W8 Complement

COMP 9020, 24T2, Jiapeng Wang

Please note that the form of my tutorial is based on my own experience, and thus differs from Simon's lectures and Epp's textbook. I hope this arrangement will better help you understand the concepts and methods related to functions.

Regarding the knowledge discussed yesterday, there are a few points that need emphasis:

1. Total, functional, injective (or one-to-one), surjective (or onto), and bijective are terms describing *relations* that satisfy certain specific conditions. In contrast, injection (or injective function, one-to-one function), surjection (or surjective function, onto function), and bijection are terms describing *functions* that meet specific criteria. Please note the distinction in terminology.
2. For certain special functions, there are additional properties. For example, linear functions, exponential functions, and logarithmic functions have unique properties. These are detailed on page 449 of the textbook, but are not covered in my notes.

Due to limited tutorial time, please allow me to supplement with the following text:

Inverse

For a relation $R : S \to T$, we have the corresponding inverse relation $R^{\leftarrow} : T \to S$, satisfying: the domain and co-domain of $R^{\leftarrow}$ are the co-domain and domain of R respectively, and if $(s, t) \in R$, then $(t, s) \in R^{\leftarrow}$. Note the definition of the inverse relation and the method for proving the equality of relations, so it is not difficult to prove (*please think it by yourself follow the hints*) that

- *the inverse relation of $R^{\leftarrow}$ is equal to R.*

Additionally, consider the definitions of total, functional, injective, and surjective (*think about what their definitions are*). It is not hard to deduce that **total relations and surjective relations are inverses**, and **functional relations and injective relations are inverses**.

Now, consider the inverse relation of a function f. Since $f : X \to Y$ is total and functional, its inverse relation $f^{\leftarrow} : Y \to X$ must be surjective and injective. However, note that $f^{\leftarrow}$ is not necessarily a function!
When the inverse relation of a function is also a function, we denote $f^{\leftarrow}$ as f^{-1}. Since f^{-1} is a function, it is also total and functional. Similarly, we can deduce that f is also surjective and injective; therefore,

- **if a function's inverse is also a function, then both functions must be bijective!**

In other words,

- **if a function has an inverse that is also a function, then this function must be bijective!**

In the explanation above, please pay special attention to the distinction between $f^{\leftarrow}$ and f^{-1}.

Composition

Due to time constraints, this topic was not discussed in depth, so here is a brief supplement.

First, the prerequisite for the composition of functions $g \circ f$ is that the range of f is a subset of the domain of g. If this condition is not met, the operation of function composition cannot be performed.

Second, please refer to my notes on the conclusions related to identity functions, inverse functions, and the composition of Inj/Sur/Bij functions. These can be directly proven using definitions. For example, to prove the transitivity of injective:

- If $f : X \to Y$ and $g : Y \to Z$ are both injective, then $g \circ f$ is injective.
  - **Proof.** Suppose $f : X \to Y$ and $g : Y \to Z$ are both injective.
  - *[We must show that $g \circ f$ is injective.]*
  - Suppose x_1 and x_2 are elements of X such that
  - $(g \circ f)(x_1) = (g \circ f)(x_2)$.
  - *[We must show that $x_1 = x_2$.]*
  - By definition of composition of functions,
  - $g(f(x_1)) = g(f(x_2))$.
  - Since g is injective,
  - $f(x_1) = f(x_2)$.
  - And since f is injective,
  - $x_1 = x_2$.
  - *[as was to be shown].*
  - Hence $g \circ f$ is injective.

Finally, note the properties of composite functions attached in my notes, especially that composite functions *do **NOT*** necessarily have the commutative property!

- Define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by the rules $F(x) = 3x$ and $G(x) = \left\lfloor \frac{x}{3} \right\rfloor$ for every real number x.
- (a) Find $(G \circ F)(6)$, $(F \circ G)(6)$, $(G \circ F)(1)$, and $(F \circ G)(1)$.
  - **Proof.**
  - $(G \circ F)(6) = G(F(6)) = G(18) = \left\lfloor \frac{18}{3} \right\rfloor = 6$
  - $(F \circ G)(6) = F(G(6)) = F\left(\left\lfloor \frac{6}{3} \right\rfloor\right) = F(2) = 3 \cdot 2 = 6$
  - $(G \circ F)(1) = G(F(1)) = G(3) = \left\lfloor \frac{3}{3} \right\rfloor = 1$
  - $(F \circ G)(1) = F(G(1)) = F\left(\left\lfloor \frac{1}{3} \right\rfloor\right) = F(0) = 3 \cdot 0 = 0$
- (b) Is $G \circ F = F \circ G$? Explain.
  - **Proof.** No, because $(G \circ F)(1) \neq (F \circ G)(1)$.

Iteration

Regarding function iteration, it is essentially the composition of the same function, so I will not elaborate further. But please note the difference between these two notations: $f^2(x)$ and $(f(x))^2$.