Recap of Course Outline

Week	Main Content	Note
W1	Introduction	Speaking Mathematically
W2	Logic	Thinking Mathematically
W3	Number Theory	Direct Proof; Contradiction and Contraposition
W4	Mathematical Induction; Recursion	Induction Proof
W5	Set Theory	Element Proof; Algebraic Proof
W6	FLEX WEEK	FLEX WEEK
W7	Properties of Functions	Corresponding Method
W8	Properties of Relations	Classifying Method
W9	Probability	Counting Method
W10	Graph Theory	Modeling Method

Proposition Logic

1. Definitions

1.1. Statements

- A **statement** (or **proposition**) is a sentence that is true or false but not both.

> 1. *In logic and analytic philosophy, an **atomic sentence** (**simple/ basic sentence**) is a type of declarative sentence which is either true or false (may also be referred to as a proposition, statement or truthbearer) and which **cannot be broken down** into other simpler sentences.
> 2. If such sentences are to be statements, however, they must have well-defined **truth values**—they must be either true or false
>
> - A **universal statement** says that a certain property is true for all elements in a set.
> - A **conditional statement** says that if one thing is true then some other thing also has to be true.
> - An **existential statement** says that there is at least one thing for which the property is true.

Name	Note
universal conditional statement	Both universal and conditional
universal existential statement	First part: universal, Second part: existential
existential universal statement	First part: existential, Second part: universal

1.2. Compound Statements

- The symbol $\sim$ (!) denotes *not*, $\wedge$ (&&) denotes *and*, and $\vee$ (||) denotes *or*.
- If p and q are statement variables, **conjunction**: $p \wedge q$, **disjunction**: $p \vee q$, **negation**: $\sim p$.
- A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables.
- The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

1.3. Logical Equivalence

- Two **statement forms** are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.
- Two **statements** are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

- A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables.

> A statement whose form is a tautology is a **tautological statement**.
>
> - A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables.
>   A statement whose form is a contradiction is a **contradictory statement**.

Theorem Logical Equivalences (P49, 11 laws)

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6.	*Double negative law:*	${\sim}({\sim}p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* **t** *and* **c**:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

1.4. Conditional Statements

- The **conditional** of q by p: "If p then q" or "p implies q" and is denoted p $\rightarrow$ q.
  (p: **hypothesis** (or antecedent), q: **conclusion** (or consequent).)
- A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default**.
-

$p \rightarrow q$	**denote**
contrapositive	$\sim q \rightarrow \sim p$
converse	$q \rightarrow p$
inverse	$\sim p \rightarrow \sim q$

- p **only if** q means "if not q then not p", or equivalently, "if p then q".
- The **biconditional** of p and q is "p if, and only if, q" ("p, iff q") and is denoted p $\leftrightarrow$ q.

> precedence: $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\rightarrow$.

> *r is a **sufficient condition** for s means "if r then s."
> *r is a **necessary condition** for s means "if not r then not s." (, or equivalently, "if s then r.")

- **Well-formed formula (wff) of Propositional Logic**:

> - Any simple statement p, q, r, . . . is a well-formed formula wff.
> - If A and B are wffs of Propositional Logic, then:
> - 1. $\sim$ A is a wff.
> - 2. A $\wedge$ B is a wff.
> - 3. A $\vee$ B is a wff.
> - 4. A $\rightarrow$ B is a wff.

- 5. A ↔ B is a wff.
- Nothing else is a well-formed formula of propositional logic.
- *If w is a wff, then (w) is a wff.

1.5. Arguments and Propositional Calculus

- An **argument** is a sequence of statements. An **argument form** is a sequence of statement forms.

> 1. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or assumptions or hypotheses). The final statement or statement form is called the **conclusion**.
> 2. The symbol ∴, which is read "therefore," is normally placed just before the conclusion.
>
> - To say an argument form is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting **premises are all true**, then the conclusion is also true. To say that an argument is valid means that its form is valid.
> - A row of the truth table in which all the premises are true is called a **critical row**.
> - A **fallacy** is an error in reasoning that results in an invalid argument.
>   Three common fallacies:
>   **using ambiguous premises**, **circular reasoning**, and **jumping to a conclusion**.
> - An argument is called **sound** if, and only if, it is valid and all its premises are true.
> - The symbolic analysis of ordinary compound statements is called the **statement calculus** (or the **propositional calculus**).

