```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import scipy.io as sio
        import pickle
```

Testat2_GaussChristoffel - Jens Pfeifle

Allgemeine Funktionen

```
In [2]: def save_obj(obj, name ):
            with open('obj/'+ name + '.pkl', 'wb') as f:
                pickle.dump(obj, f, pickle.HIGHEST_PROTOCOL)

        def load_obj(name ):
            with open('obj/' + name + '.pkl', 'rb') as f:
                return pickle.load(f)
```

```
In [3]: def integrate_polynomial(p, a, b):
            """
            Calculate the exact integral of a polynomial p over bounds (a,b)
            The polynomial is given by a coefficient vector, as in:
            [3,2,1] -> 3 + 2x + x³
            """
            t_integrals = np.array([(1/(i+1)) * (b**(i+1) - a**(i+1))
                                    for i in range(len(p))])
            return sum(p * t_integrals)
```

```
In [4]: # tests
        p = np.array([ 0, 1, 0, 0, -1 ])
        assert integrate_polynomial(p, 0, 1) == 0.3
        p = np.array([ 0, 1])
        assert integrate_polynomial(p, 0, 1) == 0.5
        assert integrate_polynomial(p, -1, 1) == 0.0
        p = np.array([ 0, 0,0,1])
        assert integrate_polynomial(p, 0, 1) == 0.25
        assert integrate_polynomial(p, -1, 1) == 0.0
        print("integratepolynomial - tests passed")

        integratepolynomial - tests passed
```

```
In [5]: def add_polynomials(p1, p2):
            """
            Calculate the sum of two polynomials given by coefficient vectors
            i.e. for p(x) = 3 + 2x + x³:
            coefficients = [3,2,1]
            The order (desceding or ascending exponent) doesnt matter,
            but must be the same for both input polynomials.
            """
            p3 = np.zeros(max([len(p1), len(p2)]))
            for i, coeff in enumerate(p1):
                p3[i] += coeff
            for i,coeff in enumerate(p2):
                p3[i] += coeff
            return p3
```

In [6]:
```
def compare_polynomials(p1,p2):
    if not (len(p1) == len(p2)):
        return False
    for n in range(len(p1)):
        if not p1[n] == p2[n]:
            return False
    else:
        return True
```

In [7]:
```
#tests
p1 = np.array([ 1, 2, 3])
p2 = np.array([ 0 ])
result = add_polynomials(p1,p2)
expected = [1.,2.,3.]
assert compare_polynomials(result, expected)

p1 = np.array([ 1, 2, 3])
p2 = np.array([ 1,1,1 ])
result = add_polynomials(p1,p2)
expected = [2.,3.,4.]
assert compare_polynomials(result, expected)

p1 = np.array([ 1, 2, 3])
p2 = np.array([ -1, -2,-3 ])
result = add_polynomials(p1,p2)
expected = [0.,0.,0.]
assert compare_polynomials(result, expected)

print("add_polynomials - tests passed")
```

```
add_polynomials - tests passed
```

Gauss-Christoffel-Quadratur

Legendre-Polynome Berechnen

In [8]:
```
t_pow = lambda i: np.append(np.zeros(i), 1)  # returns t**i as polynomial

"""
Calculate Legendre-Polynomials through Gram-Schmidt-Orthogonalisation
"""
def legendre_polynomials(N = 10, tmin=-1.0, tmax = 1.0):
    #A1
    legendre = [np.array([1.0])]
    for i in range(1,N):
        alpha_i = []
        #A2
        sum_alpha_p = np.array([0])
        for j in range(0,i):
            alpha_i.append(integrate_polynomial(np.convolve(t_pow(i), legend
re[j]), tmin, tmax))
            sum_alpha_p = add_polynomials(sum_alpha_p, alpha_i[j] * legendre
[j])
        # A3
        p_star = add_polynomials(t_pow(i), -1*sum_alpha_p)
        # A4
        norm_p_star = np.sqrt(integrate_polynomial(np.convolve(p_star, p_sta
r), tmin, tmax))
        p = p_star / norm_p_star
        legendre.append(p)
    return legendre
```

