Assignment #3. Advanced numerical solvers based on spectral methods.

In this assignment, the spectral element methods will be put to practice for solving a two-dimensional linear advection problem with a rotating Gaussian hill as test case. The goal is to connect the continuous formulation of the advection equation to its weak form, a skew-symmetric split formulation, and a practical spectral element discretization in space. You will then implement the semi-discrete system and integrate it in time using an ODE solver of your own choice, and finally assess the numerical accuracy and convergence properties of your implementation.

Governing equations

We consider the linear advection equation in conservative form on a two-dimensional domain $\Omega \subset \mathbb{R}^2$,

$$\frac{\partial u}{\partial t} + \nabla \cdot \big(\mathbf{a}(x,y)\, u\big) = 0, \qquad (x,y) \in \Omega, \quad t > 0, \tag{1a}$$

$$u(x,y,0) = u_0(x,y), \qquad (x,y) \in \Omega, \tag{1b}$$

where $u(x,y,t)$ is a scalar quantity (e.g. a passive tracer) and $\mathbf{a}(x,y)$ is a prescribed velocity field. In this assignment we focus on a divergence-free, solid-body rotational flow given by

$$\mathbf{a}(x,y) = \big(a_x(x,y), a_y(x,y)\big) = \big(2\pi y,\, -2\pi x\big). \tag{2}$$

Note that the flow-field is divergence free

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} = 0, \tag{3}$$

and so the flow field is incompressible.

Rotating Gaussian hill. We define a Gaussian initial condition centered at (x_0, y_0):

$$u_0(x,y) = \exp\left(-\frac{(x-x_0)^2 + (y-y_0)^2}{2\sigma^2}\right), \tag{4}$$

where $\sigma > 0$ controls the width of the Gaussian hill.

Under the solid-body rotation defined above, the center of the Gaussian follows a circular trajectory around the origin. The trajectory $(x_c(t), y_c(t))$ satisfies

$$\frac{dx_c}{dt} = a_x(x_c, y_c) = 2\pi y_c, \tag{5a}$$

$$\frac{dy_c}{dt} = a_y(x_c, y_c) = -2\pi x_c, \tag{5b}$$

with initial conditions $x_c(0) = x_0$, $y_c(0) = y_0$. Solving this system gives

$$x_c(t) = x_0 \cos(2\pi t) + y_0 \sin(2\pi t), \tag{6a}$$

$$y_c(t) = y_0 \cos(2\pi t) - x_0 \sin(2\pi t). \tag{6b}$$

An exact solution of (1a)–(1b) is then given by a Gaussian that is simply transported along this trajectory,

$$u_{\text{exact}}(x,y,t) = \exp\left(-\frac{(x-x_c(t))^2+(y-y_c(t))^2}{2\sigma^2}\right). \tag{7}$$

This solution will be used both to prescribe inflow boundary data (when needed) and to assess the accuracy of your numerical solution.

We introduce also a measure of the energy of the solution. For a scalar field $u(x,y,t)$ on a domain Ω, the *energy* is defined as

$$E(t) = \frac{1}{2}||u(\cdot,t)||^2_{L^2(\Omega)}\ , \tag{8}$$

where the L^2 norm in two space dimensions is defined as

$$||u||^2_{L^2(\Omega)} = \int_\Omega |u(x,y)|^2 dxdy. \tag{9}$$

So, in the context of the advection equation this energy definition is useful to measure what we can refer to as the total amplitude of the solution and can be the basis for energy stability analysis, where the rate of change of energy dE/dt can be used to determine if a scheme is energy stable ($dE/dt \leq 0$), energy conserving ($dE/dt = 0$) or unstable ($dE/dt > 0$).

Exercises

1) **Skew-symmetric split form of the advection equation.**

   Starting from the conservative advection equation (1a)

   $$\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{a}u) = 0,$$

   and using that $\nabla\cdot\mathbf{a} = 0$, show that the equation can be written in the following skew-symmetric (split) form:

   $$\frac{\partial u}{\partial t} + \frac{1}{2}\left[\nabla\cdot(\mathbf{a}u) + \mathbf{a}\cdot\nabla u\right] = 0. \tag{10}$$

   a) Derive (10) from (1a) using the product rule from vector calculus

   $$\nabla\cdot(\mathbf{a}u) = \mathbf{a}\cdot\nabla u + (\nabla\cdot\mathbf{a})u$$

   to rewrite the advection term, then take the average of the the two PDEs in the conservative form and the new form and introduce the assumption $\nabla\cdot\mathbf{a} = 0$.

2) **Weak formulations: conservative and split form.**

Let $v(x, y)$ be a sufficiently smooth test function. Consider both the original conservative form (1a) and the split form (10).

a) Derive the weak form of the conservative advection equation: multiply (1a) by v, integrate over the domain Ω, and use integration by parts to move spatial derivatives onto the test function. Show that this yields a volume contribution and a boundary term:

$$\int_{\Omega} v \frac{\partial u}{\partial t} \, d\Omega - \int_{\Omega} \nabla v \cdot (\mathbf{a}u) \, d\Omega + \int_{\partial\Omega} v \, (\mathbf{a} \cdot \mathbf{n}) \, u \, ds = 0, \tag{11}$$

where $\mathbf{n}$ denotes the outward unit normal on the boundary $\partial\Omega$.

b) Repeat the derivation for the skew-symmetric form (10), and write down the resulting weak form in terms of volume integrals and a boundary term. Carefully show how the split between $\nabla \cdot (\mathbf{a}u)$ and $\mathbf{a} \cdot \nabla u$ is reflected at the weak formulation level.

c) In the context of a spectral element method, briefly explain how the domain Ω is decomposed into elements $\Omega = \bigcup_K K$, and how the weak form is enforced element-wise on each K.

