Jiapeng Wang, COMP9020 Tutorial W10

Some notes first:

1. Please complete the "My Experience" survey on Moodle actively.
2. The Week 11 tutorial will be held at the same time (Wednesday, 4-6 pm), but **the location might change**. I will post the details on Webcoms3 this weekend.
3. Problem Sets 7 and 8 will not be covered in our tutorial. The solutions will be posted on the ED forum or Webcoms3. If you have any specific questions, please contact me via email.

Discussion about Problem Set 6

Common Mistakes

1. Given functions $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined as $f(x) = x^2 + 3$ and $g(x) = \sqrt{x+2}$ (the principal square root of $x+2$), compute functions $f \circ g$ and $g \circ f$.

   What are the domains, codomains and ranges of both those functions? For both functions prove or disprove if they are injective and/or surjective. After doing so, argue whether they can be inverted.

2. Given sets $A = \{a, b, c\}$ and $B = \{\alpha, \beta\}$, compute the set A^B. How may you interpret that set?

3. Use the definition of function composition to prove that it is associative as mentionned in the lectures. Explain in your own words why it is natural for function composition to be associative.

4. Given a function $f : \mathbb{R} \to \mathbb{R}$ explain what is wrong if we define it as $f(x) = \frac{x^2+1}{x-1}$. How would you need to modify the domain the definition for this to be a function? Does your modified function have an inverse? Explain why. (*hint: There are many ways to go about this, if you try one way and it is too hard, consider the alternatives.*)

5. For any functions $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, we can define $F \times G$ as follows:

   $$(F \times G)(x) = F(x) \times G(x) \quad \text{for all } x \in \mathbb{R},$$

   where $x \times y$ for $x, y \in \mathbb{R}$ denotes the standard multiplication operation on the reals.

   Left-distribitivity of function composition over multiplication would means that for any $H : \mathbb{R} \to A$, we would have $H \circ (F \times G) = (H \circ G) \times (H \circ F)$.

   Right-distributivity of function composition over multiplication would mean that for any $H : A \to \mathbb{R}$, we would have $(F \times G) \circ H = (F \circ H) \times (G \circ H)$.

   Do these laws hold? If so, prove it, if not give a counter-example.

Question 1

- Please note that the domain and co-domain are the sets we refer to when defining a function.

Question 2

- What is A^B?

Question 4

- It would be better to prove surjective by definition (for all y in codomain, there is x in domain such that f(x) = y). And to disprove, you should show a counterexample.

Question 5

- You cannot prove a general statement by showing a specific example!

Sample Answer

Answer to Q1

- $f \circ g = x + 5$ has domain $\mathbb{R}^+$ and codomain $\mathbb{R}^+$. Its range is $[5, \infty)$.
- $g \circ f = \sqrt{x^2 + 5}$ has domain $\mathbb{R}$ and codomain $\mathbb{R}$. Its range is $[\sqrt{5}, \infty)$.

$f \circ g$ is injective because setting $f(x_1) = f(x_2) \implies x_1 + 5 = x_2 + 5 \implies x_1 = x_2$. However, it fails to be surjective since $x + 5 = 0$ does not have a solution in $\mathbb{R}^+$. This means it cannot be inverted.

$g \circ f$ is not surjective since $\sqrt{x^2 + 5} = \frac{1}{2}$ has no solution. It also fails to be injective since, for example, setting $x_1 = -1$ and $x_2 = 1$ gives the same output despite the input being different. This means it is not invertible.

Answer to Q2

- The set $A^B = \{\{(a,a),(b,a)\},\{(a,a),(b,b)\},\{(a,a),(b,c)\},\{(a,b),(b,a)\},\{(a,b),(b,b)\},\{(a,b),(b,c)\},\{(a,c),(b,a)\},\{(a,c),(b,b)\},\{(a,c),$

  can be interpreted as the set of all functions from B to A.

Answer to Q3

- Function composition is defined as $(f \circ g)(x) = f(g(x))$ for all x in the domain of g.
  The associativity comes from the fact that for all x in the domain of h,
  $f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x))) = (f \circ g)(h(x)) = ((f \circ g) \circ h)(x)$.

It is natural because function composition transforms the output of one function into the input of another without changing the nature of the function. Since function composition does not affect the "inside" of a function, the order in which we group the functions should not affect the final outcome.

