02417 Times Series Analysis - Assignment 2

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1 Stability

We are given an AR(2) process $\{Y_t\}$ with residual term ε_t arising from a stochastic process $\{\varepsilon_t\} \in \mathcal{N}(0, \sigma_\varepsilon^2 = 1)$:

$$y_t + \phi_1 y_{t-1} + \phi_2 y_{t-2} = \varepsilon_t \tag{1}$$

1.1 Stationarity

To prove stationarity we must derive the *characteristic equation* of the AR(2) process. This is done by taking the Z-transform of the AR(2) process.

We first restrict ourselves to the *homogeneous* part of the equation by removing $\varepsilon(t)$ and rewrite past operations using the backshift operator, $By_t \equiv y_{t-1}$:

$$y_t + \phi_1 B y_t + \phi_2 B^2 y_t = 0 \tag{2}$$

We consider $B = z^{-1}$ where B is understood to be defined in the time-domain of the process while z belongs to the complex frequency domain. Additionally, we factor out y_t and multiply through by z^2 to arrive at the *characteristic equation*:

$$y_t\left(z^2 + \phi_1 z + \phi_2\right) = 0 \tag{3}$$

We then solve for the roots of Eq. 3:

$$z^2 + \phi_1 z + \phi_2 = 0 \tag{4}$$

This is a quadratic equation in z and can be solved using the quadratic formula:

$$z_\pm = \frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{2} \tag{5}$$

Plugging in $\phi_1 = -0.7, \phi_2 = -0.2$ from the assignment we find:

$$z_\pm \in \{-0.218, 0.918\} \tag{6}$$

Recalling the condition on stationarity to be that all roots of the process are within the unit circle, or equivalently:

$$|z| \leq 1 \qquad \forall\, z \tag{7}$$

Which is satisfied by inspection of Eq. 6.

As such, we conclude that the process is stationary for $\phi_1 = -0.7, \phi_2 = -0.2$.

1.1.1 Additional Theory

Let us provide some theory detour into why the roots of the characteristic equation actually provide information about stability.

The background is that in signal processing as an overall field, there are a variety of domain transformations. Most prominent of all the Fourier transform, which takes a signal (real or complex) and decomposes it into a sum of its base-frequencies in the form of sine and cosine waves (for $\mathbb{C} \rightsquigarrow A_f e^{ix} = A_f(\cos x + i \sin x)$, where A_f are the Fourier coefficients). The other very prominent domain is the Laplace domain, which converts time-domain functions, like our signal, into a complex frequency domain. In our case, we consider the discrete time version of the Laplace-Transform — the Z-transform:

$$Y(z) = \sum_{t=0}^{\infty} y_t z^{-t} \quad z \in \mathbb{C} \tag{8}$$

This transform maps a signal y_t into the complex plane. Intuitively, one can think of it as a signal being squeezed into the sum of a polar-coordinate sequence.

For stability, we want to find poles in the Z-plane. Those are the points for which there is a singularity in the transfer function $Y(z)$, hence, the complex values z for which $y_t z^{-t}$ is a division by 0. Intuitively those poles can be thought of as points in the Z-plane (complex plane) where the signal (after transform) decays, provided the point is within the unit-circle in the complex plane.

Now back to our AR-process, we have $\phi(B)Y_t = \varepsilon_t$ as the shorthand expression for the polynomial in B:

$$\begin{aligned} \varepsilon_t &= y_t + \phi_1 B y_t + \phi_2 B^2 y_t \\ \varepsilon_t &= \phi(B) Y_t \end{aligned} \tag{9}$$

Now we apply the transfer function to both the noise-signal ε_t and the original signal Y_t, with a minor substitution $B = z^{-1}$; because the backshift operators' Z-transform representation is $BY_t \overset{Z-trans.}{\rightarrow} z^{-1}Y(x)$:

$$\begin{aligned} && \varepsilon_t &= \phi(B) Y_t \\ &\rightsquigarrow & E(z) &= \phi(B) Y(z) \\ &\Leftrightarrow & \frac{E(z)}{\phi(z^{-1})} &= Y(z) \end{aligned} \tag{10}$$

Hence, if we want to find the poles/singularities, we need to find the roots of

$$\begin{aligned} & \phi(B) = 1 + \phi_1 B + \dots + \phi_p B^p && = 0 \\ \Leftrightarrow \quad & \phi\left(z^{-1}\right) = 1 + \phi_1 z^{-1} + \dots + \phi_p z^{-p} && = 0 \\ \Leftrightarrow \; & z^p \phi\left(z^{-1}\right) = z^p + \phi_1 z^{p-1} + \dots + \phi_p && = 0 \\ \overset{\text{p=2}}{\rightsquigarrow} \quad & \phantom{z^p \phi\left(z^{-1}\right)} = z^2 + \phi_1 z + \phi_2 && = 0 \end{aligned} \tag{11}$$

1.2 Invertibility

Invertibility refers to the ability to represent current errors as a function of past observations, which is inherently obtained for auto-regressive processes. To realise this, we simply inspect Eq. 1 and find that the error term has already been expressed as a function of past observations, thus satisfying the definition of invertibility.

