$$\boxed{\begin{gathered}\mathrm{epi}\, f = \{(\vec{x}, t) \in \mathbb{R}^{n+1} \mid \vec{x} \in \mathrm{dom}\, f,\ f(\vec{x}) \le t\} \\ f \text{ is convex function} \Leftrightarrow \mathrm{epi}\, f \text{ is convex set.}\end{gathered}}$$

③ polyhedron : $\{\vec{x} \mid \vec{a}_i^T \cdot \vec{x} \le b_i,\ i=1,2,\cdots,k\} = \{(\vec{x},t) \mid \vec{a}_i^T \cdot \vec{x} + a_i^n t \le b_i,\ i=1,2,\cdots,k\} = \{(\vec{x},t) \mid f(\vec{x}) \le t\}$.

$$a_i^n t \le -\vec{a}_i^T \vec{x} + b_i$$

$$t \ge \underbrace{\frac{\vec{a}_i^T \vec{x} - b_i}{-a_i^n}}_{f_i(\vec{x})\ (i=1,2,\cdots,k)} \quad i=1,2,\cdots,k$$

$\therefore$ if epi f is polyhedron, then f is $\longrightarrow$ piecewise affine.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

① halfspace : $\{\vec{x} \mid \vec{a}^T \cdot \vec{x} \le b\} = \{(\vec{x},t) \mid \vec{a}^T \cdot \vec{x} + b \cdot t \le k\}$

$$t \ge \underbrace{\frac{\vec{a}^T \cdot \vec{x} - k}{b}}_{f(\vec{x})\text{ : affine function.}}$$

$\therefore$ When f is affine func. $\longrightarrow$ then epi f is half space.

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② Convex cone $= S$

$$\left.\begin{array}{l} \vec{x} \in S,\ \theta\vec{x} \in S\ (\theta \ge 0) \\ \vec{x}_1, \vec{x}_2 \in S,\ \alpha \cdot \vec{x}_1 + (1-\alpha)\vec{x}_2 \in S\ (0 \le \alpha \le 1) \end{array}\right\} \text{Convex Cone}$$

epi $f = S = \{(\vec{x},t) \mid f(\vec{x}) \le t\}$ $\quad (\vec{x},t) \in S$: $f(\vec{x}) \le t$

$(\theta\vec{x}, \theta t) \in S$: $f(\theta\vec{x}) \le \theta t$ $(\theta \ge 0)$ $\xrightarrow{t = f(\vec{x})}$ $f(\theta\vec{x}) \le \theta f(\vec{x})$ (convex)

$\cancel{(\vec{x}_1, t_1), (\vec{x}_2, t_2) \in S : f(\vec{x}_1) \le t_1,\ f(\vec{x}_2) \le t_2}$

$\cancel{\alpha(\vec{x}_1, t_1) + (1-\alpha)(\vec{x}_2, t_2) \in S : f(\alpha\vec{x}_1 + (1-\alpha)\vec{x}_2) \le \alpha t_1 + (1-\alpha) t_2}$

$\cancel{= (\alpha\vec{x}_1 + (1-\alpha)\vec{x}_2,\ \alpha t_1 + (1-\alpha) t_2)}$

f : convex $\Leftrightarrow$ epi f : convex.

$\therefore$ When f is $f(\theta\vec{x}) \le \theta f(\vec{x})$ $(\theta \ge 0)$ $\longrightarrow$ then epi f is convex cone.