In [9]:
```
legendre = legendre_polynomials(3, tmin= -1.0,tmax = 1.0)
```

In [10]:
```
from scipy.special import legendre as scp_legendre
for n,p in enumerate(legendre):
    print("Koeffizienten {}. Legendre Polynom ".format(n+1))
    print("Soll: " + str(list(reversed(scp_legendre(n).coefficients))))
    print("Ist : " + str(p))
```

```
Koeffizienten 1. Legendre Polynom
Soll: [1.0]
Ist : [1.]
Koeffizienten 2. Legendre Polynom
Soll: [0.0, 1.0]
Ist : [0.         1.22474487]
Koeffizienten 3. Legendre Polynom
Soll: [-0.5, 0.0, 1.5]
Ist : [-1.05409255  0.          1.58113883]
```

Nach Absprache mit Hannes ist es für das Testat nicht nötig, bei dem Gram-Schmidt-Orthogonalisierungsverfahren die korrekten Ergebnisse zu haben. Verwende stattdessen die Legendre Polynome aus dem scipy Paket.

In [11]:
```
### ersetze berechnete Legendre-Polynome durch Legendre-Funktion von scipy.s
pecial.legendre
legendre = lambda n: np.array(list(scp_legendre(n).coefficients))
```

Stützstellen als Nullstellen der Legendre-Polynome

Transformation von [-1,1] auf [0,1]

In [12]:
```
def transform_gausspts(pts, a = -1.0,b=1.0):
    return np.array([(b-a)/2*x + (a+b)/2 for x in pts])

print("Stützstellen auf Referenzintervall [-1,1]:")
for n in range(4):
    print(np.roots(legendre(n+1)))

print("Stützstellen auf Referenzinterval [0,1]:")
for n in range(4):
    print(transform_gausspts(sorted(np.roots(legendre(n+1))),0,1))
```

```
Stützstellen auf Referenzintervall [-1,1]:
[0.]
[-0.57735027  0.57735027]
[-0.77459667  0.77459667  0.        ]
[ 0.86113631 -0.86113631  0.33998104 -0.33998104]
Stützstellen auf Referenzinterval [0,1]:
[0.5]
[0.21132487 0.78867513]
[0.11270167 0.5        0.88729833]
[0.06943184 0.33000948 0.66999052 0.93056816]
```

Speichern

In [13]:
```
MAX_PTS = 10
# auf -1,1
lambdas = {n+1: sorted(np.roots(legendre(n+1))) for n in range(MAX_PTS+1)}
# auf 0,1
lambdas = {n: transform_gausspts(l,0,1) for n,l in lambdas.items()}
save_obj(lambdas, "testat2_tau")
# to load: load_obj("testat2_tau")
```

Gewichte (Lagrange)

Berechnen

In [14]:
```python
def gauss_weights(gausspts, t_min = -1.0, t_max = 1.0):
    """
    Given an array of gauss points,
    calculate an array of Gauss weights
    on the reference interval [t_min,t_max]
    """

    npts = len(gausspts)
    coeffs = np.zeros(npts)

    for i in range(npts):

        ti = gausspts[i]

        prod = np.array([1])
        for j in range(npts):
            if not i == j:
                tj = gausspts[j]
                a = 1/(ti - tj)
                p_ = a * np.array([-tj, 1]) # a*(t-t_j)
                prod = np.convolve(prod, p_)  # polynomial multiplication
        c = integrate_polynomial(prod, t_min, t_max)
        assert c > 0, "Weights must be > 0"
        coeffs[i] = c
    return coeffs
```

Speichern

In [15]:
```python
tau = load_obj("testat2_tau")
# beachte intervall!
tmin = 0.0
tmax = 1.0

all_weights = {}
for n in range(1,max(tau.keys())):
    all_weights[n] = gauss_weights(gausspts = tau[n],t_min=tmin,t_max=tmax)
save_obj(all_weights, "testat2_lambda")
```