1.3 Autocorrelation

We recall the definition of *autocorrelation* for a *stationary process*:

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} \tag{12}$$

Where $\gamma(k)$ is the *autocovariance* for a timeshift k:

$$\gamma(k) = \mathrm{Cov}[y_t, y_{t+k}] \tag{13}$$

We consider the autocorrelation of an AR(2) process by solving for y_t in Eq. 1:

$$y_t = -\phi_1 y_{t-1} - \phi_2 y_{t-2} + \varepsilon_t \tag{14}$$

And then inserting this in Eq. 12:

$$\begin{aligned}
\rho(k) = \frac{\mathrm{Cov}[y_t, y_{t+k}]}{\gamma(0)} &= \frac{\mathrm{Cov}[y_t, -\phi_1 y_{t-1+k} - \phi_2 y_{t-2+k} + \varepsilon_t]}{\gamma(0)} \\
&= \frac{\mathrm{Cov}[y_t, -\phi_1 y_{t-1+k}]}{\gamma(0)} + \frac{\mathrm{Cov}[y_t, -\phi_2 y_{t-2+k}]}{\gamma(0)} + \cancel{\frac{\mathrm{Cov}[y_t, \varepsilon_t]}{\gamma(0)}} \\
&= -\phi_1 \frac{\mathrm{Cov}[y_t, y_{t-1+k}]}{\gamma(0)} - \phi_2 \frac{\mathrm{Cov}[y_t, y_{t-2+k}]}{\gamma(0)} \\
&= -\phi_1 \frac{\gamma(k-1)}{\gamma(0)} - \phi_2 \frac{\gamma(k-2)}{\gamma(0)} \\
\rho(k) &= -\phi_1 \rho(k-1) - \phi_2 \rho(k-2)
\end{aligned} \tag{15}$$

Notably by stationary it follows $\rho(-k) = \rho(k)$ and from Eq. 12 we find $\rho(0) = 1$, which allows us to build a recursive relation for $\rho(k)$ from:

$$\begin{aligned}
&\rho(0) = 1 \\
&\rho(1) = -\phi_1 \rho(0) - \phi_2 \rho(-1) = -\phi_1 - \phi_2 \rho(1) = \frac{-\phi_1}{1+\phi_2}
\end{aligned} \tag{16}$$

1.4 Autocorrelation Plot

Computing the sequence of autocorrelations with different lags $k \in \{0, 1, \ldots, 30\}$ is then trivial using Eq. 15, the outcome of which is shown in Figure 1.

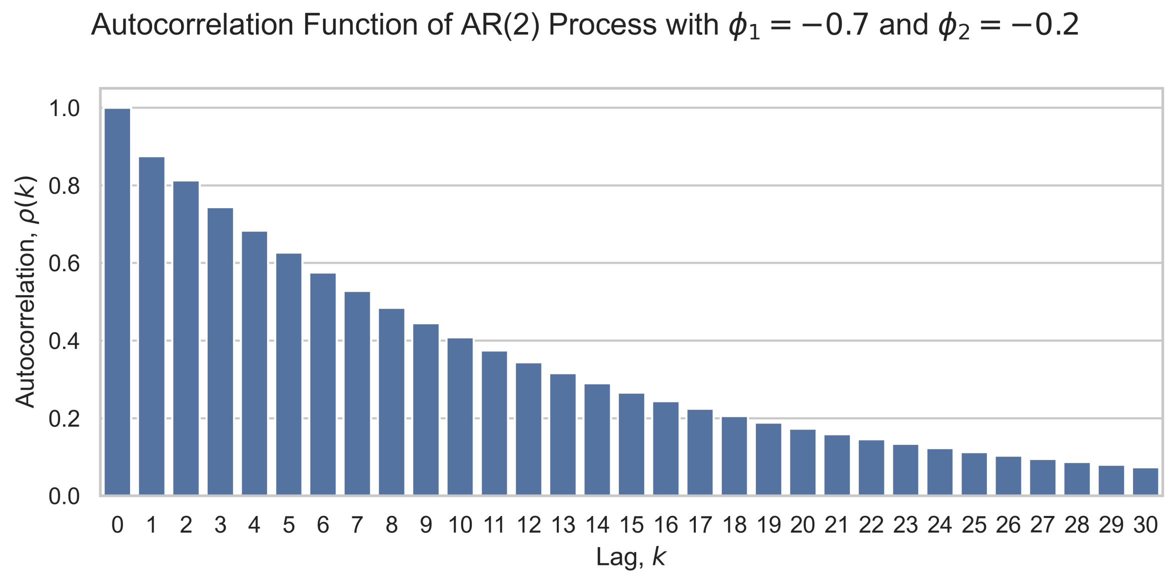

Figure 1: Autocorrelation function of AR(2) process with $\phi_1 = -0.7, \phi_2 = -0.2$. Notice exponential decay.

While the Figure 1 shows the correct autocorrelation function for the assignment description as can be trivially verified by inspection of Eq. 14, it is not beyond the realm of possibility that the assignment description is incorrect and the intended parameters were $\phi_1 = 0.7, \phi_2 = 0.2$, which we have also plotted in Figure 2.

