Neural Network Basic Assignment

Part 1.

1. $\sigma(z) = \dfrac{1}{1+e^{-z}}$ 미분

$$\sigma'(z) = \left((1+e^{-z})^{-1}\right)' = -(1+e^{-z})^{-2} \cdot (e^{-z}) \cdot (-1)$$

$$= \frac{e^{-z}}{(1+e^{-z})^2} = \frac{1+e^{-z}-1}{(1+e^{-z})^2} = \frac{1+e^{-z}}{(1+e^{-z})^2} - \frac{1}{(1+e^{-z})^2}$$

$$= \frac{1}{1+e^{-z}} - \frac{1}{(1+e^{-z})^2} = \frac{1}{1+e^{-z}}\left(1 - \frac{1}{1+e^{-z}}\right)$$

$$= \sigma(z)(1-\sigma(z))$$

2. $MSE = J(W) = \frac{1}{2}\Sigma(y_k - O_k)^2$ $\quad O_k = W_k^T X + b_k$

W_i에 대해 편미분

$$\frac{\partial}{\partial W_i} J(w) = \frac{\partial O_k}{\partial W_i} \cdot \frac{\partial}{\partial O_k}\left(\frac{1}{2}\Sigma(y_k - O_k)^2\right)$$

$$= -\frac{\partial O_k}{\partial W_i} \cdot \Sigma(y_k - O_k)$$

$$\frac{\partial}{\partial W_i} O_k = X_i^k$$

$$\therefore \frac{\partial}{\partial W_i} J(w) = -\Sigma(y_k - O_k) X_i^k$$

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3. $\log \text{likelihood} = J(w) = -\Sigma \{ y_k \log p_k + (1-y_k) \log (1-p_k) \}$

$p_k = \sigma(z_k) \qquad z_k = w_k^T x + b_k$

$$\frac{\partial J(w)}{\partial w_i} = \frac{\partial z_k}{\partial w_i} \cdot \frac{\partial p_k}{\partial z_k} \cdot \frac{\partial}{\partial p_k} J(w)$$

$$= \frac{\partial z_k}{\partial w_i} \cdot \frac{\partial p_k}{\partial z_k} \cdot \left(-\Sigma\, y_k \cdot \frac{\partial}{\partial p_k} \log p_k + (1-y_k) \cdot \frac{\partial}{\partial p_k} \log (1-p_k) \right)$$

$$= \frac{\partial z_k}{\partial w_i} \cdot \frac{\partial p_k}{\partial z_k} \cdot \left(-\Sigma \left(\frac{y_k}{p_k} - \frac{1-y_k}{1-p_k} \right) \right)$$

$$= \frac{\partial z_k}{\partial w_i} \cdot \left(-\Sigma\, y_k (1-p_k) + \Sigma (1-y_k) p_k \right)$$

$$\left(\because \quad \frac{\partial}{\partial z_k} p_k = \sigma'(z_k) = \sigma(z_k)(1-\sigma(z_k)) = p_k (1-p_k) \right)$$

$$\therefore \frac{\partial J(w)}{\partial w_i} = -\Sigma\, y_k (1-p_k) x_i^k + \Sigma (1-y_k) p_k x_i^k$$

$$\left(\because \quad \frac{\partial z_k}{\partial w_i} = x_i^k \right)$$

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4. $-\sum_{k=1}^{k} y_k \log(p_k) = -\log p_i \quad 1 \le i \le k$

일반적으로 이 Loss function은 CE로 범주형변수에서 많이 사용된다. 즉 y_k는 하나의 Class에서만 1을 값을 갖게된다.

$y = (y_1, y_2, \dots y_i, \dots y_k) = (0, 0, \dots, 1, \dots 0)$

$p = (p_1, p_2, \dots, p_i, \dots, p_k)$

이런식의 vector가 만들어질 것이고 실질적인 값을 갖는 부분은 $y_i = 1$일때 밖에 없을 것이다.

