(a) $f(\vec{x}) = \max_{i=1,2,\dots,n}(x_i) \quad (\vec{x}\in R^n)$

$f^*(\vec{y}) = \sup_{\vec{x}\in \mathrm{dom} f}\{\vec{y}^T\cdot\vec{x} - f(\vec{x})\}$ if $y_k<0$ and $x_k=-\infty$ then $f^*(\vec{y})=\infty$

so $\vec{y}\succeq 0$ for $f^*(\vec{y})$ to be finite.

1) if $\vec{y}$ is in probability simplex : $\vec{1}^T\cdot\vec{y}=1$. $\longrightarrow$ $\vec{y}^T\cdot\vec{x} - \max_i(x_i) \le 0$ mean. $\therefore f^*(\vec{y})=0$.

2) if $\vec{y}$ is not in probability simplex : $\vec{1}^T\cdot\vec{y}\ne 1$ $\longrightarrow$ $\vec{y}^T\cdot\vec{x} - \max_i(x_i)$ can be ∞

$$f^*(\vec{y}) = \begin{cases} 0 & (\vec{1}^T\cdot\vec{y}=1, \vec{y}\succeq 0) \\ +\infty & (\vec{1}^T\cdot\vec{y}\ne 1, \vec{y}\succeq 0) \end{cases}$$

3.36 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements.* $f(x) = \sum_{i=1}^r x_{[i]}$ on $\mathbf{R}^n$.

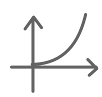
(c) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\dots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \le \cdots \le a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

(d) *Power function.* $f(x) = x^p$ on $\mathbf{R}_{++}$, where $p > 1$. Repeat for $p < 0$.

(e) *Negative geometric mean.* $f(x) = -(\prod x_i)^{1/n}$ on $\mathbf{R}^n_{++}$.

(f) *Negative generalized logarithm for second-order cone.* $f(x,t) = -\log(t^2 - x^T x)$ on $\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 < t\}$.

(d) $f(x)=x^p \quad (x>0, p>1)$



$$f^*(y) = \sup_{x\in R_{++}}\{yx-f(x)\} = \sup_{x>0}\underbrace{\{yx-x^p\}}_{g(x)} = y\cdot\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} = p\left(\frac{y}{p}\right)^{\frac{p}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$(p>1, y>0)$ $g'(x) = y - px^{p-1} = 0$, $x^* = \left(\frac{y}{p}\right)^{1/(p-1)} \longrightarrow$ maximum.

$g''(x) = -p(p-1)x^{p-2} < 0$: concave.

$(p>1, y\le 0)$ $g(x) = yx - x^p$, $x^* = 0$. $\longrightarrow$ maximum.

$$(p>1) \quad f^*(y) = \begin{cases} 0 & (y\le 0) \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & (y>0) \end{cases}$$

$(p<0, y\le 0)$ $g(x) = yx - x^p$, $g'(x) = y - px^{p-1} = 0$, $x^{p-1} = \frac{y}{p}$, $x^* = \left(\frac{y}{p}\right)^{\frac{1}{p-1}} \longrightarrow$ maximum.

$g''(x) = -p(p-1)x^{p-2} < 0$: concave.

$(p<0, y>0)$ $x^* = \infty$

$\mathrm{dom} f^* = -R_+$

$$(p<0) \quad f^*(y) = \begin{cases} \cancel{\infty \quad (y>0)} \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & (y\le 0) \end{cases}$$