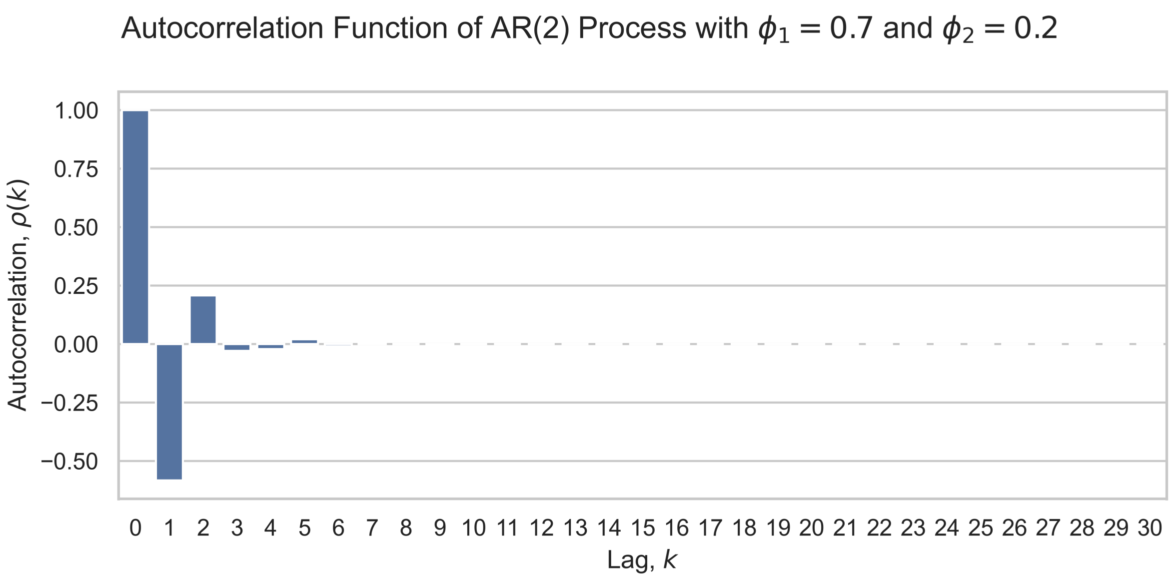

Figure 2: Autocorrelation function of AR(2) process with $\phi_1 = 0.7, \phi_2 = 0.2$. Cycling behavior and exponential decay

These two figures yield an intuition for the role of the parameters, where those that lead to a negated relation to the previous observations give rise to the cyclical behaviour found in Figure 2. This will be investigated further in the following sections.

2 Simulating seasonal processes

We are given a seasonal ARIMA model:

$$\phi(B)\Phi(B^s)\nabla^d\nabla_s^D y_t = \theta(B)\Theta(B^s)\varepsilon_t \tag{17}$$

To grasp a better understanding, it is helpful to explain the role of some variables in the model:

$$\begin{array}{rl}
s \to \text{seasonal shift} & \text{periodicity of seasonal ARIMA} \\
p \to \text{lag of time-series} & \text{for AR, polynomial order} \\
 & \phi(B)Y_t = Y_t\left(1+\phi_1 B+\phi_2 B^2 ... \phi_p B^p\right) = \varepsilon_t \\
q \to \text{lag of random noise} & \text{for MA, polynomial order} \\
 & \theta(B)\varepsilon_t = \varepsilon_t\left(1+\theta_1 B+\theta_2 B^2 ... \theta_p B^p\right) = Y_t \\
\phi \to \text{AR } (p) & \text{coeff. for the auto-regressive part} \\
\Phi \to \text{AR } (P) & \text{coeff. for the seasonal auto-regressive part} \\
\theta \to \text{MA } (q) & \text{coeff. for the moving-average part} \\
\Theta \to \text{MA } (Q) & \text{coeff. for the seasonal moving-average part} \\
 & \text{hence shift } B^s \\
\nabla^d \to \nabla^d Y_t = Y_t - Y_{t+d} & \text{difference shift of normal model} \\
\nabla_s^D \to \nabla_s^D Y_t = Y_{t+s} - Y_{t+s+d} & \text{difference shift of seasonal model}
\end{array} \tag{18}$$

In order to simulate the seasonal processes we utilise the Python library `statsmodels`, which offers an implementation of the `SARIMAX` (Seasonal AutoRegressive Integrated Moving Average with eXogenous regressors) model.

Looking at the definition of the SARIMAX implementation and comparing against Eq. 17, we confirm that the parameters are defined in a similar fashion, meaning there should not be any sign transformations needed for the parameters.

We employ initial conditions $\boldsymbol{y_t} = \boldsymbol{0}$ and avoid burn-in effects by simulating the models for $N_{\text{burn-in}} = 10000$ before simulating the process.

For all simulations, we use $n = 1000$ observations and let $\varepsilon_t \sim \mathcal{N}(0,1)$.

When computing the (partial) autocorrelation functions we use `plot_(p)acf` from `statsmodels` with $N_{\text{lags}} = 30$ and a significance level of $p = 0.05$ for the confidence intervals.

In order to understand the behaviour of the simulations, we refer to Table 6.1 in [1, p. 155], which we reproduce here:

	ACF $\rho(k)$	PACF φ_{kk}
AR(p)	Damped exponential and/or sine functions	$\varphi_{kk} = 0$ for $k > p$
MA(q)	$\rho(k) = 0$ for $k > q$	Dominated by damped exponential and/or sine functions
ARMA(p, q)	Damped exponential and/or sine functions after lag $q - p$	Dominated by damped exponential and/or sine functions after lag $p - q$s

Table 1: Reproduction of Table 6.1 from [1, p. 155] showing the expected behaviour of the autocorrelation function for different ARMA processes