$\therefore -\sum_{k=1}^{k} y_k \log(p_k) = -y_i \log p_i = -\log p_i \quad (\because y_i = 1)$

5. $CE = -\sum y_k \cdot \log(p_k) \qquad p_i = \dfrac{e^{z_i}}{\sum e^{z_k}}$

$$\frac{\partial}{\partial z_i} CE = \frac{\partial}{\partial p_k} \cdot \frac{\partial p_k}{\partial z_i} \left(-\sum y_k \log p_k\right)$$

$$= \frac{\partial p_k}{\partial z_i}\left(-\sum y_k \cdot \frac{1}{p_k}\right)$$

이때 $\dfrac{\partial p_i}{\partial z_i} = \dfrac{e^{z_i}\sum e^{z_k} - e^{z_i} e^{z_i}}{(\sum e^{z_k})^2} = \dfrac{e^{z_i}(\sum e^{z_k} - e^{z_i})}{(\sum e^{z_k})^2}$

$$= \frac{e^{z_i}}{\sum e^{z_k}} \cdot \left(1 - \frac{e^{z_i}}{\sum e^{z_k}}\right) = p_i(1-p_i)$$

if $\dfrac{\partial p_i}{\partial z_j} = \dfrac{-e^{z_i} \cdot e^{z_j}}{(\sum e^{z_k})^2} = -p_i \cdot p_j$

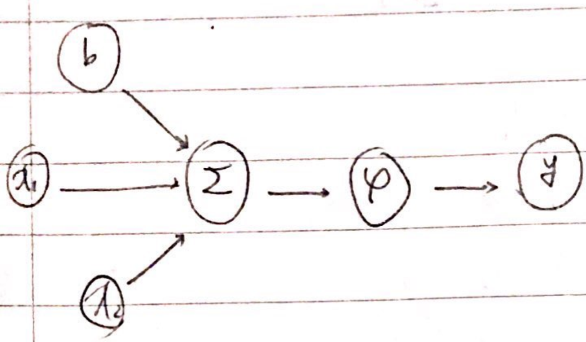
$$\frac{\partial J}{\partial z_i} = \frac{\partial p_k}{\partial z_i}\left(-\Sigma \frac{y_k}{p_k}\right) = \left(-\Sigma \frac{y_k}{p_k} \cdot \frac{\partial p_k}{\partial z_i}\right)$$

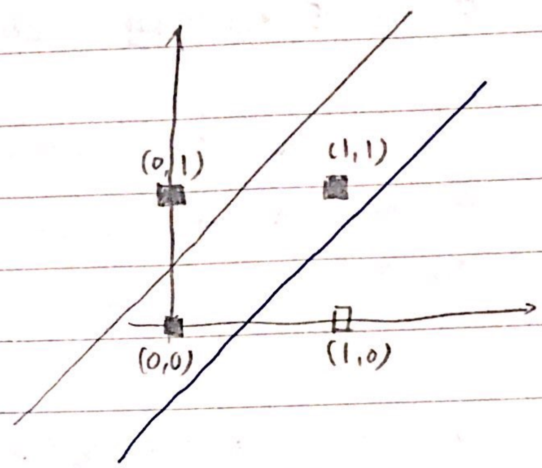
$$= \left(-\Sigma \frac{y_k}{p_k}\left(p_k(1-p_k) - p_k p_i\right)\right) = -y_k(1-p_k) + \sum_{k \neq i} y_k p_i$$

$$= -y_k + y_k p_k + \sum_{k \neq i} y_k p_i = -y_k + \Sigma\, y_k p_i = -y_k + p_i$$

$$(\because\ y = (y_0, y_1, \ldots y_k \ldots y_n) = (0, 0, \cdots, 1, \cdots 0))$$

Part 2





1. 분류를 위해선 파란선과 같이 직선이 구해져야 한다.

