- **Proposition 3** If q^2 is divisible by 3, so is q.

Proof by contraposition. We will prove the contrapositive; $i.e.$, we will prove if q is not divisible by 3, then q^2 is not divisible by 3.

By *Proposition 2*, we know that if q is not divisible by 3, then $q^2 \ mod\ 3 = 1$.

Thus q^2 is not divisible by 3. $\square$

- **Proposition 4** Prove $\sqrt{3}$ is irrational.

Proof by contradiction. Suppose not; $i.e.$, suppose $\sqrt{3} \in \mathbb{Q}$. Then $\exists m, n \in \mathbb{Z}$ with m and n relatively prime and $\sqrt{3} = \frac{m}{n}$. Then $3 = \frac{m^2}{n^2}$.

Thus m^2 is divisible by 3 so by *Proposition 3*, m is also. By definition, $m = 3k$ for some $k \in \mathbb{Z}$. Hence $m^2 = 9k^2 = 3n^2$ and so $3k^2 = n^2$. Thus n^2 is divisible by 3 and again by *Proposition 3*, n is also divisible by 3. But m, n are relatively prime, a contradiction.

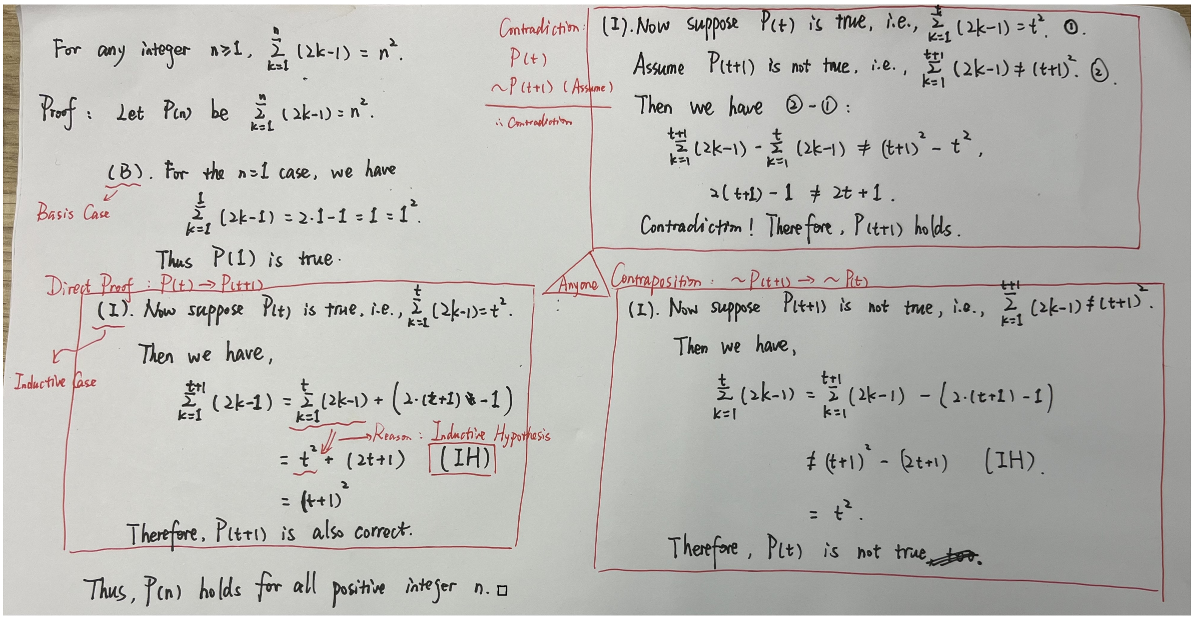
Thus $\sqrt{3} \in \mathbb{Q}$. $\square$

> Actually, if we ignore the *coprime* characteristics, we can reason that m and n should have an infinite number of factors of 3. However, since m and n are **finite** (even though they can be any arbitrary positive integers), this also leads to a contradiction. (Nevertheless, this explanation is clearly not as concise as the one involving coprimeness.)

- **Induction**

I use an example to demonstrate how to prove a proposition through mathematical induction.

In the induction cases, I explained how to use direct proof, contraposition, and contradiction. (In fact, for most problems, direct proof is often feasible and straightforward.) Please choose the specific method according to the requirements of the problem.



For any integer $n \geq 1$, $\sum_{k=1}^{n} (2k-1) = n^2$.

Proof: Let $P(n)$ be $\sum_{k=1}^{n} (2k-1) = n^2$.

(B). For the $n=1$ case, we have

Basis Case

$$\sum_{k=1}^{1} (2k-1) = 2 \cdot 1 - 1 = 1 = 1^2.$$

Thus $P(1)$ is true.

Direct Proof : $P(t) \Rightarrow P(t+1)$

(I). Now suppose $P(t)$ is true, i.e., $\sum_{k=1}^{t} (2k-1) = t^2$.

Inductive Case

Then we have,

$$\sum_{k=1}^{t+1} (2k-1) = \sum_{k=1}^{t} (2k-1) + (2 \cdot (t+1) - 1)$$

Reason : Inductive Hypothesis

$$= t^2 + (2t+1) \quad \boxed{(IH)}$$

$$= (t+1)^2$$

Therefore, $P(t+1)$ is also correct.

Thus, $P(n)$ holds for all positive integer n. $\square$

Contradiction: $P(t)$, $\sim P(t+1)$ (Assume) $\therefore$ Contradiction

(I). Now suppose $P(t)$ is true, i.e., $\sum_{k=1}^{t} (2k-1) = t^2$. ①.

Assume $P(t+1)$ is not true, i.e., $\sum_{k=1}^{t+1} (2k-1) \neq (t+1)^2$. ②.

Then we have ② - ① :

$$\sum_{k=1}^{t+1} (2k-1) - \sum_{k=1}^{t} (2k-1) \neq (t+1)^2 - t^2,$$

$$2(t+1) - 1 \neq 2t + 1.$$

Contradiction! Therefore, $P(t+1)$ holds.

Anyone

Contraposition : $\sim P(t+1) \rightarrow \sim P(t)$

(I). Now suppose $P(t+1)$ is not true, i.e., $\sum_{k=1}^{t+1} (2k-1) \neq (t+1)^2$.

Then we have,

$$\sum_{k=1}^{t} (2k-1) = \sum_{k=1}^{t+1} (2k-1) - (2 \cdot (t+1) - 1)$$

$$\neq (t+1)^2 - (2t+1) \quad (IH).$$

$$= t^2.$$

Therefore, $P(t)$ is not true, ~~too~~.