Answer to Q4

- The issue with the definition of this function is that it fails to satisfy totality. Indeed the function is not defined for $x = 1$.
  To make this function work we can redefine the domain as $\mathbb{R} \setminus \{1\}$. The functions thus obtained do not have an inverse. One way to show how is by showing that it fails to be surjective, and can therefore not be bijective, which invertible functions are.
  We choose to show it is not surjective
  $\frac{x^2+1}{x-1} = 0 \implies x^2 + 1 = 0$
  Which is not solvable in $\mathbb{R}$ which implies that there is no x for which $f(x) = 0$ and thus the function is not surjective and therefore not invertible.

Answer to Q5

- Function composition is not left-distributive over multiplication. Setting all functions to $F(x) = F(x) = H(x) = 2x$ gives an immediate counter-example of this.
  $H \circ (F \times G)(x) = H((F \times G)(x)) = H(F(x) \times G(x)) = H(2x \times 2x) = H(4x^2) = 8x^2$
  This is different from
  $((H \circ G) \times (H \circ F))(x) = (H \circ G)(x) \times (H \circ F)(x) = H(2x) \times H(2x) = 4x \times 4x = 16x^2$
  On the other hand, function composition is right-distributive over multiplication, which we can show as follows:
  $\forall a \in A,$
  $((F \times G) \circ H)(a)$
  $= (F \times G)(H(a)) \quad \text{(Definition of function composition)}$
  $= F(H(a)) \times G(H(a)) \quad \text{(Definition of function multiplication)}$
  $= (F \circ H)(a) \times (G \circ H)(a) \quad \text{(Definition of function composition)}$
  $= ((F \circ H) \times (G \circ H))(a) \quad \text{(Definition of function multiplication)}$

Counting Mathods

Basic Counting Rules

Principles

- Union Rule: If S and T are disjoint |S ∪ T| = |S| + |T|
- Product Rule (or 'Multiplication Rule'): |S × T| = |S| · |T|

Common strategies

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g. |S| + |T| = |S ∪ T| + |S ∩ T|)
- Find a bijection to a set that can be counted

Exercise 5.1 Suppose there are k elements in set A. How many elements in $\mathcal{P}(A)$?

Exercise 5.2 How many different Boolean functions with k arguments (i.e., $f : \{0,1\}^k \rightarrow \{0,1\}$)?

Permutation and Combination

A **permutation** of a set of objects is an *ordering* of the objects in a row.

An **r-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.

- $P(n, r) = \frac{n!}{(n-r)!}$.

Exercise 6.1

a. How many different ways can three of the letters of the word BYTES be chosen and written in a row?

b. How many different ways can this be done if the first letter must be B?

A **combination** of a set of objects is an *disordering* of the objects in a row.

Let n and r be nonnegative integers with $r \leq n$. An r-combination of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$, read "n choose r," denotes the number of subsets of size r (or r-combinations) that can be formed from a set of n elements.

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Exercise 6.2 Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Exercise 6.3 Prove that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k b^{n-k}$.

Exercise 6.4 Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Inclusion/Exclusion Rule

This rule tells us how to count the number of elements in a union of sets when they are non-disjoint.

Inclusion/Exclusion Rule: If A, B, and C are finite sets, then:

$N(A \cup B) = N(A) + N(B) - N(A \cap B)$

and

$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$

And so on for arbitrarily large but finite numbers of sets.

Exercise 7 A professor in a discrete mathematics class passes out a form asking students to check all the mathematics and computer science courses they have recently taken. She found that, out of a total of 50 students in the class,

- 30 took precalculus;
- 18 took calculus;
- 26 took Python;
- 9 took both precalculus and calculus;
- 16 took both precalculus and Python;
- 8 took both calculus and Python;
- 47 took at least one of the three courses.

> Note that when we write "30 students took precalculus," we mean that the total number of students who took precalculus is 30, and we allow for the possibility that some of these students may have taken one or both of the other courses. If we want to say that 30 students took precalculus *only* (and not either of the other courses), we will say so explicitly.

a. How many students did not take any of the three courses?

b. How many students took all three courses?

c. How many students took precalculus and calculus but not Python? How many students took precalculus but neither calculus nor Python?