2.1

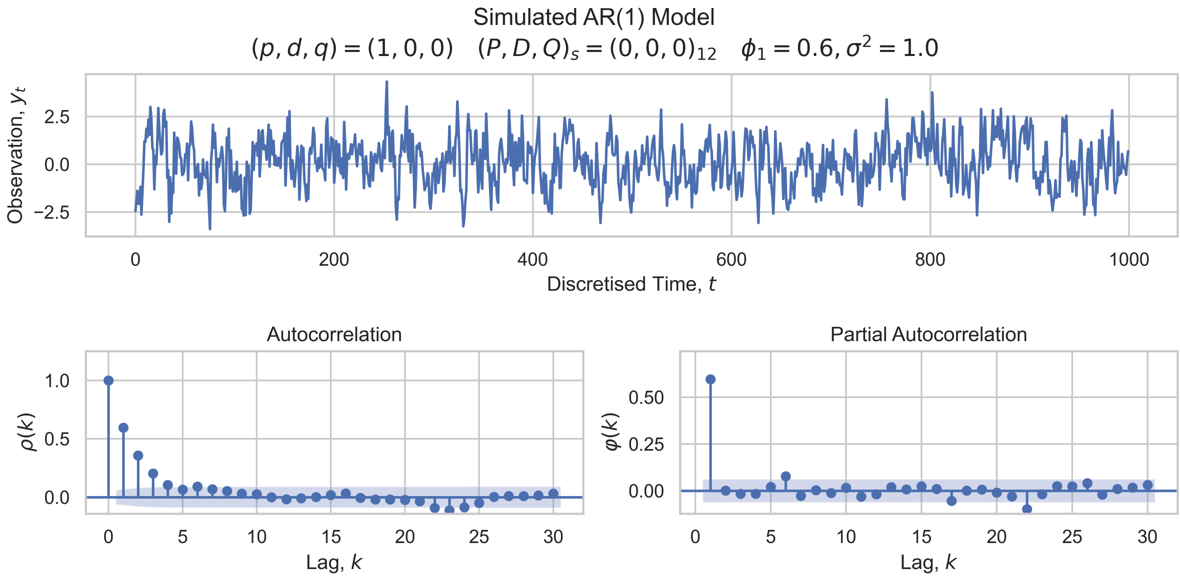

Figure 3: Simulated AR(1) process with $\phi_1 = 0.6$.

In Figure 3 we note the exponential decay of the autocorrelation function and single significant value in the partial autocorrelation function. We also find that the exponential decay of the autocorrelation function matches the expected shape from Eq. 15.

2.2

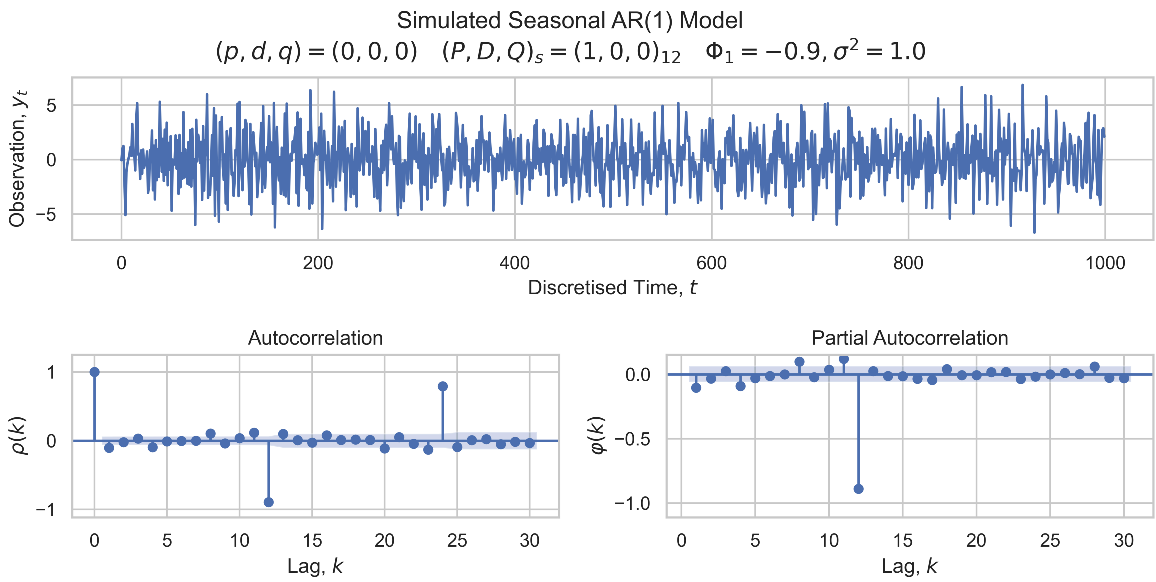

Figure 4: Simulated Seasonal AR(1) process with seasonality $s = 12$ and $\Phi_1 = -0.9$.

In Figure 4 we note that the autocorrelation function is periodic with period $s = 12$ while still decaying exponentially. Because of the negative sign of Φ_1 we find that the value of $\rho(k)$ cycles around 0.

Additionally, we find that the significant point in the partial autocorrelation function has shifted to $k = 12$. Additional points sticking out of the confidence interval are assumed to be statistical noise as would be expected with a confidence level $p = 0.05$.

