(a) $\vec{x} \in K_{m+}, \theta \ge 0.$ $\theta\vec{x} \in K_{m+} \Rightarrow K_{m+}$ is a cone.

$\vec{x}, \vec{y} \in K_{m+}$

$\theta\vec{x} + (1-\theta)\vec{y} = \begin{bmatrix} \theta x_1 + (1-\theta)y_1 \\ \theta x_2 + (1-\theta)y_2 \\ \vdots \end{bmatrix} \in K_{m+} \Rightarrow K_{m+}$: Convex cone.

① K_{m+} is closed (contain its boundary) $\rightarrow$ $\underbrace{x_1 \ge x_2, x_2 \ge x_3, \dots, x_{n-1} \ge x_n, x_n \ge 0}_{\text{polyhedra}}$

② K_{m+} is solid (non-empty interior) $\rightarrow$ $x_1 > x_2, x_2 > x_3, \dots, x_{n-1} > x_n, x_n > 0.$

③ K_{m+} is pointed (no line) $\rightarrow$ if $\vec{x} \in K_{m+}$, then $-\vec{x} \in K_{m+}$ only if $\vec{x} = \vec{0}$

2.33 *The monotone nonnegative cone.* We define the *monotone nonnegative cone* as

$$K_{m+} = \{x \in \mathbf{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that K_{m+} is a proper cone.

(b) Find the dual cone K_{m+}^*. *Hint.* Use the identity K_{m+}: proper cone $\rightarrow K_{m+}^*$: proper cone

$$\begin{aligned} \sum_{i=1}^n x_i y_i &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots \\ &\quad + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n). \end{aligned}$$

(b) $K_{m+}^* = \{ \vec{y} \mid \vec{y}^T \cdot \vec{x} \ge 0 \text{ for } \forall \vec{x} \in K_{m+} \}$

$$\sum_{i=1}^n x_i y_i = \underset{\oplus}{(x_1 - x_2)}y_1 + \underset{\oplus}{(x_2 - x_3)}(y_1 + y_2) + \underset{\oplus}{(x_3 - x_4)}(y_1 + y_2 + y_3) + \cdots + \underset{\oplus}{(x_n - 0)}(y_1 + y_2 + \cdots + y_n) \ge 0.$$

$$= \vec{a}^T \cdot \begin{bmatrix} y_1 \\ y_1 + y_2 \\ \vdots \\ y_1 + \cdots + y_n \end{bmatrix} \ge 0. \quad \Longleftrightarrow \quad \sum_{i=k}^n y_i \ge 0 \text{ for } k = 1, 2, \dots, n$$

$$\therefore K_{m+}^* = \{ \vec{y} \mid \sum_{i=k}^n y_i \ge 0 \text{ for } k = 1, 2, \dots, n \}$$