Jiapeng Wang, COMP9020 Tutorial W5

The First Mathematics Crisis: Take $\sqrt{3}$ as an example

1. Let the domain be the domain of integers $\mathbb{Z}$, and let Ex be x is an even number. Prove the following statement:
   $\forall x \ (Ex \to E(x^2))$.
2. If q is not divisible by 3, then $q^2 \bmod 3 = 1$.

> From Proposition 1 and 2, can you find any patterns? Is the following proposition also true?
>
> - If q is not divisible by 5, then $q^2 \bmod 5 = 1$.

3. If q^2 is divisible by 3, so is q.
4. Prove that $\sqrt{3}$ is irrational.

Recap

> If $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
> It is not true that $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
> $\therefore$ ___.

Common mistakes in Problem Set 1

Question 1-3

1. Partial translation.
   - There are no restrictions on the scope of 'everyone'/'someone'/'all students' in your answer.
2. Wrong answer.
   - Please distinguish between these two wffs: ExPx-->Pa; Ex(Px-->Pa)

> 1. Translate the following into fully symbolic predicate logic - "Either Alice is hungry or everyone is hungry".
> 2. Translate the following into fully symbolic predicate logic - "If someone is tall then Alice is tall".
> 3. Translate the following into fully symbolic predicate logic - "If all students are studying hard, then everyone is happy".
> 4. Is the following *wff* a logical truth? If so, then justify your answer in your own words. If not, then provide a countermodel:
>   - $\exists x Px \leftrightarrow (\exists x Px \to \exists x Qx)$
> 5. Is the following argument valid? If so, justify your answer in your own words. If not, then provide a countermodel:
>
>   1. $\forall x(Px \to Qx)$
>   2. $\sim Qa$
>
> $\therefore \sim Pa$

Question 4

1. Correct answer but no countermodel provided.

- Please note that merely assuming the truth value of a condition is not a correct proof. You need to construct an actual model.

2. Please note that a countermodel should include 'the domain', 'the extension', and 'the reference'. When providing a countermodel, you are expected to mention these elements.

Question 5

1. Please indicate the corresponding law or rule at each step of your derivation.

Mathematical Induction

Definition

Sequence

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

We typically represent a sequence as a set of elements written in a row. In the sequence denoted $a_m, a_{m+1}, a_{m+2}, \cdots, a_n$.

- Each individual element a_k (read “a sub k”) is called a **term**.
- The k in a_k is called a **subscript** or **index**,
- m (which may be any integer) is the subscript of the **initial term**,
- n (which must be an integer that is greater than or equal to m) is the subscript of the **final term**.
- The notation $a_m, a_{m+1}, a_{m+2}, \cdots$ denotes an **infinite sequence**.
- An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

Summation and Product Notation

If m and n are integers and $m \leq n$, the symbol $\sum\limits_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \cdots, a_n$.

- We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write $\sum\limits_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$.
- We call k the **index** of the summation,
- m the **lower limit** of the summation,
- n the **upper limit** of the summation.

If m and n are integers and $m \leq n$, the symbol $\prod\limits_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \cdots, a_n$.

- We write $\prod\limits_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$.

Some Properties

If $a_m, a_{m+1}, a_{m+2}, \cdots$ and $b_m, b_{m+1}, b_{m+2}, \cdots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum\limits_{k=m}^{n} a_k + \sum\limits_{k=m}^{n} b_k = \sum\limits_{k=m}^{n} (a_k + b_k)$

2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$
3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Factorial and “n Choose r” Notation

For each positive integer n, the quantity **n factorial** denoted $n!$, is defined to be the product of all the integers from 1 to n: $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$.

- **Zero factorial**, denoted $0!$, is defined to be 1: $0! = 1$.

Let n and r be integers with $0 \leq r \leq n$. The symbol $\binom{n}{r}$ is read “**n choose r**” and represents the number of subsets of size r that can be chosen from a set with n elements.

- We have $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Mathematical Induction

Principle

Let Pn be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. Pa is true.
2. For every integer $k \geq a$, if Pk is true then $P(k+1)$ is true.

Then the statement $\text{for every integer } n \geq a, Pn$ is true.

Methodology

Method of Proof by Mathematical Induction

Consider a statement of the form, “For every integer $n \geq a$, a property $P(n)$ is true.” To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis**.*]*

Then

show that $P(k+1)$ is true.

Example

1. For every integer $n \geq 1$, $\sum_{k=1}^{n}(2k-1) = \sum_{k=0}^{n-1}(2k+1) = n^2$.
2. For every integer $n \geq 1$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
3. For every integer $n \geq 0$ and every real number $r \neq 1$, $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$.
4. For any integer n $n \geq 1$, $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

Recursion

Principle

A **recurrence relation** for a sequence $a_0, a_1, a_2, \cdots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \cdots, a_{k-i}$, where i is an integer with $k - i \geq 0$.

The **initial conditions** depend on the properties of i:

> 1. If i is a fixed integer, the initial conditions for such a recurrence relation specify the values of $a_0, a_1, a_2, \cdots, a_{i-1}$.
> 2. If i depends on k, the initial conditions specify the values of $a_0, a_1, \cdots, a_m$, where m is an integer with $m \geq 0$.

Example

(**Catalan Numbers**) For each integer $n \geq 1$, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

1. Find C_1, C_2, and C_3.
2. Show that this sequence satisfies the recurrence relation $C_k = \frac{4k-2}{k+1}C_{k-1}$ for every integer $k \geq 2$.

Fibonacci Sequence

In mathematics, the **Fibonacci sequence** is a sequence in which each number is the sum of the two preceding ones. Numbers that are part of the Fibonacci sequence are known as Fibonacci numbers, commonly denoted F_n ($n \geq 0$). The sequence commonly starts from 0 and 1.

1. Define the Fibonacci numbers F_n by the recurrence relation.
2. Find F_5, F_6, and F_7.
3. *Prove that $F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$, where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$.
4. Prove the following properties of Fibonacci sequence:
   - $\sum\limits_{i=1}^{n} F_i = F_{n+2} - 1$.
   - $\sum\limits_{i=1}^{n} F_i^2 = F_n F_{n+1}$.
   - $\sum\limits_{i=1}^{n} F_{2i-1} = F_{2n}$.
   - $\sum\limits_{i=1}^{n} F_{2i} = F_{2n+1} - 1$.
   - $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$.

     > $F_{2n+1} = F_{n+1}^2 + F_n^2$.
   - $F_{n+1} F_{n-1} = F_n^2 + (-1)^n$.
5. *Prove the following properties of Fibonacci sequence:
   - $F_n = \sum\limits_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$.
   - $(F_n F_{n+3})^2 + (2F_{n+1} F_{n+2})^2 = F_{2n+3}^2$.

     > $2F_{n+1} F_{n+2} = F_{n-1} F_{n+2} + 2F_n F_{n+1} + F_{2n+1}$.

- $gcd(F_n, F_{n-1}) = 1$.
- $gcd(F_n, F_m) = F_{gcd(n,m)}$.

> $n \mid m \Leftrightarrow F_n \mid F_m$.