Pigeonhole Principle

In mathematics, the **pigeonhole principle** (or 'Dirichlet's drawer principle') states that

- if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

Exercise 8.1 If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, what is the minimum number of socks that the Martian must pull out of the drawer to guarantee they have a pair?

Exercise 8.2 Prove that if a positive integer can be expressed as the product of two positive integers, then the smaller factor must be less than or equal to the square root of this number.

Graph Theory

Recap Tutorial W1

A **graph** G consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

Definitions

Let G be a graph, and let v and w be vertices in G.

A **walk** from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form
$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$,
where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1, 2, \ldots, n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk** from v to v consists of the single vertex v.

A **trail** from v to w is a walk from v to w that does not contain a repeated edge.

A **path** from v to w is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

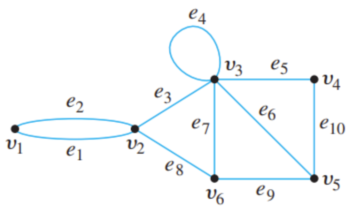
A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Exercise 9 In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$ b. $e_1 e_3 e_5 e_5 e_6$ c. $v_2 v_3 v_4 v_5 v_3 v_2$

d. $v_2 v_3 v_4 v_5 v_6 v_2$ e. $v_1 e_1 v_2 e_1 v_1$ f. v_1



Connectedness

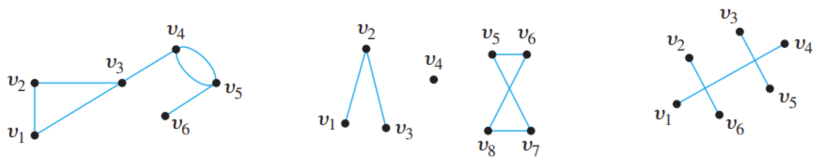
Connected

Let G be a graph. Two **vertices v and w of G are connected** if, and only if, there is a walk from v to w.

The **graph G is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically:

G is connected $\iff$ $\forall$ vertices v and w in G, $\exists$ a walk from v to w.

Exercise 10.1 Which of the following graphs are connected?



(a) (b) (c)

Lemma 1 Let G be a graph.

- a. If G is connected, then any two distinct vertices of G can be connected by a path.
- b. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.
- c. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

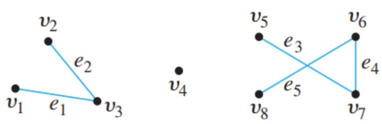
Subgraph and Connected Component

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

A graph H is a **connected component** of a graph G if, and only if,

1. H is a subgraph of G;
2. H is connected; and
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Exercise 10.2 Find all connected components of the following graph G.



Euler and Hamiltonian Circuits

Euler circuit

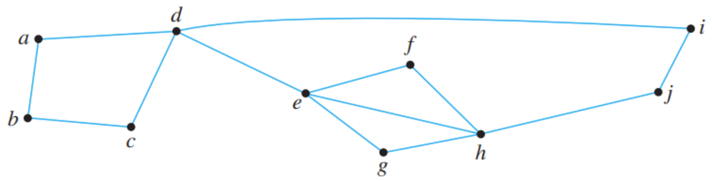
Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem 1 If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Theorem 2 If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Theorem 3 A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

Exercise 11.1 Use *Theorem 3* to check that the graph below has an Euler circuit. Then use the algorithm from the proof of the theorem to find an Euler circuit for the graph.

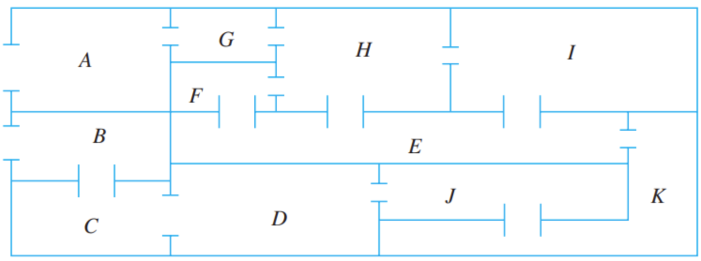


Euler trail

Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail from v to w** is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

Corollary 1 Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Exercise 11.2 The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room A, ends in room B, and passes through every interior doorway of the house exactly once? If so, find such a trail.



*Hamiltonian circuit

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Proposition 1 If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Matrices and Trees

Note: Please refer to the Second Lecture slides in Week 10 and textbook sections 10.2 and 10.4...