$$\varphi = \begin{cases} 0 & z < 0 \\ 1 & z > 0 \end{cases}$$

$1 = b = x_0$ 라고 가정 $\qquad \varphi(x_0 w_0 + w_1 x_1 + w_2 x_2) = y$

$$\varphi(w_0 + w_1 \times 0 + w_2 \times 0) = 1$$

$$\varphi(w_0 + w_1 + w_2 \times 0) = -1$$

$$\varphi(w_0 + w_1 + w_2) = 1$$

$$\varphi(w_0 + w_1 \cdot 0 + w_2) = 1$$

$w_0 = 0.4 \qquad w_1 = -0.6 \qquad w_2 = 0.4$

$\therefore \varphi = \begin{cases} 0 & z < 0 \\ 1 & z > 0 \end{cases}$ 일 때 $\quad w_0 = 0.4 \quad w_1 = -0.6 \quad w_2 = 0.4$

$x_0 = b = 1$

($\because$ 일반적으로 $b = 1$로 많이 둔다)

2. $\eta = 0.05$ 라고 가정, 같은 선까지 끌어올리기

$$w_0 \leftarrow w_0 + 0.05\,(0-1)\times 1$$
$$w_1 \leftarrow w_1 + 0.05\,(0-1)\times 0$$
$$w_2 \leftarrow w_2 + 0.05\,(0-1)\times 0$$

$$w_0 \leftarrow w_0 + 0.05\,(0-1)\times 1$$
$$w_1 \leftarrow w_1 + 0.05\,(0-1)\times 1$$
$$w_2 \leftarrow w_2 + 0.05\,(0-1)\times 1$$

$$w_0 = 0.4 - 0.1 = 0.3$$
$$w_1 = -0.6 - 0.05 = -0.65$$
$$w_2 = 0.4 - 0.05 = 0.35$$

$\therefore$ $w_0 = 0.3$ $\quad w_1 = -0.65$ $\quad w_2 = 0.35$

3. $\eta = 0.05$ $\quad w_0 = 0.4$ $\quad w_1 = -0.6$ $\quad w_2 = 0.4$

$x_0^{(i)}$	$x_1^{(i)}$	$x_2^{(i)}$	$y^{(i)}$	$w^T x^{(i)}$	$y^{(i)} - w^T x^{(i)}$
1	0	0	0	0.4	-0.4
1	0	1	1	0.8	0.2
1	1	0	0	-0.2	0.2
1	1	1	0	0.2	-0.2

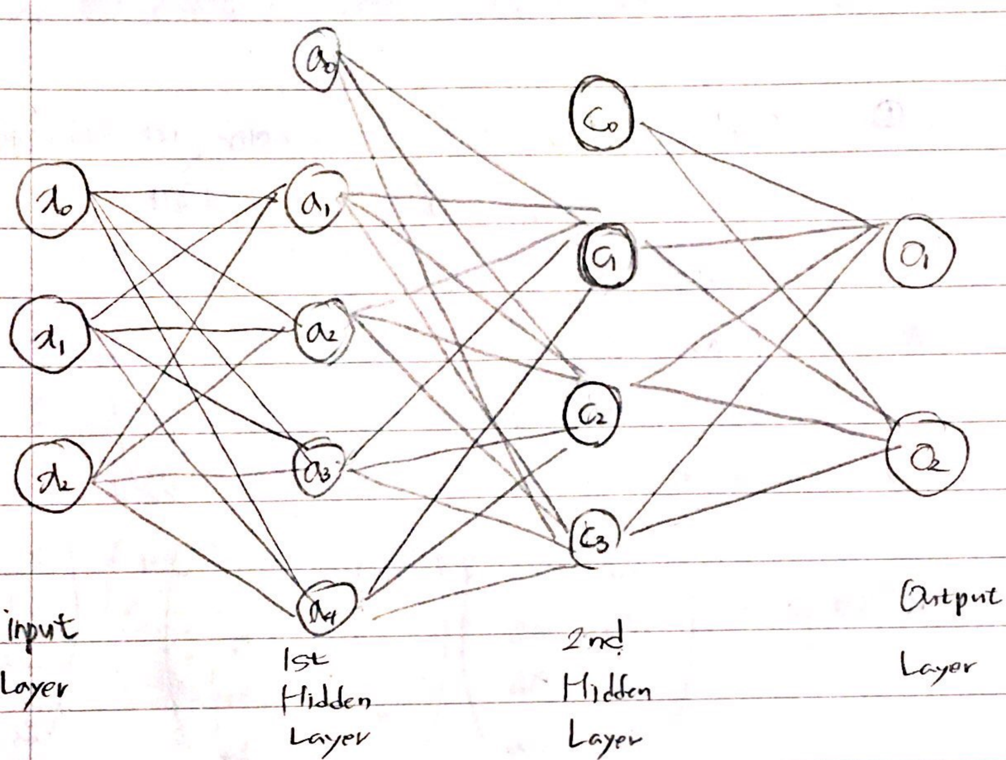
$$w_j \leftarrow w_j + \eta\,(y^{(i)} - w^T x^{(i)})\,x_j^{(i)}$$

