Note 5. MIMO Systems – Capacity, Diversity-Multiplexing Tradeoff

Channel Capacity(1/3)

- **Entropy** $H(X)$
  - The measure of the uncertainty of a random variable
  - Amount of information gained when X is measured
  - The randomness of X
- Let X be a discrete random variable taking values in a set $A_x = \{x_1, \ldots, x_m\}$ with probability $p(X = X_i) = p_i$

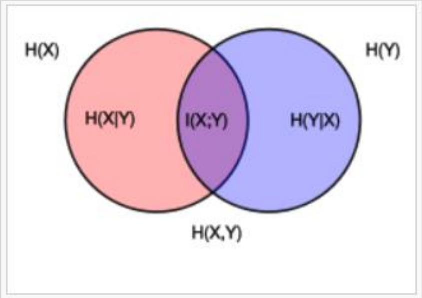
$$H(X) = E[-\log_2 p(X)] = -\sum_{x \in A_x} p(X = x) \log_2 p(X = x) \quad (\text{bits})$$

- A certain event that occurs with probability 1 provides no information
- An unlikely event provides a very large amount of information

- Coin toss example
  - Head with probability ½ , Tail with probability ½ → Entropy?
  - Head with probability 1, Tail with probability 0 → Entropy?

Channel Capacity(2/3)

- **Mutual information** $I(X;Y)$
  - A measure of the amount of information that one random variable contains about another random variable
  - It quantifies how much information Y tells about X
  - Definition
    : The mutual information of X and Y is the relative entropy between the joint distribution $p(x,y)$ and the product of the marginals $p(x)p(y)$



$$I(X;Y) = \sum_{x \in A_x} \sum_{y \in A_y} P(x,y) \log_2 \frac{P(x,y)}{P(x)P(y)}$$

$\Rightarrow$ Lemma 1 : $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$

$\Rightarrow$ Lemma 2 : If X and Y are independent , $P(X,Y) = P(X)P(Y) \rightarrow I(X,Y) = 0$

→ Y tells no information at all about X

Channel Capacity(3/3)

- **Channel capacity** *C* of a channel with input X and output Y
  - Definition

    The maximum mutual information between X and Y,
    where the maximum is taken over all possible input distributions

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} \sum_{x,y} P(x,y) \log\left(\frac{P(x,y)}{P(x)P(y)}\right)$$

→ **"The maximum data rate that can be attained over a given channel"**

  - Example of entropy : $X \sim N\left(0,\sigma^2\right)$

$$H(X) = -E\left[\log_2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}}\right] = -E\left[\log_2 \frac{1}{\sqrt{2\pi}\sigma} + \left(-\frac{1}{2\sigma^2}\right)\left(X^2\right)\log_2(e)\right]$$

$$= \left(\frac{1}{2\sigma^2}\right)E\left(X^2\right)\log_2(e) + \frac{1}{2}\log_2 2\pi\sigma^2 = \frac{1}{2}\log_2(e) + \frac{1}{2}\log_2 2\pi\sigma^2 = \frac{1}{2}\log_2 2\pi e\sigma^2$$

  - Example of channel capacity: $X \sim N\left(0,\sigma_x^2\right)$, $Z \sim N\left(0,\sigma_z^2\right)$, X & Z are independent

$$Y = X + Z$$

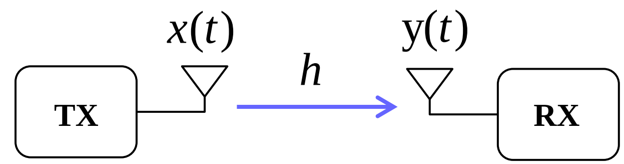
$$\rightarrow Y \sim N\left(0,\sigma_x^2 + \sigma_z^2\right)$$

$$C = I(X;Y) = H(Y) - H(Y \mid X) = H(Y) - H(X + Z \mid X) = H(Y) - H(Z \mid X)$$

$$= H(Y) - H(Z) = \frac{1}{2}\log_2 2\pi e\sigma_y^2 - \frac{1}{2}\log_2 2\pi e\sigma_z^2 = \frac{1}{2}\log_2\left(\frac{\sigma_x^2}{\sigma_z^2}\right)$$

Capacity of SISO Channel

- SISO channel



$$y(t) = h\sqrt{P}x(t) + n(t)$$

  - $x(t)$: Channel input at time t (with normalized power)
  - $y(t)$: Corresponding channel output
  - $n(t)$: white Gaussian noise random process

- SISO channel capacity

$$C = \log_2\left(1 + \frac{P}{\sigma^2}|h|^2\right) \quad \text{[bits/s/Hz]}$$

P : transmit power

σ^2 : noise variance

- SISO-AWGN channel capacity $(|h|^2 = 1)$

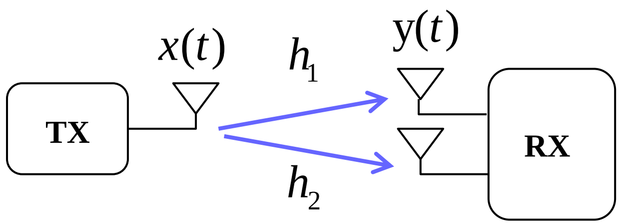
$$C = \log_2\left(1 + \frac{P}{\sigma^2}\right) \quad \text{[bits/s/Hz]}$$

Capacity of SIMO Channel

- SIMO channel capacity

$$\mathbf{h} = \begin{bmatrix} h_1 & h_2 & \dots & h_M \end{bmatrix}^T$$

$$\|\mathbf{h}\|_F^2 = h_1^2 + h_2^2 + \cdots + h_M^2$$



- Capacity of the SIMO channel with no CSIT at the TX

$$C_{SIMO} = \log_2\left(1 + \frac{P}{\sigma^2}\|\mathbf{h}\|_F^2\right)$$

- Each element of $\mathbf{h}$ is i.i.d. Gaussian random variable with zero mean and unit variance.
  - As M increases → $\|\mathbf{h}\|_F^2 = h_1^2 + h_2^2 + \cdots + h_M^2 \approx M$

$$C_{SIMO} = \log_2\left(1 + \frac{P}{\sigma^2}\|\mathbf{h}\|_F^2\right) \cong \log_2\left(1 + \frac{P}{\sigma^2}M\right)$$

