Jiapeng Wang, COMP9020 Tutorial W4

Common mistakes in Problem Set 1

1. Question 1-3

1. Not be decompose into basic statements.
    - Alice is not hungry.
    - They (Alice and Bob) both study hard.
2. The definition of 'only if'.
    - Alice and Bob will pass COMP9020 **only if** they both study hard.

1. Translate the following into fully symbolic propositional logic - “Either Alice is hungry or she is not”.
2. Translate the following into fully symbolic propositional logic - “If Bob is tall then Alice is too”.
3. Translate the following into fully symbolic propositional logic - “Alice and Bob will pass COMP9020 only if they both study hard”.
4. Is the following argument valid? If so, provide a natural deduction proof via the rules in TABLE 2.3.1 on p.76 of the textbook. If not, then provide a counterexample via either a truth-table or the short method of truth-value assignments demonstrated in the lectures.
    - 1. $p \wedge \sim p$
    - $\therefore q$
5. What is the logical form for any statement that we can add to the set of premises of a deductively valid argument in order to make the argument invalid? Explain your answer in your own words.

2. Question 4

1. Not provide a natural deduction proof **via the rules in TABLE 2.3.1 on p.76 of the textbook**.
2. Jumping conclusion.
    - p $\rightarrow$ q

3. Question 5

1. The definition of 'validity' of the argument.
2. The differences between 'validity' and 'soundness'.

My answer to Q5

There is no such logical form because there is no such statement. A valid argument cannot be invalidates by the addition of new premises of any sort.

> (***Definition***) *An argument processes the property of validity if and only if*
>
> - *it is impossible for all of the argument premises to be true and the conclusion of the argument to be false.*

Suppose we have a valid argument A with (n-1) premises $P_1, P_2, \cdots P_{n-1}$ and a conclusion Q . Let K be $\sim ((P_1 \wedge P_2 \wedge \cdots P_{n-1}) \wedge \sim Q)$.

Since A is valid, from the definition, we have K is true.

Now we add a new statement P_n to the set of premises of this argument. Let A_1 be this new argument. Then

$$\begin{aligned} \sim ((P_1 \wedge P_2 \wedge \cdots \wedge P_{n-1} \wedge P_n) \wedge \sim Q) &\equiv \sim (((P_1 \wedge P_2 \wedge \cdots \wedge P_{n-1}) \wedge \sim Q) \wedge P_n) \\ &\equiv K \vee \sim P_n \end{aligned}$$

Since K is true, it implies that $K \vee \sim P_n$ is true. Therefore, it is impossible for $P_1, P_2, \cdots P_n$ to be true and Q to be false, i.e., A_1 is still valid.

Thus, there is no such a logical form for any statement that we can add to the set of premises of a deductively valid argument in order to make the argument invalid. QED

Definitions in Number Theory

1. n is **even** iff $n = 2k$ for some integer k.
2. n is **odd** iff $n = 2k + 1$ for some integer k.
3. Where n, r, and s are integers such that $n > 1$ and $r > 0$ and $s > 0$:
    - n is **prime** iff $\forall r \forall s((n = rs) \rightarrow ((r = 1 \wedge s = n) \vee (r = n \wedge s = 1)))$.
    - n is **composite** iff $\exists r \exists s((n = rs) \wedge ((1 < r < n) \wedge (1 < s < n)))$.
4. r is **rational** iff $(r \in \mathbb{R}) \wedge \exists x \exists y\ (((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \wedge (y \neq 0)) \wedge (r = \frac{x}{y}))$.
5. If n and d are integers then
    - n is **divisible** by d if, and only if, n equals d times some integer and $d \neq 0$. The notation $d \mid n$ is read “d divides n.”
    - Let a and b be integers that are not both zero. The **greatest common divisor** of a and b, denoted $gcd(a, b)$, is that integer d with the following properties:
        1. d is a common divisor of both a and b.
        2. For every integer c, if c is a common divisor of both a and b, then c is less than or equal to d.
    - Integers a and b are said to be **relatively prime** if, and only if, their greatest common divisor is 1.
6. **(The Quotient Remainder Theorem)** For any integer n and positive integer d, there exist unique integers q and r such that: $n = dq + r \text{ and } 0 \leq r < d.$
    - Where n is an integer, and d is a positive integer:
        - n **div** d: the integer quotient obtained when n is divided by d.
        - n **mod** d: the nonnegative integer remainder obtained when n is divided by d.
    - If n and d are integers, and d > 0, then:
      $n\ div\ d = q \text{ and } n\ mod\ d = r \quad \text{iff} \quad n = dq + r$