2.3

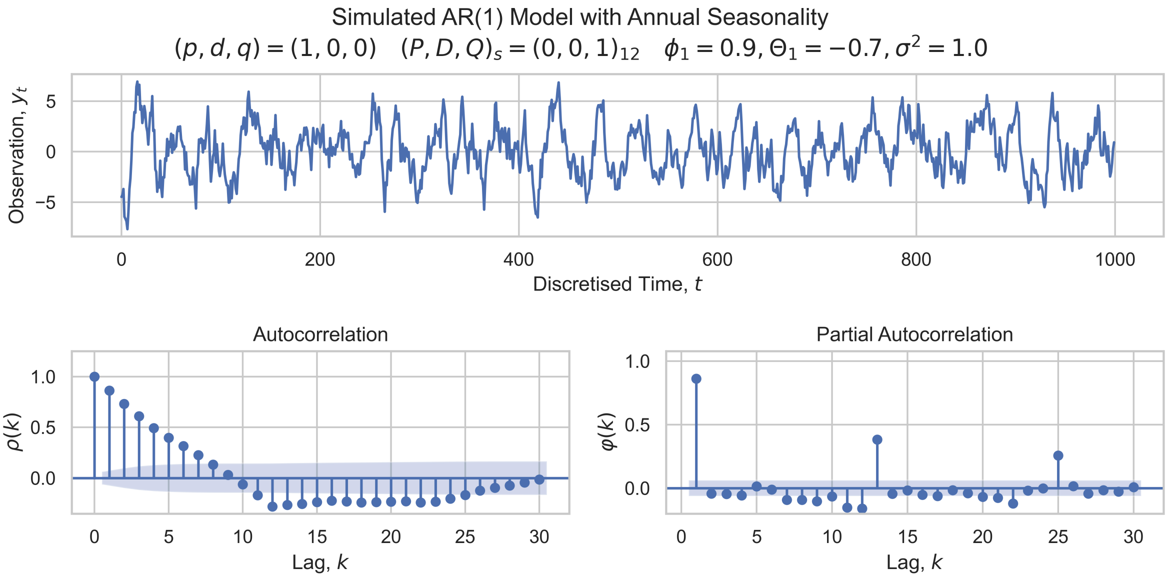

Figure 5: Simulated Seasonal ARMA process with seasonality $s = 12$ and parameters $\phi_1 = 0.9$ and $\Theta_1 = -0.7$.

Referring to Figure 5, we note that the AR part of the process contributes to a single significant value in the partial autocorrelation function at $k = 1$ that is then convolved with a decaying 'Dirac brush' with periodicity $s = 12$.

The autocorrelation function can be observed to decay away as expected for the AR(1) part of the model, but interestingly also exhibits some periodicity with period $s = 12$ arising from the MA(1) seasonal component.

2.4

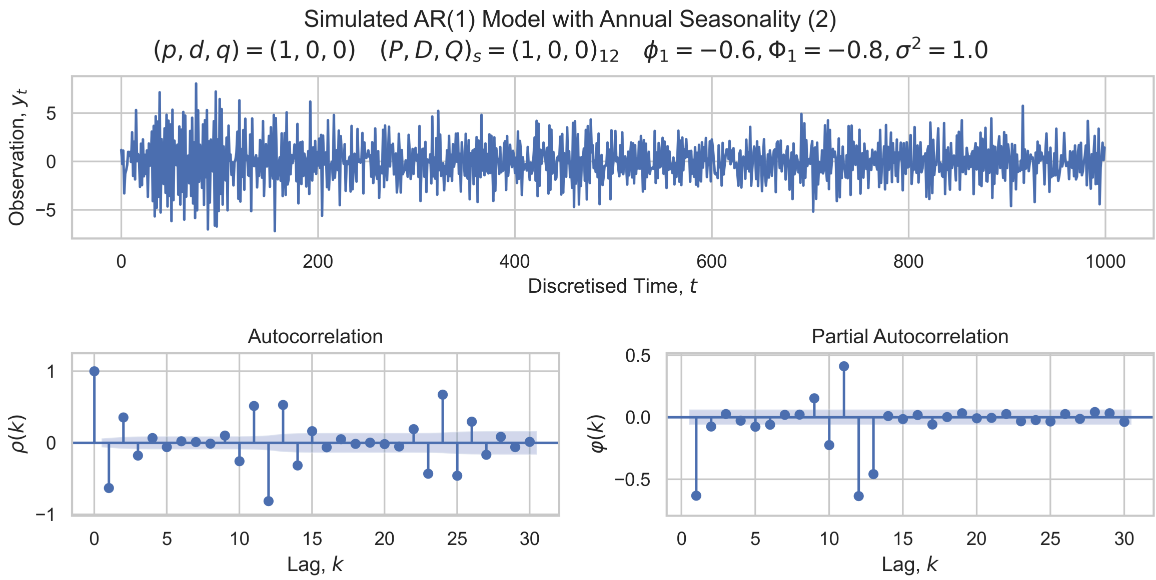

Figure 6: Simulated Seasonal AR process with seasonality $s = 12$ and parameters $\phi_1 = -0.6$ and $\Phi_1 = -0.8$.

The picture emerging from Figure 6 gets more muddied for this more complex model, though using Table 1 we would expect the autocorrelation function to have a exponentially decaying sine envelope that is repeated every $s = 12$ observations, also decaying for each repetition.

For the partial autocorrelation we would expect to see a single significant value at $k = 1$ and then additional significant values around $k = 12$. Due to the interaction between the regular and seasonal AR(1) processes, we would expect there to be several significant values, though the exact number is difficult to deduce using intuition alone.

