THEORETICAL PHYSICS

Quantum Field Theory
SI2410
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Preparation Questions

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1 SEMINAR 1: FUNCTIONAL INTEGRALS AND INTRODUCTION TO RENORMALIZATION

1.1 WHAT IS THE RELATIONSHIP BETWEEN THE FUNCTIONAL INTEGRAL FORMALISM AND THE N-POINT CORRELATION FUNCTION?

General generating functional is given by

$$Z[J] = \int D\phi \exp\left[i\int d^4x[L + J(x)\phi(x)]\right] \tag{1.1}$$

where L is the Lagrangian density and $J(x)\phi(x)$ is a source term.
The n-point correlation function is then given by

$$\langle 0|T\phi(x_1)...\phi(x_n)|0\rangle = \frac{1}{Z_0}\prod_{i=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}Z[J]\right) \tag{1.2}$$

where $Z_0 = Z[J=0]$. This yields

$$\langle 0|T\phi(x_1)...\phi(x_n)|0\rangle = \frac{\int D\phi\phi(x_1)...\phi(x_n)\exp\left[i\int d^4xL\right]}{\int D\phi\exp\left[i\int d^4xL\right]} \tag{1.3}$$

1.2 Given a Lagrangian density, how can the Feynman rules of a theory be computed using functional integral formalism?

The Feynman rules can be computed by Taylor expanding

$$\exp[iS] = \exp\left[i\int d^4xL_0 + L_{int}\right] \tag{1.4}$$

where S is the action, L_0 is the free Lagrangian density and L_{int} is the interaction Lagrangian density. In the case of $\lambda\phi^4$-theory we have the Lagrangian

$$L = L_0 - \frac{\lambda}{4!}\phi^4 \tag{1.5}$$

and by Taylor expansion

$$\exp iS = \exp\left[i\int d^4x\left(L_0 - \frac{\lambda}{4!}\phi^4\right)\right] = \exp\left[i\int d^4xL_0\right]\left(1 - i\int d^4x\frac{\lambda}{4!}\phi^4 + ...\right) \tag{1.6}$$

where L_0 gives the propagators and all terms of order higher than two yields interaction. In this case we read of the vertex factor in momentum space as

$$-i\lambda(2\pi)^4\delta^4\left(\sum p\right) \tag{1.7}$$

1.3 What complications arise when using the functional integral formalism to quantize the electromagnetic field? How is it solved?

In QED the functional integral,

$$\int DAe^{iS[A]} \tag{1.8}$$

is badly defined because we are redundantly integrating over a continuous infinity of physically equivalent field configurations. This happens when S[A] = 0 due to gauge invariance,

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x) \tag{1.9}$$

and thus $A_\mu = \frac{1}{e}\partial_\mu\alpha(x)$ is equal to $A_\mu = 0$. Through the Faddeev-Popov procedure we introduce some gauge fixing function G(A) = 0, this would be G(A) = $\partial_\mu A^\mu$ in the Lorentz gauge. We can insert a functional delta function $\delta(G(A))$ into our functional integral by using,

$$1 = \int D\alpha\,\delta(G(A^\alpha))det\left[\frac{\delta G}{\delta\alpha}\right] \tag{1.10}$$

and a general G given by, $G(A) = \partial^\mu A_\mu - \omega(x)$ where ω is some scalar function. This gives the photon propagator as,

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{1.11}$$

where the choice of ξ gives our gauge.

1.4 What are the properties of Grassmann numbers and how are they used to quantize spinor fields?

The Grassman numbers are anticommuting numbers with the following properties,

$$\begin{aligned} \theta\eta &= -\eta\theta \\ \theta^2 &= 0 \\ \int 1 d\theta &= 0 \\ \int \theta d\theta &= 1 \\ \int \frac{\partial f}{\partial\theta} d\theta &= 0 \end{aligned} \tag{1.12}$$

To quantize spinor fields a grassman valued source field is introduced into $Z[\eta,\bar{\eta}]$ such that,

$$Z[\eta,\bar{\eta}] = \int D\psi \int D\bar{\psi} e^{i\int d^4x[L+\bar{\eta}\psi+\bar{\psi}\eta]}. \tag{1.13}$$

A Grassman field is given by,

$$\eta(x) = \sum_i \eta_i\phi_i \tag{1.14}$$

where η_i is a grassman number and ϕ_i is a field basis and in the equation above the base is 4-spinor Dirac fields.

