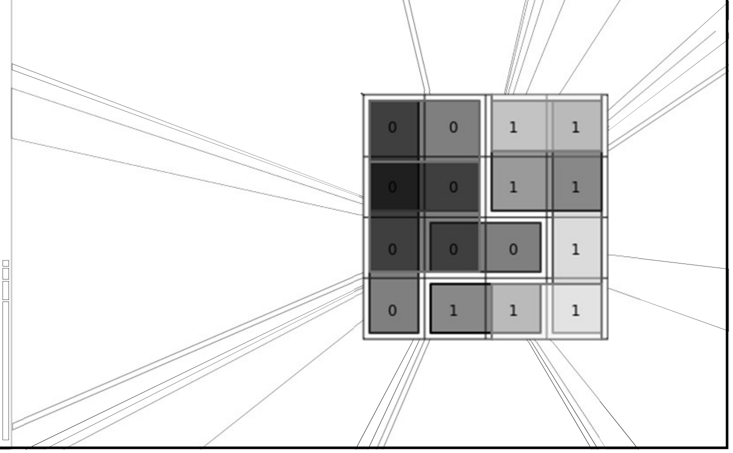
Reducing boolean expressions

- What is considered the "simplest" expression?
  - In this case, "simple" denotes the lowest gate cost (G) or the lowest gate cost with NOTs (GN).
  - To calculate the gate cost, simply add all the gates together (as well as the cost of the NOT gates, in the case of the GN cost).



G(Y) = 5
GN(Y) = 8

Karnaugh maps



Reducing boolean expressions

- How do we find the "simplest" expression for a circuit?
  - Technique called Karnaugh maps (or K-maps).
  - Karnaugh maps are a 2D grid of minterms, where adjacent minterm locations in the grid differ by a single literal.
  - Values of the grid are the output for that minterm.

	$\overline{B}\cdot\overline{C}$	$\overline{B}\cdot C$	$B\cdot C$	$B\cdot\overline{C}$
$\overline{A}$	0	0	1	0
A	1	0	1	1

Karnaugh maps

- Karnaugh maps can be of any size, and have any number of inputs.
  - i.e. the 4-input example here.
- Since adjacent minterms only differ by a single value, they can be grouped into a single term that omits that value.

	$\overline{C}\cdot\overline{D}$	$\overline{C}\cdot D$	$C\cdot D$	$C\cdot\overline{D}$
$\overline{A}\cdot\overline{B}$	m_0	m_1	m_3	m_2
$\overline{A}\cdot B$	m_4	m_5	m_7	m_6
$A\cdot B$	m_{12}	m_{13}	m_{15}	m_{14}
$A\cdot\overline{B}$	m_8	m_9	m_{11}	m_{10}

Using Karnaugh maps

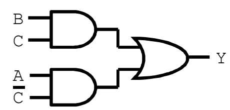
- Once Karnaugh maps are created, draw boxes over groups of high output values.
  - Boxes must be rectangular, and aligned with map.
  - Number of values contained within each box must be a power of 2.
  - Boxes may overlap with each other.
  - Boxes may wrap across edges of map.

	$\overline{B}\cdot\overline{C}$	$\overline{B}\cdot C$	$B\cdot C$	$B\cdot\overline{C}$
$\overline{A}$	0	0	1	0
A	1	0	1	1

Using Karnaugh maps

	$\overline{B}\cdot\overline{C}$	$\overline{B}\cdot C$	$B\cdot C$	$B\cdot\overline{C}$
$\overline{A}$	0	0	1	0
A	1	0	1	1

- Once you find the minimal number of boxes that cover all the high outputs, create boolean expressions from the inputs that are common to all elements in the box.
- For this example:
  - Vertical box: $B\cdot C$
  - Horizontal box: $A\cdot\overline{C}$
  - Overall equation: $Y = B\cdot C + A\cdot\overline{C}$



Karnaugh maps and maxterms

- Can also use this technique to group maxterms together as well.
- Karnaugh maps with maxterms involves grouping the zero entries together, instead of grouping the entries with one values.

	$C+D$	$C+\overline{D}$	$\overline{C}+\overline{D}$	$\overline{C}+D$
$A+B$	M_0	M_1	M_3	M_2
$A+\overline{B}$	M_4	M_5	M_7	M_6
$\overline{A}+\overline{B}$	M_{12}	M_{13}	M_{15}	M_{14}
$\overline{A}+B$	M_8	M_9	M_{11}	M_{10}

Quick Exercise

	$\overline{C}\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\overline{B}$	0	0	1	1
$\overline{A}B$	1	1	0	0
AB	1	1	0	0
$A\overline{B}$	0	0	0	0

$$F = B\cdot\overline{C} + \overline{A}\cdot\overline{B}\cdot C$$