Additional Exercise W11

COMP9024 24T2, 7th Aug, 2024

Jiapeng Wang

Some Instructions:

1. There will be *no* multiple-choice or fill-in-the-blank questions on the final exam this semester.
2. Both *problem-solving* and *explaining-your-understanding* may be assessed.
3. All the topics we assess come from the *lectures*, but we may introduce new definitions to evaluate your ability to apply this knowledge.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For every integer $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

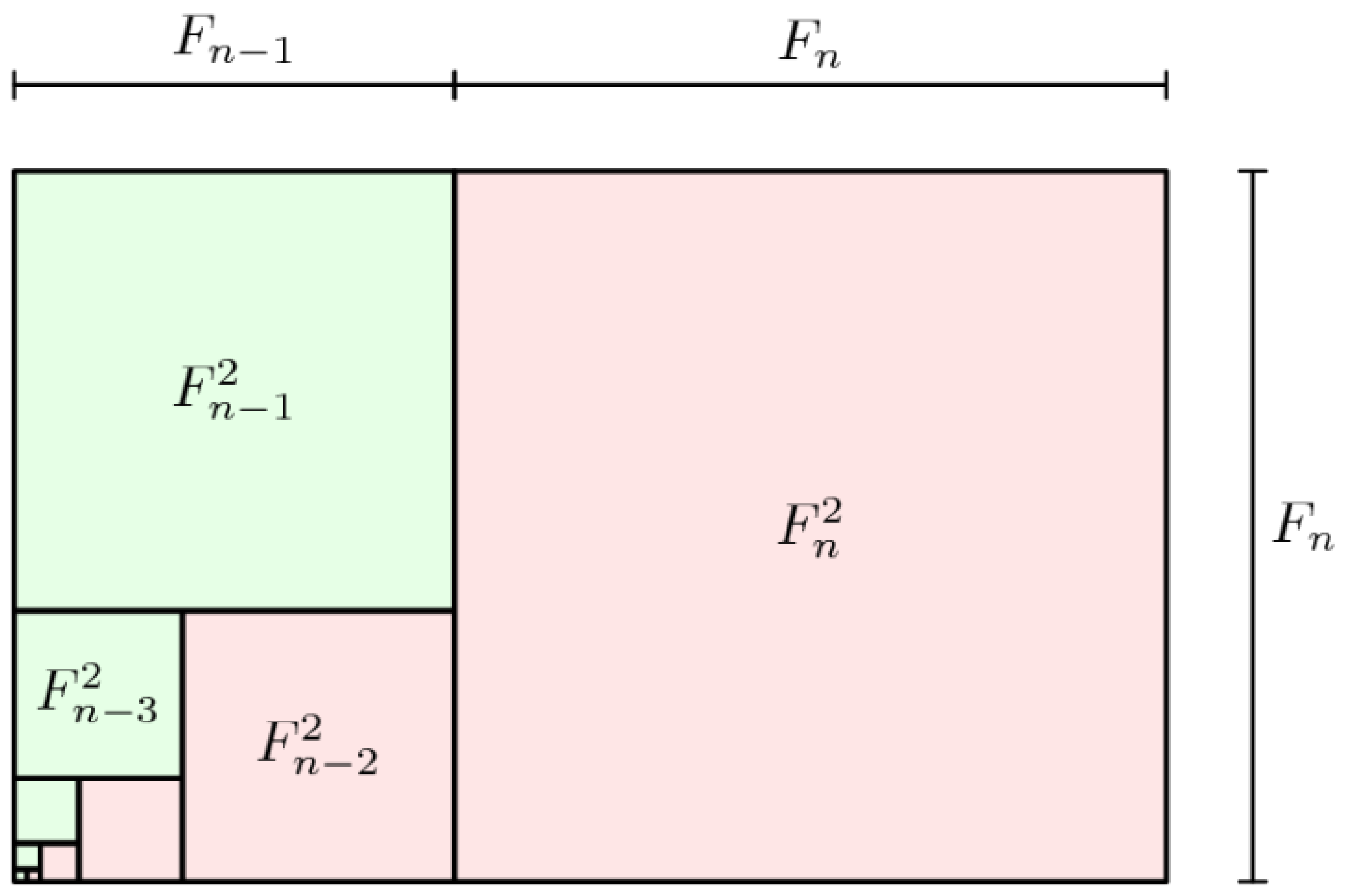
Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

> **suppose** that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.
>
> *[This supposition is called the* **inductive hypothesis**.*]*

Then

> **show** that $P(k+1)$ is true.