2.5

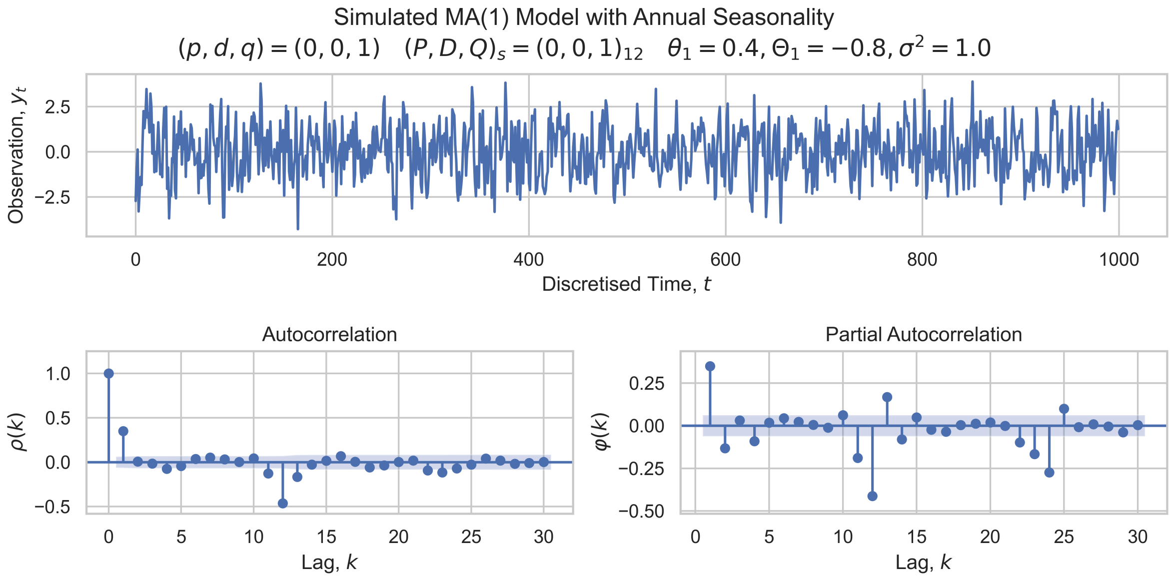

Figure 7: Simulated Seasonal MA model with seasonality $s = 12$ and parameters $\theta_1 = 0.4, \Theta_1 = -0.8$.

From Table 1 we find that we would expect an exponential decay of the partial autocorrelation function for an MA(q) process with a limited number of significant values in the autocorrelation function, which is also reflected by Figure 7. With moderate difficulty we observe a single significant value in the initial part of the autocorrelation function as we would expect from the regular MA(1) process. This is then repeat once around $k = 12$ by the seasonal MA(1) process. We note that both the $k = 0$ and $k = 1$ values from the regular MA(1) process are convolved with the single expected peak from the seasonal MA(1) process to produce 3 significant values at $k \in \{11, 12, 13\}$.

The parital autocorrelation function is more difficult to interpret, but here we would expect a decaying exponential envelope, potentially over a sine function. While the the *inner* exponential function understood to arise from the regular MA(1) process with $\theta_1 = 0.4$ decays very quickly as observed around $k = 1$, we observe the expected periodicity of $s = 12$ effectuated by the seasonal MA(1) component.

2.6

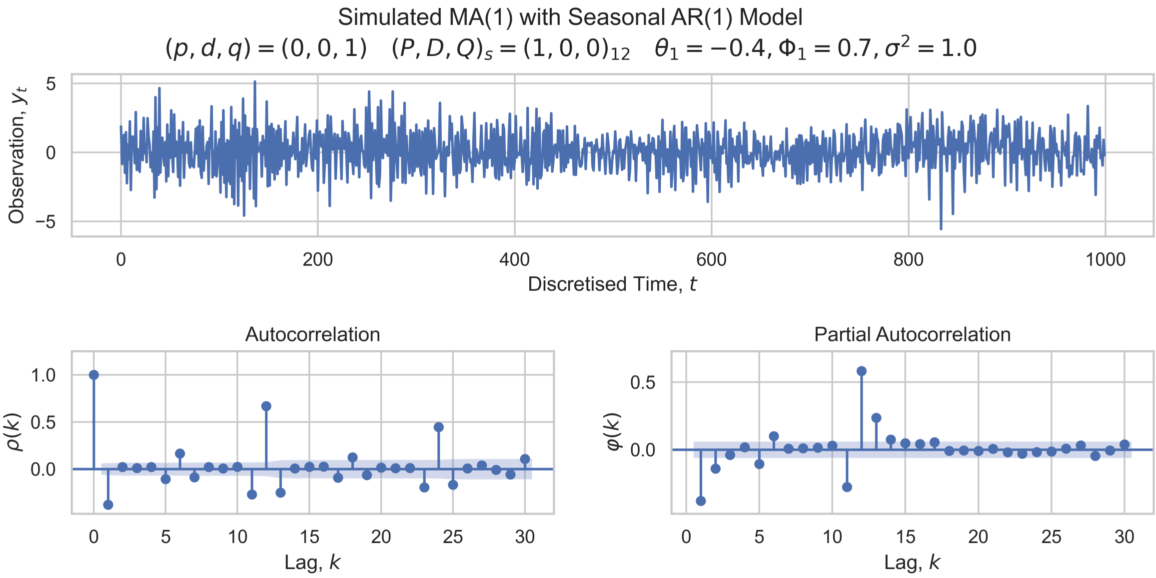

Figure 8: Simulated Seasonal ARMA model with seasonality $s = 12$ and parameters $\theta_1 = -0.4, \Phi_1 = 0.7$.

Lastly, in Figure 8 we observe an exponential decay in the partial autocorrelation function which we attribute to the regular MA(1) progress. We also observe it repeated at $k = 12$, though not at $k = 24$, as would be expected by the AR(1) seasonal component.

In the autocorrelation function we find a decaying exponential envelope over a 3-element sequence of alternating signs repeated at $k = 12$ and $k = 24$, which again is consistent with the rules outlined in Table 1.

2.7 Summary of Identifications

We have perhaps cheated a bit by using the rules from Table 1 to identify the processes, as the conclusions outlined in Table 1 should have been deduced and presented here.