$$w_0 \leftarrow w_0 + 0.05\,(-0.2) = 0.4 - 0.01 = 0.39$$
$$w_1 \leftarrow w_1 + 0.05\,(0) = -0.6$$
$$w_2 \leftarrow w_2 + 0.05\,(0) = 0.4$$

$\therefore \quad w_1 = -0.6 \quad w_2 = 0.4 \quad w_0 = 0.34$

Part 3

1.



2. 첫 Layer 가중치 개수 : $3 \times 4 = 12$

두번째 Layer 가중치개수 : $5 \times 3 = 15$

세번째 Layer 가중치 개수 : $4 \times 2 = 8$

$\therefore$ 전체 Layer 개수 : $12 + 15 + 8 = 35$

3. 첫 번째 활성함수 : $ReLU = \max(0, x)$

두 번째 활성함수 : $Sigmoid = \sigma(z) = \dfrac{1}{1+e^{z}}$

세 번째 활성함수 : $Softmax = S(z) = \left(\dfrac{e^{z_1}}{\sum_{k=1}^{K} e^{z_k}}, \dfrac{e^{z_2}}{\sum_{k=1}^{K} e^{z_k}}, \ldots, \dfrac{e^{z_k}}{\sum_{k=1}^{K} e^{z_k}} \right)$

$$P_i = \frac{e^{z_i}}{\sum_{k=1}^{K} e^{z_k}}$$

input : $X^i = (x_1^{(i)}, x_2^{(i)})^T$ ex) X^2는 2×2 차원

즉, 여기서 i는 전체 데이터 수를 의미. 편의상 i 생략 후 진행

1st Weight : $w_i^{1} = (w_{i1}^1, w_{i2}^1)^T$ 여기서 1의 레이어 위치

$W^1 = (w_1^1, w_2^1, w_3^1, w_4^1)^T$ 윗 첨자

$$\therefore \quad W^1 = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \\ w_{41} & w_{42} \end{pmatrix} \qquad b^1 = \begin{pmatrix} b_1^1 \\ b_2^1 \\ b_3^1 \\ b_4^1 \end{pmatrix}$$

$$z^1 = W^1 \cdot x + b^1 = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \\ w_{41} & w_{42} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1^1 \\ b_2^1 \\ b_3^1 \\ b_4^1 \end{pmatrix} = \begin{pmatrix} z_1^1 \\ z_2^1 \\ z_3^1 \\ z_4 \end{pmatrix}$$

ex) $z_1^1 = w_{11}x_1 + w_{12}x_2 + b_1^1 = w_1^{1T}X + b_1^1$

1차 활성함수 처리 $ReLU = Max(0, z^1) = a = (a_1, a_2, a_3, a_4)^T$

이때 a에도 전체 데이터 수를 나타내는 첨자가 필요하나 역시 생략.

