$f(X) = \log\det(X)$ is concave on $\mathrm{dom} f \in S^n_{++}$

pf.> $g(t) = f(X + t\cdot Y)$ $\quad \{t \mid \underbrace{X + t\cdot Y \in \mathrm{dom} f}_{S^n_{++}} \text{ for any } \underbrace{X \in \mathrm{dom} f}_{S^n_{++}},\ Y \in S^n\}$

$$X + tY \underset{\uparrow}{=} X^{1/2}\left(I + t\cdot X^{-1/2}\cdot Y\cdot X^{-1/2}\right)\cdot X^{1/2}$$

$X \succ 0$: invertible.

$$\det(X+tY) \underset{\uparrow}{=} \det(X^{1/2})\cdot\det(I + t\cdot \underbrace{X^{-1/2}\cdot Y\cdot X^{-1/2}}_{S})\cdot\det(X^{1/2}) = \det(X)\cdot\det(I+t\cdot S) \xrightarrow{\log} \log\det(X) + \log\det(I+t\cdot S)$$

$\det ABC = \det A\cdot\det B\cdot\det C$

$\underline{I + t\cdot S = X^{-1/2}(X+tY)X^{-1/2} \succ 0}$ $\quad$ sym. $S = Q\Lambda Q^T$, $\quad \det(S) = \det Q\cdot\det\Lambda\cdot\det Q^T = \det(\Lambda)$

$\det Q^T(I+t\cdot S)Q = \det(I + t\cdot\Lambda)$

$$\therefore g(t) = \log\det(X) + \log\det(I + t\cdot\Lambda) = \log\det(X) + \log\prod_{i=1}^{n}(1+t\cdot\lambda_i) = \log\det(X) + \underbrace{\sum_{i=1}^{n}\log(1+t\cdot\lambda_i)}$$

sum of concave funcs.
$\rightarrow$ concave w.r.t t

3.18 Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

~~(a)~~ $f(X) = \mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(b) $g(t) = f(X+tY)$ $\quad \{t \mid \underbrace{X+tY \in \mathrm{dom} f}_{S^n_{++}},\ \underbrace{X \in \mathrm{dom} f}_{S^n_{++}},\ Y \in S^n\}$

$$X + tY = X^{1/2}\left(I + tX^{-1/2}\cdot Y\cdot X^{-1/2}\right)\cdot X^{1/2} \rightarrow \det(X)\cdot\det(I + t\underbrace{X^{-1/2}YX^{-1/2}}_{S}) = \det(X)\cdot\det(I+t\cdot S)$$

$$\underset{\uparrow}{=} \det(X)\cdot\det(I+t\cdot\Lambda) = \det(X)\cdot\prod_{i=1}^{n}(1+t\cdot\lambda_i)$$

$S = Q\Lambda Q^T$ (sym.), $\underline{I + t\cdot S = X^{-1/2}(X+tY)X^{-1/2} \succ 0}$

$$\rightarrow g(t) = f(X+tY) = \left\{\det(X)\cdot\prod_{i=1}^{n}(1+t\cdot\lambda_i)\right\}^{1/n} = (\det X)^{1/n}\cdot\boxed{\left\{\prod_{i=1}^{n}(1+t\cdot\lambda_i)\right\}^{1/n}}$$

$1 + t\cdot\lambda_i > 0$, $\det X \succ 0$ $(X \in S^n_{++})$

Geometric Mean : Concave on $\mathbb{R}^n_{++}$

$\therefore g(t)$ is concave on $\{t \mid X+tY \in S^n_{++}, X \in S^n_{++}, Y \in S^n\} \Leftrightarrow f(X)$ is concave on $X \in \mathrm{dom} f = S^n_{++}$.