- Capacity ↑ → M (# of Rx antennas) increases
- Knowledge of channel at the Tx → No gain in terms of capacity

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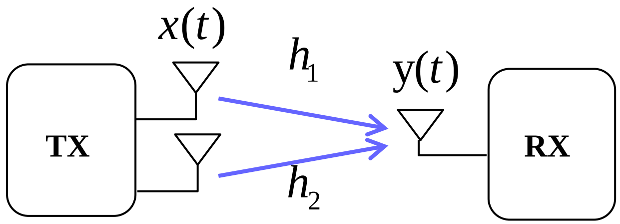
Capacity of MISO Channel without CSIT

- MISO channel capacity

$$\mathbf{h} = \begin{bmatrix} h_1 & h_2 & \dots & h_N \end{bmatrix}$$

$$\|\mathbf{h}\|_F^2 = h_1^2 + h_2^2 + \cdots + h_N^2$$



- Capacity of the MISO channel with no CSIT
  - CSIT - Channel State Information at Transmitter

$$C_{MISO} = \log_2\left(1 + \frac{P}{N\sigma^2}\|\mathbf{h}\|_F^2\right)$$

  - Each element of $\mathbf{h}$ is i.i.d. Gaussian random variable with zero mean and unit variance
    - As N increases → $\|\mathbf{h}\|_F^2 = h_1^2 + h_2^2 + \cdots + h_N^2 \approx N$

$$C_{MISO} = \log_2\left(1 + \frac{P}{\sigma^2}\right)$$

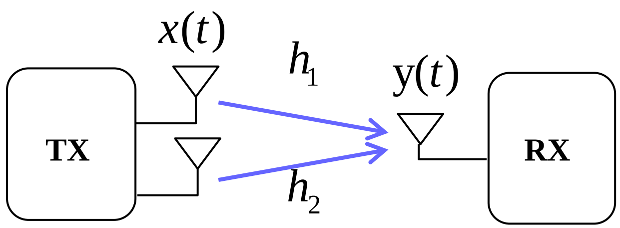
- Capacity of MISO channel without CSIT is the same as that of a SISO channel

Capacity of MISO Channel with CSIT

- MISO channel capacity

$$\mathbf{h} = \begin{bmatrix} h_1 & h_2 & \dots & h_N \end{bmatrix}$$

$$\|\mathbf{h}\|_F^2 = h_1^2 + h_2^2 + \cdots + h_N^2$$



  - Capacity of the MISO channel with CSIT
    - When CSI is available at the transmitter (CSIT)
      - Maximal-Ratio Transmission becomes possible.
      - That is, $\left(\mathbf{h}^H / \|\mathbf{h}\|\right)\mathbf{x}$ can be transmitted instead of $\mathbf{x}$.

$$\mathbf{y} = \sqrt{P}\mathbf{h}\mathbf{x} + \mathbf{n}$$

No CSIT

$$\rightarrow$$

$$\mathbf{y} = \sqrt{P}\mathbf{h}\frac{\mathbf{h}^H}{\|\mathbf{h}\|}\mathbf{x} + \mathbf{n} = \sqrt{P}\|\mathbf{h}\|\mathbf{x} + \mathbf{n}$$

MRT with CSIT

$$C_{MISO} = \log_2\left(1 + \frac{P}{\sigma^2}\|\mathbf{h}\|_F^2\right) = C_{SIMO}$$

- Capacity of MISO channel with CSIT is the same as that of a SIMO channel

Deterministic MIMO Channel Capacity

- MIMO signal model
  - $\mathbf{y} = \sqrt{\frac{P}{N}}\mathbf{Hx} + \mathbf{n}$, where $\mathbf{R}_{xx} = E\left\{\mathbf{xx}^H\right\}$ and $Tr\left(\mathbf{R}_{xx}\right) = N$
  - $\mathbf{n}$ is ZMCSCG(Zero-mean circular symmetric complex Gaussian)

- Capacity of the MIMO channel
  - Capacity : $C = \max_{f(\mathbf{x})} I(\mathbf{x};\mathbf{y})$
  - Mutual information : $I(\mathbf{x};\mathbf{y}) = H(\mathbf{y}) - H(\mathbf{y}\,|\,\mathbf{x}) = H(\mathbf{y}) - H\left(\sqrt{\frac{P}{N}}\mathbf{Hx} + \mathbf{n}\,|\,\mathbf{x}\right)$

$$= H(\mathbf{y}) - \cancel{H\left(\sqrt{\frac{P}{N}}\mathbf{Hx}\,|\,\mathbf{x}\right)} - H(\mathbf{n}\,|\,\mathbf{x}) = H(\mathbf{y}) - H(\mathbf{n})$$

$\left(\because\ \mathbf{x}\text{ and }\mathbf{n}\text{ are independent}\right)$

$$C = \max_{f(\mathbf{x})} I(\mathbf{x};\mathbf{y}) = \max_{\mathbf{R}_{xx}} \log_2\left(\det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{HR}_{xx}\mathbf{H}^H\right)\right) \text{ bps/Hz}$$

Channel Unknown to TX

- No Channel Knowledge at Transmitter (Known to RX)
  - Choose the signals to be independent and equal-powered at the Tx

    $\rightarrow \quad \mathbf{R}_{xx} = \mathbf{I}_N \quad \text{or} \quad \mathbf{R}_{xx}[i] = \gamma_i \mathbf{I}_N$

- Capacity

$$C = \log_2\left(\det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H\right)\right) = \log_2 \det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{H}\mathbf{H}^H\right)$$

