Notice

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Predicate Logic

Foremost: Predicate Logic is an *extension* of Propositional Logic not a replacement. It retains the central tenet of Propositional Logic:

> 1. sentences express propositions;
> 2. propositions denote truth-conditions.

1. Definition

1.1. Predicate and Model

- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

> *Denoted as Px, or P(x).
>
> - The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.
> - The **extension** of a predicate is the set of values that satisfy the predicate.
>   *If Px is a predicate and x has domain $\mathbb{D}$, the **truth set** of Px is the set of all elements of $\mathbb{D}$ that make Px true when they are substituted for x. The truth set of Px is denoted $\{x \in \mathbb{D} \mid Px\}$.
> - A **Model** $\mathbb{M}$ for a statement in predicate logic consists of three things:
>   - the **domain**, the **referents** of value(s), the **extension** of the predicate(s). Where $\mathbb{M} \models Pa$ means that Pa is true in model $\mathbb{M}$.

1.2. Quantifier

- The symbol $\forall$ is called the **universal quantifier**.

  > Depending on the context, it is read as "for every," "for each," "for any," "given any," or "for all".

- The symbol $\exists$ is called the **existential quantifier**.

  > Denoted as "there exists".

- The universal and existential quantifiers are **duals**:
  1. $\forall x Px \text{ iff } \sim \exists x \sim Px.$
  2. $\exists x Px \text{ iff } \sim \forall x \sim Px.$

  > - *In mathematics, a **duality** translates concepts, theorems or mathematical structures into other concepts, theorems or structures in a one-to-one fashion, often (but not always) by means of an involution operation: if the dual of A is B, then the dual of B is A.
  > - *A duality is a pair of related concepts that display a one-to-one translation symmetry, usually (not always) as the result of some form of involution operator.

- Quantifiers must not contain names. **Variables only!**

- We say that the variable x is **bound** by the quantifier that controls it and that its **scope** begins when the quantifier introduces it and ends at the end of the quantified statement.
- A wff with one (or more) free variables is an **open** formula.

  > An open formula is **NOT** a statement.

- A wff with no free variables is a **closed** formula.

  > A closed formula **IS** a statement.

Name	Note
universal statement	$\forall x Px.$
existential statement	$\exists x Px.$

- A **universal statement** says that a certain property is true for all elements in a set.
- A **conditional statement** says that if one thing is true then some other thing also has to be true.
- An **existential statement** says that there is at least one thing for which the property is true.

*Name	Note1
universal conditional statement	Both universal and conditional
universal existential statement	First part: universal, Second part: existential
existential universal statement	First part: existential, Second part: universal

1.3. Pradical Logical Equivalence

Some useful logical equivalencies:

- $\forall x(Px \to Qx) \equiv \sim\exists x(Px \wedge \sim Qx)$ **(UETransform)**
- $\sim\forall x(\_) \equiv \exists x \sim(\_)$ **(NUQI/E)**
- $\sim\exists x(\_) \equiv \forall x \sim(\_)$ **(NEQI/E)**

1.4. Implicit Quantification

- Let Px and Qx be predicates and suppose the common domain of x is $\mathbb{D}$.
  - The notation $Px \Rightarrow Qx$ means that every element in the truth set of Px is in the truth set of Qx, or, equivalently, $\forall x(Px \to Qx),\ (\text{or } \forall x, Px \to Qx)$.
  - The notation $Px \Leftrightarrow Qx$ means that Px and Qx have identical truth sets, or, equivalently,
    $\forall x(Px \leftrightarrow Qx),\ (\text{or } \forall x, Px \leftrightarrow Qx)$.
- **Contrapositive**, **Converse** and **Inverse**

$\forall x(Px \to Qx)$	**denote**
contrapositive	$\forall x(\sim Qx \to \sim Px)$
converse	$\forall x(Qx \to Px)$
inverse	$\forall x(\sim Px \to \sim Qx)$

- ***Sufficient Condition** and **Necessary Condition**

$\forall x, Rx\ (\ \triangle\ )\ Sx$	**Definition**	**Equivalence**
is **sufficient** condition for	$\forall x(Rx \to Sx)$	-
is **necessary** condition for	$\forall x(\sim Rx \to \sim Sx)$	$\forall x(Sx \to Rx)$
only if	$\forall x(\sim Sx \to \sim Rx)$	$\forall x(Rx \to Sx)$

- ***wffs** of predicate logic:
    1. If P is a predicate and t is the term, then Pt is a wff;
    2. If α is a wff and x is an individual variable, then $\exists x\alpha$ and $\forall x\alpha$ are wffs;
    3. If α and β are wffs, then $\sim \alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$ are wffs.
    4. Nothing else is a wff of predicate logic.

1.5. Argument and Predicate Calculus

- An argument in predicate logic is **valid** iff there is **no** model where all of the premises of the argument are true and the conclusion is false.
- An argument in predicate logic is **invalid** iff there **is** a model where all of the premises of the argument are true and the conclusion is false.

  > Such a model is called a **countermodel** for the argument in question.

- The symbolic analysis of predicates and quantified statements is called the **predicate calculus**.
- **Inference rules**

NAME	RULE	EXAMPLE
UQElim (Universal Quantifier Elimination/ Instantational)	**∀xPx** **∴ Pc**	All humans are motal. Therefore, Jiapeng is motal.
EQElim Existential Quantifier Elimination/ Instantational)	**∃xPx** **∴ Pc for some element c**	There is someone who is a tutor. Let's call him Jiapeng and say that Jiapeng is a tutor.
UQInt (Universal Quantifier Introduction/ generalization)	**Pc for an arbitrary c** **∴ ∀xPx**	-
EQInt Existential Quantifier Introduction/ generalization)	**Pc for some element c** **∴ ∃xPx**	Jiapeng is a tutor. Therefore, someone is a tutor.

