(a). $\lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{\ln x \cdot x^\alpha}{1} = \log x.$ $(x>0)$

(b) $u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$ $(0<\alpha\leq 1,\ x\geq 0)$

$u_\alpha'(x) = \frac{\alpha x^{\alpha-1}}{\alpha} = x^{\alpha-1} > 0$

$u_\alpha''(x) = (\alpha-1)x^{\alpha-2} > 0$ $(-2<\alpha-2\leq -1,\ x>0)$ $\Rightarrow$ concave func.

$\alpha=0$: $u_0(x) = \log x$ $(x>0)$ $\Rightarrow$ concave func.

$u_0'(x) = \frac{1}{x}$ $(x>0)$

Examples

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha\to 0} u_\alpha(x)$.

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation*.