  - Using the eigen decomposition, $\mathbf{H}\mathbf{H}^H = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$

$\mathbf{Q}$: Unitary Matrix

$\mathbf{\Lambda}$: Diagonal Matrix with Eigenvaules of $\mathbf{H}\mathbf{H}^H$

$$\begin{aligned} C &= \log_2 \det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H\right) \\ &= \log_2 \det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{\Lambda}\right) \\ &= \log_2\left(\prod_{i=1}^{r}\left(1 + \frac{P}{N\sigma^2}\lambda_i\right)\right) = \sum_{i=1}^{r}\log_2\left(1 + \frac{P}{N\sigma^2}\lambda_i\right) \end{aligned}$$

$$\det\left(\mathbf{I}_m + \mathbf{AB}\right) = \det\left(\mathbf{I}_n + \mathbf{BA}\right)$$

$$r = rank\left(\mathbf{H}\right) = rank\left(\mathbf{H}\mathbf{H}^H\right)$$

  - Sum of the capacities of r SISO channels: channel gain $\sqrt{\lambda_i}$, TX power $\frac{P}{N}$

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Singular Value Decomposition

- SVD(Singular Value Decomposition)
  - SVD of $M \times N$ complex matrix $\mathbf{H}$ with $rank\left(\mathbf{H}\right) = r$

$$\mathbf{H} = \mathbf{U} \sum \mathbf{V}^H$$

- $\mathbf{U} : M \times M$, $\mathbf{V} : N \times N$ unitary matrix, $\Sigma : M \times N$ matrix

$$\mathbf{U}^H \mathbf{U} = \mathbf{I}_M, \mathbf{V}^H \mathbf{V} = \mathbf{I}_N, \ \sum_{1,1} = \sigma_1, \ldots, \sum_{r,r} = \sigma_r, \ \sigma_i \geq \sigma_{i+1}, \ \text{other} \ \sum_{i,n} = 0$$

- Singular value σ_i : square root of eigenvalue λ_i of $\mathbf{HH}^H$

$$\mathbf{HH}^H = \mathbf{Q \Lambda Q}^H,$$

Eigen-decomposition

$$\mathbf{\Lambda}_{1,1} = \lambda_1, \ldots, \mathbf{\Lambda}_{r,r} = \lambda_r, \ \lambda_i \geq \lambda_{i+1}, \ \text{other} \ \mathbf{\Lambda}_{i,n} = 0$$

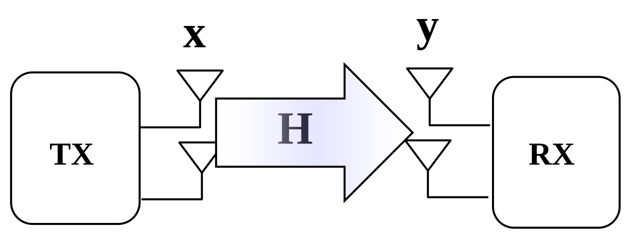
$$\Rightarrow \lambda_i = \begin{cases} \sigma_i^2 & i = 1, 2, \ldots, r \\ 0 & i = r+1, \ldots, \min(M, N) \end{cases}$$

Channel Known to TX

- Channel known to the transmitter (and RX)
  - If the channel gain matrix **H** known to both TX & RX

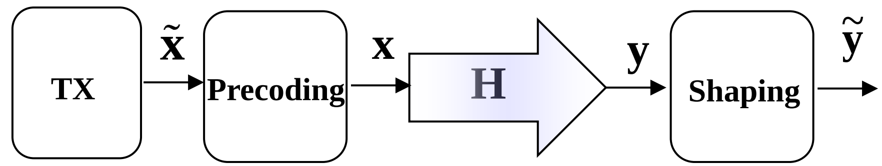
    → Parallel decomposition of the MIMO channel is available by singular value decomposition(SVD)

Basic MIMO Transmission Model



$$\mathbf{y} = \sqrt{\frac{P}{N}}\mathbf{H}\mathbf{x} + \mathbf{n}$$

Transmit precoding & receiver shaping using SVD

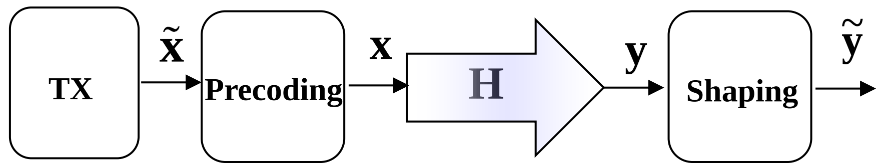


$$\mathbf{y} = \sqrt{\frac{P}{N}}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{H}(\mathbf{V}\tilde{\mathbf{x}}) + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\mathbf{V}\tilde{\mathbf{x}} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\boldsymbol{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}$$

$$\tilde{\mathbf{y}} = \mathbf{U}^H(\sqrt{\frac{P}{N}}\mathbf{U}\boldsymbol{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}) = \sqrt{\frac{P}{N}}\boldsymbol{\Sigma}\tilde{\mathbf{x}} + \mathbf{U}^H\mathbf{n} = \sqrt{\frac{P}{N}}\boldsymbol{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}$$

SVD Precoding for MIMO (1/3)

Transmit precoding & receiver shaping using SVD



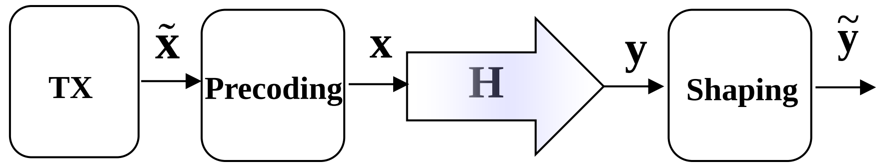
$$\mathbf{y} = \sqrt{\frac{P}{N}}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{H}(\mathbf{V}\tilde{\mathbf{x}}) + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{V}\tilde{\mathbf{x}} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}$$

$$\tilde{\mathbf{y}} = \mathbf{U}^H\left(\sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}\right) = \sqrt{\frac{P}{N}}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{U}^H\mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}$$