Methods of Proof

1. CP Rule (Conditional Proof Rule):

$$
\begin{aligned}
&P\ (assume)\\
&\vdots\\
&Q\\
&---\\
&P \rightarrow Q
\end{aligned}
$$

2. Classical Proof

METHOD	RULE
Universal Generalization	Modus Ponens Rule
Proof by Exhausting	Conditional Proof Rule Proof by Division of Cases
Contraposition	Modus Tollens Rule
Contradiction	Contradiction Rule

3. Constructive and Non-contructive Proof

(Example) Prove that:

- A power of an irrational number to an irrational exponent may be rational.

The first crisis in the history of mathematics

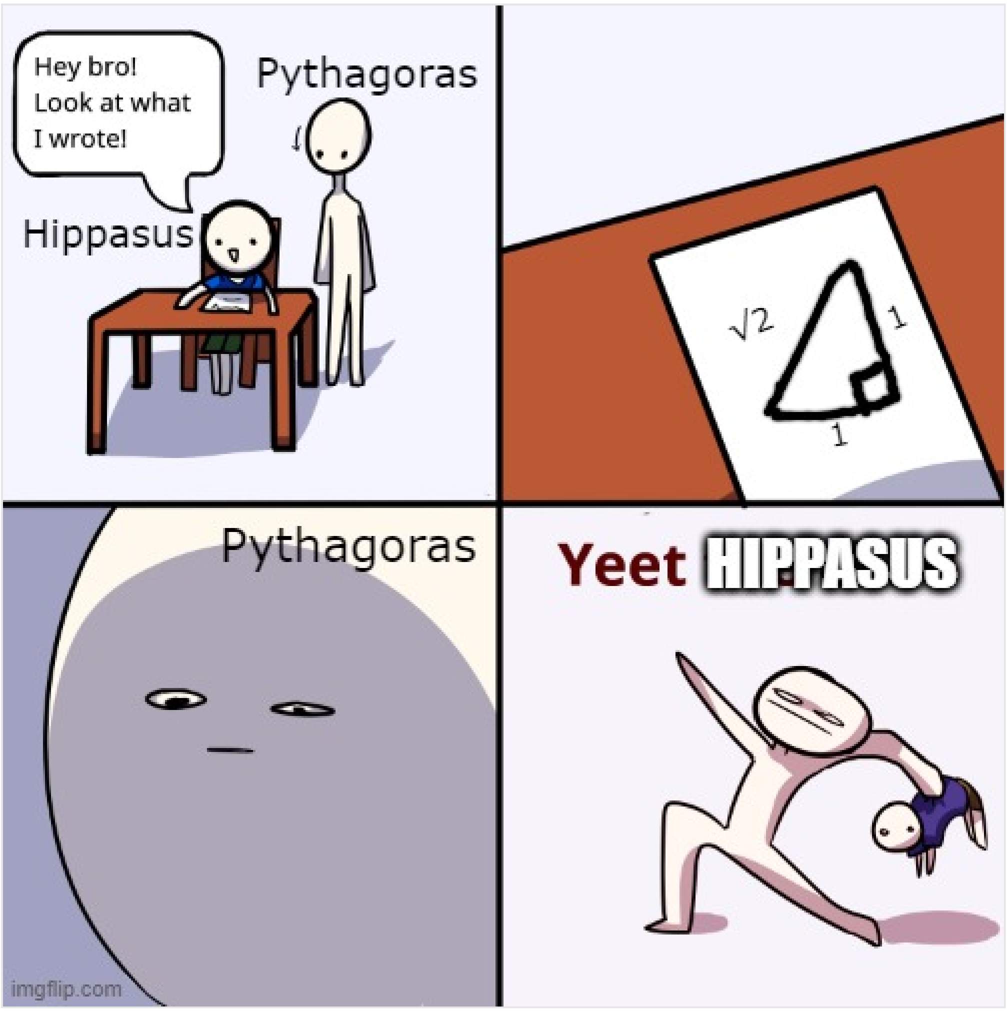
1. Summary

1.1. Background

The first crisis occurred in ancient Greece between 580-568 BC, the mathematician Pythagoras established the Pythagoras School. This school integrates religion, science and philosophy, the number of which is fixed, the knowledge is kept secret, and all inventions are attributed to the leader of the school. At that time, people's understanding of rational numbers is still very limited, the concept of irrational numbers is even more ignorant, the Pythagorean School of Numbers, originally refers to integers, they do not think of fractions as a number, but only as the ratio of two integers, they mistakenly **believe that all phenomena in the universe are attributed to the ratio of integers or integers**.

1.2. Crisis

Hibersos, a member of the school of thought, finds by logical reasoning that the diagonal length of a square with an L is **neither an integer nor a ratio of integers**. Hibersos's discovery was considered "absurd" and a violation of common sense. It not only seriously violates the Pythagoras School's Creed, but also shocks the Greek's traditional ideas at that time. The Greek mathematicians at that time deeply disturbed, legend Hibersos because this discovery was thrown into the sea drowned, this is the first mathematical crisis.



> Notice that: The crisis wasn't that the side length was root 2. They already knew this. The crisis was that they then couldn't find a scale factor that made all 3 sides integer lengths, or in other words, they couldn't find a rational equal to root 2.

1.3. Solution