However, we can add additional commentary - for instance, we realise that more complex processes would be very difficult to identify simply by inspection of the autocorrelation and partial autocorrelation functions. Instead, we propose that a model is iteratively built in order to identify the parameters of such processes. Here a single AR(1) or MA(1) model may be extended appropriately by first fitting the model to a realisation of the process and then inspecting the residuals for any remaining autocorrelation. A new simple model may be fit on the residuals and combined with the original model to construct a better fitting model, which may then again be improved iteratively by inspection of the residuals.

The seasonality parameter will generally be relatively easy to deduce from autocorrelations, especially if the process only features a single periodicity with little variance in the periods.

Overall it is probably best to use parameter estimation techniques. In a real world scenario, one would likely have the time-series as actual data and not just as a plotted graph. There are two main ways to estimate especially ϕ and θ for ARMA models:

1. OLS regression for AR models $\hat{\boldsymbol{\phi}} = \left(X^T X\right)^{-1} X^T \mathbf{y_{t-k}}$ with a feature matrix X constructed from the $t - k$ to the $t - N$ th samples, with order of the AR model k
2. MLE for MA models to estimate $\hat{\boldsymbol{\theta}}$ based on a likelihood function that assumes Gaussian $\boldsymbol{\varepsilon}_t$

Since none of the given model display any trends, but rather are all stationary, a de-trending via differencing on real world data may be appropriate. Otherwise combining a standard trend model like OLS, RLS or WLS as a prior model, could be useful.

3 Identifying ARMA(p,q) Models

Now, using the rules established and tested in Section 2, we are able to infer the order of the 3 processes presented in the assignment description.

Additionally, we use the same function used to generate the realisations in Section 2 to inspect a realisation of the proposed ARMA structure for each of the 3 given processes.

We note that we are informed that the processes are ARMA(p,q) models and as such not expected to feature any seasonal or differencing components.

3.1 Process 1

In the given plots of the first process, we observe no significant values in the autocorrelation nor partial autocorrelation function. As such, we conclude that the process is an ARMA(0,0) process, which is equivalent to a white noise process. From the timeseries plot, we find that the variance is likely unitary. Recreating a realisation of such a process, we obtain Figure 9.

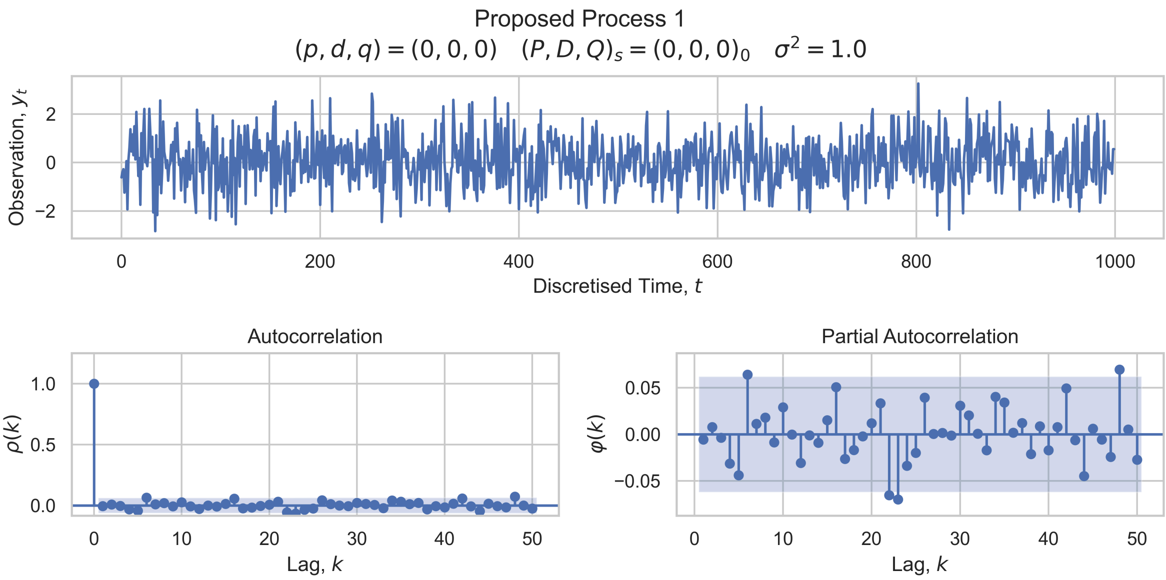

Figure 9: Simulated ARMA(0,0) process with $\varepsilon_t \sim \mathcal{N}(0, 1)$

3.2 Process 2

For the second process we find two significant values in the partial autocorrelation function and a double-exponential decay in the autocorrelation function, which we immediately identify as an ARMA(2,0) process in accordance with Table 1.

A realisation of such a process is shown in Figure 10.