- Parallel Decomposition of the MIMO Channel
  - Obtained by a transformation : **Transmit precoding & receiver shaping**
  - Transmit precoding
    - Linear transformation on the channel input vector $\tilde{\mathbf{x}}$ as $\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$
  - Receiver shaping
    - Multiplying the channel output $\mathbf{y}$ with $\mathbf{U}^H$
  - Transmit precoding and receiver shaping
    - Transform the MIMO channel into r parallel single-input single-output (SISO) channels with input $\tilde{\mathbf{x}}$ and output $\tilde{\mathbf{y}}$

SVD Precoding for MIMO (2/3)

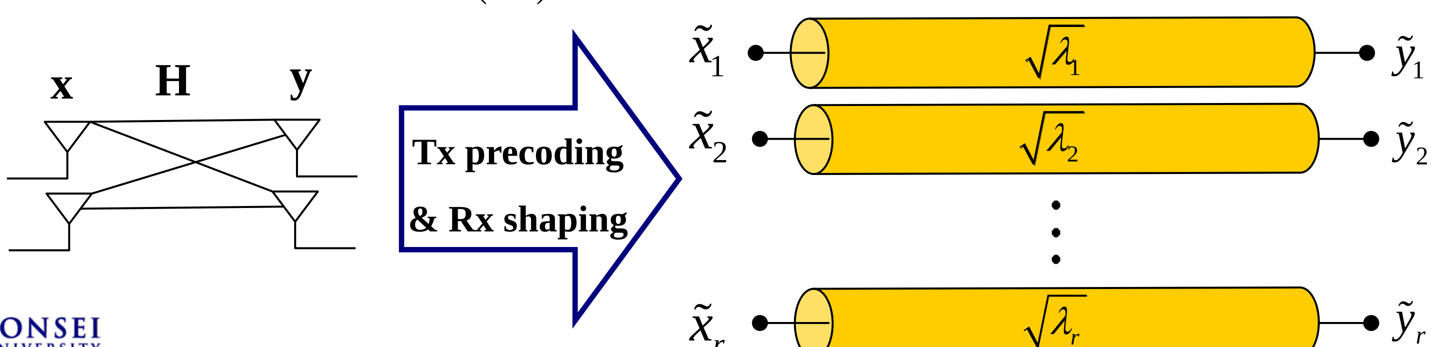
Transmit precoding & receiver shaping using SVD



$$\mathbf{y} = \sqrt{\frac{P}{N}}\mathbf{H}\mathbf{x} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{H}(\mathbf{V}\tilde{\mathbf{x}}) + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{V}\tilde{\mathbf{x}} + \mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}$$

$$\tilde{\mathbf{y}} = \mathbf{U}^H(\sqrt{\frac{P}{N}}\mathbf{U}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{n}) = \sqrt{\frac{P}{N}}\mathbf{\Sigma}\tilde{\mathbf{x}} + \mathbf{U}^H\mathbf{n} = \sqrt{\frac{P}{N}}\mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}$$

- Parallel Decomposition of the MIMO Channel
  - Multiplication by a unitary matrix does not change the distribution of the noise $\rightarrow$ $\mathbf{n}$ and $\tilde{\mathbf{n}} = \mathbf{U}^H\mathbf{n}$ are identically distributed.
  - The transmit precoding and receiver shaping transform the MIMO channel into $r = rank(\mathbf{H})$ parallel independent channels



For i-th channel,

Input: $\tilde{x}_i$

Output: $\tilde{y}_i$

Noise: $\tilde{n}_i$

Channel gain: $\sigma_i = \sqrt{\lambda_i}$

SVD Precoding for MIMO (3/3)

- Parallel Decomposition of the MIMO Channel(cont'd) - example
  - Find the equivalent parallel channel model for a MIMO channel with channel gain matrix

$$\mathbf{H} = \begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0.6 & 0.8 \end{bmatrix}$$

  - Solution:
    - The SVD of H is given by

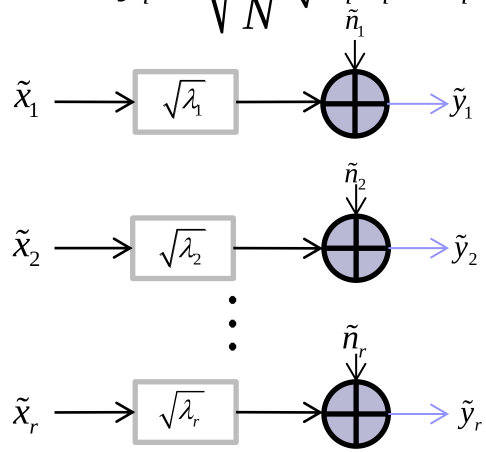
$$\mathbf{H} = \begin{bmatrix} -0.555 & 0.3764 & -0.7418 \\ -0.3338 & -0.9176 & -0.2158 \\ -0.7619 & -0.1278 & 0.6349 \end{bmatrix} \begin{bmatrix} 1.333 & 0 & 0 \\ 0 & 0.5129 & 0 \\ 0 & 0 & 0.0965 \end{bmatrix} \begin{bmatrix} -0.2811 & -0.7713 & -0.5710 \\ -0.5679 & -0.3459 & 0.7469 \\ -0.7736 & 0.5342 & -0.3408 \end{bmatrix}$$

    - There are 3 nonzero singular values, $r = 3$
    - Leading to three parallel channels, with channel gains $\sigma_1 = 1.3333$, $\sigma_2 = 0.5129$, and $\sigma_3 = 0.0965$
    - The channels have diminishing gain, with a very small gain on the 3rd channel
    - This last channel will either have a high error probability or a low capacity