Quadraturfunktion Gauss

In [16]:
```
def quad_Gauss(f, a, b, N):
    """
    Approximate the integral of any (callable) function f
    on the interval a,b
    using Gauss Quadrature with N support points.
    Weights/support points are loaded for the reference interval [0,1]
    and must be transformed to the integration interval.
    """
    try:
        weights = load_obj("testat2_lambda")[N]
    except KeyError as e:
        print("Weights for N={} not available".format(N))
        return

    try:
        support_points = load_obj("testat2_tau")[N]
    except KeyError as e:
        print("Gausspts for N={} not available".format(N))
        return

    # transform support points to [a,b] to get function values
    support_points = [a + (b-a)*t for t in support_points]
    function_values = [f(t) for t in support_points]

    #print("weights: \t {}".format(weights))
    #print("supports: \t {}".format(support_points))
    #print("function: \t {}".format(function_values))

    return (b-a) * sum(weights*function_values)
```

Newton-Cotes

Quadraturfunction NC (Gewichte werden geladen)

In [17]:
```
def quad_NC(f, a, b, N):
    """
    Approximate the integral of any (callable) function f
    on the interval a,b
    using Newton-Cotes Quadrature with N support points.
    The weights are loaded for the reference interval [0,1]
    and must be transformed to the function interval.
    """
    try:
        weights = load_obj("ub8_nc_weights")[N]
    except KeyError as e:
        print("Weights for N={} not available".format(N))
        return
    weights = weights[:N]  # only the first N, rest are 0
    support_points = np.linspace(a,b,N)

    function_values = [f(ti) for ti in support_points]

    return (b-a) * sum(weights*function_values)
```

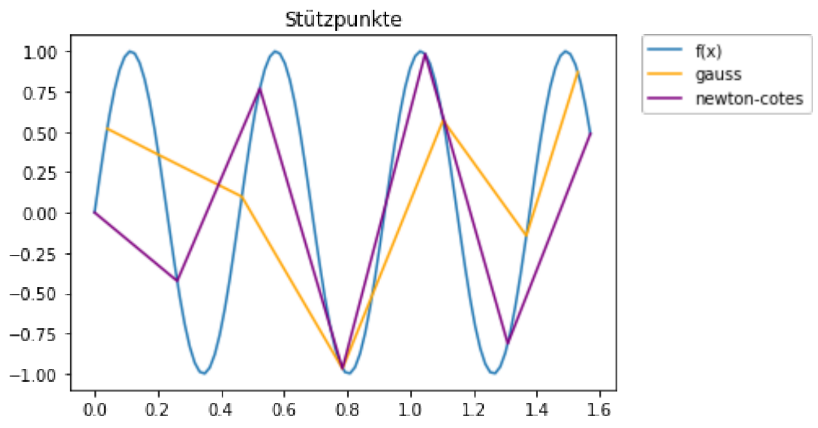
Integration

Definitionen

In [18]:
```
omega = 13.6753
PI    = np.arccos(-1)
Tmin  = 0.0
Tmax  = 0.5*np.pi

def f(t):
    return np.sin(omega*t)
```

Plot function and integration points

In [19]:
```
def transform_gausspts(pts, a = -1.0,b=1.0):
    return np.array([(b-a)/2*x + (a+b)/2 for x in pts])


N = 7  # anzahl Stüzstellen für Plot

# Newton-Cotes Stützpunkte
tau_NC = np.linspace(Tmin,Tmax,N)

# Gauss Stützpunkte
tau = load_obj("testat2_tau")
tau_G = tau[N]
tau_G = Tmin + (Tmax - Tmin)*tau_G # transformation

T = np.linspace(Tmin, Tmax, 100)
plt.plot(T, f(T),label='f(x)')
plt.plot(tau_G,f(tau_G),label='gauss',color='orange')
plt.plot(tau_NC,f(tau_NC),label='newton-cotes',color='purple')
plt.title("Stützpunkte")
# Put a legend to the right of the current axis
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()
```



Exaktes Integral

"Exact" -> scipy's quad function.

