Jiapeng Wang, COMP9020 24T2 Tutorial W7

Answer to Problem Set 3

1. In Week 3, we introduced **numerical identity**, denoted with '='. We assumed the **principle of substitution** for numerical identity. What is the principle of substitution and why is it legitimate?
2. In your own words, explain the relationship between **rational numbers** and **real numbers**. Use examples in your explanation.
3. Provide a proof that -22 is an **even number**.
4. Let the domain be the domain of integers $\mathbb{Z}$, and let Ox be **x is an odd number**. Now consider the following statement:
$$\forall x \forall y ((Ox \wedge Oy) \rightarrow O(x+y)) \tag{1}$$
Is this statement true or false? If true, then provide a proof. If false, then provide a dis-proof (a proof that there exists a counterexample).
5. Let the domain be the set of integers $\mathbb{Z}$. Do there exist n and d such that:
$$n \; div \; d = q, \quad n \; mod \; d = r, \quad \text{and } q = r \tag{2}$$
If not, then why not? If so, then find values for n and d such that $q = r$.

1. The POS is just the principle which states that we may switch one thing for another if they are **identical**. If they are identical then *they are one and the same thing*. This is why the POS is legitimate.
2. A rational number is a real number that can be represented as the quotient $\frac{n}{m}$ of two integers n, m such that $m \neq 0$. All rational numbers are real numbers, not all real numbers are rational numbers. The non-rational real numbers are irrational numbers.

Set Theory

*Recap W1: The Language of Sets

definition

(*Wikipedia*) In mathematics, a set is a collection of different things; these things are called **elements** or members of the set and are typically mathematical objects of any kind: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets.

- If S is a set, the notation $x \in S$ means that x is an element of S.
- The notation $x \notin S$ means that x is not an element of S.

notations

- Set-Roster Notation: $\{1, 2, 3\}$, $\{1, 2, 3, \ldots, 100\}$, $\{1, 2, 3, \ldots\}$.
- Set-Builder Notation: $\{x \in S \mid P(x)\}$.

the axiom of extension

A set is completely determined by *what its elements are* — not the order in which they might be listed or the fact that some elements might be listed more than once.

> Sets A and B are equal if, and only if, they have exactly the same elements.

relation - subset

$A \subseteq B$: $\forall x \; (x \in A \rightarrow x \in B)$. (*What is the negation?*)

> * Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

operation - cartesian product

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Element Proof

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Exercise 1 Define sets A and B as follows:

- $A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$
- $B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$.

1. Outline a proof that $A \subseteq B$.
2. Proof that $A \subseteq B$.
3. Disprove that $B \subseteq A$.

Set Equality $A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$.

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Exercise 2 Define sets A and B as follows:

- $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a\}$
- $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some integer } b\}$.

Is $A = B$?

Operations on Sets

(*Textbook*) Most mathematical discussions are carried on within some context. For example, in a certain situation *all sets being considered* might be sets of real numbers. In such a situation, the set of real numbers would be called a **universal set** or a universe of discourse for the discussion.

Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A.

Symbolically:

$$\begin{aligned} A \cup B &= \{x \in U \mid x \in A \text{ or } x \in B\} \\ A \cap B &= \{x \in U \mid x \in A \text{ and } x \in B\} \\ B - A &= \{x \in U \mid x \in B \text{ and } x \notin A\} \\ A^c &= \{x \in U \mid x \notin A\}. \end{aligned}$$

***Interval Notation**

There is a convenient notation for subsets of real numbers that are intervals.

Interval Notation

Given real numbers a and b with $a \leq b$:

$$\begin{aligned} (a, b) &= \{x \in \mathbf{R} \mid a < x < b\} \qquad & [a, b] &= \{x \in \mathbf{R} \mid a \leq x \leq b\} \\ (a, b] &= \{x \in \mathbf{R} \mid a < x \leq b\} \qquad & [a, b) &= \{x \in \mathbf{R} \mid a \leq x < b\}. \end{aligned}$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$\begin{aligned} (a, \infty) &= \{x \in \mathbf{R} \mid x > a\} \qquad & [a, \infty) &= \{x \in \mathbf{R} \mid x \geq a\} \\ (-\infty, b) &= \{x \in \mathbf{R} \mid x < b\} \qquad & (-\infty, b] &= \{x \in \mathbf{R} \mid x \leq b\}. \end{aligned}$$

Exercise 3 Let the universal set be $\mathbb{R}$, the set of all real numbers, and let
$A = (-1, 0] \text{ and } B = [0, 1)$
Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

relation - disjoint

A and B are disjoint ⇔ $A \cap B = \emptyset$.

*Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or pairwise disjoint or nonoverlapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \ldots\ A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$

> A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,
>
> 1. A is the union of all the A_i;
> 2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

***Exercise 4** Let $\mathbb{Z}$ be the set of all integers and let

- $T_0 = \{n \in \mathbb{Z} \mid n = 3k, \text{ for some integer } k\}$,
- $T_1 = \{n \in \mathbb{Z} \mid n = 3k + 1, \text{ for some integer } k\}$, and
- $T_2 = \{n \in \mathbb{Z} \mid n = 3k + 2, \text{ for some integer } k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Properties of Sets

Translation between set language and logic language

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

Some Relation

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Exercise 5 (*Inclusion of Intersection*) For all sets A and B, $A \cap B \subseteq A$.