$W_i^2 = (W_{i1}^2, W_{i2}^2, W_{i3}^2, W_{i4}^2)^T$

$W^2 = (W_1^2, W_2^2, W_3^2)^T$

$$\therefore \quad W^2 = \begin{pmatrix} W_{11}^2 & W_{12}^2 & W_{13}^2 & W_{14}^2 \\ W_{21}^2 & W_{22}^2 & W_{23}^2 & W_{24}^2 \\ W_{31}^2 & W_{32}^2 & W_{33}^2 & W_{34}^2 \end{pmatrix} \qquad b^2 = \begin{pmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{pmatrix}$$

$$z^2 = W^2 a + b^2 = \begin{pmatrix} w_{11}^2 & w_{12}^2 & w_{13}^2 & w_{14}^2 \\ w_{21}^2 & w_{22}^2 & w_{23}^2 & w_{24}^2 \\ w_{31}^2 & w_{32}^2 & w_{33}^2 & w_{34}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \begin{pmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{pmatrix} = \begin{pmatrix} z_1^2 \\ z_2^2 \\ z_2^2 \end{pmatrix}$$

2번째 활성함수 Sigmoid $= \sigma(z^2) = c = (c_1, c_2, c_3)^T$

편의를 위해 데이터셋 개수 나타내는 첨자생략

$$w_i^3 = (w_{i1}^3, w_{i2}^3, w_{i3}^3)^T$$

$$W^3 = (w_1^3, w_2^3)^T$$

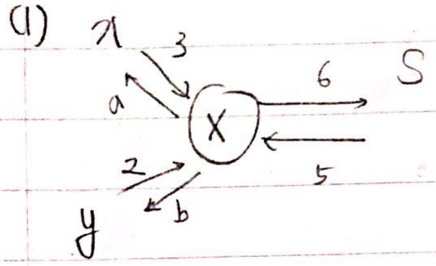
$$W^3 = \begin{pmatrix} w_{11}^3 & w_{12}^3 & w_{13}^3 \\ w_{21}^3 & w_{22}^3 & w_{23}^3 \end{pmatrix} \qquad b^3 = \begin{pmatrix} b_1^3 \\ b_2^3 \end{pmatrix}$$

$$z^3 = W^3 c + b^3 = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} b_1^3 \\ b_2^3 \end{pmatrix} = \begin{pmatrix} z_1^3 \\ z_1^3 \end{pmatrix}$$

3번째 활성함수 : Softmax $= S(z^3) = O = (O_1, O_2)^T$

Part 4

1. (1)



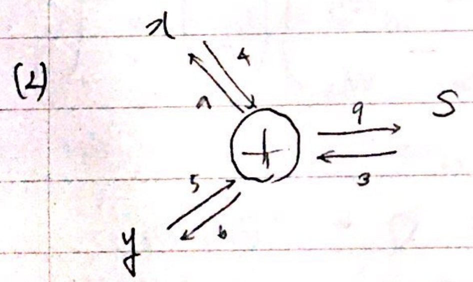
$$\frac{\partial J}{\partial x} = \frac{\partial S}{\partial x} \cdot \frac{\partial J}{\partial S} = 5 \cdot \frac{\partial}{\partial x} S = 5 \cdot \frac{\partial}{\partial x}(xy) = 5y$$

$$= 10 \qquad \therefore\ a = 10$$

$$\frac{\partial J}{\partial y} = \frac{\partial S}{\partial y} \cdot \frac{\partial J}{\partial S} = 5 \cdot \frac{\partial}{\partial y} S = 5 \cdot \frac{\partial}{\partial y}(xy) = 5x$$

$$= 15$$

(2)



$$\frac{\partial J}{\partial x} = \frac{\partial S}{\partial x} \cdot \frac{\partial J}{\partial S} = 3 \cdot \frac{\partial}{\partial x}(x+y) = 3$$

$$a = 3$$

$$\frac{\partial J}{\partial y} = \frac{\partial S}{\partial y} \cdot \frac{\partial J}{\partial S} = 3 \cdot \frac{\partial}{\partial x}(x+y) = 3$$

$$b = 3$$

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(3) $x \underset{b}{\overset{\ln 3}{\rightleftarrows}} \sigma \underset{4}{\overset{a}{\rightleftarrows}} S$

$$\sigma(z) = \frac{1}{1+e^{-z}} \qquad \sigma(\ln 3) = \frac{1}{1+e^{-\ln 3}} = \frac{1}{1+e^{\ln\frac{1}{3}}} = \frac{1}{\frac{4}{3}}$$

