Jiapeng Wang, COMP9020 Tutorial W9

Common mistakes in Problem Set 5

> 1. When Epp uses the term 'universal set', she is referring to something to which we have been referring throughout COMP9020 with a **different** term. What is this different term? Hint - if you can see what Epp actually means by her use of 'universal set', then this should be obvious.
> 2. Explain the relationship between **deductive validity** and the **subset relation**.
> 3. Consider the following two sets, A and B:
>
> $$A = \{m \in \mathbb{Z} : m = 3r - 2 \text{ for some } r \in \mathbb{Z}\}$$
>
> $$B = \{n \in \mathbb{Z} : n = 2s - 2 \text{ for some } s \in \mathbb{Z}\}$$
>
> Is it the case that $B \subseteq A$? If yes, provide a proof. If no, then provide a counterexample with a detailed justification.
>
> 4. Take the "universal set" U to be:
>
> $$\{1, 2, 3, 4, 5, 6\}$$
>
> Consider the following two sets, $P \subseteq U$ and $Q \subseteq U$:
>
> $$P = \{1, 2, 3, 4\}$$
>
> $$Q = \{3, 4, 5, 6\}$$
>
> Let it be the case that $P \cap Q \subseteq O$ where $O \nsubseteq P \cap Q$ but $O \subseteq Q$. Find O. Explain your reasoning.
>
> 5. Is it possible to recover the truth-function f_3^2 from the truth-functions f_1^1 and f_2^2? If so, then show how. If not, then explain why not/provide a counterexample.

Question 2

- Your answer generally reflects your correct understanding, but it is not comprehensive.

> In general, both are partial orders. In particular, the subset relation tracks satisfaction in models for valid arguments.

Question 4

- Partial correct. *Find O* means find all the cases of set O!

Question 5

- What do f_1^1, f_2^2, and f_3^2 mean?

Relations

Property and Relation

One-place predicates are **properties**
n-place predicates (where *n* > 1) are **relations**

- 2-place relations (binary relation) include **next to** and **greater than** etc.
- 3-place relations (ternary relation) include...?

Excitingly, **relations** can possess **properties**!

- Example: **next to** is **symmetric** but **non-transitive**.
- Example: **greater than** is **transitive** but **non-symmetric**.

We write **a** bears relation **R** to **b** as **aRb**
Three place relations are written **Rabc** but let us not speak of these...

Exercise 1 Define a relation L from $\mathbb{R}$ to $\mathbb{R}$ as follows: For all real numbers x and y,

$x\,L\,y \iff x < y$

a. Is $57\,L\,53$? b. Is $(-17)\,L\,(-14)$? c. Is $143\,L\,143$? d. Is $(-35)\,L\,1$?

Directed Graph of a Relation

A **relation on a set** A is a relation from A to A.

When a relation R is defined on a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph**. Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to each related point. As with an ordinary arrow diagram,

> For all points x and y in A,
>
> - there is an arrow from x to $y \iff x\,R\,y \iff (x, y) \in R$.

If a point is related to itself, a *loop* is drawn that extends out from the point and goes back to it.

Exercise 2 Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A$,

$x\,R\,y \iff 2 \mid (x - y)$.

Draw the directed graph of R.

Properties of Relations

Reflexive, Symmetric and Transitive

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x\,R\,x$.
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

> What do *non-reflexive*, *non-symmetric*, and *non-transitive* means?

Exercise 3.1 Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$,
$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$,
$T = \{(0, 1), (2, 3)\}$.

a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

c. Is T reflexive? symmetric? transitive?

Exercise 3.2 Define a relations E, L on $\mathbb{R}$ as follows: For all real numbers x and y,

$x\,E\,y \iff x = y$.
$x\,L\,y \iff x < y$.

a. Is E reflexive? symmetric? transitive?

b. Is L reflexive? symmetric? transitive?

Antireflexive and Antisymmetric

1. R is **antireflexive** if, and only if, for every $x \in S$: $(x, x) \notin R$.
2. R is **antisymmetric** if, and only if, for every $x, y \in S$: If (x, y) and $(y, x) \in R$ then $x = y$.

> Are *non-reflexive* and *non-symmetric* the same as *antireflexive* and *antisymmetric*?

Exercise 3.3 Are the relations E and L defined in *Exercise 3.2* antireflexive? antisymmetric?

Equivalence and Partial Order

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

Exercise 4.1 Define a relations L_e on $\mathbb{R}$ as follows: For all real numbers x and y,

$x \, L_e \, y \iff x \leq y.$

Let the relations E and L be the same definations as those in *Exercise 3.2*.

a. Is E an equivalence relation? a partial order relation?

b. Is L an equivalence relation? a partial order relation?

c. Is L_e an equivalence relation? a partial order relation?

Notation for partial order: Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Exercise 4.2 Show that the *subset* relation is a partial order.

Partition and Class

Partition

A **partition** in a set A is a (finite or infinite) collection of subsets of A such that these subsets are **nonempty**, **mutually exclusive**, and **jointly exhaustive**.

Example: Let $A = \{1, 2, 3\}$, and Let $A_1 = \{1, 2\}$ and $A_2 = \{3\}$

A_1 and A_2 form a **partition** on A, since $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$.

Given a partition on A, the **relation induced by the partition**, R, is defined in A as:

$\forall x \forall y : x \in A \wedge y \in A, \, xRy \iff$ there is a subset A_i of the partition s.t. $x \in A_i$ and $y \in A_i$.

Equivalence Class

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of** a, denoted $[a]$ and called the **class of** a for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$[a] = \{x \in A \mid x\, R\, a\}$

The procedural version of this definition is

$\text{for every } x \in A,\ x \in [a] \iff x\, R\, a.$

Counting Mathods

Basic Counting Rules

Principles

- Union rule ("or"): If S and T are disjoint |S ∪ T| = |S| + |T|
- Product rule ("followed by"): |S × T| = |S| · |T|

Common strategies

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g. |S| + |T| = |S ∪ T| + |S ∩ T|)
- Find a bijection to a set that can be counted

Exercise 5.1 Suppose there are k elements in set A. How many elements in $\mathcal{P}(A)$?

Exercise 5.2 How many different Boolean functions with k arguments (i.e., $f : \{0, 1\}^k \to \{0, 1\}$)?

Permutation and Combination

A **permutation** of a set of objects is an *ordering* of the objects in a row.

An **r-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.

- $P(n, r) = \frac{n!}{(n-r)!}$.

Exercise 6.1

a. How many different ways can three of the letters of the word BYTES be chosen and written in a row?

b. How many different ways can this be done if the first letter must be B?

A **combination** of a set of objects is an *disordering* of the objects in a row.

Let n and r be nonnegative integers with $r \le n$. An r-combination of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$, read "n choose r," denotes the number of subsets of size r (or r-combinations) that can be formed from a set of n elements.

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Exercise 6.2 Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Exercise 6.3 Prove that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k b^{n-k}$.

Exercise 6.4 Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?