Assignment #2. Spectral methods for the solution of differential equations.

In this assignment, the fundamental routines of the spectral methods toolbox you have developed during assignment #1 in Matlab/Octave/Python is to be applied for the first time for the solution of differential equations.

Boundary Value Problems

a) Consider the second-order boundary value problem

$$-\epsilon \frac{d^2}{dx^2}u - \frac{d}{dx}u = 1, \quad u(0) = u(1) = 0$$

where ϵ is a small positive parameter which control the width of a 'boundary layer' on the left side of the domain for $x \in [0, 1]$. Confirm that the exact solution to the problem is

$$u(x) = \frac{e^{-x/\epsilon} + (x-1) - e^{-1/\epsilon}x}{e^{-1/\epsilon} - 1}$$

Derive and implement two numerical schemes for solving this problem using, respectively, a spectral Legendre Tau Method (LTM) and a Legendre Collocation Method (LCM). Consider the relative differences in accuracy for $\epsilon = 0.1, 0.01, 0.001$ as a function of modes included in a series expansions for the solution. Discuss pros and cons for using these methods.

Remark:

It is given that if $u(x)$ (square-integrable) and derivatives of $u(x)$ are represented in terms of Legendre series

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n L_n(x), \quad \frac{d^q}{dx^q}u(x) = \sum_{n=0}^{\infty} \hat{u}_n \frac{d^q}{dx^q} L_n(x) = \sum_{n=0}^{\infty} \hat{u}_n^{(q)} L_n(x)$$

where the coefficients of the derivatives $\hat{u}_n^{(q)}$ and those of the solution $\hat{u}_n$ are related through

$$\hat{u}_n^{(1)} = (2n+1) \sum_{\substack{p=n+1 \\ n+p \text{ odd}}}^{\infty} \hat{u}_p, \qquad \hat{u}_n^{(2)} = \left(n + \tfrac{1}{2}\right) \sum_{\substack{p=n+2 \\ n+p \text{ even}}}^{\infty} (p(p+1) - n(n+1))\hat{u}_p, \quad n \geq 0$$

Furthermore, the three-term recurrence relation for Legendre polynomials exist

$$\hat{u}_n^{(q-1)} = \frac{\hat{u}_{n-1}^{(q)}}{2n-1} - \frac{\hat{u}_{n+1}^{(q)}}{2n+3}, \quad n \geq 1$$

and may be used for the construction of a sparse coefficient matrix for the Tau method.

b) Derive, implement and verify spectral accuracy for a spectral method for solving the second-order Boundary Value Problem for irrotational flow around a cylinder[1] stated in polar coordinates (r, θ) as

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0, \quad (r, \theta) \in [r_1, \infty[\times [0, 2\pi], \quad r_1 > 0$$

where $\phi(x, y)$ a scalar velocity potential function which describes the velocity field $(u, v) = \nabla \phi$. The exact solution is given as $\phi(r, \theta) = V_\infty (r + \frac{r_1^2}{r}) \cos(\theta)$ and can be used for imposing boundary conditions in a finite domain $(r, \phi) \in [r1, r2] \times [0, 2\pi]$, $r_1 < r_2 < \infty$. V_∞ is the magnitude of the far-field flow velocity ($V_\infty = 1$ can be assumed) and r_1 the cylinder radius.

Time-dependent problems

In this part, the goal is to develop a spectral model for solving the famous Korteweg-de Vries (KdV) equation and then assess the properties and performance of the implemented model.

In the following, consider the following KdV equation

$$\partial_t u + 6u\partial_x u + \partial_{xxx} u = 0, \quad -\infty < x < \infty, \quad t > 0$$

An exact traveling wave solution (referred to as a *soliton*) to the KdV equation is given as

$$u(x,t) = f(x - ct), \quad f(x) = \tfrac{1}{2}c\,\mathrm{sech}^2\left(\tfrac{1}{2}\sqrt{c}(x - x_0)\right)$$

where x_0 is the center of soliton at $t = 0$. The soliton travels to the right with speed $c > 0$ with a shape defined by $f(x)$. The solitary wave solution has the property that

$$\lim_{x\to\infty} u(x,t) = 0$$

for all times t.