$\therefore\ a = \frac{3}{4}$

$$\frac{\partial J}{\partial x} = \frac{\partial S}{\partial x}\cdot\frac{\partial J}{\partial S} = 4\cdot\frac{\partial S}{\partial x} = 4\sigma(x)(1-\sigma(x))$$

$$= 4\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{4}$$

$\therefore\ b = \frac{3}{4}$

(4) $x \underset{a}{\overset{-2}{\rightleftarrows}} \text{ReLU} \underset{2}{\overset{b}{\rightleftarrows}} S$

$\text{ReLU} = \text{Max}(0, x) = \text{Max}(0, -2) = 0$

$\therefore\ b = 0$

$$\frac{\partial J}{\partial x} = \frac{\partial S}{\partial x}\cdot\frac{\partial J}{\partial S} = 2\cdot\frac{\partial S}{\partial x} = 0$$

$$\left(\because\ \text{ReLU}' = \begin{cases} 0 & (x<0) \\ 1 & (x\geq 0) \end{cases}\right)$$

$\therefore\ a = 0$

2. $$\frac{\partial J}{\partial b^3} = \frac{\partial z^3}{\partial b^3} \cdot \frac{\partial O}{\partial z^3} \cdot \frac{\partial J}{\partial O} = \frac{\partial O}{\partial z^3} \cdot \frac{\partial J}{\partial O}$$

$$\left(\because \quad ex) \; z = w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_k x_k + b \qquad \frac{\partial z}{\partial b} = 1 \right)$$

$$\frac{\partial O}{\partial z^3} \cdot \frac{\partial J}{\partial O} = \delta^3 = \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix} = \begin{pmatrix} p_1 - y_1 \\ p_2 - y_2 \end{pmatrix}$$

$$b^3 \Leftarrow b^3 - \underset{\text{상수}}{\underset{\downarrow}{\eta}} \times \delta^3 \times c_0 = b^3 - \eta \times \underset{\text{vector}}{\underset{\downarrow}{\begin{pmatrix} \delta_1^3 \, c_0 \\ \delta_2^3 \, c_0 \end{pmatrix}}}$$

$$\frac{\partial J}{\partial b^2} = \frac{\partial z^2}{\partial b^2} \cdot \frac{\partial c}{\partial z^2} \cdot \frac{\partial z^3}{\partial c} \cdot \frac{\partial J}{\partial z^3} = \frac{\partial c}{\partial z^2} \cdot \frac{\partial z^3}{\partial c} \cdot \frac{\partial J}{\partial z^3}$$

$$\frac{\partial z^3}{\partial c} = \begin{pmatrix} w_{11}^3 & w_{12}^3 & w_{13}^3 \\ w_{21}^3 & w_{22}^3 & w_{23}^3 \end{pmatrix} \qquad \frac{\partial J}{\partial z^3} = \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix}$$

$$\frac{\partial c}{\partial z^2} = f'(z^2) = \begin{pmatrix} f'(z_1^2) \\ f'(z_2^2) \\ f'(z_3^2) \end{pmatrix} = \begin{pmatrix} \sigma(z_1^2)(1-\sigma(z_1^2)) \\ \sigma(z_2^2)(1-\sigma(z_2^2)) \\ \sigma(z_3^2)(1-\sigma(z_3^2)) \end{pmatrix}$$

$$\therefore \frac{\partial c}{\partial z^2} \cdot \frac{\partial z^3}{\partial c} \cdot \frac{\partial J}{\partial z^3} = \delta^2 = \begin{pmatrix} f'(z_1^2) \\ f'(z_2^2) \\ f'(z_3^2) \end{pmatrix} \circ \begin{pmatrix} w_{11}^3 & w_{21}^3 \\ w_{12}^3 & w_{22}^3 \\ w_{13}^3 & w_{23}^3 \end{pmatrix} \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix}$$

$$\therefore b^2 \Leftarrow b^2 - \eta \delta^2 a_0 = b^2 - \eta \times \begin{pmatrix} \delta_1^2 \, a_0 \\ \delta_2^2 \, a_0 \\ \delta_3^2 \, a_0 \end{pmatrix}$$

