(c) $\mathbf{prob}(\alpha \le x \le \beta) = \sum_{i=k}^{k+t} p_i \longrightarrow$ concave (o) $\longrightarrow$ quasi-concave (o)

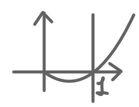
convex (o) $\longrightarrow$ quasi-convex (o)

$x_{k-1} < \alpha \le x_k < x_{k+t} \le \beta < x_{k+t+1}$

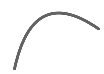
(d) $\sum_{i=1}^{n} p_i \log p_i = [p_1\ p_2 \cdots p_n]\begin{bmatrix}\log p_1 \\ \log p_2 \\ \vdots \\ \log p_n\end{bmatrix} = \vec{p}^T \log \vec{p} \approx x \log x$

negative entropy

$f' = \log x + 1$, $f'' = \frac{1}{x} > 0$ in $\mathbf{R}_{++}$ } convex(o) $\longrightarrow$ quasi-convex(o); concave(x), quasi-concave(x)



(e) $\mathrm{Var}(x) = E[x^2] - E[x]^2 = \sum_i x_i^2 p_i - \left(\sum_i x_i p_i\right)^2 \approx ax - bx^2\ (b>0) \rightarrow$ concave (o) $\longrightarrow$ quasi-concave



convex(x), quasi-convex(x)

(f) $\mathrm{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \le \beta) \ge 0.25\} = t.$

~~$a_k < t \le a_{k+1}$ $\mathbf{prob}(x \le a_{k+1}) \ge 0.25$~~

~~$a_k \le t < a_{k+1}$ $\mathbf{prob}(x \le a_k) \ge 0.25$~~ } ~~linear in $\vec{p}$. $\therefore$ concave(o) $\longrightarrow$ quasi-concave(o)~~

~~convex(o) $\longrightarrow$ quasi-convex(o)~~

By 2.15, sublevel & super level sets of quartile(x) in p are convex $\Rightarrow$ quasi-convex(o), quasi-concave(o).

But quartile(x) is not continuous $\Rightarrow$ convex(x), concave(x)

3.24 *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \dots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}_+^n \mid \mathbf{1}^T p = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbf{E}\, x$.

(a) $E[x] = \sum_{i=1}^{n} x_i \cdot p_i \rightarrow$ linear func. $\rightarrow$ convex (o) $\rightarrow$ quasi-convex (o)

concave (o) $\rightarrow$ quasi-concave(o)

(b) $\mathbf{prob}(x \ge \alpha)$.

(c) $\mathbf{prob}(\alpha \le x \le \beta)$.

(d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

(e) $\mathbf{var}\, x = \mathbf{E}(x - \mathbf{E}\, x)^2$.

(f) $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \le \beta) \ge 0.25\}$.

(g) The cardinality of the smallest set $\mathcal{A} \subseteq \{a_1, \dots, a_n\}$ with probability $\ge 90\%$. (By cardinality we mean the number of elements in $\mathcal{A}$.)

(h) The minimum width interval that contains 90% of the probability, *i.e.*,

$$\inf\{\beta - \alpha \mid \mathbf{prob}(\alpha \le x \le \beta) \ge 0.9\}.$$

(b) $\mathbf{prob}(x \ge \alpha) = \sum_{i=k}^{n} p_i = \begin{bmatrix}0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1\end{bmatrix}^T \begin{bmatrix}p_1 \\ p_2 \\ \vdots \\ p_n\end{bmatrix} \rightarrow$ linear func. $\rightarrow$ convex (o) $\rightarrow$ quasi-convex (o)

concave (o) $\rightarrow$ quasi-concave(o)

$x_k \ge \alpha \ge x_{k-1}$

(g) Cardinality of the smallest set $A \subseteq \{a_1, a_2, \cdots, a_n\}$ with prob. $\geq 90\%$

f: integer $\Rightarrow$ not continuous $\therefore$ convex(X), concave(X)

$f(\vec{p}) = \min\{k : \text{s.t. } A \in \{1, 2, \cdots, n\},\ |A| = k,\ \sum_{\tau \in A} p_\tau \geq 0.9\}$

let $p_{(1)} \geq p_{(2)} \geq p_{(3)} \geq \cdots \geq p_{(n)}$, $S_k(\vec{p}) = \sum_{\tau=1}^{k} p_{(\tau)}$

then $f(\vec{p}) = \min\{k : S_k(\vec{p}) \geq 0.9\}$

3.24 *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}^n_+ \mid \mathbf{1}^T p = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbf{E}\, x$.

(b) $\mathbf{prob}(x \geq \alpha)$.

(c) $\mathbf{prob}(\alpha \leq x \leq \beta)$.

(d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

(e) $\mathbf{var}\, x = \mathbf{E}(x - \mathbf{E}\, x)^2$.

(f) $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(g) The cardinality of the smallest set $\mathcal{A} \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$. (By cardinality we mean the number of elements in $\mathcal{A}$.)

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