1.5 What is the superficial degree of divergence and how can it be computed? Use QED as an example.

The superficial degree of divergence is for a Feynmann diagram's integral defined as,

$$D = (\text{power of momenta in numerator} - \text{power of momenta in denominator}) \tag{1.15}$$

and tells us something about the divergence of a given diagram. In general d-dimensional QED this can be written as,

$$D = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-1}{2}N_e \tag{1.16}$$

where V is the number of vertices and N_i the number of i = photon, electron external lines. D does not tell us everything about the divergence, in QED the three diagrams (c,f and g in Fig. 1.1) always diverge, D can only tell us if a given diagram diverge or not if the diagram does not contain any of those three. However for any theory we can always remove the divergent subdiagrams and D holds.

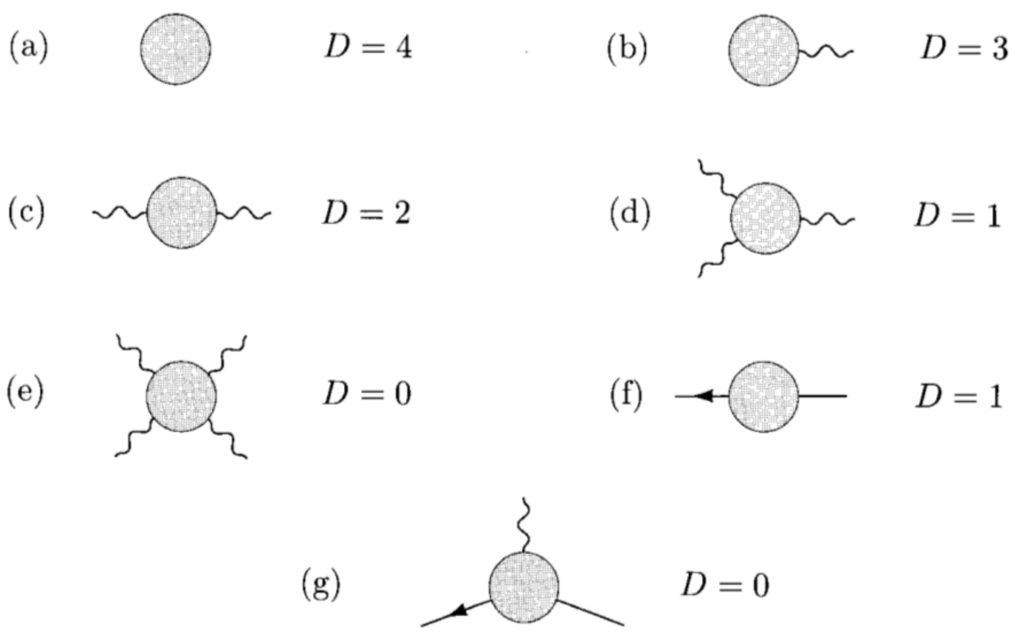

Figure 1.1: The seven QED amplitudes whose superficial degree of divergence (D) is ≥ 0.

1.6 How does renormalized perturbation theory relate the bare and physical masses?

In renormalized perturbation theory we begin by introducing new fields to rescale the original Lagrangian which only contains bare parameters. The next step is to introduce counterterms containing the physical parameters which splits the Lagrangian into two parts where one part now contains all infinities. The third step is to specify renormalization conditions which then fixes the physical parameters such as mass, charge or field strength. Finally one computes the Feynman amplitudes while maintaining the renormalization condition which yields the counter terms.

The common conditions to specify the physical mass is where the pole of the propagator is, in $\lambda\phi^4$-theory this simply implies that $m^2 = p^2$ since the propagator is given by $\frac{i}{p^2 - m^2 + i\epsilon}$. This will define the physical mass m as a function of the bare mass m_0 and the mass-counterterm δ_m.

2 Renormalization and spontaneous symmetry breaking

2.1 What is spontaneous symmetry breaking? Give a concrete example of a spontaneously broken theory.

Spontaneous symmetry breaking occurs when the potential term in the Lagrangian has more than one minima which is asymmetric with respect to the Lagrangian. One needs to re-parametrize the Lagrangian around the minima in order to achieve this but this brakes the symmetry of the Lagrangian, i.e. the physical system is asymmetric with respect to an arbitrary minima.

Consider the linear sigma model which is invariant under $\phi^i \to R^{ij}\phi^j$ implying that it has N symmetries with Lagrangian,

$$L = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2 \tag{2.1}$$

where the potential term is

$$V\left(\phi^i\right) = -\mu^2\left(\phi^i\right)^2 + \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2 \tag{2.2}$$

which has it's minimas at $\left(\phi_0^i\right)^2 = \frac{\mu^2}{\lambda}$. Re-parametrizing the Lagrangian around one of this minimas breaks the N symmetries and thus the symmetries are spontaneously broken.

2.2 What is Goldstone's theorem and its implications?

Goldstone's theorem states that every spontaneously broken symmetry give rise to a massless particles. Thus Goldstone's theorem sets an upper bound for the amount of massive particles in our theory.