Capacity with SVD Precoding

- Channel known to the transmitter

$$\tilde{\mathbf{y}} = \sqrt{\frac{P}{N}}\boldsymbol{\Sigma}\tilde{\mathbf{x}} + \tilde{\mathbf{n}} \Rightarrow \tilde{y}_i = \sqrt{\frac{P}{N}}\sqrt{\lambda_i}\tilde{x}_i + \tilde{n}_i, \quad i = 1, 2, \ldots, r$$



- Capacity of MIMO channel with CSIT

$$C = \max_{\mathbf{R}_{xx}} \log_2\left(\det\left(\mathbf{I}_M + \frac{P}{N\sigma^2}\mathbf{H}\mathbf{R}_{xx}\mathbf{H}^H\right)\right) \xrightarrow{\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}} C = \sum_{i=1}^{r}\log_2\left(1 + \frac{P}{N\sigma^2}\gamma_i\lambda_i\right)$$

Capacity of MIMO channel = sum of the individual parallel SISO channel capacities

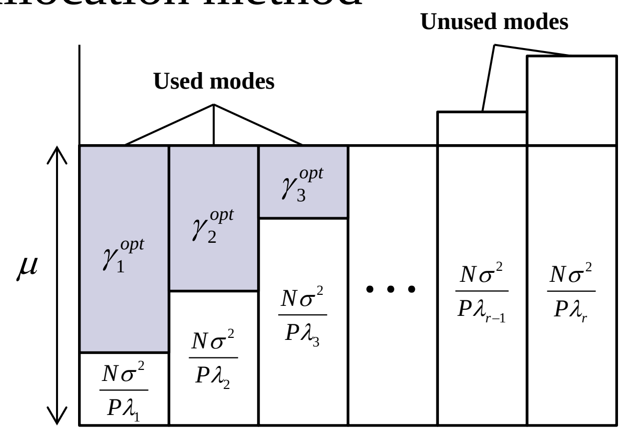
- $\gamma_i = E\left\{\left|x_i\right|^2\right\}$: the transmit energy in the i^{th} sub-channel and $\sum_{i=1}^{r}\gamma_i = N$

- $\rightarrow C = \max_{\sum_{i=1}^{r}\gamma_i = N} \sum_{i=1}^{r}\log_2\left(1 + \frac{P}{N\sigma^2}\gamma_i\lambda_i\right)$ ⇒ **Optimal energy allocation**

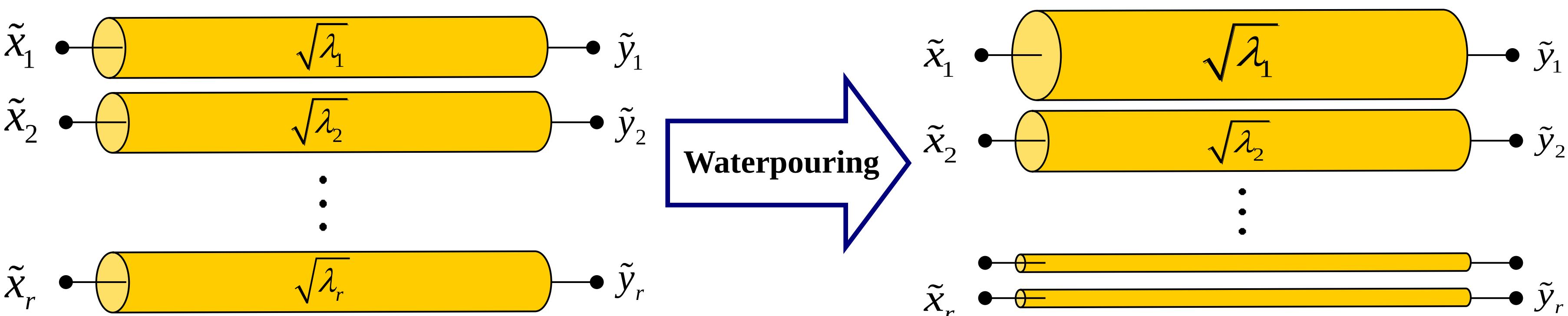
: Water-pouring (water-filling) algorithm

Water-Pouring algorithm (1/2)

- Waterpouring algorithm
  - Optimal power allocation method



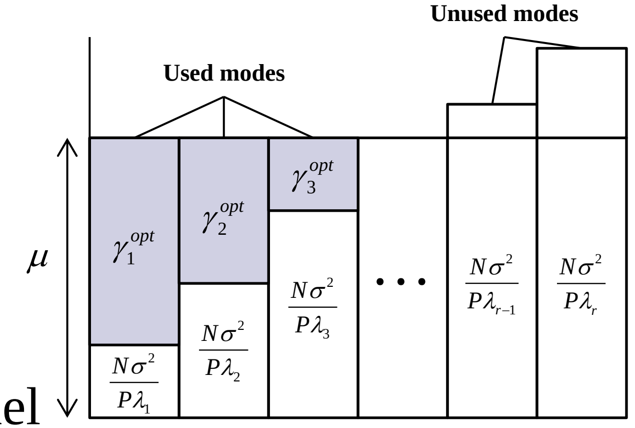
Optimal waterpouring power allocation strategy
: power allocated to each spatial sub-channel is non-negative



Water-Pouring algorithm (2/2)

- Water-pouring power allocation
  - 1) Calculate μ with $p = 1$

$$\mu = \frac{N}{r-p+1}\left[1+\frac{\sigma^2}{P}\sum_{i=1}^{r-p+1}\frac{1}{\lambda_i}\right]$$



  - 2) Calculate the power for the i-th channel

$$\gamma_i^{opt} = \left(\mu - \frac{N\sigma^2}{P\lambda_i}\right)_+ ,i = 1,\ldots, r-p+1,\ (x)_+ = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

    - If $\gamma_i^{opt} = 0$, set $p = p+1$ and go back to step 1)