$\int_{t=0}^{\pi/2} f(t)dt = \int_{t=0}^{\pi/2} sin(wt)dt =$

In [20]:
```
from scipy import integrate
result, errorbound = integrate.quad(f, Tmin, Tmax)
print("result = {}".format(result))
print("max error = {}".format(errorbound))

I_ex = result
```

```
result = 0.13694223655230858
max error = 6.026545148335449e-12
```

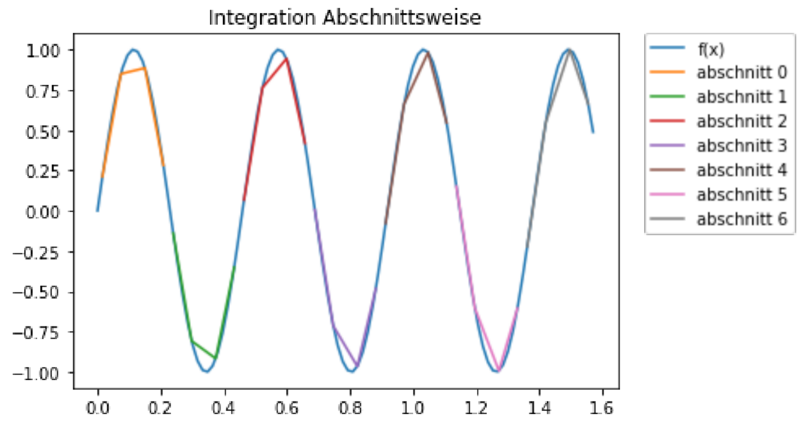
Approximiertes Integral

In [21]:
```
quadfunc = quad_NC
print("Newton-Cotes:")
for n in range(2,8):
    I = quadfunc(f, Tmin, Tmax, n)
    e = abs(I_ex - I)
    print("o={}, I={} \t Fehler = {}".format(n, I, e))

quadfunc = quad_Gauss
print("Gauss-Christoffel:")
for n in range(1,11):
    I = quadfunc(f, Tmin, Tmax, n)
    e = abs(I_ex - I)
    print("o={}, I={} \t Fehler = {}".format(n, I, e))
```

```
Newton-Cotes:
o=2, I=0.38343926342917667      Fehler = 0.24649702687686809
o=3, I=-0.8855165839223643      Fehler = 1.022458820474673
o=4, I=1.1279553948515686       Fehler = 0.9910131582992601
o=5, I=-0.803924390927535       Fehler = 0.9408666274798436
o=6, I=-0.44276678415863785     Fehler = 0.5797090207109464
o=7, I=-0.8660547701464941      Fehler = 1.0029970066988028
Gauss-Christoffel:
o=1, I=-1.5199945075981345      Fehler = 1.656936744150443
o=2, I=-1.5148718618357326      Fehler = 1.6518140983880412
o=3, I=-0.29642079887855327     Fehler = 0.43336303543086185
o=4, I=1.385715363716855        Fehler = 1.2487731271645466
o=5, I=-0.7277322732999928      Fehler = 0.8646745098523014
o=6, I=0.4370722347045538       Fehler = 0.3001299981522452
o=7, I=0.0714919237498817       Fehler = 0.06545031280242689
o=8, I=0.14690429874386962      Fehler = 0.009962062191561039
o=9, I=0.13581468201915053      Fehler = 0.0011275545331580483
o=10, I=0.13704131124035962     Fehler = 9.907468805103736e-05
```

e) Unterteilte Integration mit Gauss

In [23]:
```
M = 7 # Anzahl Abschnitte
N = 4 # Anzahl Stützstellen

fig, ax = plt.subplots(1,1)

T = np.linspace(Tmin, Tmax, 100)

# plot function
ax.plot(T, f(T),label='f(x)')

#split into parts
parts = np.linspace(Tmin,Tmax,M+1)

I = 0
for m in range(M):
    start = parts[m]
    end = parts[m+1]

    # integration
    I +=  quad_Gauss(f, start, end, N)

    #plots
    # Gauss Stützpunkte
    tau = load_obj("testat2_tau")
    tau_G = tau[N]
    tau_G = start + (end - start)*tau_G # transformation
    ax.plot(tau_G,f(tau_G),label='abschnitt {}'.format(m))


# Put a legend to the right of the current axis
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)

plt.title("Integration Abschnittsweise")
plt.show()

print("{} Abschnitte, {} Stützstellen:".format(M,N))
print("I = {}".format(I))
print("e = {}".format(abs(I_ex - I)))
```



```
7 Abschnitte, 4 Stützstellen:
I = 0.13694135921169162
e = 8.773406169615416e-07
```