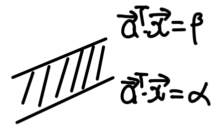
(d) $\{\vec{x} \mid \underbrace{\|\vec{x}-\vec{x}_0\|_2 \le \|\vec{x}-\vec{y}\|_2}$ for $\forall \vec{y} \in S \subseteq \mathbb{R}^n\}$

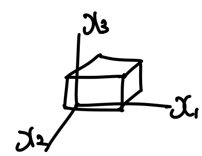
$-2\vec{x}^T\cdot\vec{x}_0 + \|\vec{x}_0\|_2^2 \le -2\vec{x}^T\cdot\vec{y} + \|\vec{y}\|_2^2$

$2\vec{x}^T(\vec{y}-\vec{x}_0) \le \|\vec{y}\|_2^2 - \|\vec{x}_0\|_2^2$

$\vec{x}^T(\vec{y}-\vec{x}_0) \le \frac{1}{2}(\|\vec{y}\|_2^2 - \|\vec{x}_0\|_2^2) = k.$

∴ intersection of all halfspaces for $\vec{y} \in S$. ⇒ ∴ Convex.



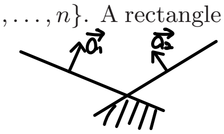


2.12 Which of the following sets are convex?

Polyhedra.

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \le a^T x \le \beta\}$. (O)

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$. (O)

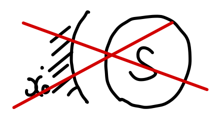


(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$. (O)

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$. ~~(X)~~ (O)



(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$ (X)

Counter ex) $S = \{-1, 1\}$, $T = \{0\}$ in $\mathbb{R}^1$.

∴ Convex (X)

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex. (O)

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \le \theta\|x - b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$. (O)

(f) $\{\vec{x} \mid \vec{x} + S_2 \subseteq S_1\} = T$ where $S_1, S_2 \subseteq \mathbb{R}^n$, S_1: Convex set.

$\vec{x}_1, \vec{x}_2 \in T$, $\vec{s}_2 \in S_2$

$\vec{x}_1 + \vec{s}_2 \in S_1$: Convex

$\vec{x}_2 + \vec{s}_2 \in S_1$: Convex.

$\alpha(\vec{x}_1 + \vec{s}_2) + (1-\alpha)(\vec{x}_2 + \vec{s}_2) \in S_1$: Convex

$= \underbrace{\alpha\vec{x}_1 + (1-\alpha)\vec{x}_2 + \vec{s}_2}_{\in T}$ ∴ T is Convex set.

(g) $S = \{\vec{x} \mid \|\vec{x}-\vec{a}\|_2 \le \theta\|\vec{x}-\vec{b}\|_2\}$

$\|\vec{x}\|_2^2 - 2\vec{a}^T\cdot\vec{x} + \|\vec{a}\|_2^2 \le \theta^2\{\|\vec{x}\|_2^2 - 2\vec{b}^T\cdot\vec{x} + \|\vec{b}\|_2^2\}$

$(1-\theta^2)\|\vec{x}\|_2^2 - 2(\vec{a} - \theta^2\cdot\vec{b})^T\vec{x} \le \theta^2\|\vec{b}\|_2^2 - \|\vec{a}\|_2^2$

$(\theta \ne 1)$ $p\|\vec{x} - \vec{k}\|_2^2 \le q$ $(\theta = 1)$: halfspace

$\|\vec{x} - \vec{k}\|_2^2 \le q/p$

: L2 Norm Ball.