1.6. An Example

1. **"All lions are fierce."**
2. **"Some lions do not drink coffee."**

$\therefore$ **"Some fierce creatures do not drink coffee."**

Let's let Lx be "x is a lion," Fx be “x is fierce,” and Cx be “x drinks coffee.”

Then, translate statements below into fully symbolic Predicate Logic.

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

Now fill in the blanks to complete the inference.

STATEMENT	SYMBOL	REASON
1. Some lions do not drink coffee.		
2. Let's call him Lambert and say that Lambert is a lion that doesn't drink coffee		
3. If Lambert is both a lion and a non-coffee drinker, then we can simplify and say Lambert is a lion		
4. If Lambert it both a lion and a non-coffee drinker, then we can simplify and say Lambert does not drink coffee		
5. All lions are fierce.		
6. If all lions are fierce and Lambert is a lion, then Lambert is fierce.		
7. If Lambert is a lion, then Lambert is fierce. And we know that Lambert is a lion is true, then we can say that Lambert is fierce.		
8. Lambert is fierce and doesn't drink coffee.		
9. Therefore, we have shown that if Lambert is fierce and does not drink coffee, then there is some fierce creature who does not drink coffee.		

2. Exercise

2.1. Translate English into fully symbolic predicate logic.

1. The truth value of every statement is true or not true but not both.
2. If both you and I know the answer, then **all humans** know it.
3. For any integer x, there exists an integer y satisfying $x < 0$ and $y > 0$.
4. If every student in the class is fine, then there exists a robot who can dance.
5. Let: Ax: “x lives in La Perouse,” Bx: “x is a COMP 9020 student,” Cx: “x gets a good grade,” Dx: “x majors in computer science.” The domain: all UNSW students. Write the following statements symbolically in predicate logic:
    1. All COMP 9020 students get good grades.
    2. No COMP 9020 student lives in La Perouse.
    3. COMP 9020 students who do not live in La Perouse major in computer science.

2.2. Determine whether the statement forms are logically equivalent.

1.
    - $\sim \forall x P x$
    - $\forall x \sim P x$

2.
- $\exists x P x$
- $\exists x \sim P x$

3.
- $\forall\ x(P x \rightarrow Q x)$
- $\forall x P x \rightarrow \forall x Q x$

4.
- It is impossible for all of the argument premises to be true and the conclusion of the argument to be false.
- At least one of the argument premise is false or the conclusion of the argument is true.

5.
- $\forall x \exists y\ (x R y)$
- $\exists y \forall x\ (x R y)$

6. Rewrite the statement formally using quantifiers and variables, and then write a negation for the statement:
   - Everybody trusts somebody.
   - Somebody trusts everybody.

2.3. Determine whether the argument forms are valid.

1. $\forall m, n \in \mathbb{Z}^{+},\ m \cdot n \geq m+n.$
2. There exists an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6.
3.
   - All healthy people eat an apple a day.
   - Keisha eats an apple a day.
   - $\therefore$ Keisha is a healthy person.
4.
   - If a graph has no edges, then it has a vertex of degree zero.
   - This graph has at least one edge.
   - $\therefore$ This graph does not have a vertex of degree zero.
5.
   - If a product of two numbers is 0, then at least one of the numbers is 0.
   - For a particular number x, neither $(2x+1)$ nor $(x-7)$ equals 0.
   - $\therefore$ The product $(2x+1)(x-7)$ is not 0.

An Example -- Answer

Let’s let Lx be "x is a lion," Fx be “x is fierce,” and Cx be “x drinks coffee.”

Then,

STATEMENT	SYMBOL
"All lions are fierce."	$\forall x(Lx \to Fx)$
"Some lions do not drink coffee."	$\exists x(Lx \land \sim Cx)$
"Some fierce creatures do not drink coffee."	$\exists x(Fx \land \sim Cx)$

Now we have,

STATEMENT	SYMBOL	REASON
1. Some lions do not drink coffee.	$\exists x(Lx \land \sim Cx)$	Premise 2/ Given
2. Let's call him Lambert and say that Lambert is a lion that doesn't drink coffee	$L_{Lambert} \land \sim C_{Lambert}$	EQElim from (1)
3. If Lambert is both a lion and a non-coffee drinker, then we can simplify and say Lambert is a lion	$L_{Lambert}$	Spec. from (2)
4. If Lambert it both a lion and a non-coffee drinker, then we can simplify and say Lambert does not drink coffee	$\sim C_{Lambert}$	Spec. from (2)
5. All lions are fierce.	$\forall x(Lx \to Fx)$	Premise 1/ Given
6. If all lions are fierce and Lambert is a lion, then Lambert is fierce.	$L_{Lambert} \to F_{Lambert}$	UQElim from (5)
7. If Lambert is a lion, then Lambert is fierce. And we know that Lambert is a lion is true, then we can say that Lambert is fierce.	$F_{Lambert}$	MP Rule from (3) and (6)
8. Lambert is fierce and doesn't drink coffee.	$F_{Lambert} \land \sim C_{Lambert}$	Conj. from (4) and (7)
9. Therefore, we have shown that if Lambert is fierce and does not drink coffee, then there is some fierce creature who does not drink coffee.	$\exists x(Fx \land \sim Cx)$	EQInt from (8)

QED.