Table Valid Argument Forms (P76, 9 Forms)

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

2. Exercise

2.1. Translate English into fully symbolic propositional logic.

1. An integer x is equal to 2.
2. An integer y is not equal to 2.
3. If x is equal to 2, then the square of x is equal to 4.
4. *An integer x is inequal to y, if, and only if, x is equal to 2 and y is not equal to 2.
5. It is not hot but it is sunny.
6. It is neither hot nor sunny.

7. Let p, q, and r symbolize "$0 < x$," "$x < 3$," and "$x = 3$," respectively. Write the following inequalities symbolically:
   a. $x \leq 3$ b. $0 < x < 3$ c. $0 < x \leq 3$

2.2. Write truth tables for the statement forms.

1. $\sim p \wedge q$.
2. $p \wedge (q \vee r)$.

2.3. Determine whether the statement forms are logically equivalent.

(By the truth table)

1. All the laws in P49 of textbook could be proven by the truth table.
2. $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$.
3. $p \rightarrow q$ and $\sim p \vee q$.
4. $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$.
5. $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$.

(By equivalence laws)

1. $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$.
2. $(p \wedge \sim q) \wedge (\sim p \vee q) \equiv c$.

2.4. Determine whether the argument forms are valid.

(By the truth table)

1. (*vacuously true*)
   c
   $\therefore p$
2. (*converse error*)
   $p \rightarrow q$
   q
   $\therefore p$
3. (*inverse error*)
   $p \rightarrow q$
   $\sim p$
   $\therefore \sim q$
4. $p \rightarrow q$
   $q \rightarrow p$
   $\therefore p \vee q$

(By the valid argument forms)

1. a. $\sim p \vee q \rightarrow r$
   b. $s \vee \sim q$
   c. $\sim t$
   d. $p \rightarrow t$
   e. $\sim p \wedge r \rightarrow \sim s$
   f. $\therefore q$
2. a. $p \rightarrow q$
   b. $r \vee s$
   c. $\sim s \rightarrow \sim t$

d. $\sim q \vee s$
e. $\sim s$
f. $\sim p \wedge r \rightarrow u$
g. $w \vee t$
h. $\therefore u \wedge w$

2.5. Others

1. Let the symbol $\oplus$ denote *exclusive or*; so $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$. Hence the truth table for $p \oplus q$ is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Find simpler statement forms that are logically equivalent to $p \oplus p$ and $(p \oplus p) \oplus p$.
- Is $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$? Justify your answer.
- Is $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$? Justify your answer.

2. "Do you mean that you think you can find out the answer to it?" said the March Hare.
"Exactly so," said Alice.
"Then you should say what you mean," the March Hare went on.
"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you
know."
"Not the same thing a bit!" said the Hatter.
"Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
—from "A Mad Tea-Party" in *Alice in Wonderland*, by Lewis Carroll
The Hatter is right. "I say what I mean" is not the same thing as "I mean what I say." Rewrite each of these two sentences in if-then form and explain the logical relation between them.

Use modus ponens or modus tollens to fill in the blanks in the arguments of 3–4 so as to produce valid inferences.

3. If $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
It is not true that $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
$\therefore$ ____.
4. If they were unsure of the address, then they would have telephoned.
____.
$\therefore$ They were sure of the address.

Some of the arguments in 5-7 are valid, whereas others exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

5. Sandra knows Java and Sandra knows C++.
   ∴ Sandra knows C++.
6. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
   Neither of these two numbers is divisible by 6.
   ∴ The product of these two numbers is not divisible by 6.
7. If I get a Christmas bonus, I'll buy a stereo.
   If I sell my motorcycle, I'll buy a stereo.
   ∴ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You are visiting this island and have the following encounters with natives.

8. Two natives A and B address you as follows:
   A says: Both of us are knights.
   B says: A is a knave.
   What are A and B?
9. Another two natives C and D approach you but only C speaks.
   C says: Both of us are knaves.
   What are C and D?
10. You then encounter natives E and F.
    E says: F is a knave.
    F says: E is a knave.
    How many knaves are there?
11. Finally, you meet a group of six natives, U, V, W, X, Y, and Z, who speak to you as follows:
    U says: None of us is a knight.
    V says: At least three of us are knights.
    W says: At most three of us are knights.
    X says: Exactly five of us are knights.
    Y says: Exactly two of us are knights.
    Z says: Exactly one of us is a knight.
    Which are knights and which are knaves?