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$$\frac{\partial J}{\partial b^1} = \frac{\partial z^1}{\partial b^1} \cdot \frac{\partial a}{\partial z^1} \cdot \frac{\partial z^2}{\partial a} \cdot \frac{\partial J}{\partial z^2} = \frac{\partial a}{\partial z^1} \cdot \frac{\partial z^2}{\partial a} \cdot \delta^2$$

$$\frac{\partial a}{\partial z^1} = f'(z^1) = \begin{pmatrix} f'(z_1^1) \\ f'(z_2^1) \\ f'(z_3^1) \\ f'(z_4^1) \end{pmatrix} = \begin{pmatrix} 0 \text{ or } 1 \\ 0 \text{ or } 1 \\ 0 \text{ or } 1 \\ 0 \text{ or } 1 \end{pmatrix}$$

$$\frac{\partial z^2}{\partial a} = \begin{pmatrix} w_{11}^2 & w_{12}^2 & w_{13}^2 & w_{14}^2 \\ w_{21}^2 & w_{22}^2 & w_{23}^2 & w_{24}^2 \\ w_{31}^2 & w_{32}^2 & w_{33}^2 & w_{34}^2 \end{pmatrix} \qquad \delta^2 = \begin{pmatrix} \delta_1^2 \\ \delta_2^2 \\ \delta_3^2 \end{pmatrix}$$

$$\therefore \delta^1 = \begin{pmatrix} f'(z_1^1) \\ f'(z_2^1) \\ f'(z_3^1) \\ f'(z_4^1) \end{pmatrix} \circ \begin{pmatrix} w_{11}^2 & w_{21}^2 & w_{31}^2 \\ w_{12}^2 & w_{22}^2 & w_{32}^2 \\ w_{13}^2 & w_{23}^2 & w_{33}^2 \\ w_{14}^2 & w_{24}^2 & w_{34}^2 \end{pmatrix} \begin{pmatrix} \delta_1^2 \\ \delta_2^2 \\ \delta_3^2 \end{pmatrix}$$

$$b^1 \Leftarrow b^1 - 2\delta^1 \alpha.$$

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3. $\delta_1^2 = \dfrac{\partial J}{\partial z_1^2} = \dfrac{\partial O}{\partial z_1^2} \cdot \dfrac{\partial J}{\partial O} = p_i - y_i$

$(\because$ Loss Function = Cross Entropy $)$

Output Layer = Softmax

$$\delta_1^2 = p_1 - y_1 = 0.7 - 1 = -0.3$$
$$\delta_2^2 = p_2 - y_2 = 0.2 - 0 = 0.2$$
$$\delta_3^2 = p_3 - y_3 = 0.1 - 0 = 0.1$$

$$\delta_1^1 = \frac{\partial J}{\partial z_1^1} = \frac{\partial a_1}{\partial z_1^1} \cdot \left(\sum \frac{\partial z_k^2}{\partial a_1} \cdot \frac{\partial O_k}{\partial z_k^2} \cdot \frac{\partial J}{\partial O_k} \right)$$

$$= f'(z_1^1) \sum W_{k1}^2 \delta_k^2$$

$$= f'(z_1^1) (-0.3 \times 0.9 + 0.2 \times 0.3 + 0.1 \times 0.1)$$

$$= f'(z_1^1) (-0.27 + 0.06 + 0.01) = f'(z_1^1)(-0.2)$$

$$\left(f(z_1^1) = \frac{1}{1+e^{-z_1^1}} = 0.6 = \frac{1}{\frac{5}{3}} = \frac{1}{1+\frac{2}{3}} \qquad e^{-z_1^1} = \frac{2}{3} \qquad z_1^1 = \ln\frac{3}{2} \right)$$

$$= f(z_1^1)(1 - f(z_1^1))(-0.2) = 0.6 \times 0.4 \times -0.2$$

$$\therefore \delta_1^1 = -0.048$$

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W_{11}'의 변화량 $= -\eta \cdot \delta_1' \cdot x_1 = -0.05 \times 0.5 \times (-0.048)$

$= 0.0012$