Capacity with SVD / Water-Pouring

- MIMO capacity with CSIT and CSIR (r parallel channels)

$$C = \max_{\sum_{i=1}^{r}\gamma_i = N} \sum_{i=1}^{r} \log_2\left(1 + \frac{P}{N\sigma^2}\gamma_i\lambda_i\right) \xrightarrow{\frac{P}{\sigma^2}\lambda_i = c_i,} C = \max_{\sum_{i=1}^{r}\gamma_i = P} \sum_{i=1}^{r} B\log_2\left(1 + \frac{\gamma_i}{N}c_i\right)$$

$$\gamma_i^{opt} = \left(\mu - \frac{N\sigma^2}{P\lambda_i}\right)_+ = \left(\mu - \frac{N}{c_i}\right)_+ = \left(\frac{N}{c_0} - \frac{N}{c_i}\right)_+ \Leftrightarrow \frac{\gamma_i^{opt}}{N} = \left(\frac{1}{c_0} - \frac{1}{c_i}\right)_+$$

$$\mu = \frac{N}{c_0}$$

$$\frac{\gamma_i^{opt}}{N} = \begin{cases} \frac{1}{c_0} - \frac{1}{c_i}, & \text{if } c_i \geq c_0 \\ 0, & \text{otherwise} \end{cases}$$

  - Resulting MIMO capacity

$$C = \max_{\sum_{i=1}^{r}\gamma_i = P} \sum_{i=1}^{r} \log_2\left(1 + \frac{\gamma_i^{opt}}{N}c_i\right) = \sum_{i=1}^{r} \log_2\left(1 + \left(\frac{1}{c_0} - \frac{1}{c_i}\right)c_i\right) = \sum_{i=1}^{r} \log_2\left(\frac{c_i}{c_o}\right)$$

Capacity of Random MIMO Channels

- Wireless channel
  - **Not deterministic, but time-varying random channel**
  - Two commonly used statistics : Ergodic capacity, Outage capacity
  - Ergodic capacity → Fast Fading Channel
    - Since **H** is random, capacity is also random
    - Ensemble average over channel realizations

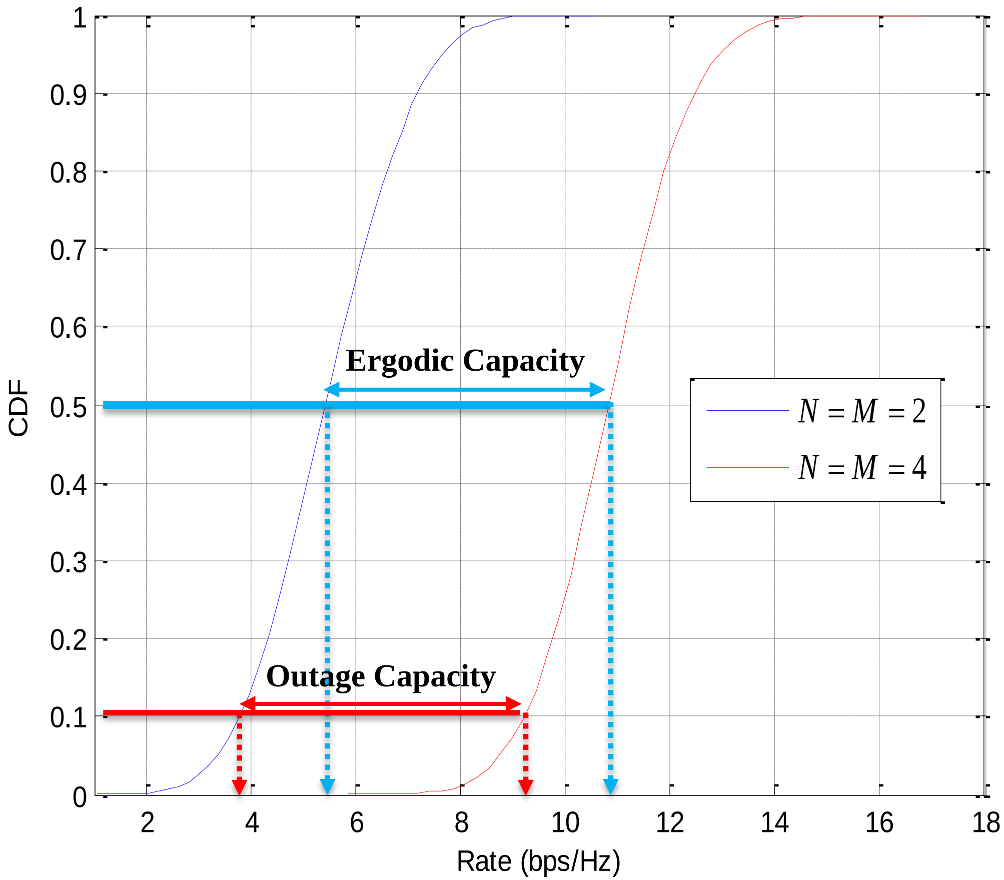
$$\bar{C} = \mathbf{E}\{C\}$$

  - Outage capacity → Slow Fading Channel
    - Information rate guaranteed for (100 - q)% of the channel realizations

