Jiapeng Wang, COMP9020 Tutorial W8

Common mistakes in Problem Set 4

> 1. A **sequence** is a **function**. What then, is the **domain** and **co-domain** of a sequence?
> 2. The **Generalised Law of Distribution** is as follows:
>
> $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$$
>
> Explain in your own words why it is that this law is permissible. It might help in your answer if you imagine yourself explaining this law to someone for whom it is novel and surprising.
>
> 3. Where $n = 3$ and $r = 2$, compute the following:
>
> $$\sqrt{\binom{n \times n}{r}}$$
>
> 4. Here is a **recursively defined sequence** $a_0, a_1, a_2 \dots$ for all integers $k \geq 2$:
>
> $$a_k = a_{k-1} + ka_{k-2} + 1 \qquad \text{(recurrence relation)} \qquad (1)$$
> $$a_0 = 1 \quad \text{and} \quad a_1 = 2 \qquad \text{(initial conditions)} \qquad (2)$$
>
> **Find** a_5, a_6, **and** a_7
>
> **Note:** You must show your working.
>
> 5. Prove the following by **mathematical induction**. For the inductive step, do **not** use the method of direct proof. Instead, you must use **proof by contradiction:**
>
> **For all integers** $n \geq 1$, it is the case that:
>
> $$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
>
> **Note:** you must show all steps of the proof, justify each step, and identify when the **explicit contradiction** is reached. You must explain also why it is that the contradiction in question is in fact a contradiction.

Question 1

- The domains or co-domains of different sequences may be different. For example, not all sequence subscripts are all natural numbers starting from 1.
- It does not specify which variables are input or output when you use definition of domains and co-domains from function.

> **Answer**:
> Domain = Addresses of the sequence;
> Co-domain = Values recorded at the addresses

Question 2

Distributive Law (in elementary arithmetic): $\forall x, y, z \in \mathbb{R}, x \cdot (y + z) = x \cdot y + x \cdot z$.

- Explain the translation of the summation symbol correctly, but fail to explain the reason for the generalized distributive law.

> *A good answer* will explain this for a one-element sequence, and then a two-element sequence, then generalise.

Question 3

- The definition of 'n choose r'.

Question 5

- Failure to provide proof as required (*explicit contradiction*).

Properties of Functions

*Recap W1: The Language of Relation

definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written $x \, R \, y$, if, and only if, (x, y) is in R. The set A is called the **domain** of R and the set B is called its **co-domain**.

The notation for a relation R may be written symbolically as follows:

$$x \, R \, y \quad \text{means that} \quad (x, y) \in R.$$

The notation $x \, R \, y$ means that x is not related to y by R:

$$x \, R \, y \quad \text{means that} \quad (x, y) \notin R.$$

*Domain: Total and Functional

A binary relation $R \subseteq S \times T$ is:

- **Total**: For all $s \in S$ there is **at least** one $t \in T$ such that $(s, t) \in R$.
- **Functional**: For all $s \in S$ there is **at most** one $t \in T$ such that $(s, t) \in R$.

Function

Definition of Function

A **function f from a set X to a set Y**, denoted $f : X \to Y$, is a relation from X, the domain of f, to Y, the co-domain of f, that satisfies two properties:

1. every element in X is related to some element in Y, and
2. no element in X is related to more than one element in Y.
   Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f : x \to y$. The unique element to which f sends x is denoted $f(x)$, and is called f **of** x, or

- the output of f for the input x, or
- the value of f at x, or
- the image of x under f.

The set of all values of f taken together is called the **range of f** or the **image of X under f**. Symbolically:

$$\text{range of } f = \text{image of } X \text{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \in X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. When x is an element such that $f(x) = y$, then x is called a **preimage of y** or an **inverse image of y**. The set of all inverse images of y is called the **inverse image of y**. Symbolically:

$$\text{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

> *A (binary) function is a relation **both** *total* and *functional*.

Function Equality

If $F : X \to Y$ and $G : X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for every $x \in X$.

> The *domain* and *co-domain* are **critical aspects** of a function's definition.

Exercise 1.1 Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For every x in J_3,

$$f(x) = (x^2 + x + 1) \ mod \ 3 \quad \text{and} \quad g(x) = (x+2)^2 \ mod \ 3.$$

Does $f = g$?

Exercise 1.2 Let $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ be functions. Define new functions $F + G : \mathbb{R} \to \mathbb{R}$ and $G + F : \mathbb{R} \to \mathbb{R}$ as follows: For every $x \in \mathbb{R}$,

$$(F+G)(x) = F(x) + G(x) \quad \text{and} \quad (G+F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Examples of Functions

The Identity Function on a Set

Given a set X, define a function I_X from X to X by

$$I_X(x) = x \quad \text{for each } x \in X.$$

The function I_X is called the **identity function on** X because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

*Sequences

The formal definition of sequences specifies that an infinite sequence is a function defined on the set of integers that are greater than or equal to a particular integer. For example, the sequence denoted

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates $0 \to 1, 1 \to -\frac{1}{2}, 2 \to \frac{1}{3}, 3 \to -\frac{1}{4}, 4 \to \frac{1}{5}$, and, in general, $n \to \frac{(-1)^n}{n+1}$. In other words, $f : \mathbb{Z}_{\text{nonneg}} \to \mathbb{R}$ is the function defined as follows:

Send each integer $n \geq 0$ to $f(n) = \frac{(-1)^n}{n+1}$.

Boolean function

An (n-place) **Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$.

Thus $f : \{0, 1\}^n \to \{0, 1\}$.