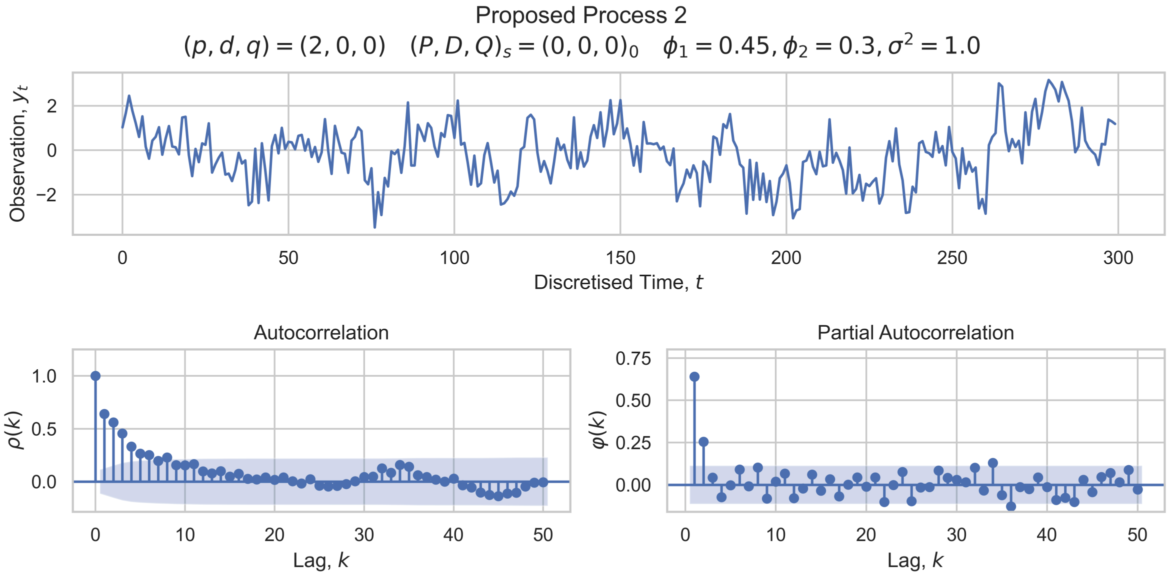

Figure 10: Simulated ARMA(2,0) process with $\phi_1 = 0.45, \phi_2 = 0.3$

3.3 Process 3

The third process is rather more difficult to identify. Squinting slightly, we find the autocorrelation function to appear as if it is decaying exponentially with two distinct characteristic decay constants, which would suggest an AR order of $p = 2$. Finding the envelope of the absolute partial autocorrelation function to resemble a decaying exponential, we find that the MA order may be $q = 1$.

The cyclical nature of the partial autocorrelation function suggests that one of the autoregressive parameters must be negative.

Fiddling a bit with the magnitude of the parameters, we obtain Figure 11, which matches the the plot of the third process in the assignment description well.

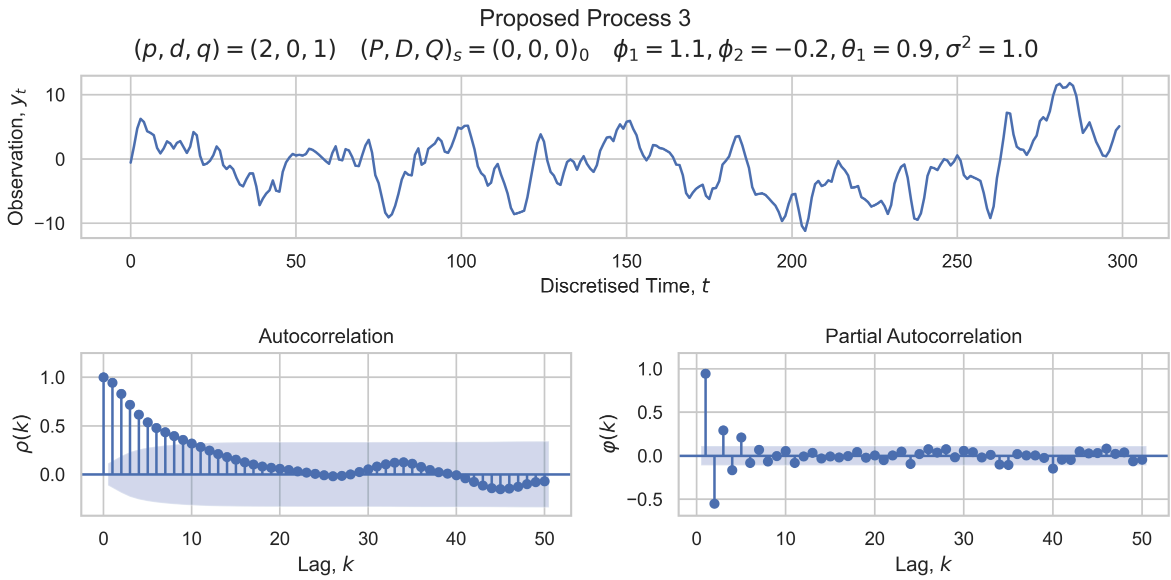

Figure 11: Simulated ARMA(2,1) process with $\phi_1 = 1.1, \phi_2 = -0.2, \theta_1 = 0.9$

3.3.1 Commentary:

In addition, it would be a good idea to start some residual analysis on the given models. As mentionend in Ex 2.7: in a real world scenario, we would have a time-series as actual data.

The main difficulty is the estimate the order of models, so the parameters (p, d, q). From there, building a model can be started done by fitting and identifying parameters numerically. Ideally, we can start with models of lower order, thus lower complexity and then analyse the residuals between model and data. As metrics, we suggest AIC and BIC, as those also penalize complexity of a model, which prevents overfitting. This will be especially useful when combining with regression methods (OLS, WLS, RLS).

Furthermore, the given chart indicates some notion of auto-regressive seasonality. Yet, the period of seasonality would be too long to actually fit a model of reasonable complexity (as the ACF and PACF do not allow estimations for such long periods; we are talking about 100 periods towards the end of the series). Therefore, we stick to the simpler ARMA(2,0,1) model. As it does fit the model well.

Bibliography

[1] H. Madsen, *Time Series analysis*. Chapman & Hall/CRC, 2008.