The crisis was solved by introducing the concept of irreducible quantities into geometry. Two geometric segments, if there is a third segment to be able to simultaneously measure them, it is said that these two segments are accessible, otherwise known as irreducible. On one side and diagonal lines of a square, there is no third segment that can simultaneously measure them, so they are irreducible. It is clear that the so-called mathematical crisis ceases to exist, as long as the existence of irreducible amounts is accepted so that the geometry is no longer limited by integers. The study of irreducible quantity began in the Eudox of 4th century BC, and its results were absorbed by Euclid, some of whom were received in his "geometrical original".

——Yang Jinzhou, Three crises in the history of mathematics work

2. Main Proof: Take $\sqrt{3}$ as an example

1. Let the domain be the domain of integers $\mathbb{Z}$, and let Ex be x is an even number. Prove the following statement:
   $\forall x\ (Ex \rightarrow E(x^2))$.
2. If q is not divisible by 3, then $q^2 \mod 3 = 1$.

> From Proposition 1 and 2, can you find any patterns? Is the following proposition also true?
>
> - If q is not divisible by 5, then $q^2 \mod 5 = 1$.

3. If q^2 is divisible by 3, so is q.
4. Prove that $\sqrt{3}$ is irrational.

Recap

> If $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
> It is not true that $\sqrt{2} = \frac{a}{b}$ for some integers a and b.
> $\therefore$ ____.

Answer of Main Proof

1. (UG) *If $m \in \mathbb{Z}$ is even, then m^2 is even.

Proof. Suppose $m \in \mathbb{Z}$ is even. By definition of an even integer, there exists $n \in \mathbb{Z}$ such that $m = 2n$.
Thus we get $m^2 = (2n)^2 = 4n^2 = 2(2n^2)$
and we have m^2 is also even. $\square$

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2. (PE) If q is not divisible by 3, then $q^2 \ mod \ 3 = 1$.

Proof. If $3 \nmid q$, we know $q \ mod \ 3 = 1$ or $q \ mod \ 3 = 2$.

Case 1: $q \ mod \ 3 = 1$. By definition, $q = 3k + 1$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned} q^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

and we have $q^2 \ mod \ 3 = 1$.

Case 2: $q \ mod \ 3 = 2$. By definition, $q = 3k + 2$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned} q^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

and in this case we again have $q^2 \ mod \ 3 = 1$.

In either case $q^2 \ mod \ 3 = 1$, so the result is proven. $\square$