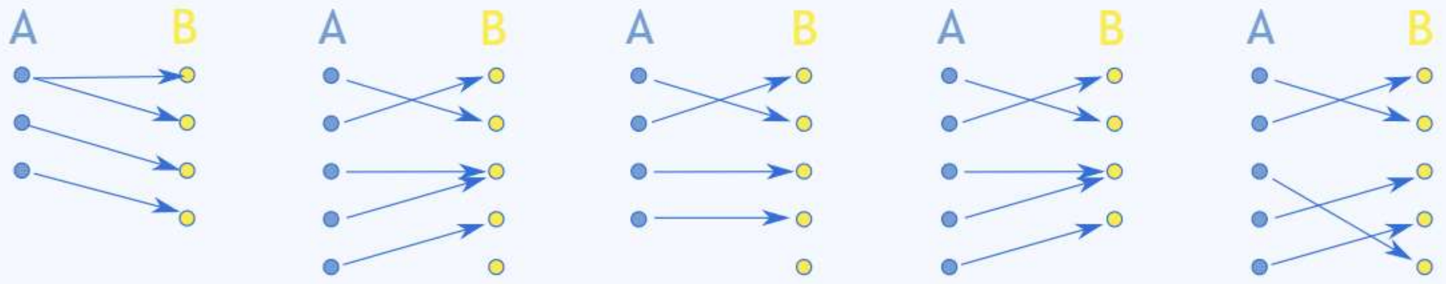
The identity $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$, where F_i is the ith Fibonacci number.

A binary relation $R \subseteq S \times T$ is:

Definition

(Fun)	functional	For all $s \in S$ there is at most one $t \in T$ such that $(s,t) \in R$
(Tot)	total	For all $s \in S$ there is at least one $t \in T$ such that $(s,t) \in R$
(Inj)	injective	For all $t \in T$ there is at most one $s \in S$ such that $(s,t) \in R$
(Sur)	surjective	For all $t \in T$ there is at least one $s \in S$ such that $(s,t) \in R$
(Bij)	bijective	Injective and surjective

A function is a way of matching the members of a set "A" **to** a set "B":



NOT a Function — *A has many B*

General Function — *B can have many A*

Injective (not surjective) — *B can't have many A*

Surjective (not injective) — *Every B has some A*

Bijective (injective, surjective) — *A to B, perfectly*

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. **(basis step)**
2. For every integer $k \geq b$, if $P(i)$ is true for each integer i from a through k, then $P(k+1)$ is true. **(inductive step)**

Then the statement

$$\text{for every integer } n \geq a,\ P(n)$$

is true. (The supposition that $P(i)$ is true for each integer i from a through k is called the **inductive hypothesis**. Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Euclid's proof

1. Assume there are a finite number n of primes, listed as $[p_1, ..., p_n]$.
2. Consider the product of all the primes in the list, plus one: $N = (p_1 \times ... \times p_n) + 1$.
3. By construction, N is not divisible by any of the p_i.
4. Hence it is either prime itself (but not in the list of all primes), or is divisible by another prime not in the list of all primes, contradicting the assumption.
5. *q.e.d.*

The Quotient Remainder Theorem states that for any integer n and positive integer d, there exist unique integers q and r such that:

$$n = dq + r \quad \text{and} \quad 0 \leq r < d$$

Where n is an integer, and d is a positive integer:

n div d $=$ the integer quotient obtained when n is divided by d.

n mod d $=$ the nonnegative integer remainder obtained when n is divided by d.

If n and d are integers, and $d > 0$, then:

$$n \; div \; d = q \quad \text{and} \quad n \; mod \; d = r \quad iff \quad n = dq + r$$

Where q and r are integers and $0 \leq r < d$:

$$23 \; div \; 6 = 3 \quad \text{and} \quad 23 \; mod \; 6 = 5 \quad iff \quad 23 = (6 \cdot 3) + 5$$

You can compute the day of the week with *div* and *mod*! See example 4.5.3 in the textbook :)

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

Recap of Course Outline

Week	Main Content	Note
W1	Introduction	Speaking Mathematically
W2	Logic	Thinking Mathematically
W3	Number Theory	Direct Proof; Contradiction and Contraposition
W4	Mathematical Induction; Recursion	Induction Proof
W5	Set Theory	Element Proof; Algebraic Proof
W6	FLEX WEEK	FLEX WEEK
W7	Properties of Functions	Corresponding Method
W8	Properties of Relations	Classifying Method
W9	Probability	Counting Method
W10	Graph Theory	Modeling Method

Something else to say:

- The **hasty generalization** fallacy.
- Always show your point first.