$$P(C \le C_{out,q}) = q\ \%$$

Ergodic Capcity / Outage Capacity

- Ergodic/Outage capacity



Simulation environment: No CSIT

$$SNR = 10dB$$

$$N = M = 2 \text{ or } 4$$

: **Ergodic Capacity**

: **10% Outage Capacity**

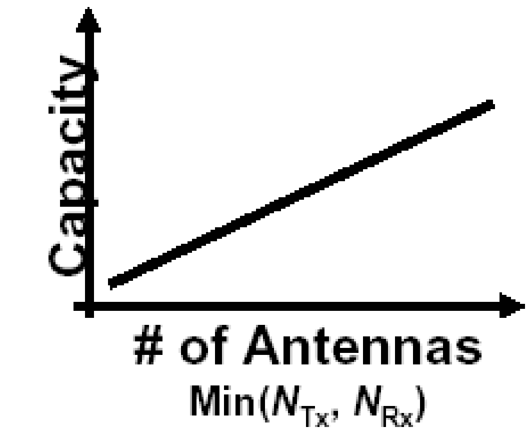
Diversity and Multiplexing Gains

- Two types of gains in MIMO systems
  - Spatial Multiplexing Gain

  - Increase ***data rates*** through ***multiplexing***

    ⇒ Independent signaling paths that can be used to send independent data

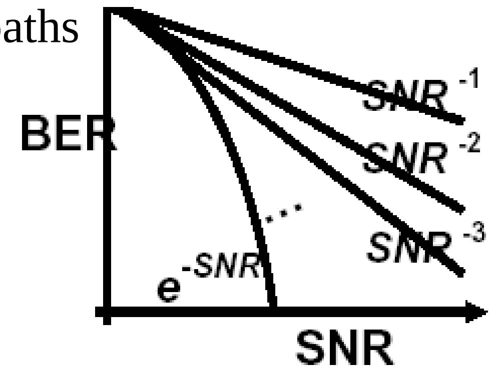
  → Determined by the numbers of antennas



  - Diversity Gain

  - Improve ***performance*** through ***diversity***

  → Determined by the number of independent paths



DMT (D/M Tradeoff) (1/3)

- Fundamental tradeoff between diversity and multiplexing [1]
  - Diversity gain
    - For $N \times M$ MIMO systems ($M \times N$ MIMO channel matrix), the *maximum achievable diversity gain* is *MN* at high SNR.
    - If a diversity order is *d*, then the error probability is given by $P_e \propto \mathrm{SNR}^{-d}$.
      - It shows how fast the error probability can be decreased with the SNR for *a fixed rate*.
  - Multiplexing gain
    - Capacity of the system increases as $C \propto r \log(\mathrm{SNR})$, where $r=\min(M,N)$ denote the number of spatial degrees of freedom, in other words, *multiplexing gain*.
      - It shows how fast the data rate can be increased with the SNR for *a fixed error probability*.

[1] L. Zheng and D. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," IEEE Transactions on Information Theory, vol. 49, pp. 1073–1096, May 2003.

DMT (D/M Tradeoff) (2/3)

- Fundamental tradeoff between diversity and multiplexing
  - To achieve the maximum diversity gain, one needs to communicate at a fixed rate R, which becomes vanishingly small compared to the fast fading capacity at high SNR ($\min(N,M)\cdot\log\mathrm{SNR}$).
    - Spatial multiplexing benefit is sacrificed to maximize the reliability.
- Formulation

A diversity gain $d^*(r)$ is achieved at multiplexing gain r if

$$R = r\log\mathrm{SNR}$$

and

$$p_{out}(R) \approx \mathrm{SNR}^{-d^*(r)},$$

or more precisely,

$$\lim_{\mathrm{SNR}\to\infty}\frac{\log p_{out}(r\log\mathrm{SNR})}{\log\mathrm{SNR}} = -d^*(r)$$

The curve $d^*(r)$ is the diversity-multiplexing tradeoff of the slow fading channel

DMT (D/M Tradeoff) (3/3)

- Fundamental tradeoff between diversity and multiplexing
  - A diversity-multiplexing tradeoff for any space-time coding scheme can be formulated with outage probability replaced by error probability.
    - [1, *Lemma* 5] Outage probability provides a lower bound on the error probability for channel

A space-time coding scheme is a family of codes, indexed by the signal-to-noise ratio SNR. It attains a mulplexing gain r and a diversity gain d if the data rate scales as

$$R = r \log \mathrm{SNR}$$

and the error probability scales as

$$p_e \approx \mathrm{SNR}^{-d},$$

or more precisely,

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log p_e}{\log \mathrm{SNR}} = -d$$

DMT of General MIMO Channel

- For a $M \times N$ MIMO channel matrix $\mathbf{H}$ with the following assumption
  - All elements of $\mathbf{H}$ are i.i.d. random complex Gaussian variable with zero mean & unit variance.
  - Therefore, $\mathbf{H}$ is a full rank matrix, i.e., $\mathrm{rank}(\mathbf{H}) = \min(M, N)$

- [1, *Theorem* 2] Optimal trade off curve of $M \times N$ MIMO channel

$$d^*(r) = (M - r)(N - r), \quad r = 0, \ldots, \min(M, N)$$

  - Example) 2x2 Channel with full rank: three DMT points - (0,4), (1,1), (2,0)
    - 1) If the system(scheme) utilizes the channel with the multiplexing gain r = 0, then the achievable diversity gain is $d^*(0) = (M - 0)(N - 0) = 4$.
      - As SNR increases → Error prob. is $P_e \propto \mathrm{SNR}^{-4}$(rapid decreasing)
      - As SNR increases → Increased Data rate is $R = 0 \cdot \log \mathrm{SNR}$ (not increasing)
    - 2) If the system(scheme) utilizes the channel with the multiplexing gain r = 1, then the achievable diversity gain is $d^*(1) = (M - 1)(N - 1) = 1$.
      - As SNR increases → Error prob. is $P_e \propto \mathrm{SNR}^{-1}$(decreasing)
      - As SNR increases → Increased Data rate is $R = 1 \cdot \log \mathrm{SNR}$ (increasing)
    - 3) If the system(scheme) utilizes the channel with the multiplexing gain r = 2, then the achievable diversity gain is $d^*(2) = (M - 2)(N - 2) = 0$.
      - As SNR increases → Error prob. is $P_e \propto \mathrm{SNR}^{0}$ (no decreasing)
      - As SNR increases → Increased Data rate is $R = 2 \cdot \log \mathrm{SNR}$(rapid increasing)

DMT Examples for MIMO Systems (1/2)

Repetition	Alamauti
$\begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$	$\begin{bmatrix} x_1 & x_2^* \\ x_2 & -x_1^* \end{bmatrix}$