In the linear sigma model with $N(N-1)/2$ symmetries, $N-1$ symmetries are broken and the same amount of massless particles are created.

(This implies that spontaneously broken theories can only have one massive particle, e.g. LSW-theory where the only massive particle is the Higgs boson. The Z- and $W^\pm$-bosons are massless but are given mass by the Higgs mechanism.)

2.3 What are the reasons that spontaneously broken theories are renormalizable? (Assuming that the un-broken theory is.)

Assuming that the unbroken theory is renormalizable the broken theory will also remain renormalizable since the structure of the divergent parts of the Feynman diagrams are unaffected [p.360]. The reparametrization of the Lagrangian will yield several more new divergent diagrams but they all have the same structure and thus all infinities can be absorbed into the same counterterm which renders the broken theory renormalizable as well.

2.4 What are the basics of Wilson's approach to renormalization?

The method of Wilson's approach to renormalization follows the following steps:

- Introduce high momenta cut-off Λ
- Separate fields into $\hat{\phi}(k)$ where $b\Lambda < |k| < \Lambda$, 0 otherwise and $\phi(k)$ where $|k| < b\Lambda$, 0 otherwise and $0 < b < 1$.
- Integrate out $\hat{\phi}(k)$ to obtain an effective Lagrangian L_{eff}
- Rescale L_{eff} to obtain recursion relations for all coefficients in the Lagrangian
- Iterate through the space of all possible Lagrangians until we find a fixpoint

During this procedure some parameters will tend towards zero and thus is not relevant for the theory.

2.5 What is the Callan-Symanzik equation and how does it relate to the running of physical quantities?

Consider a massless theory and introduce the renormalization condition that $p^2 = -M^2$ when the dressed (all self-interactions included) propagator is equal to zero. Then the Green's function satisfies the Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}(x_i; M, \lambda) = 0 \tag{2.3}$$

It asserts that there exits two universal functions $\beta(\lambda)$ and $\gamma(\lambda)$, related to the shifts in the coupling constants and field strength, that compensates for the shift in the renormalization scale M.

The solution of Eq. (2.3) can be written as

$$G^{(2)}(p,\lambda) = \hat{G}\left(\bar{\lambda}(p;\lambda)\right)\exp\left(-\int_{p'=M}^{p'=p} d\log(p'/M)2\left[1-\gamma\left(\bar{\lambda}(p;\lambda)\right)\right]\right) \tag{2.4}$$

where $\hat{G}$ is an unknown function through series expansion and $\bar{\lambda}$ is the running coupling constant and it can be shown that it has to satisfy,

$$\frac{d}{d\log(p/M)}\bar{\lambda}(p,\lambda) = \beta(\bar{\lambda}), \;\; \bar{\lambda}(M;\lambda) = \lambda \tag{2.5}$$

Eq. (2.5) is normally called the renormalization group equation and solving this yields the running constant as function of the momenta and the original coupling constant.

2.6 Describe how you would chose suitable renormalization conditions.

- Maintain original Feynman rules, e.g. propagator, vertex etc.
- Fix the particle mass at the physical mass as opposed to the bare mass, and same for the coupling constant.
- Set

3 Non-Abelian gauge theory

3.1 Why does the requirement of gauge invariance imply the existence of a gauge field?

In is possible to have a gauge invariant Lagrangian with vector fields, e.g. QED, but when we are interested in creating theories with higher order vector interactions, specifically containing derivatives of vector fields $\partial_\mu A^\mu$ we run into trouble. The derivative yields extra terms which breaks the gauge invariance. This can be remedied by introducing the covariant derivative

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x) \tag{3.1}$$

The extra term in Eq. (3.1) exactly cancels the extra term from the "ordinary" derivative, thus making the replacement $\partial_\mu \to D_\mu$ remedies our problem, but at the same time forcing us to introduce a new vector field A_μ.

3.2 What representation do the Yang-Mills gauge field strengths transform according to?

In the general $SU(N)$ group the covariant derivative transforms as

$$D_\mu \to V D_\mu V^\dagger \tag{3.2}$$

where $V = \exp\left(i f^{abc} t^c\right)$, f^{abc} are structure constants and t^c are the generators of the group. Since $\left[D_\mu, D_\nu\right] = -igF^a_{\mu\nu} t^a$ this implies that the Yang-Mills field tensor has the infinitesimal transformation

$$F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu} \tag{3.3}$$

and that the quantity $-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2$ appearing in the Yang-Mills Lagrangian is gauge invariant.

