Exercises

Definition of convexity

2.1 Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

(k) Assume that $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$ with $\theta_i \geq 0$, $\theta_1 + \theta_2 + \cdots + \theta_k = 1$

(k+1) Let $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k = y_k \in C$ and $x_{k+1} \in C$

$\alpha \cdot y_k + (1-\alpha) x_{k+1} \in C$ $(0 \leq \alpha \leq 1)$ by definition of convex set.

$\alpha\theta_1 x_1 + \alpha\theta_2 x_2 + \cdots + \alpha\theta_k x_k + (1-\alpha) x_{k+1} = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \beta_{k+1} x_{k+1}$

with $\beta_i \geq 0$, $\beta_1 + \beta_2 + \cdots + \beta_k = 1$.

k=2 (base case) holds by definition of convexity.

$\therefore$ By induction on k, the statement holds!