Jiapeng Wang, COMP9020 24T2, Additional Tut W11

Exercises

Exercise 1 In mathematics, the **Fibonacci sequence** is a sequence in which each number is the sum of the two preceding ones. Numbers that are part of the Fibonacci sequence are known as Fibonacci numbers, commonly denoted F_n ($n \geq 0$). The sequence commonly starts from 0 and 1.

1. Define the Fibonacci numbers F_n by the recurrence relation.
2. Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$.

Exercise 2 A 4-letter word is selected at random from Σ^4, where $\Sigma = \{a, b, c, d, e\}$.

1. What is the probability that the letters in the word are distinct?
2. What is the probability that there are no vowels in the word?
3. What is the probability that the word begins with a vowel?

The Well-formed formula (*wff*) of Propositional Logic can be inductively defined as follows:

- Each propositional variable is, on its own, a formula.
- If φ is a formula, then $\sim \varphi$ is a formula.
- If φ and ψ are formulas, and $\bullet$ is any binary connective, then $(\varphi \bullet \psi)$ is a formula. Here $\bullet$ could be the usual operators $\vee$, $\wedge$, $\rightarrow$, or $\leftrightarrow$.

Exercise 3 Let p, q, r be propositional variable. Is the following sequences of symbols a *wff*? Why?

1. $(((p \rightarrow q) \wedge (r \rightarrow s)) \vee (\sim q \wedge \sim s))$.
2. $((p \rightarrow q) \rightarrow ((qq))p)$.

Exercise 4.1 Is the inverse of a bijective function also a function? Explain your idea.

Exercise 4.2 Let the composition $g \circ f$ be a bijection. Do you think $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$? Prove or give a counterexample.

Exercise 5 Let propositions P_n be defined for integers n. Let a be some integer in particular. Now assume that the following two claims are true:

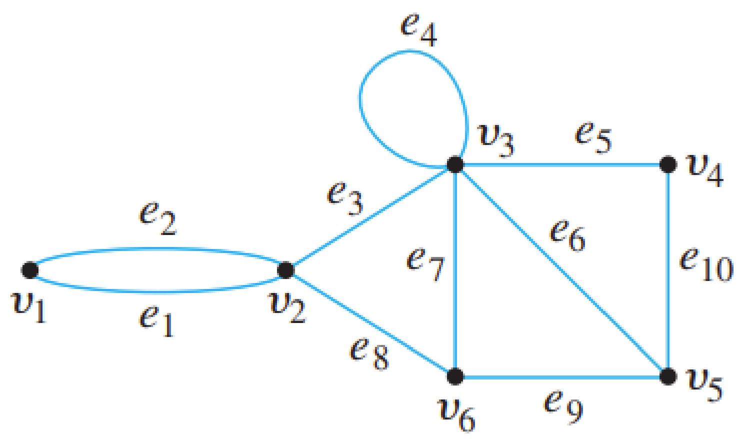
1. P_a is true.
2. For all integers $k \geq a$, if
    1. $P(k)$ is true then $P(k+2)$ is true, and
    2. if $P(k+1)$ is true then $P(k+3)$ is true.

In this case, is P_n also true for all integers $n \geq a$? Explain your idea.

Exercise 6 In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$ b. $e_1e_3e_5e_5e_6$ c. $v_2v_3v_4v_5v_3v_2$

d. $v_2v_3v_4v_5v_6v_2$ e. $v_1e_1v_2e_1v_1$ f. v_1



Exercise 7 Let P be the set of all people who have ever lived and define a relation R on P as follows:

- For every $r, s \in P$, $r \; R \; s \iff r$ is an ancestor of s or $r = s$.

Is R a partial order relation? Prove or give a counterexample.

Exercise 8 Assuming the following propositions are true:

1. Every integer greater than 1 is either a prime number or can be factored into a product of prime numbers.
2. If a number is a product of a prime number plus 1 (i.e., $n = kp + 1$), then this number does not have this prime number as a factor.

Prove that there are infinitely many prime numbers.

Comprehensive Exercise - Modulo

Let m and n be integers and let d be a positive integer.

We say that m is congruent to n **modulo** d $\left(\text{and write } m \equiv n \pmod d\right) \iff m = n + kd$ for some integer k.

(Please use this statement as the definition of *modulo* in this part.)

Exercise 1 Prove that

- if $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $ac \equiv bd \pmod n$.

Exercise 2 Prove that

- if $a \equiv b \pmod n$, then for any positive integer k, $a^k \equiv b^k \pmod n$.

Let d be a positive integer. Define a relation $\mathcal{M}_d$ from $\mathbb{N}$ to $\mathbb{N}$ as follows:

- For all integers m and n, $m\,\mathcal{M}_d\,n \iff m \equiv n \pmod d$.

Exercise 3.1 Is $\mathcal{M}_d$ an equivalence relation?

Exercise 3.2 Is $\mathcal{M}_d$ a partial order relation?

Let d be a positive integer. Let D be the set of integer numbers from 1 to d, i.e., $D = \{0, 1, 2, \ldots, d-1\}$.

Define a function $f : \mathbb{N} \to D$ by the rules $f(n) = n \bmod d$.

Exercise 4 Is $f(n) \equiv n \pmod d$?

Exercise 5.1 Is f injective?

Exercise 5.2 Is f surjective?

Let the domain be the set of integer numbers from 1 to 100, i.e., $\mathbb{D} = \{1, 2, \ldots, 100\}$.

Let P, Q be the extensions of predicates in $\mathbb{D}$ which are $n \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 3$ respectively.

Let T be the extension of predicate in $\mathbb{D}$ which are $n \equiv 0 \pmod 6$.

Exercise 6.1 How many elements in P?

Exercise 6.2 How many elements in Q?

Exercise 6.3 How many elements neither in P nor Q?

Exercise 6.4 Does $T = P \cap Q$? Prove or or give a counterexample.