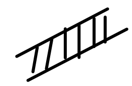
2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

$\alpha \le \sum_{i=1}^{n} p_i \cdot f(a_i) \le \beta$



linear in $\vec{p}$

(a) $\alpha \le \mathbf{E} f(x) \le \beta$, where $\mathbf{E} f(x)$ is the expected value of $f(x)$, *i.e.*, $\mathbf{E} f(x) = \sum_{i=1}^{n} p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.) (o)

(b) $\mathbf{prob}(x > \alpha) \le \beta$. (o) $\quad \sum_{i=k}^{n} p_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \cdot \vec{p} \le \beta$

(c) $\mathbf{E}\,|x^3| \le \alpha\, \mathbf{E}\,|x|$. (o)

(d) $\mathbf{E}\, x^2 \le \alpha$. (o)

(e) $\mathbf{E}\, x^2 \ge \alpha$. (o)

(f) $\mathbf{var}(x) \le \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}\,x)^2$ is the variance of x. ~~(o)~~ (X)

(g) $\mathbf{var}(x) \ge \alpha$. (o)

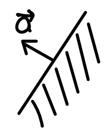
(h) $\mathbf{quartile}(x) \ge \alpha$, where $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \le \beta) \ge 0.25\}$. (o)

(i) $\mathbf{quartile}(x) \le \alpha$. (o)

(c) $$E|x^3| = \sum_{i=1}^{n} p_i \cdot |a_i^3|, \qquad E|x| = \sum_{i=1}^{n} p_i \cdot |a_i|$$

$$\sum_{i=1}^{n} p_i \cdot |a_i^3| \le \alpha \cdot \sum_{i=1}^{n} p_i \cdot |a_i|$$

$$p_1(|a_1^3| - \alpha|a_1|) + p_2(|a_2^3| - \alpha|a_2|) + \cdots = \vec{a}^T \cdot \vec{p} \le 0.$$

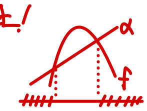


(d) $$E[x^2] = \sum_{i=1}^{n} p_i \cdot a_i^2 \le \alpha.$$

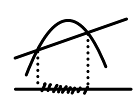
(f) $$Var(x) = E[(x - E(x))^2] = E(x^2) - E(x)^2 = \sum_{i=1}^{n} p_i \cdot a_i^2 - \left(\sum_{i=1}^{n} p_i \cdot a_i\right)^2 \le \alpha.$$

$$S = \{\vec{p} \mid \underbrace{\vec{a}^T\vec{p}}_{\text{linear}} - \underbrace{(\vec{b}^T\vec{p})^2}_{\text{concave}} \le \alpha\} \rightarrow S' = \{\vec{x} \mid f(\vec{x}) \le \alpha,\ f: \text{concave}\}$$

S' is not convex set!



(g) $$S = \{\vec{p} \mid \underbrace{\vec{a}^T\vec{p}}_{\text{linear}} - \underbrace{(\vec{b}^T\vec{p})^2}_{\text{concave}} \ge \alpha\} \rightarrow S' = \{\vec{x} \mid f(\vec{x}) \ge \alpha,\ f: \text{concave}\}$$



$\therefore S'$ is convex set!

(h) let $a_k < \alpha \le a_{k+1}$ $\quad \underbrace{\mathrm{prob}(x \le a_{k+1}) \ge 0.25}_{\text{linear in } \vec{p}}$

(i) let $a_k \le \alpha < a_{k+1}$ $\quad \underbrace{\mathrm{prob}(x \le a_k) \ge 0.25}_{\text{linear in } \vec{p}}$