Set Identities

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a)}\ A \cup B = B \cup A \quad \text{and} \quad \text{(b)}\ A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a)}\ (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b)}\ (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets A, B, and C,

$$\text{(a)}\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b)}\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For every set A,

$$\text{(a)}\ A \cup \varnothing = A \quad \text{and} \quad \text{(b)}\ A \cap U = A.$$

5. *Complement Laws:* For every set A,

$$\text{(a)}\ A \cup A^c = U \quad \text{and} \quad \text{(b)}\ A \cap A^c = \varnothing.$$

6. *Double Complement Law:* For every set A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For every set A,

$$\text{(a)}\ A \cup A = A \quad \text{and} \quad \text{(b)}\ A \cap A = A.$$

8. *Universal Bound Laws:* For every set A,

$$\text{(a)}\ A \cup U = U \quad \text{and} \quad \text{(b)}\ A \cap \varnothing = \varnothing.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a)}\ (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b)}\ (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a)}\ A \cup (A \cap B) = A \quad \text{and} \quad \text{(b)}\ A \cap (A \cup B) = A.$$

11. *Complements of U and $\varnothing$:*

$$\text{(a)}\ U^c = \varnothing \quad \text{and} \quad \text{(b)}\ \varnothing^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

***Exercise 6** (*Distributive law*) For all sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Figure:

Suppose A, B, and C are sets.

> Show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
> Suppose $x \in A \cup (B \cap C)$.
> $\vdots$
> Thus $x \in (A \cup B) \cap (A \cup C)$.
> Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

> Show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
>
> Suppose $x \in (A \cup B) \cap (A \cup C)$.
>
> $\vdots$
>
> Thus $x \in A \cup (B \cap C)$.
>
> Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Details: Suppose $x \in A \cup (B \cap C)$

Case 1: Since $x \in A$, then both statements $x \in A \cup B$ and $x \in A \cup C$ are true by definition of $\cup$. Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of $\cap$.

Case 2: Since $x \in B \cap C$, then $x \in B$ and $x \in C$ by definition of $\cap$.

- Case 2.1: Since $x \in B$, then $x \in A \cup B$ by definition of $\cup$.
- Case 2.2: Since $x \in C$, similarly, then $x \in A \cup C$ by definition of $\cup$.

Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of $\cup$.

The Power Set

Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Exercise 7

1. Find the power set of the set $\{x, y\}$. That is, find $\mathcal{P}(\{x, y\})$.
2. Suppose there are k elements in set A. How many elements in $\mathcal{P}(A)$?

The Empty Set

If E is a set with no elements and A is any set, then $E \subseteq A$. (How to prove?)

Exercise 8.1 (*Uniqueness of the Empty Set*) There is only one set with no elements.

> **Element Method for Proving a Set Equals the Empty Set**
>
> To prove that a set X is equal to the empty set $\varnothing$, prove that X has no elements. To do this, suppose X has an element and derive a contradiction.

***Exercise 8.2** (*Universal Bound Law*) For any set A, $A \cap \emptyset = \emptyset$.

Boolean Logic

*Boolean Algebra

Definition and Axioms for a Boolean Algebra

Definition and Axioms for a Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a+b$ and $a \cdot b$ are in B and the following axioms are assumed to hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a+b=b+a \quad \text{and} \quad \text{(b) } a \cdot b=b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a+b)+c=a+(b+c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c=a \cdot(b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a+(b \cdot c)=(a+b) \cdot(a+c) \quad \text{and} \quad \text{(b) } a \cdot(b+c)=(a \cdot b)+(a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for each a in B,

$$\text{(a) } a+0=a \quad \text{and} \quad \text{(b) } a \cdot 1=a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\bar{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a+\bar{a}=1 \quad \text{and} \quad \text{(b) } a \cdot \bar{a}=0.$$

We can prove that a Boolean Algebra holds the following **properties**.

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Laws:* For all a and x in B, if $a+x=1$ and $a \cdot x=0$ then $x=\bar{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a+x=a$ for every a in B, then $x=0$, and if there exists y in B such that $a \cdot y=a$ for every a in B, then $y=1$.
3. *Double Complement Law:* For every $a \in B$, $\overline{(\bar{a})}=a$.
4. *Idempotent Laws:* For every $a \in B$,

$$\text{(a) } a+a=a \quad \text{and} \quad \text{(b) } a \cdot a=a.$$

5. *Universal Bound Laws:* For every $a \in B$,

$$\text{(a) } a+1=1 \quad \text{and} \quad \text{(b) } a \cdot 0=0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\text{(a) } \overline{a+b}=\bar{a} \cdot \bar{b} \quad \text{and} \quad \text{(b) } \overline{a \cdot b}=\bar{a}+\bar{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$\text{(a) } (a+b) \cdot a=a \quad \text{and} \quad \text{(b) } (a \cdot b)+a=a.$$

8. *Complements of* 0 *and* 1:

$$\text{(a) } \bar{0}=1 \quad \text{and} \quad \text{(b) } \bar{1}=0.$$

*__Exercise 9__ (*Uniqueness of the Complement Law*) For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \bar{a}$.

Boolean Function

(*Wikipedia*) In mathematics, a **Boolean function** is a function whose arguments and result assume values from a two-element set (usually {true, false}, {0,1} or {-1,1}).

> Alternative names are *switching function*, used especially in older computer science literature, and *truth function* (or logical function), used in logic. Boolean functions are the subject of Boolean algebra and switching theory.

A Boolean function takes the form $f : \{0,1\}^k \to \{0,1\}$ (or f^k), where $\{0,1\}$ is known as the **Boolean domain** $\mathbb{B}$ and k is a non-negative integer called the **arity** of the function.

Exercise 10

1. What is the Boolean function of $\sim, \vee, \wedge, \to$?
2. Let $f^3 = F(W, X, Y) = W + XY$. The output here would be 1 when W = 1 or XY = 1 or when both of these are 1.

   Draw the truth table of f^3.
3. *How many different Boolean functions with k arguments?