Sets of Functions

- The notation B^A represents the set of all functions from a set A to a set B.
- Formally, if A and B are sets, then B^A is the set of all functions f such that $f : A \to B$.
- This notation emphasizes the idea that functions can be viewed as elements of a set, just like numbers or other mathematical objects.
- Example: If $B = \{0, 1\}$ and $A = \{x, y\}$, then B^A consists of all functions from A to B, such as:
  - $f_1 : \{x \mapsto 0, y \mapsto 0\}$
  - $f_2 : \{x \mapsto 0, y \mapsto 1\}$
  - $f_3 : \{x \mapsto 1, y \mapsto 0\}$
  - $f_4 : \{x \mapsto 1, y \mapsto 1\}$
- Properties:
  - B^A forms a set whose elements are functions.
  - The size of B^A (cardinality) is $|B|^{|A|}$ if both A and B are finite sets.

Well Defined

It can sometimes happen that what appears to be a function defined by a rule is not really a function at all. There are **two** distinct reasons why this description does not define a function. For almost all values of x,

1. either there is no y that satisfies the given equation,
2. or there are two different values of y that satisfy the equation.

In general, we say that a "function" is **not well defined** if it fails to satisfy *at least one* of the requirements for being a function.

Exercise 2 Are these functions below well defined?

- For each real number x, $f(x)$ is the real number y such that $x^2 + y^2 = 1$.
- $\forall x \in \mathbb{R}, f(x) = \frac{1}{x}$.

> Note that the phrase *well-defined function* is actually **redundant**;
> ——for a function to be well defined really means that it is worthy of being called a function.

Ways to Define (Classes of) Functions

1. Finite Sets
2. Algebraic Structure
3. Induction/Recursion
4. . . .

Co-domain: Injective and Surjective

Definition

A binary relation $R \subseteq S \times T$ is:

- **Injective (One-to-One)**: For all $t \in T$ there is at most one $s \in S$ such that $(s, t) \in R$.
- **Surjective (Onto)**: For all $t \in T$ there is at least one $s \in S$ such that $(s, t) \in R$.

> **Bijective**: Both *injective* and *surjective*.

Translate into logic language:

- $F : X \to Y$ is injective $\iff$ $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$.
- $F : X \to Y$ is surjective $\iff$ $\forall y \in Y, \exists x \in X$ such that $F(x) = y$.

(What is the negation?)

Exercise 3 Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}$ by the rules

$$f(x) = 4x - 1 \text{ (for all } x \in \mathbb{R}) \quad \text{and} \quad g(n) = n^2 \text{ (for all } n \in \mathbb{Z}).$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

Exercise 4 Define $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{Z} \to \mathbb{Z}$ by the rules

$$f(x) = 4x - 1 \text{ (for each } x \in \mathbb{R}) \quad \text{and} \quad h(n) = 4n - 1 \text{ (for each } n \in \mathbb{Z}).$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

> **Thinking Exercise** *For any sets A, B, C, there is a bijection between $C^{A \times B}$ and $(C^B)^A$.

Property of Functions

Property of Functions	Definition
*Partial function	a binary relation that is (Fun)
Function	a binary relation that is (Fun) and (Tot)
Injection	a function that is (Inj)
Surjection	a function that is (Sur)
Bijection	a function that is (Bij)

Inverse Function

Suppose $F : X \to Y$ is a **bijection**. Then there is a function $F^{-1} : Y \to X$ that is defined as follows: Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

Or, equivalently,

$$F^{-1}(y) = x \iff y = F(x).$$

> If X and Y are sets and $F : X \to Y$ is bijective, then $F^{-1} : Y \to X$ is also bijective.

Operation: Composition of Functions

Let $f : X \to Y$ and $g : Y' \to Z$ be functions with the property that the range of f is a subset of the domain of g.

Define a new function $g \circ f : X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for each } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x."

The function $g \circ f$ is called the **composition of f and g**.

- **Composition with an Identity Function**
  If f is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then
  $f \circ I_X = f \quad \text{and} \quad I_Y \circ f = f.$
- **Composition of a Function with Its Inverse**
  If $f : X \to Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \to X$, then
  $f^{-1} \circ f = I_X \quad \text{and} \quad f \circ f^{-1} = I_Y.$
- **Composition of Inj/Sur/Bij Functions**
  If $f : X \to Y$ and $g : Y \to Z$ are both inj/sur/bij functions, then $g \circ f$ is inj/sur/bij.

Exercise 5 Define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by the rules

$$F(n) = 3x \quad \text{and} \quad G(n) = \lfloor x/3 \rfloor \quad \text{for every real number } x.$$

a. Find $(G \circ F)(6)$, $(F \circ G)(6)$, $(G \circ F)(1)$, and $(F \circ G)(1)$.

b. Is $G \circ F = F \circ G$? Explain.

Properties of Composition

- **Associativity:** For any functions $f : C \to D$, $g : B \to C$, and $h : A \to B$,

$$f \circ (g \circ h) = (f \circ g) \circ h$$

- **Identity:** For any set A, let $\mathrm{id}_A : A \to A$ be the identity function defined by $\mathrm{id}_A(x) = x$ for all $x \in A$. For any function $f : A \to B$,

$$f \circ \mathrm{id}_A = f \quad \text{and} \quad \mathrm{id}_B \circ f = f$$

- **Commutativity:** In general, function composition is not commutative, meaning $f \circ g \neq g \circ f$ in most cases.

Operation: Function Iteration

(*Wikipedia*) In mathematics, an iterated function is a function that is obtained by composing another function with itself two or several times. The process of repeatedly applying the same function is called iteration. In this process, starting from some initial object, the result of applying a given function is fed again into the function as input, and this process is repeated.

Repeatedly applying a function to its own output:

- $f^1(x) = f(x)$
- $f^2(x) = f(f(x))$
- $f^n(x) = f(f^{n-1}(x))$

***Exercise 6** Define the function $h(x) = \frac{x+1}{x-1}$. Find $h(h(x))$.