It can be shown that the solitary wave solution fulfills the KdV equation and that the three fundamental quantities mass (M), momentum (V) and energy (E)

$$M = \int_{-\infty}^{\infty} u dx, \quad V = \int_{-\infty}^{\infty} u^2 dx, \quad E = \int_{-\infty}^{\infty} (\tfrac{1}{2}u_x^2 - u^3) dx$$

are all conserved (invariant) with respect to time.

c) Develop (in writing) a spectral method of Fourier type and implement it for solving the KdV equation. This includes selecting an appropriate time-stepping method (give heuristic expression for defining acceptable stable time steps for arbitrary number of nodes), verifying that the implemented method is correct and description of how this has been done in sufficient details.

d) Test your solver using the solution of a single solitary wave of varying heights, e.g. $c \in [0.25, 0.5, 1]$. Make plots of the evolution of estimated errors $||u - \mathcal{I}_N u||_2$ and $||u - \mathcal{I}_N u||_\infty$ as well as the quantifies $\tilde{M} \approx M$, $\tilde{V} \approx V$ and $\tilde{E} \approx E$ for all runs you do. How well does the solver mimic the physical solution and properties hereof? Do a small investigation of the importance of choice of domain size and number of nodes with respect to accuracy and relate to choice of c. Make sure to comment and discuss relevant details.

e) Describe and discuss possible sources of aliasing errors in your model. Investigate the effects of aliasing errors by spectral harmonic analysis of changes in magnitudes of coefficients of different Fourier modes of the solution in time (use FFT). Do you see problems with aliasing errors? If so, test if you can stabilize the model via some appropriate de-aliasing strategy?

f) Simulate a collision of two solitons which are initially well apart in order to represent the initial state by superposition of the two single solitons initially defined by $(x_0, c) = (-40, 0.5)$ and $(x_0, c) = (-15, 0.25)$. Make a space-time plot for $x \in [-L_x, L_x]$ (choose L_x) and time interval $t \in [0, 120]$ in which a collision of the solitons will take place.

g) Carry out a scalability analysis for your implemented model (fx. measure performance as CPU time/time step for varying number of nodes N). What do you expect? Discuss what you have done to make your implementation efficient and what could be done to further improve it.

h) Consider the linear advection equation

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + a\frac{\partial \varphi}{\partial x} &= 0, \quad 0 < x < 2\pi, \quad t > 0 \\ \varphi(x,0) &= \varphi_0(x), \quad 0 \leq x \leq 2\pi \\ \varphi(0,t) &= g_l(t), \quad t \geq 0, \quad a > 0, \end{aligned}$$

Assume real-valued constant $a > 0$. It is possible to show that the exact solution can be represented as

$$\varphi(x,t) = f(x - at)$$

where $f(y)$ is an arbitrary function. The boundary condition is defined at the left boundary as $g_l(t) = u(0,t)$ when $a > 0$ and the right boundary as $g_r(t) = u(2\pi, t)$.

Solve this IVP as a Boundary Value Problem (BVP) by formulating it as a problem in multiple dimensions when solving PDEs, i.e. with dimensions for both space (x) and time (t).

Hence, we can utilize spectral differentiation matrices defined for multiple dimensions for formulate a polynomial collocation scheme that solve the advection equation using global expansions in a finite xt-domain as a BVP solver

$$\tilde{\mathcal{L}}\mathbf{u} = \tilde{\mathbf{f}}$$

where $\tilde{\mathcal{L}}$ is the modified system matrix and right-hand side vector that takes into account initial condition (IC) and spatial boundary conditions (BCs).

Define

$$\mathcal{L} = \mathcal{D}_t + a\mathcal{D}_x, \quad \mathbf{f} = 0.$$

where $D_t = D_t \otimes I$ and $D_x = I \otimes D_x$.

The modified operators is obtained by updating the scheme for nodes at the $t = 0$ boundary to incorporate the initial condition and the $x = 0$ boundary to incorporate the left boundary condition. Update procedure eliminates rows in L and inserts a '1' on the diagonal, and then adds the IC/BC-conditions to $\mathbf{f}$.

Demonstrate convergence and plot snapshots of the solutions for the report.

Communicate your results in a written report. Make sure to describe and comment on important details and findings. Include sufficiently details for the reader to both reproduce and understand the reported results and conclusions.

Enjoy!

Allan P. Engsig-Karup

Deadline for this assignment is Wednesday, 5 Nov, 2025, 23:59.