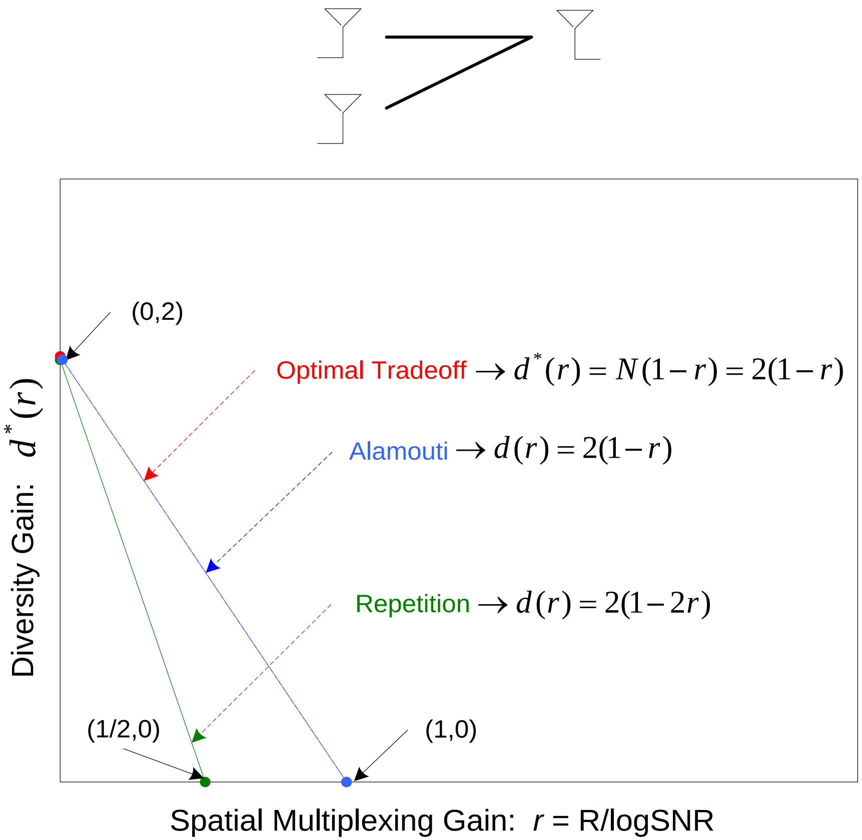
- 2x2 MIMO Rayleigh channel
  - Diversity-Multiplexing tradeoff of various schemes
    - Diversity schemes: Repetition, Alamouti
    - Multiplexing schemes: V-BLAST(ML), V-BLAST(nulling)
    - Optimal DMT of 2x2 MIMO channel
      - Maximum diversity gain of 4 and 2 degrees of freedom
      - Piecewise linear curve consisting of two linear segments

	Classical Diversity gain	Degrees of Freedom utilized	DMT
Repetition	4	1/2	$4-8r, \quad r \in [0,1/2]$
Alamouti	4	1	$4-4r, \quad r \in [0,1]$
V-BLAST (ML)	2	2	$2-r, \quad r \in [0,2]$
V-BLAST (nulling)	1	2	$1-r/2, \quad r \in [0,2]$
Channel itself	4	2	$4-3r, \quad r \in [0,1]$ $2-r, \quad r \in [1,2]$

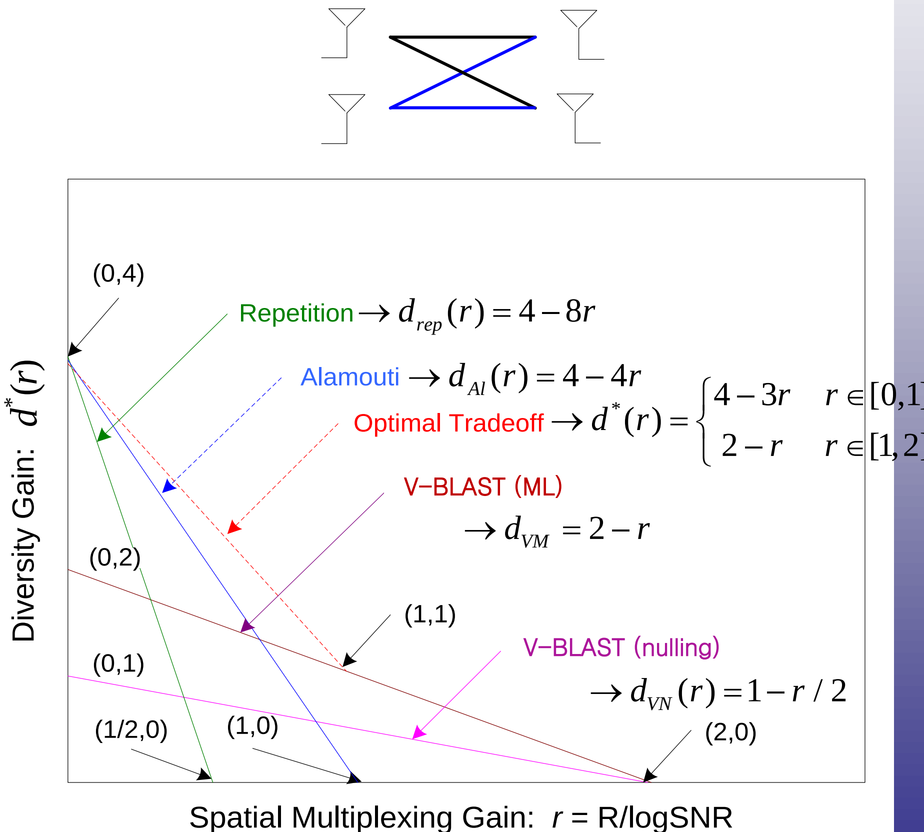
DMT Examples for MIMO Systems (2/2)

- DMT of the 2x1 MISO and 2x2 MIMO channel

2 X 1 Channel



2 X 2 Channel



Thank You!