3.3 What are the Yang-Mills equivalents of the Maxwell equations?

The Yang-Mills equivalents of the Maxwell equations are obtained by applying the Euler-Lagrange equations to the Yang-Mills Lagrangian

$$L_{YM} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}\left(i\not{D} - m\right)\psi \tag{3.4}$$

and are found to be

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = -g j^a_\nu \tag{3.5}$$

where $j^a_\nu = \bar{\psi}\gamma_\nu t^a \psi$.

3.4 What are the Feynman rules for the Yang-Mills Lagrangian?

The Feynman rules for the Yang-Mills Lagrangian can be found by expanding the non-Abelian filed tensor in Eq. (3.4) and give rise the Feynman rules for the vertices shown in Fig. 3.1. The propagators for the fermion and gauge boson respectively are given by

$$\langle \psi_{i\alpha}(x)\bar{\psi}_{j\beta}(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{k} - m}\right)_{\alpha\beta} \delta_{ij} e^{-ik\cdot(x-y)} \tag{3.6}$$

and

$$\left\langle A^a_\mu(x) A_\nu(b)\right\rangle = \int \frac{d^4k}{(2\pi)^4}\left(\frac{-ig_{\mu\nu}}{k^2}\right)\delta^{ab} e^{-ik\cdot(x-y)} \tag{3.7}$$

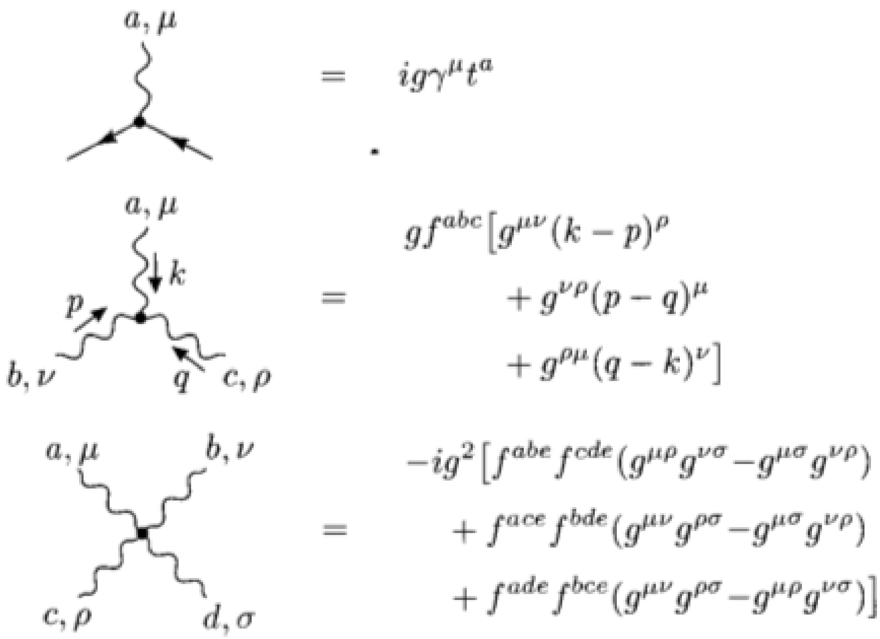

Figure 3.1: The Feynman rules for the Yang-Mills Lagrangian.

3.5 WHY ARE THE COUPLING CONSTANTS OF THE DIFFERENT NON-LINEAR TERMS NECESSARILY EQUAL IN YANG-MILLS THEORY?

There are two ways to approach this question and the first is more hands on: all interaction terms, i.e. non-linear terms, come from the square of the non-Abelian field strength tensor which was a necessity due to non-Abelian gauge invariance. This implies that there is only one coupling constant involved and thus by construction they all have the same coupling constant.

The other more technical argument is that we expect the Feynman amplitudes to satisfy Ward identities which corresponds to conservation of symmetry currents. This implies that unphysical polarization states are not produced in scattering processes. If one calculates the simple case of fermion-antifermion annihilation into a pair of gauge bosons there is actually Feynman diagrams containing both the fermion-antifermion-boson vertex and the three-boson vertex. Calculations show that the Ward identity can only be satisfied if the coupling constants of the different vertices are equal.

In the same way the diagram for boson-boson scattering includes both the three- and four-boson vertex and the Ward identity can only be satisfied if the coupling constants are equal. These two scattering processes ensures that the coupling constants are equal for the interactions in Fig. 3.1.

3.6 WHAT ARE FADDEEV-POPOV GHOST FIELDS?

There is a flaw in the previous argument: we had to assume that a gauge boson was transverse and while this is true in QED this it not necessarily true in a Yang-Mills theory. To remedy this Faddeev and Popov introduced ghost fields and ghost terms in the Lagrangian. These ghost field have the wrong relation between spin and statistics and thus only show up as virtual particles and never as physical observable particles, however they also have the consequence of cancelling the unwanted unphysical polarization states.