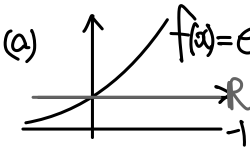
(a)  $f(x)=e^{x}-1$: convex (o) $\longrightarrow$ $\underline{\text{quasi-convex (o)}}$ \} quasi-linear

concave (x) $\underline{\text{quasi-concave (o)}}$

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x)=e^x-1$ on $\mathbf{R}$.

(b) $f(x_1,x_2)=x_1x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1,x_2)=1/(x_1x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1,x_2)=x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1,x_2)=x_1^2/x_2$ on $\mathbf{R}\times\mathbf{R}_{++}$.

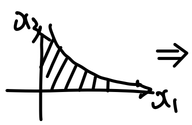
~~(f)~~ $f(x_1,x_2)=x_1^{\alpha}x_2^{1-\alpha}$, where $0\le\alpha\le1$, on $\mathbf{R}^2_{++}$.

(b) $f(x_1,x_2)=x_1x_2$ $\quad \nabla_{\vec{x}} f(\vec{x}) = [x_2, x_1]^T$

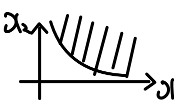
$$\nabla^2_{\vec{x}} f(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \vec{z}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{z} = [z_2, z_1]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1z_2$$

$\therefore \nabla^2_{\vec{x}} f$: indefinite $\Rightarrow$ convex (x), concave (x)

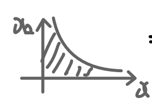
$\{\vec{x} \mid f(\vec{x}) = x_1x_2 \le \alpha,\ \vec{x}\in\mathbb{R}^2_{++}\}$

$x_2 \le \frac{\alpha}{x_1}$  $\Rightarrow$ quasi-convex (x)

$\{\vec{x} \mid f(\vec{x}) = x_1x_2 \ge \alpha,\ \vec{x}\in\mathbb{R}^2_{++}\}$

$x_2 \ge \alpha/x_1$  $\Rightarrow$ quasi-concave (o)

(c) $f(x_1,x_2) = 1/x_1x_2$ on $\mathbb{R}^2_{++}$ $\quad \{\vec{x}\in \mathrm{dom} f \mid f(\vec{x}) \ge \alpha\}$ for $\forall\alpha$

$1/x_1x_2 \ge \alpha$, $x_2 \le \frac{1}{\alpha x_1}$  $\Rightarrow$ quasi-concave (x)

$$\nabla_{\vec{x}} f(\vec{x}) = \left[-\frac{1}{x_1^2}\frac{1}{x_2}, -\frac{1}{x_1}\frac{1}{x_2^2}\right]^T$$

$$\nabla^2_{\vec{x}} f(\vec{x}) = \begin{bmatrix} \frac{2}{x_1^3}\cdot\frac{1}{x_2}, & \frac{1}{x_1^2}\frac{1}{x_2^2} \\ \frac{1}{x_1^2}\frac{1}{x_2^2}, & \frac{1}{x_1}\cdot\frac{2}{x_2^3} \end{bmatrix} = H \in S_{++} \succeq 0 \Rightarrow \text{convex (o)} \longrightarrow \text{quasi-convex (o)}$$

concave (x)

Symmetric $\rightarrow$ use Sylvester's Criterion.

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

~~(f)~~ $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbf{R}^2_{++}$.

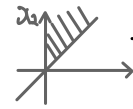
(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$

$\nabla_{\vec{x}} f(\vec{x}) = \left[\frac{1}{x_2}, -\frac{x_1}{x_2^2} \right]^T$

$\nabla^2_{\vec{x}} f(\vec{x}) = \begin{bmatrix} 0 & , -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & , +x_1 \cdot \frac{2}{x_2^3} \end{bmatrix} \begin{matrix} \nsucceq 0 \rightarrow \text{Convex(X)} \\ \npreceq 0 \rightarrow \text{Concave(X)} \end{matrix}$

Symmetric

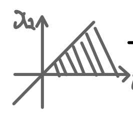
$S_\alpha = \{ \vec{x} \mid \underbrace{f(\vec{x}) \le \alpha}_{x_1/x_2 \le \alpha,\ x_1 \le \alpha x_2} \ \forall \alpha \}$



→ quasi-convex (O)

quasi-linear.

$S_\alpha = \{ \vec{x} \mid \underbrace{f(\vec{x}) \ge \alpha}_{x_1/x_2 \ge \alpha,\ x_1 \ge \alpha x_2} \ \forall \alpha \}$



→ quasi-convex (O)

(e) $f(x_1, x_2) = x_1^2/x_2$ $(x_1 \in \mathbf{R},\ x_2 \in \mathbf{R}_{++})$

$\nabla_{\vec{x}} f(\vec{x}) = \left[2x_1/x_2, -\frac{x_1^2}{x_2^2} \right]^T$

$\nabla^2_{\vec{x}} f(\vec{x}) = \begin{bmatrix} 2/x_2 & , -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & , \frac{2x_1^2}{x_2^3} \end{bmatrix} \succeq 0$ $\quad \frac{2}{x_2} > 0,\ \frac{2x_1^2}{x_2^3} > 0$

$\frac{4x_1^2}{x_2^4} - \frac{4x_1^2}{x_2^4} = 0.$

Symmetric.

$\vec{z}^T H \vec{z} = \left(\frac{2}{x_2}\right) z_1^2 - \frac{4x_1}{x_2^2} z_1 z_2 + \left(\frac{2x_1^2}{x_2^3}\right) z_2^2$

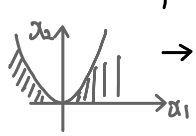
$= \frac{2}{x_2} \left\{ z_1^2 - \frac{2x_1}{x_2} z_1 z_2 + \frac{x_1^2}{x_2^2} z_2^2 \right\}$

$= \frac{2}{x_2} \left(z_1 - \frac{x_1}{x_2} z_2 \right)^2 \ge 0.$

⇒ Convex(O) → quasi-convex(O)

concave(X)

$S_\alpha = \{ \vec{x} \mid \underbrace{f(\vec{x}) \ge \alpha}_{x_1^2/x_2 \ge \alpha,\ \alpha \cdot x_2 \le x_1^2} \ \forall \alpha.\ x_1 \in \mathbf{R}, x_2 \in \mathbf{R}_{++} \}$



→ quasi-concave (X)