Implications of Black-Scholes model when assuming non geometric brownian motion

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Abstract

1 Suggestion to project

European options is a contract that gives the buyer the right to but not the obligation to purchase a number of shares at a given time (expiration date) and at a given price (strike-price). Merton presents a partial differential equation known as the Black-Scholes model in [1] in equation (44) that can be used to model the price of a European option,

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + (rS - D)\frac{\partial W}{\partial S} - \frac{\partial W}{\partial t} - rW = 0. \tag{1}$$

The model makes the following assumptions:

1. The underlying stock moves like geometric-brownian motion.
2. The volatility of the underlying stock is constant, the risk free rate is constant and known.
3. Transaction costs are zero.
4. The underlying stock can be traded at all times.

The first assumption implies that the returns of the underlying stock are log-normally distributed, this assumption is empirically known to be false. The distribution of returns in real life have fatter tails than the best-fit normal distribution.

Is this model used by market participants? No, at least not under the assumptions above. By observing the implied volatility for options of the same underlying stock and the same strike price but at different expiration dates, we can see that the implied volatility is different, in other words not constant over time, suggesting a clustering of some sort of volatility. Moreover, a publicly available model known by all market participants would never be able to beat the market in an efficient market. This begs the question whether the model is useful at all. Most likely it is, and market participants have proprietary models that resemble (1).

In this article we seek to find a stochastic process $X(t, ...)$ that fits our empirical observations about the distribution of returns that is better than geometric-brownian motion, thus, disregarding assumption 1.). We then want to study what implications the suggested stochastic process would have for pricing an option. A criterion for the distribution is that the variance has to be finite, otherwise the pricing of an option will be very difficult if not impossible. That is $Var\left[X(t+\tau, ...) - X(t, ...)\right] < \infty$. Moreover we want our stochastic process to exhibit a strong markov property and be a martingale, or maybe a martingale with constant drift proportional to time.

2 Kommentarer kring projektet

2.1 Varför just detta projektet

- Jag har inte hittat några publikationer på denna slags frågeställning som attackerar antagande 1.) och sedan undersöker vilka implikationer det får. Det kanske innebär att detta slags problem är en *"dead-end"* eftersom ingen verkar ha publicerat kring det.
- Jag har däremot hittat flera publikationer som undersöker övriga antaganden.
- Just på grund av att jag inte hittat några andra publikationer som attackerar antagande 1.) och undersöker vad det innebär om det är fel, tycker jag denna frågeställning är intressant.
- Jag har lyckats hitta en publikation av en kandidatstudent från Chicago som kritiserar antagande 1.) Noah Fisher. STOCHASTIC CALCULUS AND THE BLACK-SCHOLES-MERTON MODEL. Dock djupdyker han ej i vilka implikationer detta får. Fisher bekräftar endast att antagandet är felaktigt.

2.2 Förslag till struktur

1. Hitta en stokastisk process som modellerar aktie-priser bättre än Brownian-motion. *(Empiriskt)*
2. Klarställa vad det innebär för Black-Scholes modellen. *(Teoretiskt)*

References

[1] Robert C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.