Upwind numerical flux at the boundary.

To impose boundary conditions in a stable manner, we introduce an upwind numerical flux for the normal flux $f_n = (\mathbf{a} \cdot \mathbf{n})u$ on $\partial\Omega$. Let u^- denote the interior trace of u at the boundary, and let u^{bc} denote the prescribed boundary data, e.g. from the exact solution u_{exact}. Define the scalar normal velocity

$$a_n = \mathbf{a} \cdot \mathbf{n}. \tag{12}$$

The upwind numerical flux is then defined by

$$f_n^*(u^-, u^{\mathrm{bc}}) = \frac{1}{2}\big(|a_n| + a_n\big)\, u^- + \frac{1}{2}\big(|a_n| - a_n\big)\, u^{\mathrm{bc}}. \tag{13}$$

a) Explain the meaning of the notation u^-, u^{bc}, and a_n.

b) Show that if $a_n > 0$ (outflow), then $f_n^* = a_n u^-$, whereas if $a_n < 0$ (inflow), then $f_n^* = a_n u^{\mathrm{bc}}$.

c) Rewrite the boundary integral in your weak form so that f_n is replaced by f_n^*, and discuss how this allows you to impose inflow boundary conditions for the rotating hill problem.

3) **Semi-discrete spectral element formulation and global matrices.**

In a spectral element method, the domain is partitioned into elements K, and within each element the approximate solution u_h is represented in a high-order polynomial basis. For simplicity you may assume a nodal Lagrange basis $\{\ell_j\}_{j=1}^{N_p}$ on a reference element, mapped to each physical element via an isoparametric mapping.

a) Starting from your weak form for the conservative or split formulation, derive the element-level semi-discrete system for the expansion coefficients $u_j^K(t)$:

$$M^K \frac{d\mathbf{u}^K}{dt} = \mathbf{r}^K(\mathbf{u}), \tag{14}$$

where M^K is the element mass matrix and $\mathbf{r}^K$ collects contributions from volume terms and boundary fluxes on element K.

b) Assemble the global semi-discrete system by enforcing continuity of the spectral element approximation across element interfaces (or by appropriate grid connectivity). Show that the system can be written as

$$M \frac{d\mathbf{U}}{dt} = \mathbf{R}(\mathbf{U}), \tag{15}$$

where $\mathbf{U}(t)$ is the global vector of degrees of freedom and M is the global mass matrix obtained by assembling the element mass matrices.

c) Identify and explicitly define the global matrices that appear in your formulation, e.g. mass matrix M, advection matrices in the x- and y-directions (S_x, S_y), and any coefficient-weighted matrices associated with the split form. Write down a matrix-vector expression for the right-hand side $\mathbf{R}(\mathbf{U})$ suitable for implementation in a Matlab or Julia function.

4) **Numerical experiments: rotating hill on a square or circular mesh.**

In this part you will implement the spectral element discretization and perform numerical experiments for the rotating Gaussian hill problem.

a) Construct either (i) a structured high-order spectral element mesh on a square domain that contains the circular trajectory of the hill, or (ii) an unstructured or structured spectral element mesh approximating a circular domain with optionally curvilinear boundary elements (if time permits).

b) Implement the semi-discrete system $M\, d\mathbf{U}/dt = \mathbf{R}(\mathbf{U})$ in code, and couple it with a suitable ODE solver for time integration (e.g. explicit Runge–Kutta schemes). Use the exact solution u_{exact} to prescribe inflow boundary data and to evaluate the numerical error at selected final times.

c) Perform convergence tests by refining the mesh in two different ways:

- *h-refinement:* refine the mesh by increasing the number of elements while keeping the polynomial degree N fixed;
- *p-refinement:* increase the polynomial degree N inside each element while keeping the mesh fixed.

For each refinement strategy, compute the error in a suitable norm (e.g. discrete L^2-norm) and estimate the corresponding convergence rates.

d) Compare and discuss the trade-offs between h- and p-type convergence for this problem. Comment on accuracy per degree of freedom and on computational cost (e.g. runtime or number of time steps required for stability).

e) Summarize your findings in a short written report. The report should clearly describe the continuous problem, the derived spectral element discretization, your implementation details, and the observed convergence behaviour for the rotating Gaussian hill test case.

Communicate your results in a written report. Make sure to describe and comment on important details and findings. Include sufficiently details for the reader to both reproduce and understand the reported results and conclusions.

Enjoy!

Allan P. Engsig-Karup

Deadline for this assignment is Friday, 4 Dec 2025, 14:00.
Hand in electronically at DTU Inside..